GENERALIZED KELLER–OSSERMAN CONDITIONS FOR FULLY NONLINEAR DEGENERATE ELLIPTIC EQUATIONS

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Abstract. We discuss the existence of entire (i.e. defined on the whole space) subsolutions of fully nonlinear degenerate elliptic equations, giving necessary and sufficient conditions on the coefficients of the lower order terms which extend the classical Keller–Osserman conditions for semilinear elliptic equations. Our analysis shows that, when the conditions of existence of entire subsolutions fail, a priori upper bounds for local subsolutions can be obtained.

1. Introduction

In this note, we report the recent results we obtained in [18, 19] on existence of subsolutions in the whole space and a priori estimates for local subsolutions of fully nonlinear degenerate elliptic equations with lower oder terms. In the general case, we consider second order differential inequalities of the form

$$F(x, D^2u) \geq f(u) + g(u)\,|Du|^q \tag{1.1}$$

where the zero order term $f : \mathbb{R} \to \mathbb{R}$ and the first order coefficient $g : \mathbb{R} \to \mathbb{R}$ will be assumed to be continuous monotone increasing functions, with f positive. The exponent q will be assumed to belong to $(0, 2]$, and the principal part F is a second order degenerate elliptic operator, that is a continuous function $F : \mathbb{R}^n \times \mathcal{S}_n \to \mathbb{R}$ satisfying $F(x, O) = 0$ and the (normalized) ellipticity condition

$$0 \leq F(x, X + Y) - F(x, X) \leq \mathrm{tr}(Y)\,, \quad \forall\, x \in \mathbb{R}^n\,,\ X\,,\ Y \in \mathcal{S}_n\,,\ Y \geq O\,, \tag{1.2}$$

$\mathcal{S}_n$ being the space of symmetric $n \times n$ real matrices equipped with the usual partial ordering.

The model cases for F that we have in mind are the degenerate maximal Pucci operator $\mathcal{M}^+_{0,1}$ defined by

$$\mathcal{M}^+_{0,1}(X) = \sum_{\mu_i > 0} \mu_i(X) \tag{1.3}$$

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and the k–partial laplacian given by

$$\mathcal{P}_k^+(X) = \mu_{n-k+1}(X) + \ldots + \mu_n(X)\,, \tag{1.4}$$

where $\mu_1(X) \leq \mu_2(X) \leq \ldots \leq \mu_n(X)$ are the ordered eigenvalues of the matrix X.

Let us recall that Pucci's extremal operators are explicit fully non linear operators which play a central role in the elliptic regularity theory for equations in non divergence form, see the monograph by L. Caffarelli and X. Cabré [15]. Here, we observe that operator (1.3) is maximal in the class of all degenerate elliptic operators vanishing at $X = O$. In particular, for any $1 \leq k \leq n$ and for all $X \in \mathcal{S}_n$, one has

$$\mathcal{P}_k^+(X) \leq \mathcal{M}_{0,1}^+(X)\,,$$

and if u satisfies inequality

$$\mathcal{P}_k^+(D^2u) \geq f(u) + g(u)\,|Du|^q\,, \tag{1.5}$$

or inequality (1.1), then u is also a solution of

$$\mathcal{M}_{0,1}^+(D^2u) \geq f(u) + g(u)\,|Du|^q\ . \tag{1.6}$$

As for the operators $\mathcal{P}_k^+$, we recall that they naturally arise in Riemannian geometry, in particular in the study of k–convex manifold, see J. P. Sha [41] and H. Wu [42]. In the PDEs context, they have been recently considered by R. Harvey and B. Lawson Jr [30] and L. Caffarelli, Y.Y. Li and L. Nirenberg [16], see also M.E. Amendola, G. Galise, A. Vitolo [4], G. Galise [26] and G. Galise and A. Vitolo [27]. They also arise in the level set approach to geometric evolution problems, see L. Ambrosio, H.M. Soner [3] and Y. Giga [28], or in the convex envelope problem, see A. Oberman, L. Silvestre [38]. For recent results on regularity and existence of the principal eigenfunctions we refer to I. Birindelli, G. Galise and H. Ishii [10], and to I. Birindelli, G. Galise and F. Leoni [11] for Liouville type theorems.

Inequality (1.1) includes, as a very special case, the semilinear uniformly elliptic inequality

$$\Delta u \geq f(u)\,.$$

When f is positive, this inequality has been studied independently by J.B. Keller [32] and R. Osserman [39], and it is well known that solutions in the whole space $\mathbb{R}^n$ exist if and only if f satisfies the classical condition

$$\int_0^{+\infty} \frac{dt}{\left(\int_0^t f(s)\,ds\right)^{1/2}} = +\infty\ \ . \tag{1.7}$$

Our goal is to obtain analogous results in the general case of inequality (1.1), by detecting the sharp conditions on the lower order coefficients ensuring that no entire subsolutions exist and universal upper bounds holds for local subsolutions.

Since the considered operators are in non-divergence form, the notion of solution we use is that of viscosity solution, and we refer to [15, 20] for the general existence, uniqueness and regularity theory of this class of weak solutions.

The conditions that we obtain and that extend (1.7) are different according to the sign of the function g, that is to the case that the first order term is of either "absorbing" or "reaction" type.

If $\lim_{t\to+\infty} g(t) > 0$, then the necessary and sufficient "sublinearity" condition (1.7) should be of course generalized in order to take in proper account the definitely positive first order term. We prove indeed that inequality (1.6) possesses an entire viscosity solution if and only if

$$q \leq 1 \quad \text{and} \quad \int_0^{+\infty} \frac{dt}{\left(\int_0^t f(s)\,ds\right)^{1/2} + \left(\int_0^t g^+(s)\,ds\right)^{1/(2-q)}} = +\infty\ . \tag{1.8}$$

The same condition is also proved to be necessary and sufficient for the existence of an entire viscosity solution of (1.5), provided that $g(t) \geq 0$ for all $t \in \mathbb{R}$.

In particular, we see that if $q > 1$ then no entire subsolutions can exist, independently of how slow the growth of f and g is, whereas growth restrictions both on f and g are needed in the case $q \leq 1$.

In the case $\lim_{t\to+\infty} g(t) \leq 0$, the zero and the first order terms in inequality (1.6) are competing with each other due to their opposite signs. Our analysis shows that in this case entire viscosity subsolutions exist if and only if a relaxed version of condition (1.7) involving f, g and q holds true, namely

$$\int_0^{+\infty} \frac{dt}{\left(\int_0^t e^{-2\int_s^t \left(-\frac{g(\tau)}{f(\tau)}\right)^{2/q} f(\tau)\,d\tau} f(s)\,ds\right)^{1/2}} = +\infty\ . \tag{1.9}$$

In particular, if $\lim_{t\to+\infty} g(t) < 0$, then (1.9) is proved to be equivalent to

$$\int_0^{+\infty} \left[\frac{1}{\left(\int_0^t f(s)\,ds\right)^{1/2}} + \frac{1}{f(t)^{1/q}}\right] dt = +\infty\ . \tag{1.10}$$

Note that the above condition becomes, for $q = 2$, the "subquadratic" growth condition

$$\int_0^{+\infty} \frac{dt}{f(t)^{1/2}} = +\infty\ .$$

In all cases, we also prove that when conditions (1.8), (1.9) or (1.10) respectively fail and u satisfies (1.1) in any open subset $\Omega \subset \mathbb{R}^n$, then u is universally estimated from above by an explicit function of the distance from the boundary determined by f, g and q, see Theorem 3.4.

As previous related results, let us mention H. Brezis [14] for existence and uniqueness of entire solutions of semilinear equations with $f(u) = |u|^{p-1}u$, $p > 1$, and to L. Boccardo, T. Gallouet and J.L. Vazquez [12], [13], F. Leoni [35], F. Leoni, B. Pellacci [36] and L. D'Ambrosio, E. Mitidieri [21] for subsequent extensions to more general divergence form principal parts and zero order terms.
In the fully nonlinear framework, analogous results have been more recently obtained by M.J. Esteban, P. Felmer, A. Quaas [24], G. Diaz [23] and G. Galise, A. Vitolo [27], M. E. Amendola, G. Galise, A. Vitolo [5], and for Hessian equations, involving the k-th elementary symmetric function of the eigenvalues $\mu_1(D^2u), \ldots, \mu_n(D^2u)$ by J. Bao, X. Ji [6], J. Bao, X. Ji, H. Li [7], Q. Jin, Y.Y. Li, H. Xu [31]. For application to removable singularities results see also D. Labutin [33].

As far as equations with gradient terms are concerned, the analogous "absorbing" property of superlinear first order terms in semilinear elliptic equations was singled out first by J.M. Lasry and P.L. Lions [34] and then extensively studied, see e.g. A. Porretta [40] for semilinear equations and S. Alarcón, J. García–Melián and A. Quaas [1] and P. Felmer, A. Quaas and B. Sirakov [25] for fully nonlinear uniformly elliptic equations with purely first oder terms of the form $h(|Du|)$.

As in the original papers [32], [39] our strategy for proving the above is based on comparison with radially symmetric solutions of (1.5) and (1.6). This approach requires a detailed analysis of the initial value problem for ordinary differential equations satisfied by the radial solutions. In particular, one has to study existence of entire maximal solutions and it turns out that entire solutions of (1.6) exist if and only if entire radially symmetric solutions exist. Remarkably, this fact is proved by a comparison argument which works also in the currently considered degenerate cases, without any a priori growth assumptions at infinity on u.

When the maximal solutions of the associated ODE problem are not global, we obtain the existence in balls of $\mathbb{R}^n$ of radially symmetric solutions of

$$\begin{cases} \mathcal{M}^+_{0,1}(D^2u) = f(u) + g(u)\,|Du|^q & \text{in } B \\ \qquad\qquad u = +\infty & \text{on } \partial B \end{cases}$$

and of

$$\begin{cases} \mathcal{P}^+_k(D^2u) = f(u) + g(u)\,|Du|^q & \text{in } B \\ \qquad\qquad u = +\infty & \text{on } \partial B \end{cases}$$

Boundary blowing up solutions have been extensively studied in past and recent literature, both for semilinear and fully nonlinear elliptic equations. After the seminal paper by J.M. Lasry and P.L. Lions [34], let us mention in the fully nonlinear framework the results by S. Alarcón and A. Quaas [2], M.E. Amendola, G. Galise and A. Vitolo [5], and F. Demengel and O. Goubet [22]. For a very recent result on existence of explosive solutions of fully nonlinear singular or degenerate elliptic equations, see I. Birindelli, F. Demengel and F. Leoni [9].

We finally observe that, in some cases, our results can be applied in order to obtain also lower bounds for super solutions of degenerate elliptic equations. As an illustrative example, let v be a positive supersolution of the equation

$$\mathcal{M}^-(D^2 v) + v^\theta \leq 0 \quad \text{in } \Omega\,,$$

where $\Omega \in \mathbb{R}^n$ is an open subset with nonempty boundary, and $\mathcal{M}^-$ is the degenerate inf Pucci's operator defined as

$$\mathcal{M}^-(M) = \sum_{\mu_i<0} \mu_i(M) = -\mathcal{M}^+(-M)\,.$$

It is easy to prove that the function $u = \frac{1}{v}$ then satisfies

$$\mathcal{M}^+(D^2 u) \geq u^{2-\theta}\,.$$

In the case $\theta < 1$, then our results on subsolutions yield the estimate

$$u(x) \leq \frac{C}{d(x)^{\frac{2}{1-\theta}}} \quad \text{for } x \in \Omega\,,$$

where $C > 0$ is a positive universal constant and $d(x)$ is the distance of x from the boundary $\partial\Omega$. This in turn gives the uniform lower bound

$$v(x) \geq c\, d(x)^{\frac{2}{1-\theta}} \quad \text{for } x \in \Omega\,.$$

2. Radial solutions and Comparison Principle

Let f, g be continuous non decreasing function and, for $c, q > 0$ and $a \in \mathbb{R}$, let us consider the Cauchy problem

$$\begin{cases} \varphi'' + \frac{c-1}{r}\,\varphi' = f(\varphi) + g(\varphi)\,|\varphi'|^q \\ \varphi(0) = a \\ \varphi'(0) = 0 \end{cases} \tag{2.1}$$

By a solution of (2.1) in $[0, R)$ with $0 < R \leq +\infty$, we mean here and in the sequel a function $\varphi \in C^2((0,R)) \cap C([0,R))$ satisfying, moreover,

$$0 = \varphi'(0) = \lim_{r\to 0} \varphi'(r)\,, \quad \exists \lim_{r\to 0^+} \varphi''(r) \in \mathbb{R}\,.$$

Therefore the ordinary differential equation in (2.1) has to be satisfied for $r = 0$, too.

The existence of local solutions of (2.1) follows from the standard theory of ordinary differential equations with continuous data. Some easy to prove monotonicity and convexity properties of the local solutions φ of problem (2.1) are given by the following result.

Lemma 2.1. *Let $c > 0$, $q > 0$, and $f, g : \mathbb{R} \to \mathbb{R}$ be continuous. If φ is a solution of* (2.1) *in* $[0, R)$, *then:*

(i) *if f is positive, then φ is strictly increasing;*
(ii) *if f is positive and f, g are non decreasing, and $c \geq 1$, then φ is convex and*

$$\varphi'(r) \leq \left(\frac{f(\varphi(r))}{g^-(\varphi(r))} \right)^{1/q} \text{ for all } r \in [0, R), \tag{2.2}$$

(iii) *if f is positive, f, g are non decreasing, g is non negative and $c \geq 1$, then*

$$\varphi''(r) \geq \frac{\varphi'(r)}{r} \text{ for all } r \in [0, R)\,. \tag{2.3}$$

Proof. From (2.1) it follows that

$$c\varphi''(0) = \lim_{r \to 0^+} \left(\varphi''(r) + (c-1)\, \frac{\varphi'(r)}{r} \right) = f(a) > 0\,,$$

so that φ' is increasing, hence positive, in some interval $(0, r_0)$. Actually, one has $\varphi'(r) > 0$ in the whole interval $(0, R)$, since, if not, there should be a point $r^* \in (0, R)$ satisfying $\varphi'(r^*) = 0$ and $\varphi''(r^*) \leq 0$ on the one hand, and $\varphi''(r^*) = f(\varphi(r^*)) > 0$ on the other hand. Hence (i) is proved.
Next, let us prove (ii). Since $\varphi''(0) > 0$, there exists some $r_1 > 0$ such that $\varphi'(r) > 0$ and $\varphi''(r) > 0$ for $r \in (0, r_1]$. By contradiction, let us assume that there exists $\tau > r_1$ such that $\varphi''(\tau) < 0$. Then, the function φ' has a strict local maximum point $r_0 \in (0, \tau)$ and the set $\mathcal{R} = \{r \in (0, r_0) : \varphi'(r) = \varphi'(\tau)\}$ is non empty. Let $\sigma = \min \mathcal{R}$, so that $\varphi''(\sigma) \geq 0$. Therefore, we have found $\sigma < \tau$ where $\varphi'(\sigma) = \varphi'(\tau)$ and such that

$$\varphi''(\sigma) - \varphi''(\tau) > 0\,. \tag{2.4}$$

On the other hand, equation (2.1) tested at σ and τ yields

$$\begin{aligned} \varphi''(\sigma) - \varphi''(\tau) = \ & (c-1) \left(\tfrac{\varphi'(\tau)}{\tau} - \tfrac{\varphi'(\sigma)}{\sigma} \right) \\ & + g(\varphi(\sigma))\varphi'(\sigma)^q - g(\varphi(\tau))\varphi'(\tau)^q \\ & + f(\varphi(\sigma)) - f(\varphi(\tau))\,, \end{aligned}$$

Since f, g and φ are monotone, $c \geq 1$, $\sigma < \tau$ and $\varphi'(\sigma) = \varphi'(\tau)$, this yields $\varphi''(\sigma) - \varphi''(\tau) \leq 0$, which contradicts (2.4). Therefore, φ is convex and increasing in $[0, R)$ and from equation (2.1) we immediately obtain

$$f(\varphi) + g(\varphi)(\varphi')^q \geq 0 \quad \text{in } [0, R)\,,$$

which yields (2.2). This proves (ii).
Finally, let us prove (iii). Multiplying equation (2.1) by r^{c-1} and integrating between

0 and r yields

$$r^{c-1}\varphi'(r) \le [f(\varphi(r)) + g(\varphi(r))\varphi'(r)^q] \int_0^r s^{c-1} ds = \left[\varphi''(r) + \frac{(c-1)}{r}\varphi'(r)\right] \frac{r^c}{c},$$

since φ, φ', f and g are non decreasing. Hence (2.3) is proved.

□

Let us now consider the partial differential equations

$$\mathcal{M}^+_{0,1}(D^2u) = f(u) + g(u)\,|Du|^q \tag{2.5}$$

and

$$\mathcal{P}^+_k(D^2u) = f(u) + g(u)\,|Du|^q\,. \tag{2.6}$$

We show that radially symmetric solutions of the above equations can be obtained from solutions $\varphi \in C^2([0,R))$ of problem (2.1).

Lemma 2.2. *(i) Let f and g be continuous non decreasing functions, with f positive. For any $q > 0$, if $\varphi \in C^2([0,R))$ is a solution of the Cauchy problem* (2.1) *with $c = n$, then $\Phi(x) = \varphi(|x|) \in C^2(B_R)$ is a classical solution of equation* (2.5) *in the ball B_R.*

(ii) Let f and g be continuous non decreasing functions, with f positive and g non-negative. For any $q > 0$, if $\varphi \in C^2([0,R))$ is a solution of the Cauchy problem (2.1) *with $c = k$, then $\Phi(x) = \varphi(|x|) \in C^2(B_R)$ is a classical solution of equation* (2.6) *in the ball B_R.*

Proof. A direct computation shows that, if $\Phi(x) = \varphi(|x|)$, then

$$D^2\Phi(x) = \begin{cases} \varphi''(0)\, I_n & \text{if } x = 0 \\ \frac{\varphi'(|x|)}{|x|}\, I_n + \left(\varphi''(|x|) - \frac{\varphi'(|x|)}{|x|}\right) \frac{x}{|x|} \otimes \frac{x}{|x|} & \text{if } x \neq 0 \end{cases}$$

Since $\varphi \in C^2([0,R))$, $\varphi'(0) = 0$ and $\varphi''(0) = \lim_{r\to 0} \varphi'(r)/r$, then Φ belongs to $C^2(B_R)$. Moreover, Φ is convex since φ is convex and increasing, and the very definition of operator $\mathcal{M}^+_{0,1}$ yields

$$\mathcal{M}^+_{0,1}(D^2\Phi(x)) = \varphi''(|x|) + (n-1)\,\frac{\varphi'(|x|)}{|x|}\,.$$

Therefore, if φ solves (2.1) with $c = n$, then Φ solves (2.5).

Analogously, we observe that, if $g \ge 0$ and φ solves (2.1) with $c = k$, then $\varphi''(|x|) \ge \frac{\varphi'(|x|)}{|x|}$ by Lemma 2.1 (iii). Therefore

$$\mathcal{P}^+_k(D^2\Phi(x)) = \varphi''(|x|) + (k-1)\,\frac{\varphi'(|x|)}{|x|}\,,$$

and Φ is a solution of (2.6).

□

In the next result we recall that form of comparison principle that will be needed in the sequel. It is an immediate consequence of the definition of sub/supersolution when one of the functions to be compared is smooth. For the general regularizing argument needed to compare merely viscosity sub and super solutions we refer to [20].

Proposition 2.3. *Let f, g be continuous functions, with f strictly increasing and g non decreasing, and let $F : \mathbb{R}^n \times \mathcal{S} \to \mathbb{R}$ be a continuous function satisfying* (1.2)*. Assume further that $u \in USC(B_R)$ and $\Phi \in C^2(B_R)$ satisfy*

$$F(x, D^2u) - f(u) - g(u)\,|Du|^q \geq 0 \geq F(x, D^2\Phi) - f(\Phi) - g(\Phi)\,|D\Phi|^q \quad \text{in } B_R$$

and

$$\limsup_{|x| \to R^-} \, (u(x) - \Phi(x)) \leq 0\,.$$

Then $u(x) \leq \Phi(x)$ for all $x \in B_R$.

Proof. By contradiction, suppose $u - \Phi$ has a positive maximum achieved at some interior point $x_0 \in B_R$. By using $\Phi(x) + u(x_0) - \Phi(x_0)$ as test function at x_0 in the definition of viscosity subsolution for u it follows that

$$F(x_0, D^2\Phi(x_0)) \geq f(u(x_0)) + g(u(x_0))\,|D\Phi(x_0)|^q\,,$$

which, by the strict monotonicity of f and the monotonicity of g, contradicts the fact that Φ is a supersolution. □

3. Existence of entire subsolutions and a priori upper bounds

A detailed analysis for the maximal solution of the initial value problem (2.1) leads to the following characterization of the existence of global solutions. For the proof, we refer to [19].

Theorem 3.1. *Assume that $0 < q \leq 2$, $c \geq 1$ and let f, g be continuous non decreasing functions, with f positive. Let further φ be a maximal solution of the initial value problem* (2.1)*. Then,*

(i) *if $\lim_{t \to +\infty} g(t) > 0$, then φ is globally defined in $[0, +\infty)$ if and only if*

$$q \leq 1 \quad \text{and} \quad \int_0^{+\infty} \frac{dt}{(t\,f(t))^{1/2} + (t\,g^+(t))^{1/(2-q)}} = +\infty \tag{3.1}$$

(ii) *if* $\lim_{t\to+\infty} g(t) \le 0$, *then* φ *is globally defined in* $[0,+\infty)$ *if and only if*

$$\int_0^{+\infty} \frac{dt}{\left(\int_0^t e^{-2\int_s^t \left(\frac{g^-(\tau)}{f(\tau)}\right)^{2/q} f(\tau)\,d\tau} f(s)\,ds\right)^{1/2}} = +\infty \tag{3.2}$$

(iii) *if* $\lim_{t\to+\infty} g(t) < 0$, *then* (3.2) *is equivalent to*

$$\int_0^{+\infty} \left[\frac{1}{(t\,f(t))^{1/2}} + \frac{1}{f(t)^{1/q}}\right] dt = +\infty \tag{3.3}$$

Theorem 3.1 in particular implies that either all maximal solutions of Cauchy problem (2.1) blow up for finite R, or all maximal solutions are globally defined on $[0,+\infty)$, independently on the initial datum a, but only according to the growth for $t\to+\infty$ of f and g measured through conditions (3.1), (3.2) and (3.3) respectively. This is consistent with the classical theory for subquadratic ordinary equations, see e.g. [29], since under our assumption $q \le 2$ the Nagumo growth condition holds true.

Let us discuss some implications of conditions (3.1), (3.2) and (3.3).

Condition (3.1) clearly implies to the two conditions

$$\int_0^{+\infty} \frac{dt}{(tf(t))^{1/2}} = +\infty \quad \text{and} \quad \int_{t_0}^{+\infty} \frac{dt}{(t\,g^+(t))^{1/(2-q)}} = +\infty\,,$$

for every $t_0 > 0$ such that $g^+(t_0) > 0$, even though (3.1) is not equivalent to the two conditions above. We thank the referee for a punctual remark on that. Thus, (3.1) is stronger than the usual Keller–Osserman condition

$$\int_0^{+\infty} \frac{dt}{(t\,f(t))^{1/2}} = +\infty \tag{3.4}$$

and it restricts the growth at infinity both for f and g; for instance, for power like blowing up functions $g(t) \simeq t^\alpha$ for $t\to+\infty$, then (3.1) requires $\alpha \le 1-q$, and for $q=1$ at most logarithmic growth $g(t) \simeq (\ln t)^\alpha$ with $\alpha \le 1$ is allowed.

Condition (3.3) amounts to requiring that

$$\text{either} \int_0^{+\infty} \frac{dt}{(tf(t))^{1/2}} = +\infty \quad \text{or} \quad \int_0^{+\infty} \frac{dt}{(f(t))^{1/q}} = +\infty\,,$$

and, for $q=2$, it is equivalent to the "subquadraticity" condition

$$\int_0^{+\infty} \frac{dt}{(f(t))^{1/2}} = +\infty\,.$$

For functions f having a power growth at infinity, say $f(t) \simeq t^\alpha$ for $t \to +\infty$, (3.3) means that $0 \le \alpha \le \max\{1, q\}$, but it includes also functions of the form $f(t) \simeq t\,(\ln t)^\alpha$ with arbitrary $\alpha \ge 0$ if $q > 1$, and $0 \le \alpha \le 2$ if $q \le 1$.

In the case $\lim_{t\to+\infty} g(t) = 0$, it follows that

$$\int_0^{+\infty} \left[\frac{1}{(t\,f(t))^{1/2}} + \left(\frac{g^-(t)}{f(t)} \right)^{1/q} \right] dt = +\infty \tag{3.5}$$

is a sufficient condition in order to have that all maximal solutions are globally defined in $[0, +\infty)$. On the other hand, if there exists a global maximal solution $\varphi \in C^2([0, +\infty))$, then

$$\int_0^{+\infty} \left[\frac{1}{(t\,f(t))^{1/2}} + \left(\frac{\int_0^t g^-(s)\,ds}{\int_0^t f(s)\,ds} \right)^{1/q} \right] dt = +\infty\,. \tag{3.6}$$

Conditions (3.5) and (3.6) in general are not equivalent (also for $q < 2$). An easy example occurs for $q = 2$, $f(t) \simeq t\,(\ln t)^\alpha$ and $g(t) \simeq -1/t$ for $t \to +\infty$. In this case, (3.5) is satisfied if and only if $\alpha \le 2$ whereas (3.6) holds true up to $\alpha \le 3$. We observe that, in this case, (3.2) actually requires $\alpha \le 2$.

On the other hand, if $q < 2$ and $c_1/t^\beta \le g^-(t) \le c_2/t^\beta$ for positive constants c_1, c_2 and β and for t sufficiently large, then, regardless the behaviour of f, (3.5) and (3.6) can be easily proved to be equivalent. In such a case, they both are more explicit formulations of (3.2).

Let us explicitly remark that for $g \equiv 0$, condition (3.2) reduces to the classical Keller–Osserman condition (3.4). An analogous condition is also recovered when $g > 0$ in the limit $q \to 0$. Indeed, for $q \to 0$ condition (3.1) becomes (3.4) applied to the positive non decreasing nonlinearity $f(t) + g(t)$.

Theorem 3.1 combined with the comparison result given by Proposition 2.3 yields the following characterization of existence of entire subsolutions for equation (2.5).

Theorem 3.2. *Let f, g be continuous nondecreasing functions, with f positive and strictly increasing.*

(i) *If $\lim\limits_{t\to+\infty} g(t) > 0$, then there exists $u \in USC(\mathbb{R}^n)$ entire viscosity subsolution of (2.5) if and only if condition (3.1) is satisfied.*
(ii) *If $\lim\limits_{t\to+\infty} g(t) = 0$, then there exists $u \in USC(\mathbb{R}^n)$ entire viscosity subsolution of (2.5) if and only if condition (3.2) is satisfied.*
(iii) *If $\lim\limits_{t\to+\infty} g(t) < 0$, then there exists $u \in USC(\mathbb{R}^n)$ entire viscosity subsolution of (2.5) if and only if condition (3.3) is satisfied.*

Proof. (i) If condition (3.1) is satisfied, then any maximal solution of the Cauchy problem (2.1) with $c = n$, which is globally defined in $[0, +\infty)$ by Theorem 3.1 (i), gives, by Lemma 2.2, a smooth entire (sub)solution of equation (2.5).
Conversely, assume by contradiction that there exists $u \in USC(\mathbb{R}^n)$ entire subsolution of (2.5) and (3.1) does not hold true. Let us consider a maximal solution $\varphi(r)$ of the Cauchy problem (2.1) with $c = n$ and $a < u(0)$. Then, again by Theorem 3.1 (i), $\varphi(r)$ blows up for $r = R(a) \in (0, +\infty)$. On the other hand, by Lemma 2.2 and Proposition 2.3, the functions u and $\Phi(x) = \varphi(|x|)$ can be compared in $B_{R(a)}$ and we obtain the contradiction

$$u(0) \leq \Phi(0) = a < u(0)\,.$$

In the same way, statements (ii) and (iii) follow by Theorem 3.1 (ii) and (iii) respectively.

□

By the maximality of operator $\mathcal{M}^+_{0,1}$, conditions (3.1), (3.2) and (3.3) are necessary conditions for the existence of entire viscosity solutions of inequality (1.1), respectively in the cases $\lim_{t\to+\infty} g(t) > 0$, $\lim_{t\to+\infty} g(t) = 0$ and $\lim_{t\to+\infty} g(t) < 0$.
Moreover, the same proof of Theorem 3.2 can be applied to subsolutions of equation (2.6). But, in this case, the extra assumption $g(t) \geq 0$ is needed in order to have correspondence between solutions of (2.1) and radially symmetric solutions of (2.6), and we obtain the following result.

Theorem 3.3. *Let f, g be continuous nonnegative nondecreasing functions, with f positive and strictly increasing. There exists an entire viscosity subsolution $u \in USC(\mathbb{R}^n)$ of equation* (2.6) *if and only if condition* (3.1) *is satisfied.*

The comparison argument used in the proof of Theorem 3.2 actually can be applied in order to estimate from above viscosity solutions of inequality (1.1) in any open subset $\Omega \subset \mathbb{R}^n$. If $\partial\Omega \neq \emptyset$, for $x \in \mathbb{R}^n$, we set

$$d(x) = \mathrm{dist}(x, \partial\Omega)$$

to denote the distance function from the boundary $\partial\Omega$. We then have the following result, whose proof can be found in [19].

Theorem 3.4. *Let f, g be continuous nondecreasing functions, with f positive, strictly increasing and such that*

$$\lim_{t\to+\infty} f(t) = +\infty \quad \text{if} \ \lim_{t\to+\infty} g(t) < +\infty\,. \tag{3.7}$$

Let further $\Omega \subset \mathbb{R}^n$ be an open domain with non empty boundary, and, for $q \in (0, 2]$, let $u \in USC(\Omega)$ be a viscosity solution of (1.1) *in* Ω.

(i) *Assume that* $\lim_{t\to+\infty} g(t) > 0$ *and condition* (3.1) *is not satisfied. Then, pointwisely in* Ω, *one has*

$$u(x) \le \max\{t_0\,,\ \mathcal{R}^{-1}(d(x))\}\,, \tag{3.8}$$

where $t_0 = \inf\{t \in \mathbb{R}\ :\ g(t) \ge 0\}$ *and* $\mathcal{R} : \mathbb{R} \to (0,+\infty)$ *is defined as*

$$\mathcal{R}(a) = \begin{cases} 2\left(\dfrac{n}{2-q}\right)^{1/(2-q)} \displaystyle\int_a^{+\infty} \frac{dt}{\left(\int_a^t f(s)\,ds\right)^{1/2} + \left(\int_a^t g^+(s)\,ds\right)^{1/(2-q)}} & \textit{if } q < 2 \\[2ex] \sqrt{\dfrac{n}{2}} \displaystyle\int_a^{+\infty} \frac{dt}{\left(\int_a^t e^{\frac{2}{n}\int_s^t g^+(\tau)\,d\tau} f(s)\,ds\right)^{1/2}} & \textit{if } q = 2 \end{cases}. \tag{3.9}$$

(ii) *Assume* $\lim_{t\to+\infty} g(t) = 0$ *and condition* (3.2) *is not satisfied. Then, pointwisely in* Ω, *one has*

$$u(x) \le \mathcal{R}^{-1}(d(x))\,,$$

where $\mathcal{R} : \mathbb{R} \to (0,+\infty)$ *is defined as*

$$\mathcal{R}(a) = \frac{n^{1/q}}{\sqrt{q}} \int_a^{+\infty} \frac{dt}{\left(\int_a^t e^{-2\int_s^t \left(\frac{g^-(r)}{f(r)}\right)^{2/q} f(r)\,dr} f(s)\,ds\right)^{1/2}}\,. \tag{3.10}$$

(iii) *Assume* $\lim_{t\to+\infty} g(t) < 0$ *and condition* (3.3) *is not satisfied. Then, pointwisely in* Ω, *one has*

$$u(x) \le \mathcal{R}^{-1}(d(x))\,,$$

where $\mathcal{R} : \mathbb{R} \to (0,+\infty)$ *is defined as*

$$\mathcal{R}(a) = 2\frac{n^{1/q}}{q} \int_a^{+\infty} \left[\frac{1}{\left(\int_a^t f(s)\,ds\right)^{1/2}} + \left(\frac{\int_a^t g^-(s)\,ds}{\int_a^t f(s)\,ds}\right)^{1/q} \right] dt\,. \tag{3.11}$$

Let us finally remark that hypothesis (3.7) is needed in the case $\lim_{t\to+\infty} g(t) > 0$ and $q > 1$, whereas it is a consequence of the failure of conditions (3.1), (3.2) and (3.3) in the other cases. It guarantees that the amplitude $R(a)$ of the maximal interval of existence of a maximal solution of the Cauchy problem (2.1) satisfies

$$\lim_{a\to+\infty} R(a) = 0\,.$$

In order to show its necessity, we can use the same counterexample exhibited in [40]. Indeed, let us assume $q = 2$ and $g \equiv 1$, and for any $a \in \mathbb{R}$ let $\varphi \in C^2([0, R(a)))$ be a maximal solution of

$$\begin{cases} \varphi'' + \dfrac{n-1}{r}\varphi' = f(\varphi) + (\varphi')^2 \\ \varphi(0) = a\,,\ \varphi'(0) = 0 \end{cases}.$$

Then, the radial function $\Phi(x) = \varphi(|x|)$ satisfies also

$$\begin{cases} -\Delta\Phi + f(\Phi) + |D\Phi|^2 = 0 \quad \text{in } B_{R(a)} \\ \Phi = +\infty \quad \text{on } \partial B_{R(a)} \end{cases},$$

so that, the function $\Psi(x) = e^{-\Phi(x)}$ solves

$$\begin{cases} -\Delta\Psi = f\left(\ln \dfrac{1}{\Psi}\right)\Psi \quad \text{in } B_{R(a)} \\ \Psi > 0 \quad \text{in } B_{R(a)}\,,\ \Psi = 0 \quad \text{on } \partial B_{R(a)} \end{cases}.$$

By the properties of the first eigenvalue of the laplacian associated with homogeneous Dirichlet boundary conditions, it then follows that

$$\lim_{t\to+\infty} f(t) \geq \lambda_1\left(B_{R(a)}\right)\,,$$

where $\lambda_1\left(B_{R(a)}\right)$ denotes the first Dirichlet eigenvalue of $-\Delta$ in $B_{R(a)}$. Therefore, if we denote by λ_1 the first eigenvalue of $-\Delta$ in B_1 and if $\lim\limits_{t\to+\infty} f(t) < +\infty$, then we deduce

$$R(a) \geq \sqrt{\frac{\lambda_1}{\lim_{t\to+\infty} f(t)}} \qquad \forall\, a \in \mathbb{R}\,.$$

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