# Passive and active fibre reorientation in anisotropic materials.

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**Abstract**

We present a continuum model to describe the reorientation of an anisotropic material structure, characterized by two fiber families able to modify their orientations following different evolution dynamics. The evolution equations are derived in a thermodynamically consistent way, and passive and active contributions to the reorientation process are identified. It is shown that a weaker extension of a well-known coaxiality result holds. The transversely isotropic and orthotropic cases are then recovered by imposing the proper constraint on the fiber rotation. Applications to biological experiments on cell layers under stretch are discussed, showing a good agreement between the model and the experimental results. Even though we focus on cell layers, our framework remains general and may be employed to describe reorientation in engineering materials.

*Keywords:* Fibre reorientation · Anisotropic materials · Nonlinear elasticity · Coaxiality · Cell orientation · Stress fibers

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## 1. Introduction

The ability of actively changing the internal structure in response to external stimuli is a fundamental characteristic of biological tissues, cells, and

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organisms. Such a process, which is generally referred to as *remodeling*, may be driven by growth, injuries, ageing or alterations in the chemo-mechanical environment, to cite but a few examples [1, 2, 3, 4, 5]. As a consequence of these prompts, a reorganization of the biological microstructure is often observed, since the material constantly tries to adapt to the new conditions in a dynamic way. Understanding and accurately describing the remodeling of living structures is a major challenge in biomechanics, which has been of great interest in recent years: from a mechanical viewpoint, several works have focused on different features of remodeling, such as the coupling with growth [4, 6, 7], the influence of diseases in shape alterations [8] or in material properties, and the changes in the biological structure as plastic-like irreversible distortions [9]; see also [10] and references therein for a detailed overview of remodeling and growth in soft tissues.

Among the several types of remodeling that may be triggered by external forces, reorientation is probably one of the most relevant as far as living tissues and matter are concerned. For a biological material with a fiber-reinforced structure, it can be defined as a change in the local orientation of fibers due to some external actions, leading therefore to a reorganization in the microstructure of the material. This phenomenon is frequently seen in different situations: for instance, it is well known that the bones are able to change the orientation of their internal fibers to functionally adapt to environmental stimuli [11]. In particular, many biological materials with a fiber-like microstructure are described as transversely isotropic, that is, a single preferential direction exists and influences the mechanical behaviour: significant examples are articular cartilage, tendons or skin, to cite but a few [7, 12, 13, 14, 15, 16]. Instead, it is worth mentioning that tissues may exhibit an orthotropic arrangement, which requires the description of two distinct fiber directions. A significant example can be found in the circulatory system, where the myocardium and the arteries are known to have three planes with distinct mechanical properties [17, 18, 19, 20, 21]. Another phenomenon related to orthotropy that has recently gathered some attention is the cellular response to mechanical cues coming from the surrounding microenvironment. In particular, experiments have demonstrated that cells seeded on a two-dimensional substrate which is cyclically stretched reorient their stress fibers and bodies to reach a stable configuration, characterized by a precise angle between the cell axis and the direction of stretching [22, 23, 24, 25, 26, 27, 28, 29].

A possible explanation of this behaviour, that relies on mechanical arguments, has been firstly proposed by Livne et al. [22], who suggested that cells attempt to minimize the elastic energy stored in the stretching process.

Recently, a generalization of the energy considered in [22] was presented in [30, 31], where the system composed by the cells and the substrate was described as an orthotropic material: the preferential directions were identified with the cellular actin stress fibers and with the orthogonal protein network interconnecting them [32, 33, 34, 35].

However, the nonlinear elastic description of cell orientations proposed in [30] does not explicitly include a remodeling of the internal structure, even if the material composed by substrate and cells is treated as orthotropic. Instead, cells are viewed as passive fibers whose reorientation is dragged by the deformation of the layer itself. A different point of view, even if limited to transversely isotropic materials, has been recently proposed in [36, 37], taking into account both passive and active changes in the fiber directions. The introduction of a remodeling equation, that complements the usual mechanical balances, allows to describe the evolution of fiber orientation under mechanical [36] and magnetic [37] stimuli. Moreover, in [36], it was shown that the stationary solutions of the remodeling equation are the ones that make the remodeled stress and strain tensors coaxial [38].

It seems then quite natural to provide an extension of the results presented in [36] to a more general case, where an hyperelastic material is endowed with a double fiber microstructure, as in the cellular example. Furthermore, while in [30] the two fibers were fixed to be perpendicular, it is of interest to characterize the remodeling of the different fiber families independently, in a way that each fiber can change the orientation according to its own rotation tensor.

In this paper, motivated by these observations, we propose a mechanical model that describes the reorganization of an anisotropic material structure, characterized by two fiber families that can modify their orientation following different evolution equations. Both active contributions affecting the reorientation process, like external forces or stimuli, and purely passive material remodeling can be incorporated in the proposed framework. In doing so, we are able to extend an energetic modeling of cell alignment [22, 30] using a fiber reorientation framework [36], in which the rotation of the cells due to the mechanical prompt is treated as an additional variable for the model. This allowed us to derive the reorientation equations in a rigorous and thermodynamically consistent basis, recovering the results presented in [30] as a particular case. At the same time, we provide an extension of the fiber reorientation model proposed in [36], adapting it to anisotropic materials with two preferential directions that can change their orientation in different manners, even though they can be properly coupled. The limit of two families of fibers which cannot change their relative orientation, as done for instance by Menzel [39] in the orthotropic case, is recovered within the terms of a constrained model.

The paper is organized as follows. Section 2 presents the remodeling framework in presence of a double-fibered anisotropic internal material structure, through a generalization of the model proposed in [36]. Then, in Section 3, the stationary solutions of the reorientation equations for the fibers are thoroughly studied, showing that a weaker extension of the coaxiality result by Vianello [38] holds when the fiber reinforcements are able to reorient without any constraint. Section 4 is devoted to the analysis of an in-plane situation, where the fibers are supposed to lie in a plane during the whole remodeling process. In this case, we are able to specify the general evolution equations and the related stationary solutions, focusing for simplicity on a minimal elastic energy which however allows relevant comparisons with experimental data. Section 5 is devoted to the discussion of a reduced model based on an orthotropic material structure with fixed relative orientation of the fibers, naturally emerging from the general framework, provided that a constraint on fiber rotations is included. Applications of the proposed models in the biological context of cell orientation are then presented in Section 6. In particular we show that, as expected, the constrained model is consistent with some already known results, such as the phenomenological evolution equation for cell orientation postulated by Livne et al. [22] and the stationary points of the energy derived in [30]. Moreover, the general model is employed to describe stress fibers and cell nucleus reorientation, showing a good agreement with experimental data by Roshanzadeh et al. [29]. Finally, in Section 7 we provide some discussion and conclusions, as well as future possible developments and applications of the present theory. Appendix A is dedicated to the discussion of another possible choice for the constraint between the fiber families.

## 2. Remodeling framework

Following the approach proposed in [36], we describe the reorientation within the general framework of finite elasticity with remodeling presented in [40]. We consider a material equipped with an anisotropic two families of fibers (TFF) internal structure whose change of orientation is not simply dragged by the deformation but described by additional state variables, therefore including a remodeling of the fiber structure. The relationship between fiber orientations and mechanical forces, as well as the constitutive coupling between the two fiber families which characterize the anisotropic structure, are derived on a thermodynamically consistent basis.

### *2.1. Kinematics*

Given a body, identified with a region $\mathcal{B}_r$ of the Euclidean three-dimensional space $\mathcal{E}$, and a material point $X \in \mathcal{B}_r$, we consider the time-dependent map $\chi : \mathcal{B}_r \times \mathcal{T} \to \mathcal{E}$, called the deformation[1] of the body, that assigns to each point $X \in \mathcal{B}_r$ a point $x = \chi(X,t)$ at any instant $t$ of the time interval $\mathcal{T}$ and determines the current configuration $\mathcal{B} = \chi(\mathcal{B}_r, t)$ of the body at time $t$. As standard in finite elasticity, we introduce the displacement field $\mathbf{u} : \mathcal{B}_r \times \mathcal{T} \to \mathcal{V}$, with $\mathcal{V}$ the translation space of $\mathcal{E}$, such that $x = \chi(X,t) = X + \mathbf{u}(X,t)$, and the fields

$$\mathbf{F}(X,t) = \nabla\,\chi(X,t) \qquad \text{and} \qquad \dot{\chi}(X,t) = \frac{\partial \chi}{\partial t}(X,t)\,, \tag{2.1}$$

that represent the deformation gradient and the material velocity field, respectively. Clearly, $\mathbf{F} = \mathbf{I} + \nabla\mathbf{u}$, where $\mathbf{I}$ is the identity tensor. We assume that at a certain time instant $t = t_0$, the body occupies the reference configuration, that is, $\chi(X,t_0) = X$ for each point in $\mathcal{B}_r$ and $\mathbf{u}(X,t_0) = \mathbf{0}$.
In addition to the displacement field describing the current position of the body, we are interested in the evolution of the orientation of the anisotropic TFF internal structure, which in $\mathcal{B}_r$ is based on the pair of material unit vector fields $\mathsf{a}_0 : \mathcal{B}_r \to \mathcal{V}$ and $\mathsf{b}_0 : \mathcal{B}_r \to \mathcal{V}$ (with $|\mathsf{a}_0| = |\mathsf{b}_0| = 1$); they represent the preferred directions that the internal structure endows to the material, i.e. the direction of each fiber at $X \in \mathcal{B}_r$. For practical usage, it is convenient to introduce the corresponding *structural or orientation tensors*[2] at $\mathcal{B}_r$, defined as $\mathsf{A}_0 = \mathsf{a}_0 \otimes \mathsf{a}_0$ and $\mathsf{B}_0 = \mathsf{b}_0 \otimes \mathsf{b}_0$. In the following, we will sometimes refer to the field $\mathsf{a}_0$ as the *primary structure* of the material, while $\mathsf{b}_0$ will be consequently called *secondary structure*. Both $\mathsf{A}_0$ and $\mathsf{B}_0$ contribute to the material anisotropy and can reorient as prescribed by two different rotation fields, each one interpreted as the rotation of a single fiber family. In this respect, we will consider a time-dependent tensor field $\mathbf{R}_p : \mathcal{B}_r \times \mathcal{T} \to \mathbb{R}\mathtt{ot}$ which represents the reorientation of the primary structure and changes the reference orientation from $\mathsf{a}_0(X)$ to

$$\mathbf{a}(X,t) = \mathbf{R}_p(X,t)\mathsf{a}_0(X),$$

[1] We consider deformations that are twice continuously differentiable.

[2] Typically, they are denoted *structural tensors* in the mechanics of fiber-reinforced materials [18] and *orientation tensors* in the literature on nematic liquid crystals [41].

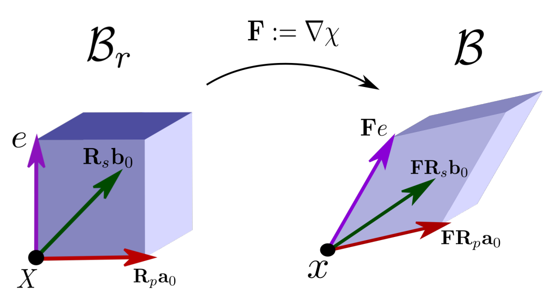


Figure 1: Schematic representation of the reorientation process at a material point. A material line element $e$ in the reference configuration $\mathcal{B}_r$ is deformed to the corresponding line element $\mathbf{F}e$ in the current configuration $\mathcal{B}$. Instead, the primary structure $\mathbf{a}_0$ firstly undergoes remodeling due to the rotation $\mathbf{R}_p$ while the secondary structure $\mathbf{b}_0$ is reoriented according to another rotation $\mathbf{R}_s$.

or equivalently $\mathbf{A}_0(X)$ to

$$\mathbf{A}(X,t) = \mathbf{R}_p(X,t)\mathbf{A}_0(X)\mathbf{R}_p^{\mathrm{T}}(X,t), \tag{2.2}$$

where $\mathbf{A} = \mathbf{a} \otimes \mathbf{a}$. Likewise, the second rotation field $\mathbf{R}_s : \mathcal{B}_r \times \mathcal{T} \to \mathbb{R}\mathtt{ot}$ describes the reorientation of the secondary structure and changes $\mathbf{b}_0(X)$ to

$$\mathbf{b}(X,t) = \mathbf{R}_s(X,t)\mathbf{b}_0(X)$$

and $\mathbf{B}_0(X)$ to

$$\mathbf{B}(X,t) = \mathbf{R}_s(X,t)\mathbf{B}_0(X)\mathbf{R}_s(X,t)^{\mathrm{T}}, \tag{2.3}$$

having denoted $\mathbf{B} = \mathbf{b}\otimes\mathbf{b}$. Each fiber family of such an anisotropic material may change its orientation as time evolves; then, the deformation gradient tensor $\mathbf{F}$ maps each remodeled fiber to the current configuration, as sketched in Fig. 1. With these definitions on hand, the state variables of the problem are the displacement and rotation fields, that is, the triple $(\mathbf{u}, \mathbf{R}_p, \mathbf{R}_s) \in \mathcal{V} \times \mathbb{R}\mathtt{ot} \times \mathbb{R}\mathtt{ot}$, and the corresponding velocity fields are identified by the time derivatives $(\dot{\mathbf{u}}, \dot{\mathbf{R}}_p\mathbf{R}_p^{\mathrm{T}}, \dot{\mathbf{R}}_s\mathbf{R}_s^{\mathrm{T}}) \in \mathcal{V} \times \mathbb{S}\mathrm{kw} \times \mathbb{S}\mathrm{kw}$, where $\mathbb{S}\mathrm{kw}$ denotes the space of skew-symmetric tensors. Finally, we denote as $(\mathbf{w}, \mathbf{W}_p, \mathbf{W}_s) \in \mathcal{V} \times \mathbb{S}\mathrm{kw} \times \mathbb{S}\mathrm{kw}$ the associated virtual velocity fields.

*Remark.* Let us note that our kinematic framework does not filter out the rotation fields $\mathbf{R}_p$ and $\mathbf{R}_s$ such that $\mathbf{R}_p\mathbf{a}_0 = \mathbf{a}_0$ and $\mathbf{R}_s\mathbf{b}_0 = \mathbf{b}_0$, which are included in the admissible rotation fields even if they maintain the energy unchanged and so are of no interest in the remodeling problem.

*2.2. Balance equations*

The balance equations of the model are delivered by the principle of virtual working, based on the choice of the external and internal virtual workings defined as continuous, linear, real-valued functionals on the space of virtual velocities. By introducing forces and torques of the model which are working-conjugate to each kinematic variable, we write

$$\mathcal{W}_e(\mathbf{w},\mathbf{W}_p,\mathbf{W}_s)=\int_{\mathcal{B}_r}(\mathbf{z}\cdot\mathbf{w}+\mathbf{Z}_p\cdot\mathbf{W}_p+\mathbf{Z}_s\cdot\mathbf{W}_s)\,dV+\int_{\partial_t\mathcal{B}_r}\mathbf{s}\cdot\mathbf{w}\,dA \tag{2.4}$$

for the external virtual working and

$$\mathcal{W}_i(\mathbf{w},\mathbf{W}_p,\mathbf{W}_s)=\int_{\mathcal{B}_r}(\mathbf{S}\cdot\nabla\mathbf{w}+\mathbf{\Sigma}_p\cdot\mathbf{W}_p+\mathbf{\Sigma}_s\cdot\mathbf{W}_s)\,dV \tag{2.5}$$

for the internal virtual working. The pair $(\mathbf{z},\mathbf{s})$ are forces per unit of (referential) volume and area, respectively, while $\mathbf{S}$ is the first Piola–Kirchhoff stress tensor. The pair of skew tensors $(\mathbf{Z}_i,\mathbf{\Sigma}_i)$, with $i=p,s$, are torques per unit of (referential) volume, and represent the outer and inner remodeling torques, as $\mathbf{W}_p$, $\mathbf{W}_s$ are skew-symmetric tensors. More specifically, the outer torques $\mathbf{Z}_p$ and $\mathbf{Z}_s$ represent external source terms which may affect fiber reorientation, for instance magnetic effects [37] or chemo-mechanical processes. On the other side, the inner remodeling torques $\mathbf{\Sigma}_p$ and $\mathbf{\Sigma}_s$ take into account the internal actions driving the reorientation of the primary and secondary material structures, respectively. It is worth remarking that, for what concerns the remodeling torques, our theory is of order zero-th, i.e., we do not take into account the gradients of the rotation fields, which instead may become relevant when the fibers are closely packed.

By enforcing the condition that the external and internal virtual working be equal for any virtual velocities $(\mathbf{w},\mathbf{W}_p,\mathbf{W}_s)\in\mathcal{V}\times\mathbb{S}\mathrm{kw}\times\mathbb{S}\mathrm{kw}$ and for any subregion $\mathcal{R}\subset\mathcal{B}_r$, we obtain the following balance equations and associated boundary conditions:

$$\mathrm{Div}\,\mathbf{S}+\mathbf{z}=\mathbf{0}\quad\text{and}\quad\mathbf{\Sigma}_p=\mathbf{Z}_p\quad\text{and}\quad\mathbf{\Sigma}_s=\mathbf{Z}_s\ \text{ in }\ \mathcal{B}_r\,, \tag{2.6}$$

$$\mathbf{u}=\widehat{\mathbf{u}}\ \text{ on }\partial_u\mathcal{B}_r\quad\text{and}\quad\mathbf{S}\,\mathbf{m}=\mathbf{s}\ \text{ on }\partial_t\mathcal{B}_r\,, \tag{2.7}$$

with $\partial_u\mathcal{B}_r$ and $\partial_t\mathcal{B}_r$ denoting parts of the boundary $\partial\mathcal{B}_r$ where displacements and tractions are respectively prescribed, and $\mathbf{m}$ denoting the unit normal to $\partial_t\mathcal{B}_r$. Equations (2.6) are the three balance equations of our theory. The first is the usual balance of mechanical forces, where inertial effects can be included in the bulk force $\mathbf{z}$, even if in the present model they are neglected since the remodeling time scale is typically much longer than the inertial one. The second and third equations are the balances of the remodeling torques, driving the reorientation of the primary and secondary structures, respectively, once the appropriate constitutive equations will be considered. In particular, they prescribe that the outer and inner remodeling torques be equal. These equations generalise the theory presented in [36], where only a transversely isotropic internal structure was considered.

The external working, that corresponds to balanced forces and torques and is evaluated on the actual velocity fields $(\dot{\mathbf{u}}, \dot{\mathbf{R}}_p\mathbf{R}_p^{\mathrm{T}}, \dot{\mathbf{R}}_s\mathbf{R}_s^{\mathrm{T}})$, identifies the external actual power $\mathcal{P}_e$ expended during the evolution of the continuum:

$$\begin{aligned}\mathcal{W}_e(\dot{\mathbf{u}}, \dot{\mathbf{R}}_p\mathbf{R}_p^{\mathrm{T}}, \dot{\mathbf{R}}_s\mathbf{R}_s^{\mathrm{T}}) &= \int_{\mathcal{B}_r} (\mathbf{z}\cdot\dot{\mathbf{u}} + \mathbf{Z}_p\cdot\dot{\mathbf{R}}_p\mathbf{R}_p^{\mathrm{T}} + \mathbf{Z}_s\cdot\dot{\mathbf{R}}_s\mathbf{R}_s^{\mathrm{T}})\,dV + \int_{\partial_t\mathcal{B}_r} \mathbf{s}\cdot\dot{\mathbf{u}}\,dA \\ &= \int_{\mathcal{B}_r} (\mathbf{S}\cdot\dot{\mathbf{F}} + \boldsymbol{\Sigma}_p\cdot\dot{\mathbf{R}}_p\mathbf{R}_p^{\mathrm{T}} + \boldsymbol{\Sigma}_s\cdot\dot{\mathbf{R}}_s\mathbf{R}_s^{\mathrm{T}}) = \mathcal{P}_e\,. \qquad (2.8)\end{aligned}$$

### 2.3. Constitutive equations and energy imbalance

The constitutive recipes to describe the behaviour of the material with the reorientable anisotropic TFF internal structure are prescribed in a thermodynamically consistent way, through the following steps. Firstly, we take into account a class of materials that admits an elastic strain energy density in the form

$$\varphi = \phi(\mathbf{E}, \mathbf{A}, \mathbf{B}) = \psi(\mathbf{E}, \mathbf{R}_p, \mathbf{R}_s)\,, \qquad (2.9)$$

dependent upon the deformation $\mathbf{F}$ through the Green–Lagrange strain tensor $\mathbf{E} = \frac{1}{2}\big(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}\big)$, and on the remodeled orientation tensors $\mathbf{A}$ and $\mathbf{B}$ or, equivalently, on the rotations $\mathbf{R}_p$ and $\mathbf{R}_s$ that transform the primary and secondary structures. More specifically, following the theory of double-fibered finite elasticity, we assume that $\phi$ and $\psi$ are isotropic functions of their arguments [42, 43].

We also assume that dissipation is only associated with the remodeling processes, and the dissipation density $\delta$ can be written as

$$\delta = \delta\left(\dot{\mathbf{R}}_p\mathbf{R}_p^{\mathrm{T}}, \dot{\mathbf{R}}_s\mathbf{R}_s^{\mathrm{T}}\right)\,. \qquad (2.10)$$

To derive constitutive equations that are consistent with the first and second laws of thermodynamics, we enforce the energy imbalance, stating that for any admissible process, characterised by the state variables $(\mathbf{u}, \mathbf{R}_p, \mathbf{R}_s)$, the time derivative of the energy must not exceed the external actual power expended on the body along the same process, i.e., the dissipation must be positive:

$$\int_{\mathcal{B}_r} \delta \, dV = \mathcal{P}_e - \int_{\mathcal{B}_r} \dot{\varphi} \, dV \geq 0. \tag{2.11}$$

Within the framework discussed so far and using Eq. (2.9), we can write the time derivative of the elastic energy as

$$\dot{\varphi} = \frac{\partial \phi}{\partial \mathbf{E}} \cdot \dot{\mathbf{E}} + \frac{\partial \phi}{\partial \mathbf{A}} \cdot \dot{\mathbf{A}} + \frac{\partial \phi}{\partial \mathbf{B}} \cdot \dot{\mathbf{B}} \,. \tag{2.12}$$

At this point, it is useful to introduce the commutator operator between two tensors $[\cdot,\cdot] : \mathbb{L}\mathtt{in} \times \mathbb{L}\mathtt{in} \to \mathbb{L}\mathtt{in}$, where $\mathbb{L}\mathtt{in}$ stands for the space of second-order tensors, defined as:

$$[\mathbf{X}, \mathbf{Y}] = \mathbf{XY} - \mathbf{YX}, \quad \forall \mathbf{X}, \mathbf{Y} \in \mathbb{L}\mathtt{in}.$$

With this definition on hand, recalling Eqs. (2.2)-(2.3) we have

$$\dot{\mathbf{A}} = [\dot{\mathbf{R}}_p \mathbf{R}_p^{\mathrm{T}}, \mathbf{A}] \quad \text{and} \quad \dot{\mathbf{B}} = [\dot{\mathbf{R}}_s \mathbf{R}_s^{\mathrm{T}}, \mathbf{B}] \,, \tag{2.13}$$

and, in general, for any $\mathbf{X}, \mathbf{Y} \in \mathbb{L}\mathtt{in}$,

$$\mathbf{X} \cdot [\dot{\mathbf{R}}_p \mathbf{R}_p^{\mathrm{T}}, \mathbf{A}] = \dot{\mathbf{R}}_p \mathbf{R}_p^{\mathrm{T}} \cdot [\mathbf{X}, \mathbf{A}] \quad \text{and} \quad \mathbf{Y} \cdot [\dot{\mathbf{R}}_s \mathbf{R}_s^{\mathrm{T}}, \mathbf{B}] = \dot{\mathbf{R}}_s \mathbf{R}_s^{\mathrm{T}} \cdot [\mathbf{Y}, \mathbf{B}] \,,$$

which allow Eq. (2.12) to be rewritten as

$$\dot{\varphi} = \frac{\partial \phi}{\partial \mathbf{E}} \cdot \dot{\mathbf{E}} + [\frac{\partial \phi}{\partial \mathbf{A}}, \mathbf{A}] \cdot \dot{\mathbf{R}}_p \mathbf{R}_p^{\mathrm{T}} + [\frac{\partial \phi}{\partial \mathbf{B}}, \mathbf{B}] \cdot \dot{\mathbf{R}}_s \mathbf{R}_s^{\mathrm{T}} \,. \tag{2.14}$$

The constitutive prescriptions for the stress $\mathbf{S}$ and for the inner remodeling actions $\mathbf{\Sigma}_p$ and $\mathbf{\Sigma}_s$ have to satisfy the imbalance principle stated above, which can be rephrased in local form requiring that

$$\delta = \mathbf{S} \cdot \dot{\mathbf{F}} + \mathbf{\Sigma}_p \cdot \dot{\mathbf{R}}_p \mathbf{R}_p^{\mathrm{T}} + \mathbf{\Sigma}_s \cdot \dot{\mathbf{R}}_s \mathbf{R}_s^{\mathrm{T}} - \dot{\varphi} \geq 0 \tag{2.15}$$

has to be positive for every realizable process [44]. By using equations (2.12)-(2.14), we reduce the inequality (2.15) to:

$$\left( \mathbf{S} - \mathbf{F} \frac{\partial \phi}{\partial \mathbf{E}} \right) \cdot \dot{\mathbf{F}} + \left( \mathbf{\Sigma}_p - [\frac{\partial \phi}{\partial \mathbf{A}}, \mathbf{A}] \right) \cdot \dot{\mathbf{R}}_p \mathbf{R}_p^{\mathrm{T}} + \left( \mathbf{\Sigma}_s - [\frac{\partial \phi}{\partial \mathbf{B}}, \mathbf{B}] \right) \cdot \dot{\mathbf{R}}_s \mathbf{R}_s^{\mathrm{T}} \geq 0 \,. \tag{2.16}$$

Since we assume that dissipation is only due to remodeling, the constitutive equation for the stress can be chosen as a standard hyperelastic law:

$$\mathbf{S} = \mathbf{F} \frac{\partial \phi}{\partial \mathbf{E}}, \qquad \mathbf{S}_c = \mathbf{F}^{-1} \mathbf{S} = \frac{\partial \phi}{\partial \mathbf{E}}, \tag{2.17}$$

where $\mathbf{S}_c$ is the second Piola-Kirchhoff stress tensor. The remainder of the dissipation inequality (2.16) can be satisfied on assuming that each inner remodeling torque can be decomposed into an elastic and a dissipative part:

$$\mathbf{\Sigma}_p = \mathbf{\Sigma}_p^{(e)} + \mathbf{\Sigma}_p^{(d)} = [\frac{\partial \phi}{\partial \mathbf{A}}, \mathbf{A}] + \mathbb{D}_p \dot{\mathbf{R}}_p \mathbf{R}_p^{\mathrm{T}}, \tag{2.18}$$

$$\mathbf{\Sigma}_s = \mathbf{\Sigma}_s^{(e)} + \mathbf{\Sigma}_s^{(d)} = [\frac{\partial \phi}{\partial \mathbf{B}}, \mathbf{B}] + \mathbb{D}_s \dot{\mathbf{R}}_s \mathbf{R}_s^{\mathrm{T}}, \tag{2.19}$$

where $\mathbb{D}_p$ and $\mathbb{D}_s$ are positive-definite fourth order tensors that represent the resistance to remodeling of the primary and secondary material structure, respectively.
By substituting the constitutive equations (2.17)-(2.19) in the balance equations (2.6), we are able to obtain two differential equations that describe the evolution of the rotation fields, and therefore of the associated fiber families. The general remodeling problem for the anisotropic TFF microstructure can then be expressed as the following system of nonlinear ordinary differential equations in $\mathcal{B}_r \times \mathcal{T}$:

$$\begin{cases} \mathbb{D}_p \dot{\mathbf{R}}_p \mathbf{R}_p^{\mathrm{T}} = \mathbf{Z}_p - [\dfrac{\partial \phi}{\partial \mathbf{A}}, \mathbf{A}]\,, \\[2ex] \mathbb{D}_s \dot{\mathbf{R}}_s \mathbf{R}_s^{\mathrm{T}} = \mathbf{Z}_s - [\dfrac{\partial \phi}{\partial \mathbf{B}}, \mathbf{B}]\,, \end{cases} \tag{2.20}$$

with initial conditions

$$\mathbf{R}_p = \mathbf{I} \quad \text{and} \quad \mathbf{R}_s = \mathbf{I} \quad \text{in} \quad \mathcal{B}_r \times \{t_0\}. \tag{2.21}$$

The problem (2.20) has to be solved together with the balance of forces $(2.6)_1$ and the constitutive prescription (2.17). We observe that the elastic terms on the right-hand side of Eqs. (2.20), represented by the commutators, include the mutual interaction between the two fiber families, as it will be explicitly shown in the case of plane remodeling. Moreover, when $\mathbf{Z}_p = \mathbf{Z}_s = \mathbf{0}$, as we shall assume in the rest of the paper, the stationary solutions of the system (2.20) are determined by solving the equations

$$[\frac{\partial \phi}{\partial \mathbf{A}}, \mathbf{A}] = \mathbf{0} \quad \text{and} \quad [\frac{\partial \phi}{\partial \mathbf{B}}, \mathbf{B}] = \mathbf{0}\,. \tag{2.22}$$

*2.4. Further constitutive prescriptions*

For the calculations carried out in the following Sections, it is convenient to express the strain energy density in terms of the invariants of the tensors

$\mathbf{E}$, $\mathbf{A}$ and $\mathbf{B}$. In doing so we introduce the following classical set of isotropic invariants of $\mathbf{E}$:

$$\mathrm{J}_1 = \mathbf{E}\cdot\mathbf{I}\,, \qquad \mathrm{J}_2 = \mathbf{E}^{\dagger}\cdot\mathbf{I}\,, \qquad \mathrm{J}_3 = \det(\mathbf{I}+2\mathbf{E})\,, \tag{2.23}$$

where $\mathbf{E}^{\dagger} = (\det\mathbf{E})\mathbf{E}^{-\mathrm{T}}$, together with the anisotropic ones [45]

$$\begin{aligned}
&\mathrm{J}_4 = \mathbf{E}\cdot\mathbf{A}\,, && \mathrm{J}_5 = \mathbf{E}^2\cdot\mathbf{A}\,,\\
&\mathrm{J}_6 = \mathbf{E}\cdot\mathbf{B}\,, && \mathrm{J}_7 = \mathbf{E}^2\cdot\mathbf{B}\,,\\
&\mathrm{J}_8 = \mathbf{E}\cdot\mathrm{sym}\,(\mathbf{A}\mathbf{B}) = (\mathbf{a}\cdot\mathbf{b})\big(\mathbf{E}\mathbf{a}\cdot\mathbf{b}\big)\,.
\end{aligned} \tag{2.24}$$

The anisotropic invariants $\mathrm{J}_4$ and $\mathrm{J}_6$ take into account the deformations along the directions of the primary and secondary internal structures, while $\mathrm{J}_5$ and $\mathrm{J}_7$ are also measures of fiber stretch but include the influence of shear on the fibers [46]; $\mathrm{J}_8$ is instead used to account for the interactions between the two fiber families.[3] If we use the representation theorem for isotropic scalar functions of three symmetric tensors (see [47, 48, 49]), we can write

$$\varphi = \phi(\mathbf{E},\mathbf{A},\mathbf{B}) = \hat{\phi}(\mathrm{J}_1,\mathrm{J}_2,\mathrm{J}_3,\mathrm{J}_4,\mathrm{J}_5,\mathrm{J}_6,\mathrm{J}_7,\mathrm{J}_8)\,. \tag{2.25}$$

On adopting the representation (2.25) and denoting with $\hat{\phi}_i := \partial\hat{\phi}/\partial\mathrm{J}_i$, $i = 1,\ldots,8$, one has:

$$\begin{aligned}
\frac{\partial\phi}{\partial\mathbf{E}} &= \hat{\phi}_1\mathbf{I} + \hat{\phi}_2(\mathrm{J}_1\mathbf{I}-\mathbf{E}) + 2\hat{\phi}_3\mathrm{J}_3\big(2\mathbf{E}+\mathbf{I}\big)^{-1} + \hat{\phi}_4\mathbf{A} + \hat{\phi}_5(\mathbf{A}\mathbf{E}+\mathbf{E}\mathbf{A})\,,\\
&+ \hat{\phi}_6\mathbf{B} + \hat{\phi}_7(\mathbf{B}\mathbf{E}+\mathbf{E}\mathbf{B}) + \frac{1}{2}\hat{\phi}_8\,(\mathbf{A}\mathbf{B}+\mathbf{B}\mathbf{A})\,,
\end{aligned} \tag{2.26}$$

$$\frac{\partial\phi}{\partial\mathbf{A}} = \hat{\phi}_4\mathbf{E} + \hat{\phi}_5\mathbf{E}^2 + \frac{1}{2}\hat{\phi}_8(\mathbf{E}\mathbf{B}+\mathbf{B}\mathbf{E})\,, \tag{2.27}$$

$$\frac{\partial\phi}{\partial\mathbf{B}} = \hat{\phi}_6\mathbf{E} + \hat{\phi}_7\mathbf{E}^2 + \frac{1}{2}\hat{\phi}_8(\mathbf{E}\mathbf{A}+\mathbf{A}\mathbf{E})\,. \tag{2.28}$$

In the following, for the sake of a lighter notation, we will drop the hat from the derivatives of the strain energy with respect to the invariants.

[3] The coupling invariant $\mathrm{J}_8$ may be defined in different ways. The definition of $\mathrm{J}_8$ in (2.24) is the one proposed in [45]; however, there are other definitions that are sometimes used in the literature. In [18], for instance, the authors use $\mathrm{J}_8 = \mathbf{a}\cdot\mathbf{E}\mathbf{b}$. In this paper we choose the former definition, which keeps the energy invariant under change of sign of either $\mathbf{a}$ or $\mathbf{b}$.

*Remark.* It is often customary to express $\hat{\phi}$ in terms of the invariants of the right Cauchy–Green strain tensor $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ rather than $\mathbf{E}$:

$$\begin{aligned} &\mathrm{I}_1 = \mathbf{C}\cdot\mathbf{I}, && \mathrm{I}_2 = \mathbf{C}^{\dagger}\cdot\mathbf{I}, && \mathrm{I}_3 = \det\mathbf{C}\,, \\ &\mathrm{I}_4 = \mathbf{C}\cdot\mathbf{A}, && \mathrm{I}_5 = \mathbf{C}^2\cdot\mathbf{A}, && \\ &\mathrm{I}_6 = \mathbf{C}\cdot\mathbf{B}, && \mathrm{I}_7 = \mathbf{C}^2\cdot\mathbf{B}, && \\ &\mathrm{I}_8 = \mathbf{C}\cdot\mathrm{sym}\,(\mathbf{A}\mathbf{B}). && && \end{aligned} \tag{2.29}$$

However, since the relationship $\mathbf{C} = 2\mathbf{E} + \mathbf{I}$ holds true, the two sets of invariants are connected by the following transformations:

$$\begin{aligned} &\mathrm{J}_1 = \frac{1}{2}\big(\mathrm{I}_1 - 3\big), && \mathrm{J}_2 = \frac{1}{4}\big(\mathrm{I}_2 - 3\big) - \frac{1}{2}\big(\mathrm{I}_1 - 3\big), && \mathrm{J}_3 = \mathrm{I}_3, \\ &\mathrm{J}_4 = \frac{1}{2}\big(\mathrm{I}_4 - 1\big), && \mathrm{J}_5 = \frac{1}{4}\big(\mathrm{I}_5 - 1\big) - \frac{1}{2}\big(\mathrm{I}_4 - 1\big), && \\ &\mathrm{J}_6 = \frac{1}{2}\big(\mathrm{I}_6 - 1\big), && \mathrm{J}_7 = \frac{1}{4}\big(\mathrm{I}_7 - 1\big) - \frac{1}{2}\big(\mathrm{I}_6 - 1\big), && \\ &\mathrm{J}_8 = \frac{1}{2}\big(\mathrm{I}_8 - \mathbf{A}\cdot\mathbf{B}\big). && && \end{aligned} \tag{2.30}$$

## 3. Characterization of the remodeling stationary solutions

In [36], it was proved that, for materials equipped with a reorientable transversely isotropic internal structure descrived by a rotation field $\mathbf{R}$ and in absence of external stimuli, the stationary solutions of the remodeling equations are those rotations which make stress $\mathbf{S}_c$ and strain $\mathbf{E}$, or equivalently $\mathbf{C}$, coaxial. In addition, those rotations render the map

$$\sigma : \mathbb{R}\mathtt{ot} \ni \mathbf{R} \mapsto \sigma(\mathbf{R}) = \psi(\mathbf{C},\mathbf{R})$$

stationary, where $\psi(\mathbf{C},\mathbf{R})$ is the elastic strain energy (see Proposition 1 of [36] and also [38]).
In this Section, it is shown that these results can be partially extended to include the anisotropic double-fibered internal structure considered here. To do so, we first note that the results in [38], often referred to as *Vianello's coaxiality theorem*, hold true whatever class of material symmetry is considered, either isotropic or orthotropic, granted that all fiber families are transformed under the same rotation acting on the body. It is then reasonable to expect that, under our more general framework in which two rotations appear, there could be a loss of equivalence between coaxiality and stationarity of the energy, as we shall prove in the following.

### 3.1. Stationarity and coaxiality

Before passing to the main result of this Section, we recall that two symmetric tensors $\mathbf{U}, \mathbf{V}$ are said to be coaxial if they commute, or equivalently if their commutator vanishes, i.e. $[\mathbf{U}, \mathbf{V}] = \mathbf{0}$. Moreover, we prove the following relation:

$$[\frac{\partial \phi}{\partial \mathbf{A}}, \mathbf{A}] + [\frac{\partial \phi}{\partial \mathbf{B}}, \mathbf{B}] = [\mathbf{E}, \mathbf{S}_c] \,. \tag{3.31}$$

By recalling (2.27) and (2.28), the left-hand side of (3.31) can be written as

$$\begin{aligned}
[\frac{\partial \phi}{\partial \mathbf{A}}, \mathbf{A}] + [\frac{\partial \phi}{\partial \mathbf{B}}, \mathbf{B}] &= \phi_4[\mathbf{E}, \mathbf{A}] + \phi_5[\mathbf{E}^2, \mathbf{A}] + \phi_6[\mathbf{E}, \mathbf{B}] + \phi_7[\mathbf{E}^2, \mathbf{B}] \\
&+ \frac{\phi_8}{2}(\mathbf{EBA} + \mathbf{BEA} - \mathbf{AEB} - \mathbf{ABE}) \\
&+ \frac{\phi_8}{2}(\mathbf{EAB} + \mathbf{AEB} - \mathbf{BEA} - \mathbf{BAE}) \\
&= \phi_4[\mathbf{E}, \mathbf{A}] + \phi_5[\mathbf{E}^2, \mathbf{A}] + \phi_6[\mathbf{E}, \mathbf{B}] + \phi_7[\mathbf{E}^2, \mathbf{B}] \\
&+ \frac{\phi_8}{2}\mathbf{E}(\mathbf{BA} + \mathbf{AB}) - \frac{\phi_8}{2}(\mathbf{AB} + \mathbf{BA})\mathbf{E} \\
&= \phi_4[\mathbf{E}, \mathbf{A}] + \phi_5[\mathbf{E}^2, \mathbf{A}] + \phi_6[\mathbf{E}, \mathbf{B}] + \phi_7[\mathbf{E}^2, \mathbf{B}] + \phi_8[\mathbf{E}, \mathrm{sym}\,(\mathbf{AB})]
\end{aligned}$$

which, in view of (2.26) and (2.17), is indeed equivalent to

$$\phi_4[\mathbf{E}, \mathbf{A}] + \phi_5[\mathbf{E}, \mathbf{AE} + \mathbf{EA}] + \phi_6[\mathbf{E}, \mathbf{B}] + \phi_7[\mathbf{E}, \mathbf{BE} + \mathbf{EB}] + \phi_8[\mathbf{E}, \mathrm{sym}\,(\mathbf{AB})] = [\mathbf{E}, \mathbf{S}_c] \,,$$

which proves Eq. (3.31). Then, at stationarity, since both the commutators on the left-hand side of (3.31) vanish, $\left[\mathbf{E}, \mathbf{S}_c\right] = \mathbf{0}$ holds, that is, the stationary solutions of (2.20) in the passive case are rotations $\left(\mathbf{R}_p^*, \mathbf{R}_s^*\right)$ which make stress and strain coaxial. This result generalizes the one derived in [36] for transversely isotropic materials. We note however that the inverse statement is not necessarily true: in fact, coaxiality of stress and strain does not imply that both commutators have to vanish, and therefore need not be equivalent to a stationary solution of the remodeling system.

Taking into account this observations, we are now in the position of proving the following result [4].

[4] Proposition 1 is formulated in terms of the Green–Lagrange strain tensor $\mathbf{E}$. However, since the principal directions of $\mathbf{E}$ and $\mathbf{C}$ coincide, it could be equivalently stated in terms of the right Cauchy–Green strain tensor $\mathbf{C}$ as was done in [36].

**Proposition 1.** *Let* $\mathbf{E}$ *be a given deformation.*

*(a)* $(\mathbf{R}_p^*, \mathbf{R}_s^*)$ *is a stationary solution of the passive remodeling system of equations if and only if* $(\mathbf{R}_p^*, \mathbf{R}_s^*)$ *is a critical point of the map*

$$\sigma : \mathbb{R}\mathtt{ot} \times \mathbb{R}\mathtt{ot} \to \mathbb{R} \tag{3.32}$$
$$(\mathbf{R}_p, \mathbf{R}_s) \mapsto \sigma(\mathbf{R}_p, \mathbf{R}_s) = \phi(\mathbf{E}, \mathbf{A}, \mathbf{B}) = \phi(\mathbf{E}, \mathbf{R}_p \mathbf{A}_0 \mathbf{R}_p^{\mathrm{T}}, \mathbf{R}_s \mathbf{B}_0 \mathbf{R}_s^{\mathrm{T}})\,,$$

*where* $\phi$ *is the strain energy density.*

*(b) If* $(\mathbf{R}_p^*, \mathbf{R}_s^*)$ *is a stationary solution of the remodelling system of equations, then the stress* $\mathbf{S}_c^* = \boldsymbol{\mathcal{S}}(\mathbf{E}, \mathbf{R}_p^*, \mathbf{R}_s^*)$ *and strain* $\mathbf{E}$ *tensors are* coaxial.

*Proof.* Statement (b) follows from the discussion carried out at the beginning of this Subsection. As a matter of fact, if $(\mathbf{R}_p^*, \mathbf{R}_s^*)$ is a stationary solution of the passive remodelling system (2.20), then

$$[\frac{\partial \phi}{\partial \mathbf{A}}, \mathbf{A}]^* = \mathbf{0} \quad \text{and} \quad [\frac{\partial \phi}{\partial \mathbf{B}}, \mathbf{B}]^* = \mathbf{0}\,,$$

where we have used a superscript $*$ to denote quantities at stationarity. Consequently, by (3.31),

$$\mathbf{0} = [\frac{\partial \phi}{\partial \mathbf{A}}, \mathbf{A}]^* + [\frac{\partial \phi}{\partial \mathbf{B}}, \mathbf{B}]^* = [\mathbf{E}, \mathbf{S}_c^*]$$

and therefore the stress and strain tensors are coaxial. To prove statement (a), we follow the procedure put forward by Vianello [38]. In this case, however, we need to exploit the canonical isomorphism between the tangent space $\mathrm{Rot}(\mathbf{R}_p, \mathbf{R}_s)$ to the product manifold $\mathbb{R}\mathtt{ot} \times \mathbb{R}\mathtt{ot}$ at $(\mathbf{R}_p, \mathbf{R}_s)$ and the product space $\mathbb{S}\mathrm{kw} \times \mathbb{S}\mathrm{kw}$, for which

$$\mathrm{Rot}(\mathbf{R}_p, \mathbf{R}_s) = \{(\mathbf{W}_p \mathbf{R}_p, \mathbf{W}_s \mathbf{R}_s) \,|\, (\mathbf{W}_p, \mathbf{W}_s) \in \mathbb{S}\mathrm{kw} \times \mathbb{S}\mathrm{kw}\}.$$

In such a case, the derivative of the energy at $(\mathbf{R}_p, \mathbf{R}_s)$ in the direction $(\mathbf{W}_p \mathbf{R}_p, \mathbf{W}_s \mathbf{R}_s)$ becomes

$$\begin{aligned}\dot{\sigma}(\mathbf{R}_p, \mathbf{R}_s) = D\sigma(\mathbf{R}_p, \mathbf{R}_s)[\mathbf{W}_p \mathbf{R}_p, \mathbf{W}_s \mathbf{R}_s] &= \frac{\partial \phi}{\partial \mathbf{R}_p} \cdot \mathbf{W}_p \mathbf{R}_p + \frac{\partial \phi}{\partial \mathbf{R}_s} \cdot \mathbf{W}_s \mathbf{R}_s \\ &= [\frac{\partial \phi}{\partial \mathbf{A}}, \mathbf{A}] \cdot \mathbf{W}_p + [\frac{\partial \phi}{\partial \mathbf{B}}, \mathbf{B}] \cdot \mathbf{W}_s\,,\end{aligned}$$

which is null for every pair $(\mathbf{W}_p, \mathbf{W}_s) \in \mathbb{S}\mathrm{kw} \times \mathbb{S}\mathrm{kw}$ if and only if

$$[\frac{\partial \phi}{\partial \mathbf{A}}, \mathbf{A}]^* = \mathbf{0} \qquad \text{and} \qquad [\frac{\partial \phi}{\partial \mathbf{B}}, \mathbf{B}]^* = \mathbf{0}, \tag{3.33}$$

i.e., if and only if $(\mathbf{R}_p^*, \mathbf{R}_s^*)$ is a stationary solution of the remodelling equations. □

Previous derivations confirm the loss of equivalence between critical points of the energy and coaxiality of stress and strain; indeed, in this anisotropic TFF context, stationarity is a stronger requirement than coaxiality, since it requires that both commutators have to vanish. In other words, the set of stationary solutions is a proper subset of the set of rotations that make the stress and strain coaxial.

## 4. In-plane remodeling

There are many situations of interest in which both fiber families are constrained to rotate in the same plane; we refer to this situation as *in–plane remodeling* and study it in more detail. By introducing an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in the vector space $\mathcal{V}$, we assume that fibers lie in the plane $\{\mathbf{e}_1, \mathbf{e}_2\}$, whereas the rotations have an axis parallel to $\mathbf{e}_3$ ($\mathbf{R}_p\mathbf{e}_3 = \mathbf{R}_s\mathbf{e}_3 = \mathbf{e}_3$). We also assume that the mobility tensors are both spherical and determined by two positive scalar constants $m_p = \mu\tau_p$ and $m_s = \mu\tau_s$, with $m_p \neq m_s$ in general, $\mu$ is a shear modulus and $\tau_p$, $\tau_s$ are two characteristic times of the remodeling processes:

$$\mathbb{D}_p = m_p\,\mathbb{I} \qquad \text{and} \qquad \mathbb{D}_s = m_s\,\mathbb{I}, \tag{4.34}$$

with $\mathbb{I}$ the fourth-order identity tensor with components $\mathbb{I}_{ijkl} = \delta_{ik}\delta_{jl}$. Taking into account that $\dot{\mathbf{a}} = \dot{\mathbf{R}}_p\mathbf{R}_p^{\mathrm{T}}\mathbf{a}$ and $\dot{\mathbf{b}} = \dot{\mathbf{R}}_s\mathbf{R}_s^{\mathrm{T}}\mathbf{b}$, the remodeling system of equations becomes

$$\begin{cases} m_p\,\dot{\mathbf{a}} = -[\dfrac{\partial \phi}{\partial \mathbf{A}}, \mathbf{A}]\mathbf{a}\,, \\ m_s\,\dot{\mathbf{b}} = -[\dfrac{\partial \phi}{\partial \mathbf{B}}, \mathbf{B}]\mathbf{b}\,. \end{cases} \tag{4.35}$$

On recalling Eq. (2.27)-(2.28), the previous system can be recast in the following form:

$$\begin{cases} m_p\,\dot{\mathbf{a}} = \big(\mathbf{A} - \mathbf{I}\big)\big(\phi_4\,\mathbf{E}\mathbf{a} + \phi_5\,\mathbf{E}^2\mathbf{a} + \frac{\phi_8}{2}\,\mathbf{B}\mathbf{E}\mathbf{a} + \frac{\phi_8}{2}\,\mathbf{E}\mathbf{B}\mathbf{a}\big)\,, \\ m_s\,\dot{\mathbf{b}} = \big(\mathbf{B} - \mathbf{I}\big)\big(\phi_6\,\mathbf{E}\mathbf{b} + \phi_7\,\mathbf{E}^2\mathbf{b} + \frac{\phi_8}{2}\,\mathbf{A}\mathbf{E}\mathbf{b} + \frac{\phi_8}{2}\,\mathbf{E}\mathbf{A}\mathbf{b}\big)\,. \end{cases} \tag{4.36}$$

Since the problem is in-plane, it is convenient to introduce a parametrization in terms of the remodeling angles $\theta_p$ and $\theta_s$, that is, we set $\mathbf{a} = \cos\theta_p\,\mathbf{e}_1 + \sin\theta_p\,\mathbf{e}_2$, $\mathbf{b} = \cos\theta_s\,\mathbf{e}_1 + \sin\theta_s\,\mathbf{e}_2$. In addition, without loss of generality, we can assume that $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are the principal strain directions of $\mathbf{E}$ associated with the principal strains $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$.
Equations (4.36) correspond therefore to the following system of scalar evolution equations for the angles:

$$\begin{cases} 2\,m_p\,\dot{\theta}_p = \left[\phi_4 + \frac{1}{2}\phi_8 + \phi_5(\varepsilon_1+\varepsilon_2)\right](\varepsilon_1-\varepsilon_2)\sin 2\theta_p + \frac{1}{2}\phi_8(\varepsilon_1+\varepsilon_2)\sin\left[2(\theta_p-\theta_s)\right], \\ 2\,m_s\,\dot{\theta}_s = \left[\phi_6 + \frac{1}{2}\phi_8 + \phi_7(\varepsilon_1+\varepsilon_2)\right](\varepsilon_1-\varepsilon_2)\sin 2\theta_s + \frac{1}{2}\phi_8(\varepsilon_1+\varepsilon_2)\sin\left[2(\theta_s-\theta_p)\right], \end{cases} \tag{4.37}$$

to be solved with the initial conditions $\theta_p(0) = \theta_{p0}$ and $\theta_s(0) = \theta_{s0}$, in which $\theta_{p0}$, $\theta_{s0}$ are the referential primary and secondary orientation angles.

The following observations are duly noted:

1. depending on the representation form of the elastic strain energy, the strain component $\varepsilon_3$ of $\mathbf{E}$ may or may not enter the equations (4.37); indeed, $\varepsilon_3$ does not appear explicitly, but it may be included in one of the energy derivatives $\phi_i$, $i = 4\ldots 8$. For the sake of simplicity, in the following we will assume $\varepsilon_3 = 0$;

2. the directions of principal strain $\mathbf{e}_1$, $\mathbf{e}_2$, that correspond to $\theta_p = k\frac{\pi}{2}$ and $\theta_s = j\,\frac{\pi}{2}$ $(k, j = 0, 1, 2...)$ are stationary solutions of the remodelling equations (4.37);

3. if $\varepsilon_1 = \varepsilon_2 = \varepsilon$, i.e., the deformation is equibiaxial, the system of equations (4.37) simplifies into

$$\begin{cases} m_p\,\dot{\theta}_p = \frac{1}{2}\phi_8\,\varepsilon\,\sin\left[2(\theta_p-\theta_s)\right] \\ m_s\,\dot{\theta}_s = \frac{1}{2}\phi_8\,\varepsilon\,\sin\left[2(\theta_s-\theta_p)\right] \end{cases} \tag{4.38}$$

   and the stationary solutions are achieved either if $\phi_8(\varepsilon, \theta_p, \theta_s) = 0$ or if $\sin 2(\theta_p - \theta_s) = 0$. The latter condition means that $\Delta\theta = \theta_s - \theta_p = k\,\frac{\pi}{2}$ $(k = 0, 1, ...)$, i.e., the fibers become either parallel or orthogonal, yet their "absolute" angle with respect to $\mathbf{e}_1$ remains undetermined. On the other hand, if $\phi_8 = 0$ identically, meaning that the energy does not depend on $\mathrm{J}_8$, then no remodeling occurs, as found also for transversely isotropic materials [36].

To solve the remodeling problem (4.37) and make comparisons with experiments, an *ansätz* on the strain energy function shall be made. In this respect, one could consider the most general form of the strain energy function which is quadratic in the deformation measures. To do so, we generalize the well-known Saint Venant–Kirchhoff model, that in its original formulation only depends on the isotropic invariants, to a double-fibered material [50, 51]. The minimal representation of that energy which provides the simplest coupling between the fibers and allows to fit data from experiments on biaxial tests is:

$$\hat{\phi}(\mathrm{J}_1,\ldots,\mathrm{J}_8) = \frac{1}{2}k_1\,\mathrm{J}_1^2 + k_2\,\mathrm{J}_2 + \frac{1}{2}k_4\,\mathrm{J}_4^2 + \frac{1}{2}k_6\,\mathrm{J}_6^2 + \\ + k_{14}\,\mathrm{J}_1\mathrm{J}_4 + k_{16}\,\mathrm{J}_1\mathrm{J}_6 + \frac{1}{2}\,k_8\mathrm{J}_8^2. \tag{4.39}$$

With these assumptions, the remodelling problem takes the following form:

$$2\,m_p\,\dot{\theta}_p = (1+r)\varepsilon_1^2\,F(\theta_p, r; k_4, k_{14}) + \frac{1}{2}\varepsilon_1^2\,k_8\,[G(\theta_p,\theta_s,r) + H(\theta_p,\theta_s,r)]\ , \tag{4.40}$$

$$2\,m_s\,\dot{\theta}_s = (1+r)\varepsilon_1^2\,F(\theta_s, r; k_6, k_{16}) + \frac{1}{2}\varepsilon_1^2\,k_8\,[G(\theta_s,\theta_p,r) + H(\theta_s,\theta_p,r)]\ , \tag{4.41}$$

where we have introduced the biaxiality ratio $r := -\varepsilon_2/\varepsilon_1$ between the transverse and longitudinal deformation, while the functions $F$, $G$ and $H$ are defined as

$$F(\theta_p, r; k_4, k_{14}) := \big(k_4(\cos^2\theta_p - r\sin^2\theta_p) + k_{14}(1-r)\big)\sin 2\theta_p\,, \tag{4.42}$$

$$G(\theta_p,\theta_s,r) := 2(1-r)\cos^2(\theta_p-\theta_s)\big(\cos\theta_p\cos\theta_s - r\sin\theta_p\sin\theta_s\big)\ \sin(\theta_p-\theta_s)\,, \tag{4.43}$$

$$H(\theta_p,\theta_s,r) := (1+r)\cos(\theta_p-\theta_s)\big(\cos\theta_p\cos\theta_s - r\sin\theta_p\sin\theta_s\big)\ \sin 2\theta_p\,. \tag{4.44}$$

A discussion of the reorientation dynamics prescribed by the equations above is postponed to Section 6, where inspired by experiments on cell layers we will be able to estimate some of the constitutive coefficients $(k_4, k_{14}, k_6, k_{16}, k_8)$.

*Remark.* The representation form of the equations (4.40) and (4.41) suggests to identify a system of elastic actions which contribute to the time rate of the fiber orientation angles $\theta_p$ and $\theta_s$. Firstly, we introduce the pair

$$\tau_p := (1{+}r)\varepsilon_1^2\,F(\theta_p, r; k_4, k_{14}) \quad \text{and} \quad \tau_s := (1{+}r)\varepsilon_1^2\,F(\theta_s, r; k_6, k_{16})\,. \tag{4.45}$$

Then, we note that

$$G(\theta_s, \theta_p, r) = -G(\theta_p, \theta_s, r)\,. \tag{4.46}$$

As a consequence, we can identify the following mutual elastic interaction between the fibers:

$$\tau_{ps} := \frac{1}{2}\varepsilon_1^2\, k_8\, G(\theta_p, \theta_s, r)\,, \tag{4.47}$$

dependent upon the constitutive coefficient $k_8$. In addition, we can define

$$\tau_{Hp} := \frac{1}{2}\varepsilon_1^2\, k_8 H(\theta_p, \theta_s, r) \quad \text{and} \quad \tau_{Hs} := \frac{1}{2}\varepsilon_1^2\, k_8 H(\theta_s, \theta_p, r)\,. \tag{4.48}$$

With this on hand, the remodeling equations are restated as

$$2m_p\,\dot{\theta}_p = \tau_p + \tau_{ps} + \tau_{Hp} \quad \text{and} \quad 2m_s\,\dot{\theta}_s = \tau_s - \tau_{ps} + \tau_{Hs}\,, \tag{4.49}$$

which have the following physical interpretation: the time evolution of the orientation of the primary (resp. secondary) angle depends on the elastic actions $\tau_p$ (resp. $\tau_s$), $\tau_{Hp}$ (resp. $\tau_{Hs}$) and $\tau_{ps}$ (resp. $-\tau_{ps}$). $\tau_p$ is defined in Eq. (4.45) and depends on the orientation $\theta_p$ (resp. $\theta_s$) of the fiber itself and on the stretches $\varepsilon_1$, $\varepsilon_2$. On the other hand, the interactions $\tau_{ps}$ and $\tau_{Hp}$ (resp. $\tau_{Hs}$) depend on both the orientations $\theta_p$ and $\theta_s$ and vanish when the two fiber families are uncoupled, that is when $k_8$ is zero.
It is of particular interest the evaluation of $\tau_p$, $\tau_s$, $\tau_{Hp}$, $\tau_{Hs}$ and $\tau_{ps}$ when the stretch is equibiaxial, i.e., for $r = -1$, and then $\varepsilon_1 = \varepsilon_2 = \varepsilon$. In such a case, Eqs. (4.43), (4.47) and (4.48) imply $\tau_p = \tau_s = \tau_{Hp} = \tau_{Hs} = 0$ and

$$\tau_{ps} = 2k_8\,\varepsilon^2 \cos^3(\theta_p - \theta_s)\,\sin(\theta_p - \theta_s)\,, \tag{4.50}$$

meaning that the only non-zero term in the right-hand side of (4.49) is $\tau_{ps}$. Since stationarity under equibiaxal stretch is attained if $\sin 2(\theta_p - \theta_s) = 0$, Eq. (4.50) states that the stationary value of the elastic interaction is zero.

## 5. Constrained reorientation

There may be practical cases in which the rotation of a fiber family induces the same rotation in the other family, in the sense that, during the remodeling process, the difference between the reoriented fiber angles remains the same. This behaviour is introduced in the modeling framework detailed above by properly constraining the elastic energy with a Lagrange multiplier. Among the many possible choices (see for instance [52]), guided by the experiments in [22] in which the *relative orientation* is fixed, we require that the primary $\mathbf{R}_p$ and secondary $\mathbf{R}_s$ rotation tensors are equal at any time instant. As such, we introduce the representation of rotation tensors through the exponential maps $\mathcal{B}_r \times \mathcal{T} \rightarrow \mathbb{S}\mathrm{kw}$ [52] given by

$$\mathbf{R}_p = \exp \boldsymbol{\Omega}_p \quad \text{and} \quad \mathbf{R}_s = \exp \boldsymbol{\Omega}_s \,. \tag{5.51}$$

Then, $\boldsymbol{\Omega}_p = \boldsymbol{\Omega}_s$ implies $\mathbf{R}_p = \mathbf{R}_s$, and the constraint expression takes the form

$$\mathbf{c}(\boldsymbol{\Omega}_p, \boldsymbol{\Omega}_s) = \boldsymbol{\Omega}_p - \boldsymbol{\Omega}_s \,, \tag{5.52}$$

with the corresponding constrained energy given by

$$\widetilde{\varphi}(\mathbf{E}, \mathbf{A}, \mathbf{B}) = \phi(\mathbf{E}, \mathbf{A}, \mathbf{B}) + \mathbf{K} \cdot \, \mathbf{c}(\boldsymbol{\Omega}_p, \boldsymbol{\Omega}_s) \,. \tag{5.53}$$

Hence, the skew tensor field $\mathbf{K}$ is the Lagrange multiplier, which physically represents the reaction needed to maintain the relative rotation between the two fiber families the same. Equation (5.52) implies

$$\dot{\mathbf{c}}(\mathbf{R}_p, \mathbf{R}_s) = \dot{\mathbf{R}}_p \mathbf{R}_p^{\mathrm{T}} - \dot{\mathbf{R}}_s \mathbf{R}_s^{\mathrm{T}} \,. \tag{5.54}$$

On substituting (5.52) and (5.53) into the dissipation inequality, we arrive at

$$\begin{aligned} &\big( - \mathbf{S} + \mathbf{F} \frac{\partial \phi}{\partial \mathbf{E}} \big) \cdot \dot{\mathbf{F}} + \dot{\mathbf{K}} \cdot \mathbf{c}(\boldsymbol{\Omega}_p, \boldsymbol{\Omega}_s) \\ &+ \big( - \boldsymbol{\Sigma}_p + [\frac{\partial \phi}{\partial \mathbf{A}}, \mathbf{A}] + \mathbf{K} \big) \cdot \dot{\mathbf{R}}_p \mathbf{R}_p^{\mathrm{T}} \\ &+ \big( - \boldsymbol{\Sigma}_s + [\frac{\partial \phi}{\partial \mathbf{B}}, \mathbf{B}] - \mathbf{K} \big) \cdot \dot{\mathbf{R}}_s \mathbf{R}_s^{\mathrm{T}} \leq 0 \,, \end{aligned}$$

which leads to the following system of *constrained* remodelling equations:

$$\left\{ \begin{aligned} &\mathbb{D}_p \dot{\mathbf{R}}_p \mathbf{R}_p^{\mathrm{T}} = -[\frac{\partial \phi}{\partial \mathbf{A}}, \mathbf{A}] - \mathbf{K} \,, \\ &\mathbb{D}_s \dot{\mathbf{R}}_s \mathbf{R}_s^{\mathrm{T}} = -[\frac{\partial \phi}{\partial \mathbf{B}}, \mathbf{B}] + \mathbf{K} \,, \\ &\mathbf{c}(\boldsymbol{\Omega}_p, \boldsymbol{\Omega}_s) = \mathbf{0} \,. \end{aligned} \right. \tag{5.55}$$

Summing up the first two equations in (5.55), we obtain an evolution equation independent of the Lagrange multiplier $\mathbf{K}$:

$$\mathbb{D}_p \dot{\mathbf{R}}_p \mathbf{R}_p^{\mathrm{T}} + \mathbb{D}_s \dot{\mathbf{R}}_s \mathbf{R}_s^{\mathrm{T}} = -[\frac{\partial \phi}{\partial \mathbf{A}}, \mathbf{A}] - [\frac{\partial \phi}{\partial \mathbf{B}}, \mathbf{B}]$$

which, at stationarity, gives

$$[\frac{\partial \phi}{\partial \mathbf{A}}, \mathbf{A}]^* + [\frac{\partial \phi}{\partial \mathbf{B}}, \mathbf{B}]^* = \mathbf{0} = [\mathbf{E}, \mathbf{S}_c]^*, \tag{5.56}$$

recovering the result in Eq. (3.31). On the other hand, by subtracting $(5.55)_1$ from $(5.55)_2$ and using (5.56), at stationarity we obtain

$$\mathbf{K}^* = [\frac{\partial \phi}{\partial \mathbf{B}}, \mathbf{B}]^* . \tag{5.57}$$

which furnishes the stationary values of the Lagrange multiplier $\mathbf{K}^*$. Equations (5.56), (5.57) together with the constraint equation $(5.55)_3$ are a system of nine equations. The nine unknowns are the six components $(a_i, b_i)$ $(i = 1, 2, 3)$ and the three independent components of $\mathbf{K} \in \mathbb{S}\mathrm{kw}$. Therefore, in this constrained case, the equivalence between coaxiality of stress and strain and stationarity of the solutions of the remodeling equation is recovered thanks to the introduction of the multiplier $\mathbf{K}$. Indeed, if $(\mathbf{R}_p^*, \mathbf{R}_s^*)$ is a pair of rotations compatible with the constraint that makes stress and strain coaxial, then (5.56) is trivially satisfied. However, the reaction term $\mathbf{K}$ and the constraint equation guarantee that $(\mathbf{R}_p^*, \mathbf{R}_s^*)$ is also a stationary solution, as it is possible to find a value of $\mathbf{K}^*$ satisfying Eq. (5.57) once $\mathbf{a}^*$ and $\mathbf{b}^*$ are known.

### *5.1. In-plane remodeling under constraint*

When in-plane remodeling is considered, we set $\mathbf{\Omega}_p = (\theta_p - \theta_{p0}) \star \mathbf{e}_3$ and $\mathbf{\Omega}_s = (\theta_s - \theta_{s0}) \star \mathbf{e}_3$, where the operator $\star : \mathcal{V} \to \mathbb{S}\mathrm{kw}$ associates to each vector $\mathbf{w}$ a skew tensor $\mathbf{W}$ whose axial vector is $\mathbf{w}$, i.e. $(\star\mathbf{w})\mathbf{u} = \mathbf{w} \times \mathbf{u}$ for any $\mathbf{u} \in \mathcal{V}$. With this notation, we have $\mathbf{K} = \kappa \star \mathbf{e}_3$. As a consequence, the constraint reduces to

$$c(\theta_p, \theta_s) = (\theta_p - \theta_s) - (\theta_{p0} - \theta_{s0}) , \tag{5.58}$$

whence the remodeling equations become

$$\begin{cases} 2\, m_p\, \dot{\theta}_p = (1+r)\, \varepsilon_1^2 F(\theta_p, r; k_4, k_{14}) + \frac{1}{2}\varepsilon_1^2\, k_8\, [G(\theta_p, \theta_s, r) + H(\theta_p, \theta_s, r)] - 2\kappa \\ 2\, m_s\, \dot{\theta}_s = (1+r)\, \varepsilon_1^2 F(\theta_s, r; k_6, k_{16}) + \frac{1}{2}\varepsilon_1^2\, k_8\, [G(\theta_s, \theta_p, r) + H(\theta_s, \theta_p, r)] + 2\kappa \\ \theta_p - \theta_s = \theta_{p0} - \theta_{s0} \,. \end{cases} \tag{5.59}$$

The constraint equation $(5.59)_3$ implies

$$\dot{\theta}_p - \dot{\theta}_s = 0$$

from which by summing and subtracting $(5.59)_1$ and $(5.59)_2$ we get

$$2(m_p + m_s)\dot{\theta}_p = (1+r)\varepsilon_1^2 \left[F(\theta_p, r; k_4, k_{14}) + F(\theta_s, r; k_6, k_{16})\right] + \frac{1}{2}\varepsilon_1^2 k_8 \left[H(\theta_p, \theta_s, r) + H(\theta_s, \theta_p, r)\right] \tag{5.60}$$

and

$$2(m_p - m_s)\dot{\theta}_p = (1+r)\varepsilon_1^2 \left[F(\theta_p, r; k_4, k_{14}) - F(\theta_s, r; k_6, k_{16})\right] + \frac{1}{2}\varepsilon_1^2 k_8 \left[2G(\theta_p, \theta_s, r) + H(\theta_p, \theta_s, r) - H(\theta_s, \theta_p, r)\right] - 4\kappa, \tag{5.61}$$

respectively, recalling (4.46). When the deformation is equibiaxial, $r = -1$ and by using the definitions of the functions $F$, $G$ and $H$, Eq. (5.60) gives

$$2(m_p + m_s)\dot{\theta}_p = 0, \tag{5.62}$$

thus $\dot{\theta}_s = \dot{\theta}_p = 0$ and no evolution occurs as happened in the transversely isotropic case [36]. Moreover, Eq. (5.61) gives

$$\kappa = k_8\, \varepsilon_1^2 \cos^3(\theta_{p0} - \theta_{s0}) \sin(\theta_{p0} - \theta_{s0}) \tag{5.63}$$

and shows that the reaction $\kappa$ is zero either for fibre initially parallel, i.e., $\theta_{p0} - \theta_{s0} = k\pi$, or orthogonal, i.e., $\theta_{p0} - \theta_{s0} = \frac{\pi}{2} + k\pi$ $(k = 0, 1, ...)$.

*Remark.* Other choices for the constraint (5.52) are indeed possible. One based on constraining the squared inner product between fiber vectors is presented and discussed in the Appendix.

## 6. Examples from Biology

In this Section, we illustrate some applications of our framework in the field of biology, which allow us to discuss interesting implications of the model and to make a comparison with experimental data. In particular, we focus on describing cell orientation under stretch: as mentioned in the Introduction, when a monolayer of cells is stretched biaxially, a reorientation happens until each cell finds a stable configuration, characterized by a certain angle with respect to the direction of greatest stretch. For more details on the biology and mechanics of this phenomenon, we refer the reader to [22, 27, 28, 30, 53] and references therein.

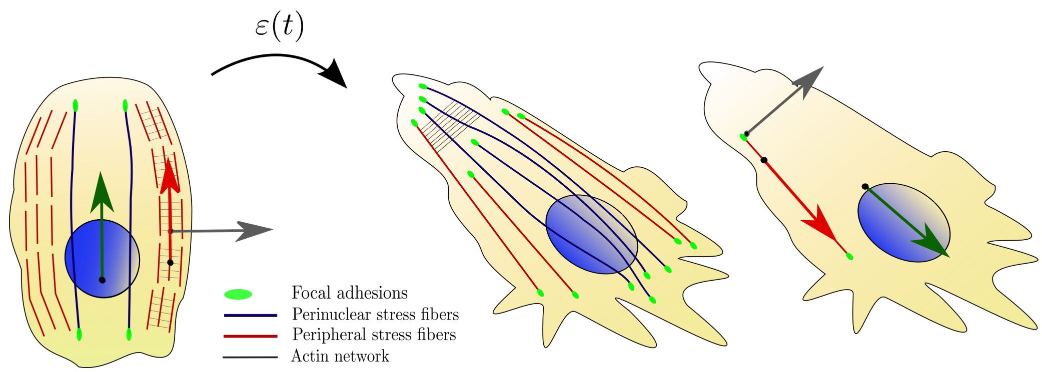


Figure 2: Sketch of the cell structure and reorientation process. For comparison with experimental assays in [22], following [30], we consider the reorientation of the actin peripheral stress fibers (in red) which determine the alignment of the cell body, while the lateral protein network (in gray) is constrained to remain orthogonal to the SF. Instead, to compare our model with results in [29], we consider the peripheral stress fibers as the primary structure, whose orientation is changed according to $\mathbf{R}_p$, while the nucleus and the perinuclear stress fibers (in blue) that drive its orientation represent the secondary structure, evolving with $\mathbf{R}_s$.

### *6.1. The case of constrained reorientation*

Livne and coworkers [22] firstly proposed a model of cell orientation based on the minimization of the stored elastic energy with respect to the angle between the cell and the stretching direction. Later, a generalization of their results using a nonlinear orthotropic energy was presented in [30], with the two fiber families representing the cell aligned stress fibers and the orthogonal linking protein network, as sketched in Figure 2. In the following, we show that our model is able to recover the results in [22, 30] as a particular case, without the need of any phenomenological justification of the evolution equation.

As a starting point, following [30], we consider the cell monolayer as an hyperelastic anisotropic material, characterized by the strain energy function defined in Eq. (4.39). Then, we introduce the two assumptions also (implicitly) done in [22]: (1) the families of fibers are orthogonal in the reference configuration and constrained to remain orthogonal during remodeling; and (2) the fibers lie in the plane of unit normal $\mathbf{e}_3$. Finally, we consider a deformation of the substrate defined by $\mathbf{E} = \mathrm{diag}(\varepsilon_1, -r\varepsilon_1, 0)$.

In doing so, we are able to apply the constrained in-plane remodeling theory derived in Section 5.1 and to write the reorientation equations using (5.59). Taking into account that $\theta_{p0} - \theta_{s0} = \pi/2$, the constraint becomes $c(\theta_p, \theta_s) = (\theta_p - \theta_s) - \pi/2 = 0$, which also implies $\dot{c}(\theta_p, \theta_s) = \dot{\theta}_p - \dot{\theta}_s = 0$.

Moreover, the orthogonality of the fibers yields

$$G(\theta_p,\theta_s,r)=G(\theta_s,\theta_p,r)=0 \quad \text{and} \quad H(\theta_p,\theta_s,r)=H(\theta_s,\theta_p,r)=0,$$

and the remodeling equations are simplified as

$$\begin{cases} 2m_p\,\dot\theta_p=(1+r)\varepsilon_1^2\,F(\theta_p,r;k_4,k_{14})-2\kappa\,, \\ 2m_s\,\dot\theta_s=(1+r)\varepsilon_1^2\,F(\theta_s,r;k_6,k_{16})+2\kappa\,. \end{cases} \tag{6.64}$$

Summing up the two equations and exploiting the constraint, we can obtain an evolution equation for the cell orientation $\theta_p$:

$$\widehat{m}\,\dot\theta_p=(1+r)\varepsilon_1^2\left[\widehat{k}_p(\cos^2\theta_p-r\sin^2\theta_p)+k_m(1-r)\right]\sin 2\theta_p\,, \tag{6.65}$$

where

$$\widehat{m}:=2(m_p+m_s)\,,\qquad \widehat{k}_p:=k_4+k_6\,,\qquad k_m:=k_{14}-k_{16}-k_6\,.$$

The stationary solutions of Eq. (6.65) can be readily identified: they correspond either to the cell being aligned with the principal directions of strain, i.e. $\theta_p^*=k\pi/2, k\in\mathbb{Z}$, or to the oblique orientations defined by

$$\cos^2\theta_p^*=1+\frac{k_m}{\widehat{k}_p}-\frac{1}{1+r}\left(1+2\frac{k_m}{\widehat{k}_p}\right)=\frac{1}{2}+\mathcal{K}\left(\frac{1}{2}-\frac{1}{1+r}\right), \tag{6.66}$$

where $\mathcal{K}:=1+2k_m/\widehat{k}_p$. Such a result is coherent with previous findings characterizing the preferential orientations of cells on a stretched substrate, both in linear elasticity [22], nonlinear elasticity [30] and viscoelasticity [31]. However, in our model we did not postulate the evolution equation (6.65) as done in previous works, but rather derived it from a more general framework lying on balance principles and thermodynamics. Clearly, the oblique stationary solutions only exist when $r\neq-1$ and the right-hand side of Eq. (6.66) has a value between 0 and 1; for details on the bifurcation analysis of the orientations, see [30].

The Lagrange multiplier $\kappa$, that represents the reaction needed to keep the fibers orthogonal, can be evaluated at stationarity through

$$4\kappa^*=(1+r)\varepsilon_1^2\left[(k_4-k_6)(\cos^2\theta_p^*-r\sin^2\theta_p^*)+(k_{14}+k_{16}+k_6)(1-r)\right]\sin 2\theta_p^*. \tag{6.67}$$

We observe that, when the cell is aligned with the principal strain directions, $\kappa^*=0$. Indeed the same holds true when $r=-1$ (see (6.65)), i.e. in the

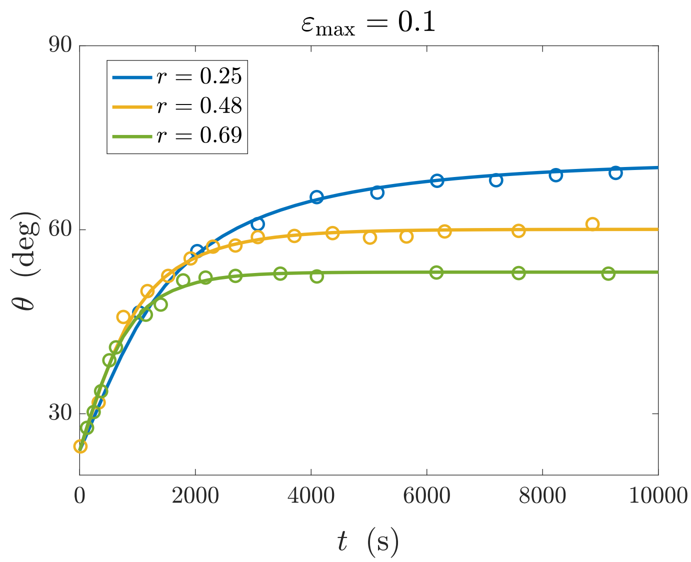


Figure 3: Evolution of the cell orientation angle $\theta_p$ with maximum strain of 10%, following Eq. (6.65), for different values of the biaxiality ratio $r$. The curves show the best fitting of the model, while the dots represent experimental data taken from [22].

case of equibiaxial deformation when no reorientation occurs. For oblique orientations, instead, we have

$$4\kappa^* = 2(1-r^2)\varepsilon_1^2 \, \frac{k_4 k_{16} + k_4 k_6 + k_6 k_{14}}{k_4 + k_6} \, \sin 2\theta_p^* \,, \tag{6.68}$$

which is in general different from zero provided that $r \neq 1$.

We conclude this Section by comparing the prediction of the constrained model with the data on cell reorientation in [22]. It is seen from Eq. (6.65), that the three parameters $\widehat{k}_p, k_m$ and $\tau$, where the latter is the characteris-

| $r$ | $k_m/\widehat{k}_p$ | $\tau$ [$s$] |
|---|---|---|
| 0.25 | 0.1594 | 7.3 |
| 0.48 | 0.2142 | 5.9 |
| 0.69 | 0.2596 | 5.7 |

Table 1: Constitutive parameters in Eq. (6.65) estimated from the fitting of experiments in [22].

tic time in $\widehat{m}$, have to be calibrated. The results of this procedure, carried out through a nonlinear least square algorithm, are shown in Figure 3. The model matches accurately the evolution of the orientation angles seen in the experiments with a relative error below 2 %. The optimal parameters used for the fitting for $r \in \{0.25, 0.48, 0.69\}$ are listed in Table 1 and are indeed coherent with experiments in [22], where the authors found a value of the ratio $k_m/\widehat{k}_p = 0.13 \pm 0.04$ and a characteristic time $\tau = 6.6 \pm 0.4$ s.
It is worth noting that the evolution of the angle in Eq. (6.65) only depends on three constitutive parameters including a characteristic time. The evaluation of the other model coefficients, necessary to estimate the Lagrange multiplier $\kappa^*$, would require further experimental data.

*Remark.* In the general case of two plane fiber families constrained to remain orthogonal, namely such that $\mathbf{A} + \mathbf{B} = \check{\mathbf{I}}$ with $\check{\mathbf{I}}$ the identity tensor in the fiber plane, it can be easily shown that the stationarity requirement

$$[\frac{\partial \phi}{\partial \mathbf{A}}, \mathbf{A}]^* + [\frac{\partial \phi}{\partial \mathbf{B}}, \mathbf{B}]^* = \mathbf{0} \tag{6.69}$$

implies

$$[\mathbf{E}, (\phi_4^* - \phi_6^*)\mathbf{A}^* + (\phi_5^* - \phi_7^*)(\mathbf{A}^*\mathbf{E} + \mathbf{E}\mathbf{A}^*)] = \mathbf{0}\,, \tag{6.70}$$

which has the following consequences:

1. If $\phi_6 = \phi_7 = 0$ identically, meaning that there is a single fiber family, the relation $[\mathbf{E}, \mathbf{S}_c^*] = \mathbf{0}$ for a transversely isotropic material found in [36] is recovered.

2. If the remodeled direction $\mathbf{a}$ is along a principal direction of strain (and then so is $\mathbf{b}$ because they are orthogonal), the commutator on the l.h.s. of (6.70) is null. Therefore, rotations that align the preferential orientations with the principal strain directions are always stationary solutions of the problem at hand.

3. If $\mathbf{a}$ is not aligned with a principal strain direction, there might be additional non-trivial solutions identified by the two conditions

$$\phi_4^*(\mathbf{E}, \mathbf{A}^*) = \phi_6^*(\mathbf{E}, \mathbf{A}^*) \quad \text{and} \quad \phi_5^*(\mathbf{E}, \mathbf{A}^*) = \phi_7^*(\mathbf{E}, \mathbf{A}^*).$$

### *6.2. The case of independent reorientation*

The complex interaction between the cell nucleus and the cytoplasm can be still captured by the proposed model in terms of changes in their relative orientation when cells are cyclically stretched.

Table 2: Constitutive parameters in Eqs. (4.40)-(4.41) identified from the experiments in [29].

| $k_{14}/k_4$ | $k_{16}/k_6$ | $k_6/k_4$ | $k_8/k_4$ | $\tau_p$ $[s]$ | $\tau_s$ $[s]$ |
|---|---|---|---|---|---|
| 26.5 | 3.3 | 9.75 | 75 | 128 | 185 |

In [29], the mechanoadaptive organization of stress fibers and nuclei in epithelial cells under cyclic stretches is considered, highlighting that these two cellular components follow different orientation dynamics. In particular, epithelial cells on a plane substrate were stretched for 2 hours at 5%, 10% and 15% maximum strains whilst measuring the reorientation of the different stress fiber subtypes. It was seen that dorsal stress fibers, transverse arcs, and peripheral stress fibers were mainly involved in the cytoplasm response whereas perinuclear cap fibers were associated with the reorientation and elongation of the nucleus (see Fig. 2).

A sketch of the experimental setup of [29] is shown in Fig. 4(a) where the coloured arrows are used to indicate the primary (red) and secondary (green) cell structures, i.e., cytoplasm and nucleus of the cell, respectively. In the experiments, the longitudinal strain $\varepsilon_1$ was controlled for 2 hours with the time history shown in inset 4(b) at two different level of maximum strain (10% and 15%); the lateral strain was also controlled in a way that $r = -\varepsilon_2/\varepsilon_1 = 0.49$ and the frequency of the deformation was set to 0.3 Hz. The corresponding reorientation velocities are shown in Fig. 4(c) for the primary $\dot{\theta}_p$ and the secondary angle $\dot{\theta}_s$. These graphs were obtained by fitting the experimental data against Eqs. (4.40)-(4.41) (with $m_p = k_4\tau_p$ and $m_s = k_6\tau_s$), through a nonlinear least square algorithm implemented in MATLAB®. The corresponding best fit parameters are listed in Table 2.

It is pointed out that the experimental results at 5% were not used for the fitting process, since they did not show any significant reorientation, probably caused by the poor interaction between cells and substrate at such a low level of strain. Finally, Figs. 4(d)-(e) show the evolution of the average orientation angles during the 2 hours test. The stretch dependence for both the primary and secondary angles is well captured by the model with a maximum relative error of 2.5% only seen for the nucleus in inset (e). The fitting is achieved with a value of the constitutive parameter $k_8$ significantly larger than $k_4$ and $k_6$, indicating a strong interaction between the two fibre families.

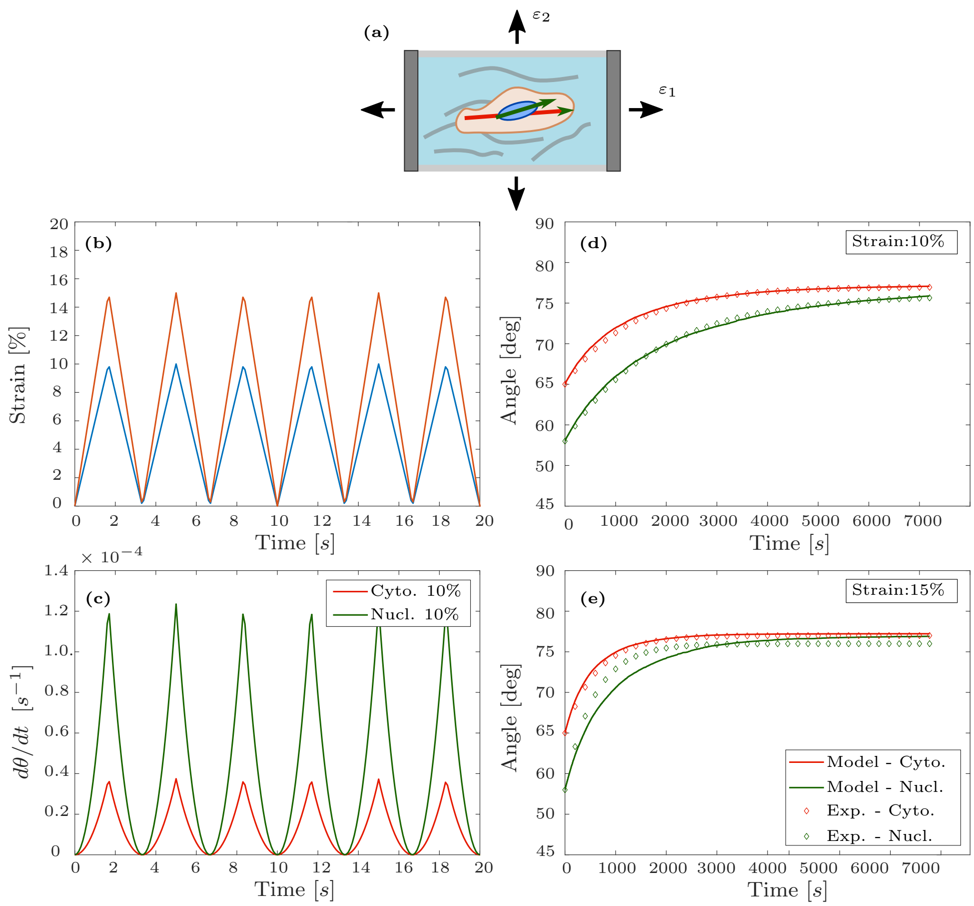


Figure 4: (a): Sketch of the experiments in [29], where the substrate is deformed with a periodic strain. (b): Time history of the longitudinal strain $\varepsilon_1$ imposed to the substrate, with maximum amplitude of 10% (blue curve) and 15% (orange curve). (c): Angular velocity of reorientation for the primary and secondary structures obtained from the model with the optimal estimated constitutive parameters in Tab. 2. (d)–(e): Evolution of the average (over a cycle) orientation angle during the test for the maximum strain 10% (d) and 15% (e), for both the cytoplasm and the nucleus, compared with experimental results in [29].

## 7. Conclusions

We presented a model to describe cell alignment under applied traction using a fiber reorientation framework, in which two different families of fibers are able to rotate independently under the action of two distinct rotation tensors. The framework accounts for the elastic coupling between the fiber families, while the introduction of a proper constraint allowed us to recover some literature results on reorientation of transversely isotropic and orthotropic materials. The stationary solutions of the remodelling equations are then thoroughly studied to find a generalization of a well-known coaxiality theorem by Vianello, that holds in the passive case; active effects, even if not studied in details in the present paper, can be naturally and straightforwardly incorporated in the modelling equations.
As a main application of the proposed framework, we studied cell reorientation under stretch. Perfect adhesion is assumed between the cell layer and the substrate and the rotations of the cells due to mechanical stimuli are viewed as additional state variables of the problem. By using experimental data available from the literature, we have found that the model is able to accurately match the experimental results on cell orientation for both the constrained and unconstrained fiber evolutions. Even though the focus is mainly on biological applications, the proposed framework is indeed general. As such, it could be readily adapted to describe engineering materials like composites with two families of fibers [54], which are engineered to reorient under different stimuli.
The thermodynamically consistent structure of the model offers advantages over the existing phenomenological approaches pursued in the literature and may open the road to future developments. These include: the possibility of having a mobility tensor dependent both on time and on strain so to describe the dynamic response seen for certain types of cells [55, 56]; the incorporation of a non-perfect adhesion between cells and substrate by inducing reorientation only when a certain amount of strain is accumulated [55] or when adhesion bonds are broken as a consequence of mechanical deformation; the inclusion of the matrix viscoelasticity, that in general induces additional characteristic times in the material response. The hierarchic role of the fiber families may also be incorporated by distinguishing between "pulled" and "dragged" reorientations, that occur when one fiber family drives the reorientation of the other through the elastic coupling.

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## Appendix A. An alternative choice for the constraint equation

Here we consider the alternative choice of the scalar constraint

$$c(\mathbf{A},\mathbf{B}) = \mathbf{A}\cdot\mathbf{B} - \mathbf{A}_0\cdot\mathbf{B}_0\,, \tag{A.1}$$

which gives

$$\dot{c}(\mathbf{A},\mathbf{B}) = \dot{\mathbf{A}}\cdot\mathbf{B} + \mathbf{A}\cdot\dot{\mathbf{B}} = [\mathbf{A},\mathbf{B}]\cdot\left(\dot{\mathbf{R}}_s\mathbf{R}_s^{\mathrm{T}} - \dot{\mathbf{R}}_p\mathbf{R}_p^{\mathrm{T}}\right). \tag{A.2}$$

Accordingly, following the procedure detailed in Section 5, the evolution equations of the constrained remodeling problem are written as

$$\left\{\begin{aligned} &\mathbb{D}_p\dot{\mathbf{R}}_p\mathbf{R}_p^{\mathrm{T}} = -[\frac{\partial\phi}{\partial\mathbf{A}},\mathbf{A}] + \kappa\,[\mathbf{A},\mathbf{B}]\,,\\ &\mathbb{D}_s\dot{\mathbf{R}}_s\mathbf{R}_s^{\mathrm{T}} = -[\frac{\partial\phi}{\partial\mathbf{B}},\mathbf{B}] - \kappa\,[\mathbf{A},\mathbf{B}]\,,\\ &\mathbf{A}\cdot\mathbf{B} = \mathbf{A}_0\cdot\mathbf{B}_0\,, \end{aligned}\right. \tag{A.3}$$

having introduced a scalar Lagrange multiplier $\kappa$. At stationarity, summing the two equations one gets

$$[\frac{\partial \phi}{\partial \mathbf{A}}, \mathbf{A}]^* + [\frac{\partial \phi}{\partial \mathbf{B}}, \mathbf{B}]^* = \mathbf{0} = [\mathbf{E}, \mathbf{S}_c]^* \tag{A.4}$$

and subtracting instead yields

$$\kappa^* \, [\mathbf{A}, \mathbf{B}]^* = [\frac{\partial \phi}{\partial \mathbf{A}}, \mathbf{A}]^* \, . \tag{A.5}$$

On assuming that $\mathbf{a} = \cos\theta_p \mathbf{e}_1 + \sin\theta_p \mathbf{e}_2$ and $\mathbf{b} = \cos\theta_s \mathbf{e}_1 + \sin\theta_s \mathbf{e}_2$, the constraint equation (A.1) gives

$$c(\mathbf{a}, \mathbf{b}) = \cos^2(\theta_p - \theta_s) - \cos^2(\theta_{p0} - \theta_{s0}) \, , \tag{A.6}$$

whence the remodeling equations become

$$\begin{cases} 2\, m_p \, \dot{\theta}_p = (1+r)\, \varepsilon_1^2 F(\theta_p, r; k_4, k_{14}) + \frac{1}{2}\varepsilon_1^2 \, k_8 \, [G(\theta_p, \theta_s, r) + H(\theta_p, \theta_s, r)] + \kappa \sin(2(\theta_s - \theta_p)) \\ 2\, m_s \, \dot{\theta}_s = (1+r)\, \varepsilon_1^2 F(\theta_s, r; k_6, k_{16}) + \frac{1}{2}\varepsilon_1^2 \, k_8 \, [G(\theta_s, \theta_p, r) + H(\theta_p, \theta_s, r)] - \kappa \sin(2(\theta_s - \theta_p)) \\ \cos^2(\theta_p - \theta_s) = \cos^2(\theta_{p0} - \theta_{s0}) \, . \end{cases} \tag{A.7}$$

For an equibiaxial test, with $r = -1$, by summing and substracting the first two equations in (A.7) one obtains

$$\begin{cases} m_p \, \dot{\theta}_p + m_s \, \dot{\theta}_s = 0 \\ m_p \, \dot{\theta}_p - m_s \, \dot{\theta}_s = \varepsilon_1^2 \, k_8 \cos^2(\theta_p - \theta_s) \sin(2(\theta_p - \theta_s)) - \kappa \sin(\theta_p - \theta_s) \\ \cos^2(\theta_p - \theta_s) = \cos^2(\theta_{p0} - \theta_{s0}) \, . \end{cases} \tag{A.8}$$

The constraint equation $\cos^2(\theta_p - \theta_s) = \pm \cos^2(\theta_{p0} - \theta_{s0})$ implies that

$$\dot{c} = \sin(2(\theta_s - \theta_p))(\dot{\theta}_p - \dot{\theta}_s) = 0 \, , \tag{A.9}$$

which has the solutions $\dot{\theta}_p = \dot{\theta}_s$ or $\sin(2(\theta_s - \theta_p) = 0$, that is $\Delta\theta = \theta_s - \theta_p = (1+k)\frac{\pi}{2}$, $\kappa = 0, 1, 2, \ldots.$ The former substituted into (A.8) gives $\dot{\theta}_p = \dot{\theta}_s = 0$ and $\kappa = \varepsilon_1^2 \, k_8 \, \cos^2(\Delta\theta_0)$. The latter gives $\dot{\theta}_p = \dot{\theta}_s = 0$ and an indeterminate reaction couple $\kappa$. In both cases, however, it is obtained that no remodeling occurs when equibiaxial strains are considered, as already seen from Eq. (5.59).