VERY SINGULAR SOLUTIONS FOR LINEAR DIRICHLET PROBLEMS WITH SINGULAR CONVECTION TERMS

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ABSTRACT. We study the existence of distributional solutions for the boundary value problems (1.1) and (1.2) if E does not belong to L^N, namely $|E| \leq \frac{|A|}{|x|}$, $A \in \mathbb{R}$. The size of A plays an important role: if $\alpha(N-2) \leq |A| < \alpha(N-1)$, we prove that if $f \in L^1(\Omega)$ there exists a distributional solution $u \in W_0^{1,q}(\Omega)$, for every $q < \frac{N\alpha}{|A|+\alpha} < \frac{N}{N-1}$, of (1.1) (the case $|A| < \alpha(N-2)$ is studied in [3]). We then use this result to prove the existence of a bounded weak solution ψ of (1.2) if $g(x) \in L^m(\Omega)$, $m > \frac{N\alpha}{N\alpha - |A|} \geq \frac{N}{2}$.

1. Introduction

In this paper we prove the existence of distributional solutions for the boundary value problems (the first with a convection term, the second with a drift term)

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) = -\operatorname{div}(u\,E(x)) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

and

$$\begin{cases} -\operatorname{div}(M(x)\nabla \psi) = E(x)\,\nabla\psi + g(x) & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where we assume that

$$f,\, g \in L^m(\Omega),\ m \geq 1,$$

and, on the singular convection or drift term, that

$$|E| \leq \frac{|A|}{|x|}, \quad A \in \mathbb{R}. \tag{1.3}$$

Here Ω is a bounded, open subset of $\mathbb{R}^N$, with $0 \in \Omega$, $N > 2$, and $M : \Omega \to \mathbb{R}^{N^2}$ is a measurable matrix such that (for α, $\beta \in \mathbb{R}^+$)

$$\alpha|\xi|^2 \leq M(x)\xi\,\xi, \qquad |M(x)| \leq \beta, \quad \text{a.e. in } \Omega,\ \forall \xi \in \mathbb{R}^N. \tag{1.4}$$

We note that, at least formally, the two above linear problems are in duality. Note also that the above boundary problems are linear, but the differential operators may be not coercive, unless one assumes that either the norm of $|E|$ in $L^N(\Omega)$ is small, or that $\operatorname{div}(E) = 0$.

In [2] and [9], the existence of u in $W_0^{1,2}(\Omega)$ and ψ in $W_0^{1,2}(\Omega)$ is proved if E belongs to $(L^N(\Omega))^N$ and f and g are $L^{\frac{2N}{N+2}}(\Omega)$, without restrictions on the norm of E.

For the problem (1.1) in [2] it is proved (despite the coercivity difficulty) the basic *a priori* estimate

$$\int_\Omega |\nabla \log(1+|u|)|^2 \leq \frac{1}{\alpha^2}\int_\Omega |E|^2 + \frac{2}{\alpha}\int_\Omega |f|; \tag{1.5}$$

on the other hand a basic *a priori* estimate for ψ is not known (see also Remark 3.4). See anyway [9] and the more recent [11], [13] for an alternative approach based on the so called slice technique.

Nevertheless, in [2] (for u) and in [4] (for ψ), it is proved, under the assumption $E \in (L^N(\Omega))^N$:

1) (*Stampacchia theory*) existence of weak solutions u, ψ belonging to $W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$, $m^{**} = \frac{Nm}{N-2m}$, if $\frac{2N}{N+2} \leq m < \frac{N}{2}$;
2) (*Caldéron-Zygmund theory*) existence of distributional solutions u, ψ belonging to $W_0^{1,m^*}(\Omega)$, $m^* = \frac{Nm}{N-m}$, if $1 < m < \frac{2N}{N+2}$;

that is the results proved by G. Stampacchia (see [16]), and in [8] ([7], if $m = 1$), under the assumption $E = 0$.

We quote also [1], [14] where the Marcinkiewicz and Lorentz regularity of u, ψ is studied by means of symmetrization arguments; for similar results on ∇u and $\nabla\psi$ see [11], [12].

Although problem (1.1) is (at least formally) the dual problem of (1.2), it is not possible to completely study problem (1.2) by a duality method, since estimate (1.5) on the solution u of (1.1) is nonlinear. A duality approach (by contradiction) is used in [6].

If E does not belong to $(L^N(\Omega))^N$, the framework changes completely: differential problems with E of form $\frac{x}{|x|^2}$ (which does not belong to L^N) are studied in [3] (where the case E in $L^2(\Omega)$ is also studied, thanks also to a very weak definition of solution) and [15].

In particular, if E satisfies (1.3), which is an assumption slightly weaker than $E \in (L^N(\Omega))^N$, the size of A plays an important role: under the assumption $f \in L^m(\Omega)$, in [3] the following existence results are proved (recall the definition of $p^* = \frac{Np}{N-p}$):

3) if $|A| < \frac{\alpha(N-2m)}{m}$, and $\frac{2N}{N+2} \leq m < \frac{N}{2}$, then $u \in W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$;
4) if $|A| < \frac{\alpha(N-2m)}{m}$, and $1 < m < \frac{2N}{N+2}$, then $u \in W_0^{1,m^*}(\Omega)$;
5) if $|A| < \alpha(N-2)$, and $m = 1$, then $\nabla u \in (M^{\frac{N}{N-1}}(\Omega))^N$ and $u \in W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$.

Similar results on ψ are proved in [15], where the main assumption is $|A| < \alpha\frac{N(m-1)}{m}$.

We point out that, in the last three statements, a better summability assumption on f does not improve the properties of u: only the size of A is important. Note that in 3),

$$\lim_{m\to\frac{N}{2}} |A| \leq \lim_{m\to\frac{N}{2}} \frac{\alpha(N-2m)}{m} = 0.$$

We observe that the strict inequality in the assumptions on A in 3) and 4) is optimal, as shown in the following example.

Example 1.1. Consider in 3) $|A| = \frac{\alpha(N-2)}{2}$, and note that if $m = \frac{2N}{N+2}$, then $\frac{N-2m}{m} = \frac{N-2}{2}$. If u is a weak solution of $-\text{div}(M(x)\nabla u) = \frac{\alpha(N-2)}{2}\,\text{div}(u\frac{x}{|x|^2}) + f(x)$ then

$$\alpha \int_\Omega |\nabla u|^2 \leq \frac{\alpha(N-2)}{2} \int_\Omega \frac{|u|}{|x|}|\nabla u| + \int_\Omega |f||u|,$$

so that the use of Young and Hardy inequalities (see (2.2) below) gives, with $B > 0$,

$$(\alpha - B)\int_\Omega |\nabla u|^2 \leq \frac{\alpha^2(N-2)^2}{16B}\int_\Omega \frac{|u|^2}{|x|^2} + \int_\Omega |f||u| \leq \frac{\alpha^2}{4B}\int_\Omega |\nabla u|^2 + \int_\Omega |f||u|.$$

Hence

$$\frac{4B\,\alpha - 4B^2 - \alpha^2}{4B}\int_\Omega |\nabla u|^2 \leq \int_\Omega |f||u|.$$

However, $4B\,\alpha - 4B^2 - \alpha^2 = -(\alpha + 2B)^2 \le 0$ for every B, so that it is not possible to prove standard *a priori* estimates.

In this paper, we will study the case $|A| \ge \alpha(N-2)$ and will prove, in Theorem 2.2 the existence of even more singular distributional solutions of (1.1) (note once more the importance of the size of A) than the solutions presented in 5) above. More precisely we prove the existence of a solution $u \in W_0^{1,q}(\Omega)$, for every $q < \frac{N\alpha}{|A|+\alpha} < \frac{N}{N-1}$, using a sharp *a priori* estimate, which is not known in the case $E \in (L^N(\Omega))^N$.

The existence result of Theorem 2.2 is then used to prove in Section 3 (by a duality method and thanks to the sharp *a priori* estimate) the existence of bounded weak solutions of (1.2) if $\alpha(N-2) \le |A| < \alpha(N-1)$ and $g(x) \in L^m(\Omega)$, with $m > \frac{N\alpha}{N\alpha - |A|}$. Note that, in this case, one always has that $m > \frac{N}{2}$. Other contributions devoted to the existence of unbounded solutions can be found in [1], [15], [11], [12].

In all the paper, we assume

$$\alpha(N-2) \le |A| < \alpha(N-1). \tag{1.6}$$

2. The problem (1.1): convection terms

Our starting point is the following Dirichlet problem, similar to the starting point of [3] (see also [2]).

$$u_n \in W_0^{1,2}(\Omega): \ -\mathrm{div}(M(x)\nabla u_n) = -A\,\mathrm{div}\left(u_n \frac{x}{\frac{1}{n} + |x|^2}\right) + f_n(x), \tag{2.1}$$

where

$$f_n(x) = \frac{f(x)}{1 + \frac{1}{n}|f(x)|}.$$

Since both $\frac{x}{\frac{1}{n}+|x|^2}$ and $f_n(x)$ are bounded functions, the existence of a weak solution u_n of (2.1) is proved in [2]; furthermore, every solution u_n is a bounded function, so that it will be possible to use as test function nonlinear compositions of u_n.

We will use the following Hardy inequality (see e.g. [10], [17])

$$\mathcal{H}\left(\int_\Omega \frac{|v|^2}{|x|^2}\right)^{\frac{1}{2}} \le \left(\int_\Omega |\nabla v|^2\right)^{\frac{1}{2}}, \quad \forall\, v \in W_0^{1,2}(\Omega), \tag{2.2}$$

where $\mathcal{H} = \frac{N-2}{2}$; we will also make use of the well known truncation

$$T_k(s) = \begin{cases} k\dfrac{s}{|s|} & \text{if } |s| \ge k, \\ s & \text{if } -k < s < k. \end{cases}$$

Our first result is an *a priori* estimate on the sequence $\{u_n\}$.

Lemma 2.1. *Assume* (1.3), (1.4), (1.6), *and*

$$f(x) \in L^1(\Omega). \tag{2.3}$$

Then the sequence $\{u_n\}$ *of solutions of* (2.1) *is bounded in* $W_0^{1,q}(\Omega)$, *for every* $q < \frac{N\alpha}{|A|+\alpha}$, *with*

$$\left(\int_\Omega |\nabla u_n|^q\right)^{\frac{1}{q}} \le C_E \,\|f\|_{L^1(\Omega)}. \tag{2.4}$$

The constant C_E *depends on* E *(see* (1.3)*)*, α, Ω.

Proof. Let

$$\theta > 1 - \alpha\frac{\mathcal{H}}{|A|}\,, \tag{2.5}$$

so that $\theta > \frac{1}{2}$ by the assumptions on A (see (1.6)). With this assumption, we have $1-\theta < \alpha\frac{\mathcal{H}}{|A|}$, that is to say $\frac{|A|(1-\theta)}{\mathcal{H}} < \alpha$ and so

$$\frac{|A|^2\,(1-\theta)^2}{\mathcal{H}^2} < \alpha^2\,.$$

Therefore, there exists $R > 1$ such that

$$R\,\frac{|A|^2\,(1-\theta)^2}{\mathcal{H}^2} < \alpha^2. \tag{2.6}$$

In the weak formulation of (2.1), we use as test function

$$D(u_n) = \frac{(1-|u_n|^{1-2\theta})^+}{2\theta-1}\frac{u_n}{|u_n|}.$$

Note that since $\theta > \frac{1}{2}$ by (2.5), we have that $1-2\theta < 0$, that $D(u_n) = 0$ where $|u_n| \leq 1$, and that $D'(u_n) = |u_n|^{-2\theta}$ where $|u_n| > 1$. We then have, thanks to Young inequality (with $0 < B < \alpha$)

$$\begin{aligned}
&\int_{\{|u_n|>1\}}\frac{|\nabla u_n|^2}{|u_n|^{2\theta}} = \alpha\int_\Omega |\nabla u_n|^2 D'(u_n)\\
&\leq |A|\int_\Omega \frac{|u_n|}{|x|}|\nabla u_n| D'(u_n) + \frac{1}{2\theta-1}\int_\Omega |f|\\
&= |A|\int_\Omega \sqrt{D'(u_n)}\frac{|u_n|}{|x|}|\nabla u_n|\sqrt{D'(u_n)} + \frac{1}{2\theta-1}\int_\Omega |f|\\
&\leq B\int_{\{|u_n|>1\}}\frac{|\nabla u_n|^2}{|u_n|^{2\theta}} + \frac{|A|^2}{4B}\int_{\{|u_n|>1\}}\Big(\frac{|u_n|^{1-\theta}}{|x|}\Big)^2 + \frac{1}{2\theta-1}\int_\Omega |f|.
\end{aligned} \tag{2.7}$$

Now we use the inequality $(a+1)^2 \leq R\,a^2 + \frac{R}{R-1}$, which holds for every $a > 0$, and with R as in (2.6). Therefore,

$$\Big(\frac{|u_n|^{1-\theta}}{|x|}\Big)^2 \leq R\Big(\frac{|u_n|^{1-\theta}-1}{|x|}\Big)^2 + \frac{R}{R-1}\frac{1}{|x|^2},$$

so that (2.7) becomes

$$\begin{aligned}
&(\alpha - B)\int_{\{|u_n|>1\}}\frac{|\nabla u_n|^2}{|u_n|^{2\theta}}\\
&\leq R\frac{|A|^2}{4B}\int_\Omega\Big(\frac{|u_n|^{1-\theta}-1}{|x|}\Big)^2 + \frac{R}{R-1}\int_\Omega\frac{1}{|x|^2} + \frac{1}{2\theta-1}\int_\Omega|f|\\
&\leq R\frac{|A|^2}{4B}\frac{(1-\theta)^2}{\mathcal{H}^2}\int_{\{|u_n|>1\}}\frac{|\nabla u_n|^2}{|u_n|^{2\theta}} + \frac{R}{R-1}\int_\Omega\frac{1}{|x|^2} + \frac{1}{2\theta-1}\int_\Omega|f|.
\end{aligned} \tag{2.8}$$

Thus, we have that

$$\Big(\alpha - B - R\frac{|A|^2}{4B}\frac{(1-\theta)^2}{\mathcal{H}^2}\Big)\int_{\{|u_n|>1\}}\frac{|\nabla u_n|^2}{|u_n|^{2\theta}} \leq \frac{R}{R-1}\int_\Omega\frac{1}{|x|^2} + \frac{1}{2\theta-1}\int_\Omega|f|\,.$$

In order to have largest possible bound on $|A|$ we are interested in maximizing the function $\psi(B) = B(\alpha - B)$, and so we choose $B = \frac{\alpha}{2}$. Therefore,

$$\Big(\frac{\alpha}{2} - R\frac{|A|^2}{2\alpha}\frac{(1-\theta)^2}{\mathcal{H}^2}\Big)\int_{\{|u_n|>1\}}\frac{|\nabla u_n|^2}{|u_n|^{2\theta}} \leq \frac{R}{R-1}\int_\Omega\frac{1}{|x|^2} + \frac{1}{2\theta-1}\int_\Omega|f|. \tag{2.9}$$

Thanks to the choice of R in (2.6), we have

$$\frac{\alpha}{2} - R\frac{|A|^2}{2\alpha}\frac{(1-\theta)^2}{\mathcal{H}^2} = \frac{1}{2\alpha}\Big(\alpha^2 - R\frac{|A|^2(1-\theta)^2}{\mathcal{H}^2}\Big) > 0,$$

so that we have

$$\int_{\{|u_n|>1\}} \frac{|\nabla u_n|^2}{|u_n|^{2\theta}} \le C\bigg(\int_\Omega \frac{1}{|x|^2} + \frac{1}{2\theta-1}\int_\Omega |f|\bigg). \tag{2.10}$$

On the other hand the use of $T_1(u_n)$ as test function in (2.1) gives (see [2], [3])

$$\int_\Omega |\nabla T_1(u_n)|^2 \le \frac{1}{\alpha^2}\int_\Omega \frac{1}{|x|^2} + \frac{2}{\alpha}\int_\Omega |f|. \tag{2.11}$$

If we put together (2.10) and (2.11), we deduce that

$$\int_{\{|u_n|\le 1\}} |\nabla u_n|^2 + \int_{\{|u_n|>1\}} \frac{|\nabla u_n|^2}{|u_n|^{2\theta}} \le C\Big(\int_\Omega \frac{1}{|x|^2} + \int_\Omega |f|\Big)$$

Let now $1 < q < \frac{N\alpha}{|A|+\alpha}$. We claim that there exists θ satisfying (2.5) such that

$$q = \frac{2N(1-\theta)}{N-2\theta}.$$

Indeed, this is equivalent to

$$\theta = \frac{N(2-q)}{2(N-q)},$$

and we have $q < \frac{N\alpha}{|A|+\alpha}$ if and only if $\theta > 1 - \frac{\alpha\mathcal{H}}{|A|}$, which is (2.5). Note that with this choice of q (depending on θ), we have that

$$\frac{2\theta q}{2-q} = \frac{Nq}{N-q} = q^* \quad \text{and} \quad q^*\Big[\frac{1}{q} - \frac{1}{2}\Big] = \frac{N(2-q)}{2(N-q)} = \theta < 1. \tag{2.12}$$

Note also that the assumptions on A imply that

$$q < \frac{N}{N-1}. \tag{2.13}$$

Then ($\mathcal{S}$ denotes the Sobolev constant and $0 < \delta < 1$)
(2.14)

$$\begin{aligned}
\int_\Omega |\nabla u_n|^q &= \int_{\{|u_n|\le 1\}} |\nabla u_n|^q + \int_{\{|u_n|>1\}} |\nabla u_n|^q \\
&= \int_{\{|u_n|\le 1\}} |\nabla u_n|^q + \int_{\{|u_n|>1\}} \frac{|\nabla u_n|^q}{|u|^{q\theta}}\,|u_n|^{q\theta} \\
&\le \Big(\int_{\{|u_n|\le 1\}} |\nabla u_n|^2\Big)^{\frac{q}{2}} |\Omega|^{1-\frac{q}{2}} + \Big(\int_{\{|u_n|>1\}} \frac{|\nabla u_n|^2}{|u_n|^{2\theta}}\Big)^{\frac{q}{2}} \Big(\int_\Omega |u_n|^{q^*}\Big)^{\frac{1}{q^*}\,q^*[1-\frac{q}{2}]} \\
&\le C\Big(\int_\Omega \frac{1}{|x|^2} + \int_\Omega |f|\Big)^{\frac{q}{2}} + \frac{C}{\mathcal{S}}\Big(\int_\Omega \frac{1}{|x|^2} + \int_\Omega |f|\Big)^{\frac{q}{2}} \Big(\int_\Omega |\nabla u_n|^q\Big)^{q^*[\frac{1}{q}-\frac{1}{2}]} \\
&\le C\Big(\int_\Omega \frac{1}{|x|^2} + \int_\Omega |f|\Big)^{\frac{q}{2}} + \delta\int_\Omega |\nabla u_n|^q + \frac{C_1}{4\delta}\,\frac{1}{\mathcal{S}^{\frac{1}{1-\theta}}}\Big(\int_\Omega \frac{1}{|x|^2} + \int_\Omega |f|\Big)^{\frac{q}{2}\,\frac{1}{1-\theta}}.
\end{aligned}$$

Therefore, since $\frac{q}{2}\frac{1}{1-\theta} = \frac{N-q}{N-2}$,

$$
\begin{aligned}
&(1-\delta)\int_\Omega |\nabla u_n|^q \\
&\le C\left(\int_\Omega \frac{1}{|x|^2} + \int_\Omega |f|\right)^{\frac{q}{2}} + \frac{C}{4\delta}\frac{1}{\mathcal{S}^{\frac{1}{1-\theta}}}\left(\int_\Omega \frac{1}{|x|^2} + \int_\Omega |f|\right)^{\frac{q}{2}\frac{1}{1-\theta}} \\
&\le C_E + C\left(\int_\Omega |f|\right)^{\frac{q}{2}} + C\left(\int_\Omega |f|\right)^{\frac{N-q}{N-2}}.
\end{aligned}
\tag{2.15}
$$

Moreover, for $t \in \mathbb{R}^+$, we have

$$
t\,u_n \in W_0^{1,2}(\Omega):\ -\mathrm{div}(M(x)\nabla(t\,u_n)) = -A\,\mathrm{div}\Big((t\,u_n)\,\frac{x}{\frac{1}{n}+|x|^2}\Big) + t\,f_n(x),
$$

so that from (2.15) it follows that the sequence $\{t\,u_n\}$ satisfies the estimate

$$
(1-\delta)\int_\Omega |\nabla u_n|^q \le C_E\,t^{-q} + C\,t^{-q/2}\left(\int_\Omega |f|\right)^{\frac{q}{2}} + C\,t^{\frac{N-q}{N-2}-q}\left(\int_\Omega |f|\right)^{\frac{N-q}{N-2}}.
$$

Now we choose $t = \|f\|_{L^1(\Omega)}^{-1}$ and we deduce that

$$
(1-\delta)\int_\Omega |\nabla u_n|^q \le C_E\,\|f\|^q_{L^1(\Omega)} + C\,\|f\|^{q/2}_{L^1(\Omega)}\|f\|^{\frac{q}{2}}_{L^1(\Omega)} + C\,\|f\|^{q-\frac{N-q}{N-2}}_{L^1(\Omega)}\,\|f\|^{\frac{N-q}{N-2}}_{L^1(\Omega)},
$$

that is (2.4). □

Once we have an *a priori* estimate, we can prove an existence result for problem (1.1).

Theorem 2.2. *Assume* (1.3), (1.4), (1.6), (2.3). *Then there exists a distributional solution* u *of* (1.1), *with* $u \in W_0^{1,q}(\Omega)$, *for every* $q < \frac{N\alpha}{|A|+\alpha}$; *that is*

$$
\int_\Omega M(x)\nabla u\nabla\varphi = \int_\Omega u\,E(x)\,\nabla\varphi + \int_\Omega f(x)\,\varphi, \quad \forall\,\varphi \in C_0^1(\Omega).
\tag{2.16}
$$

Moreover, u *satisfies the a priori estimate*

$$
\left(\int_\Omega |\nabla u|^q\right)^{\frac{1}{q}} \le C_E\,\|f\|_{L^1(\Omega)}.
\tag{2.17}
$$

Proof. In the previous lemma we have proved the boundedness of the sequence $\{u_n\}$ in $W_0^{1,q}(\Omega)$, where $1 < q < \frac{N\alpha}{|A|+\alpha}$. The reflexivity of $W_0^{1,q}(\Omega)$ and Rellich theorem imply the existence of a subsequence $\{u_{n_j}\}$ and a function $u \in W_0^{1,q}(\Omega)$ such that u_{n_j} converges to u weakly in $W_0^{1,q}(\Omega)$ and strongly in $L^\rho(\Omega)$, for every $\rho < q^*$.

Thus we can pass to the limit in the weak formulation of (2.1)

$$
\int_\Omega M(x)\nabla u_n\nabla\varphi = A\int_\Omega u_n\,\frac{x}{\frac{1}{n}+|x|^2}\nabla\varphi + \int_\Omega f_n(x)\,\varphi(x), \quad \forall\,\varphi \in C_0^1(\Omega),
$$

since ∇u_{n_j} converges weakly in $(L^q(\Omega))^N$ to ∇u (here we use the linearity of the principal part), $u_n\,\frac{x}{\frac{1}{n}+|x|^2}$ strongly converges in $(L^1(\Omega))^N$ to $u\,\frac{x}{|x|^2}$ (note that $\frac{1}{q^*}+\frac{1}{N} < 1$), and f_n strongly converges in $L^1(\Omega)$ to f.

The weak lower semicontinuity of the norm in (2.4) implies the estimate (2.17). □

Remark 2.3. Note that if $|A| = \alpha(N-2)$, then we have estimates on $u \in W_0^{1,q}(\Omega)$, for every $q < \frac{N\alpha}{|A|+\alpha} = \frac{N}{N-1}$; thus, the results of the above theorem are consistent with those of the case $|A| < \alpha(N-2)$ quoted in 5) of the introduction.

Remark 2.4. Here we consider the boundary value problem (1.1) in the radial case with $\Omega = B(0,1)$, $E = A\frac{x}{|x|^2}$, and principal part the Laplace operator $-\Delta$. We rewrite the differential operator as

$$-\Delta u + \operatorname{div}(E(x)\,u) = -\operatorname{div}(\nabla u + E(x)\,u) = -\operatorname{div}\Big(\nabla u + A\frac{x}{|x|^2}u\Big),$$

and we look for radial solutions. We claim that $u = |x|^{-A} - |x|^2$ is a solution wit datum $f = -N(A+2)$. Indeed $\nabla u = -A\frac{x}{|x|^{A+2}} - 2x$, so that we have

$$-\operatorname{div}\Big(-A\frac{x}{|x|^{A+2}} - 2x + A\frac{x}{|x|^2}\Big[|x|^{-A} - |x|^2\Big]\Big) = -\operatorname{div}(-2x - Ax) = -N(A+2) = f(x).$$

Observe that if $A < N-1$ then u belongs to $W_0^{1,q}(\Omega)$, for every $q < \frac{N}{A+1}$, as stated in Theorem 2.2, and that, despite the datum f belongs to $L^\infty(\Omega)$, there is no "elliptic-type" improvement of the summability of u.

3. The problem (1.2): drift terms

In this section, on the right hand side $g(x)$ of (1.2) we assume

$$g(x) \in L^m(\Omega), \quad m > \frac{N\alpha}{N\alpha - |A|}, \tag{3.1}$$

while on E and A we still assume (1.3) and (1.6).

In order to prove the existence of bounded weak solutions of the boundary value problem (1.2), we will merge a duality method with an approximation approach, as in [5] (see also Remark 3.3).

We start with the (nonlinear) boundary value problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla\psi_n) = A\,\dfrac{x\cdot\nabla\psi_n}{\frac{1}{n}+|x|^2}\,\dfrac{1}{1+\frac{1}{n}|\nabla\psi_n|} + g(x) & \text{in } \Omega, \\ \psi_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

The Schauder fixed point theorem guarantees the existence of a weak solution $\psi_n \in W_0^{1,2}(\Omega)$, since the secon term of the equation is bounded. Moreover the result by Stampacchia (see [16]) yields the boundedness of every ψ_n, since the second term of the equation is bounded and $g \in L^m(\Omega)$, $m > \frac{N}{2}$, a a consequence of (3.1) and (1.6): indeed,

$$m > \frac{N\alpha}{N\alpha - |A|} > \frac{N}{2}.$$

Consider now the following boundary value problem (dual of (3.2)).

$$\begin{cases} -\operatorname{div}(M(x)\nabla u_n) = -A\operatorname{div}\Big(u_n\,\dfrac{x}{\frac{1}{n}+|x|^2}\,\dfrac{1}{1+\frac{1}{n}|\nabla\psi_n|}\Big) + g_n(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

with

$$g_n(x) = \frac{g(x)}{1+\frac{1}{n}|g(x)|}.$$

Since both $f_n(x)$ and

$$\frac{x}{\frac{1}{n}+|x|^2}\,\frac{1}{1+\frac{1}{n}|\nabla\psi_n|}$$

are bounded functions as in (2.1), the existence of a solution u_n of (3.3) follows as in the previous section. Furthermore, since

$$\frac{1}{1+\frac{1}{n}|\nabla\psi_n|} \le 1\,,$$

the same techniques used in order to prove Lemma 2.1 for solutions of (2.1) work also for problem (3.3), so that the sequence u_n satisfies (2.4).

LEMMA 3.1. *Assume* (1.3), (1.4), (1.6), (3.1). *Then the sequence* $\{\psi_n\}$ *is bounded in* $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, *with*

$$\|\psi_n\|_{L^\infty(\Omega)} \le C_E\,\|g\|_{L^m(\Omega)}. \tag{3.4}$$

Proof. The use of u_n as test function in the weak formulation of (3.2) and the use of ψ_n as test function in the weak formulation of (3.3) give

$$\Big|\int_\Omega f_n\,\psi_n\Big| = \Big|\int_\Omega g\,u_n\Big|.$$

Let now q be such that $m' = q^*$; thanks to the assumptions on m, we have that $q < \frac{N\alpha}{|A|+\alpha}$. Let $p > 1$, and define $f_n = |\psi_n|^{p-2}\psi_n$; then we have, using (2.4),

$$\int_\Omega |\psi_n|^p \le \int_\Omega |g|\,|u_n| \le \|g\|_{L^m(\Omega)}\,\|u_n\|_{L^{q^*}(\Omega)} \le C_E\,\|g\|_{L^m(\Omega)}\,\|f_n\|_{L^1(\Omega)},$$

that is, recalling the choice of f_n,

$$\int_\Omega |\psi_n|^p \le C_E\,\|g\|_{L^m(\Omega)}\int_\Omega |\psi_n|^{p-1} \le C_E\,\|g\|_{L^m(\Omega)}\Big(\int_\Omega |\psi_n|^p\Big)^{\frac{1}{p'}}.$$

Thus,

$$\Big(\int_\Omega |\psi_n|^p\Big)^{\frac{1}{p}} \le C_E\,\|g\|_{L^m(\Omega)}. \tag{3.5}$$

Letting p tend to infinity in (3.5), we obtain (3.4).

We now use of ψ_n as test function in the weak formulation of (3.2); then

$$\alpha\int_\Omega |\nabla\psi_n|^2 \le |A|\int_\Omega \frac{1}{|x|}|\nabla\psi_n||\psi_n| + \int_\Omega |g(x)||\psi_n|,$$

and

$$\frac{\alpha}{2}\int_\Omega |\nabla\psi_n|^2 \le \frac{|A|}{2\alpha}(C_E\,\|g\|_{L^m(\Omega)})^2\int_\Omega \frac{1}{|x|^2} + C_E\,\|g\|_{L^m(\Omega)}\int_\Omega |g(x)|, \tag{3.6}$$

which is the desired estimate in $W_0^{1,2}(\Omega)$. □

THEOREM 3.2. *Assume* (1.3), (1.4), (1.6) *and* (3.1). *Then there exists a bounded weak solution* ψ *of the problem* (1.2), *that is*

$$\int_\Omega M(x)\nabla\psi\nabla v = A\int_\Omega \frac{x\cdot\nabla\psi}{|x|^2}v(x) + \int_\Omega g(x)\,v(x), \quad \forall\, v\in W_0^{1,2}(\Omega).$$

Moreover ψ *satisfies the a priori estimate*

$$\|\psi\|_{L^\infty(\Omega)} \le C_E\,\|g\|_{L^m(\Omega)}. \tag{3.7}$$

Proof. In the previous lemma we proved the boundedness of the sequence $\{\psi_n\}$ in $W_0^{1,2}(\Omega)$. The reflexivity of $W_0^{1,2}(\Omega)$ and Rellich theorem imply the existence of a subsequence $\{\psi_{n_j}\}$ and a function $u \in W_0^{1,2}(\Omega)$ such that ψ_{n_j} converges to ψ weakly in $W_0^{1,2}(\Omega)$ and strongly in $L^\rho(\Omega)$, for every $\rho < 2^*$. Moreover, the right hand side of (3.2) is bounded in $L^\sigma(\Omega)$, with $\sigma < \frac{2N}{N+2}$, so that we can use a result of [7] to conclude that $\nabla\psi_{n_j}(x)$ converges almost everywhere in Ω to $\nabla\psi(x)$.

Thus we can pass to the limit in the weak formulation of (3.2):

$$\int_\Omega M(x)\nabla\psi_{n_j}\nabla v = A\int_\Omega \frac{x\cdot\nabla\psi_{n_j}}{\frac{1}{n_j}+|x|^2}\,\frac{1}{1+\frac{1}{n_j}|\nabla\psi_{n_j}|}v(x) + \int_\Omega g_{n_j}(x)\,v(x),$$

which holds for every $v \in W_0^{1,2}(\Omega)$, since $\nabla\psi_{n_j}$ converges weakly in $(L^2(\Omega))^N$ to $\nabla\psi$ (here we use the linearity of the principal part), $\frac{x\cdot\nabla\psi_{n_j}}{\frac{1}{n_j}+|x|^2}\,\frac{1}{1+\frac{1}{n_j}|\nabla\psi_{n_j}|}$ strongly converges in $(L^\sigma(\Omega))^N$ to $\frac{x\cdot\nabla\psi}{|x|^2}$ (recall the above almost everywhere convergence of the gradients), and g_{n_j} strongly converges to g in $L^m(\Omega)$. Furthermore, assing to the limit in (3.4) implies (3.7). □

Remark 3.3. We recall that in [5] it is proved the existence of bounded weak solutions Ψ of the boundary value problem

$$\begin{cases} -\mathrm{div}(M(x)\nabla\Psi) + \mu\Psi = E(x)\,\nabla\Psi + G(x) & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.8}$$

under the assumptions $\mu > 0$, $G \in L^\infty(\Omega)$, $E \in (L^2(\Omega))^N$.

It is not possible a comparison between this result and the existence result for (1.2), since the assumptions $\mu > 0$, $G \in L^\infty(\Omega)$ of (3.8) are stronger and the assumption $E \in (L^2(\Omega))^N$ is weaker.

We point out that the existence of Ψ (in [5]) is proved by duality, thanks to the estimate

$$\|\mu\, w\|_{L^1(\Omega)} \le \|F\|_{L^1(\Omega)} \tag{3.9}$$

on the solution of the Dirichlet problem

$$-\mathrm{div}(M(x)\nabla w) + \mu\, w = -\mathrm{div}(w\,E) + F. \tag{3.10}$$

The estimate (3.9) looks like the classical estimate of the Dirichlet problems with $E = 0$; however, in the case $E = 0$ the estimate $\|\mu\, w\|_{L^p(\Omega)} \le \|F\|_{L^p(\Omega)}$ holds for every $p > 1$, so that we can hope for the validity of this estimate even in the case $E \neq 0$. Unfortunately the following radial example shows that such estimate is not true.

Consider the Dirichlet problem (3.10) in radial case with $N > 6$, $\Omega = B(0,1)$, $\mu = 4(N-6)$, $E = (N-4)\frac{x}{|x|^4}$, $F = -4(N-6)|x|^{6-N}$ and principal part the Laplace operator $-\Delta$. Thus, we are studying the differential problem

$$\mu\, w(r) = w''(r) + \frac{N-1}{r}w'(r) + \frac{A(N-4)w}{r^4} + \frac{A\,w'}{r^3} + F, \quad w(1) = 0.$$

The function $r^{4-N} - r^{6-N}$ is a solution. Now we note that $F \in L^\rho(\Omega)$, $\rho < \frac{N}{N-6}$ and $w \in L^\sigma(\Omega)$, $\sigma < \frac{N}{N-4}$ and, of course, $\frac{N}{N-4} < \frac{N}{N-6}$; that is the summability of the solution w is always (recall taht N is large) worse than the summability of F, except for the case L^1, proved in [3].

REMARK 3.4. In Theorem 3.2 we have proved that if $N-2 \leq |A| < N-1$, if $m > \frac{N}{N-|A|}$, and if u_n is a solution of

$$-\mathrm{div}(A(x)\nabla u_n) = A\,\frac{x}{\frac{1}{n}+|x|^2}\cdot\frac{\nabla u_n}{1+\frac{1}{n}|\nabla u_n|} + y\,,$$

with y in $L^m(\Omega)$, then u_n belongs to $L^\infty(\Omega)$, and there exists a constant $C > 0$ such that

$$\|u_n\|_{L^\infty(\Omega)} \leq C\,\|y\|_{L^m(\Omega)}\,. \tag{3.11}$$

Clearly, the same result holds for a sequence of data $\{y_n\}$ belonging to $L^m(\Omega)$, with the estimate

$$\|u_n\|_{L^\infty(\Omega)} \leq C\,\|y_n\|_{L^m(\Omega)}\,,$$

which implies that if y_n tends to zero in $L^m(\Omega)$, then u_n uniformly converges to zero in Ω.

Let now q be in $(0,1)$, and $A = -(N+q-2)$, so that $N-2 < A < N-1$. Define

$$u(r) = \frac{1}{q}(1-r^q).$$

It is easy to see that $u(r)$ is such that

$$-\Delta u = (N+q-2)\,\frac{1}{r^{2-q}} = -A\,\frac{1}{r^{2-q}} = A\,\frac{x}{|x|^2}\,\nabla u\,.$$

In other words u, *which is different from zero*, is a solution of

$$-\Delta u = A\,\frac{x}{|x|^2}\,\nabla u + 0\,.$$

This seems to be in contrast with (3.11), since one should have that the solution with datum $y = 0$ is zero. However, u is also a solution of

$$-\Delta u = A\,\frac{x}{\frac{1}{n}+|x|^2}\cdot\frac{\nabla u}{1+\frac{1}{n}|\nabla u|} + \left[A\,\frac{x}{|x|^2}\,\nabla u - A\,\frac{x}{\frac{1}{n}+|x|^2}\cdot\frac{\nabla u}{1+\frac{1}{n}|\nabla u|}\right],$$

that is of

$$-\Delta u = A\,\frac{x}{\frac{1}{n}+|x|^2}\cdot\frac{\nabla u}{1+\frac{1}{n}|\nabla u|} + y_n\,,$$

where

$$y_n = A\,\frac{x}{|x|^2}\,\nabla u - A\,\frac{x}{\frac{1}{n}+|x|^2}\cdot\frac{\nabla u}{1+\frac{1}{n}|\nabla u|}\,.$$

It is easy to see that y_n *does not belong* to $L^m(\Omega)$, with $m > \frac{N}{N-|A|}$. Indeed, in this case,

$$\frac{N}{N-|A|} = \frac{N}{N-(N+q-2)} = \frac{N}{2-q}\,,$$

while

$$y_n = (N+q-2)\,\frac{1}{r^{2-q}} + \text{an } L^\infty(\Omega) \text{ function},$$

so that y_n only belongs $L^s(\Omega)$ for every $s < \frac{N}{2-q}$, and not better, due to the fact that the $L^\infty(\Omega)$ part of y_n does not improve the singularity at the origin. In other words, y_n tends to zero, but only in $L^s(\Omega)$, with $s < \frac{N}{2-q}$. Therefore, u solves the equation

$$-\Delta u = A\,\frac{x}{\frac{1}{n}+|x|^2}\cdot\frac{\nabla u}{1+\frac{1}{n}|\nabla u|} + y_n\,,$$

with y_n that does not tend to zero in $L^m(\Omega)$ with $m > \frac{N}{N-|A|}$. Hence, the results of Theorem 3.2 does not apply, and there is no contradiction between the theorem and the example presented in this remark.

Remark 3.5. Even if (1.1) and (1.2) are problems in duality, our approach in the study of the two problems is different: for the first one we use a test function method which gives *a priori* estimates; this method is not used in the study of the second problem. The reason for this different approach is explained here: with the boundary value problem (3.2) as starting point, we will show that is not possible to find a real function $\Phi(s)$, such that the use of $\Phi(\psi_n)$ as test function yields *a priori* estimates on the sequence $\{u_n\}$ in some function space.

First of all, we consider the case $E \in (L^N(\Omega))^N$, and $g(x)$ smooth. Let ψ_n be a weak solution of the Dirichlet problem (analogous to (3.2))

$$\psi_n \in W_0^{1,2}(\Omega) : \ -\mathrm{div}(M(x)\nabla\psi_n) = \frac{E(x)\cdot\nabla\psi_n}{1+\frac{1}{n}|\nabla\psi_n|} + g(x),$$

and let $\Phi(s)$ be a real increasing Lipschitz continuous function, with $\Phi(-s) = -\Phi(s)$. If we use $\Phi(\psi_n)$ as test function in the weak formulation of (3.2), we have

(3.12)
$$\begin{aligned}
\alpha\int_\Omega |\nabla u_n|^2\,\Phi'(u_n) &\le \int_\Omega |E(x)||\nabla u_n||\Phi(u_n)| + \int_\Omega g\,\Phi(u_n)\\
&\le \|E\|_{L^N(\Omega)}\Big[\int_\Omega |\nabla u_n|^2\,\Phi'(u_n)\Big]^{\frac12}\Big[\int_\Omega\Big(\frac{\Phi(u_n)}{\sqrt{\Phi'(u_n)}}\Big)^{2^*}\Big]^{\frac{1}{2^*}} + \int_\Omega g\,\Phi(u_n)\\
&\le \frac{\|E\|_{L^N(\Omega)}}{\mathcal{S}}\Big[\int_\Omega |\nabla u_n|^2\,\Phi'(u_n)\Big]^{\frac12}\Big[\int_\Omega |\nabla u_n|^2\Big(\frac{2[\Phi'(u_n)]^2-\Phi(u_n)\,\Phi''(u_n)}{2\,\Phi'(u_n)\,\sqrt{\Phi'(u_n)}}\Big)^2\Big]^{\frac12}\\
&\quad + \int_\Omega g\,\Phi(u_n)
\end{aligned}$$

In order to have *a priori* estimates, we suppose that there exists $0 < \theta < \frac{\alpha\,\mathcal{S}}{\|E\|_{L^N(\Omega)}}$ such that

$$\Big(\frac{2[\Phi'(u_n)]^2-\Phi(u_n)\,\Phi''(u_n)}{2\,\Phi'(u_n)\,\sqrt{\Phi'(u_n)}}\Big)^2 \le \theta^2\Phi'(u_n),$$

that is

$$\{2[\Phi'(u_n)]^2-\Phi(u_n)\,\Phi''(u_n)\}^2 \le 4\theta^2\Phi'(u_n)^4,$$

which is equivalent to

$$-2\theta\Phi'(u_n)^2 \le 2[\Phi'(u_n)]^2-\Phi(u_n)\,\Phi''(u_n) \le 2\theta\Phi'(u_n)^2.$$

We prove now that it is not possible to find a nontrivial function $\Phi(s)$ satisfying the inequality

$$2[\Phi'(s)]^2-\Phi(s)\,\Phi''(s) \le 2\theta\Phi'(s)^2.$$

Indeed, this inequality is equivalent to

$$0 \le \frac{d}{ds}[\log(\Phi'(s)) - 2(1-\theta)\log(\Phi(s))]. \tag{3.13}$$

We thus have the following chain of equivalences: (3.13) is equivalent to

$$\log\Big(\frac{\Phi'(s)}{\Phi(s)^{2(1-\theta)}}\Big) \text{ is increasing, which is equivalent to } \frac{\Phi'(s)}{\Phi(s)^{2(1-\theta)}} \text{ is increasing,}$$

and the last one is equivalent to

$$\frac{[\Phi(s)]^{1-2(1-\theta)}}{1-2(1-\theta)} \text{ is convex.}$$

We note now that θ may be small, due to the fact that we do not want to impose bounds on the norm of E in $L^N(\Omega)$ (such bounds will yield a coercive differential operator); therefore, we are interested in the cases when θ is close to 0, i.e., in the cases in which $1-2(1-\theta)<0$. Under this assumption, we have that $\Lambda(s)=\frac{1}{[\Phi(s)]^{2(1-\theta)-1}}$ is concave; however,

$$\lim_{s\to 0^+}\Lambda(s)=\lim_{s\to 0^+}\frac{1}{[\Phi(s)]^{2(1-\theta)-1}}=+\infty\,,$$

which contradicts the fact that Λ is concave.

Now, we consider the case in which (1.3) holds, with $g(x)$ a smooth function. Let ψ_n be a weak solution of the Dirichlet problem (3.2), and use $\Phi(\psi_n)$ as test function. Then, if $\mathcal{H}$ is the Hardy constant, we have

$$\begin{aligned}
\alpha\int_\Omega|\nabla u_n|^2\,\Phi'(u_n) &\le |A|\int_\Omega\frac{|\nabla u_n|}{|x|}\Phi(u_n)\\
&\le |A|\bigg[\int_\Omega|\nabla u_n|^2\,\Phi'(u_n)\bigg]^{\frac12}\bigg[\int_\Omega\frac{1}{|x|^2}\bigg(\frac{\Phi(u_n)}{\sqrt{\Phi'(u_n)}}\bigg)^2\bigg]^{\frac12}+\int_\Omega g\,\Phi(u_n)\\
&\le \frac{|A|}{\mathcal{H}}\bigg[\int_\Omega|\nabla u_n|^2\,\Phi'(u_n)\bigg]^{\frac12}\bigg[\int_\Omega|\nabla u_n|^2\bigg(\frac{2[\Phi'(u_n)]^2-\Phi(u_n)\,\Phi''(u_n)}{2\,\Phi'(u_n)\,\sqrt{\Phi'(u_n)}}\bigg)^2\bigg]^{\frac12}\\
&\quad+\int_\Omega g\,\Phi(u_n)
\end{aligned}$$

Thus we have again (3.12), even if with different constants; in this case, the condition to impose on θ in order to have *a priori* estimates is

$$\theta\frac{|A|}{\mathcal{H}}<\alpha. \tag{3.14}$$

If $|A|\ge(N-2)\alpha$, then $\frac{|A|}{\mathcal{H}}\ge 2\alpha$; in order for (3.14) to hold, one must choose $\theta<\frac12$; in this case, however, $1-2(1-\theta)<0$, which yields a contradiction as above. In other words, if $|A|\ge(N-2)\alpha$, which is the assumption of Theorem 3.2 above, it is not possible to obtain estimates on the solution ψ_n using test functions depending on the solution itself.

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References

[1] M.F. Betta, V. Ferone, A. Mercaldo: Regularity for solutions of nonlinear elliptic equations; *Bull. Sci. Math.*, **118** (1994), 539–567.

[2] L. Boccardo: Some developments on Dirichlet problems with discontinuous coefficients; *Boll. Unione Mat. Ital.*, **2** (2009) 285–297.

[3] L. Boccardo: Dirichlet problems with singular convection terms and applications; *J. Differential Equations*, **258** (2015) 2290–2314.

[4] L. Boccardo: Stampacchia-Calderon-Zygmund theory for linear elliptic equations with discontinuous coefficients and singular drift; to appear on *ESAIM, Control, Optimization and Calculus of Variations.*

[5] L. Boccardo: Finite energy weak solutions to some Dirichlet problems with very singular drift; preprint.

[6] L. Boccardo, S. Buccheri, G.R. Cirmi: Two linear noncoercive Dirichlet problems in duality; *Milan J. Math.*, **86** (2018), 97–104.
[7] L. Boccardo, T. Gallouët: Nonlinear elliptic and parabolic equations involving measure data; *J. Funct. Anal.*, **87** (1989), 149–169.
[8] L. Boccardo, T. Gallouët: Nonlinear elliptic equations with right hand side measures; *Comm. Partial Differential Equations*, **17** (1992), 641–655.
[9] G. Bottaro, M.E. Marina: Problema di Dirichlet per equazioni ellittiche di tipo variazionale su insiemi non limitati; *Boll. Unione Mat. Ital.*, **8** (1973), 46–56.
[10] H. Brezis, J.L. Vazquez: Blow-up solutions of some nonlinear elliptic problems; *Rev. Mat. Univ. Complut. Madrid*, **10** (1997), 443–469.
[11] S. Buccheri: Elliptic boundary value problems with measurable coefficients and explosive boundary conditions; Ph.D. Thesis, 2019.
[12] S. Buccheri: Gradient estimates for nonlinear elliptic equations with first order terms; in preparation.
[13] S. Buccheri: The Bottaro-Marina slice method for distributional solutions to elliptic equations with drift term; in preparation.
[14] T. Del Vecchio, M.R. Posteraro: An existence result for nonlinear and noncoercive problems; *Nonlinear Analysis TMA*, **31** (1998), 191–206.
[15] T. Leonori, F. Petitta: Existence and regularity results for some singular elliptic problems; *Adv. Nonlinear Stud.*, **7** (2007) 329–344.
[16] G. Stampacchia: Le probléme de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus; *Ann. Inst. Fourier* (Grenoble), **15** (1965), 189–258.
[17] J.L. Vazquez, E. Zuazua: The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential; *J. Funct. Anal.*, **173** (2000), 103–153.

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