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IFAC PapersOnLine 54-14 (2021) 329–334

Feedback linearization of nonlinear time-delay systems over a time window via discontinuous control

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Abstract: Feedback linearization is worked out for nonlinear time-delay systems and it is shown that even if the problem can not be solved for all time, it may still be solved over some time windows. The solution then reduces to a discontinuous state feedback. It is foreseen that such piecewise feedback linearization can be instrumental for stabilization by a discontinuous control, although it is not the scope of this paper. The approach used herein may be used to address other classical control problems. It takes advantage of the delays as delays duplicate somehow the number of independent control values at some given time instant.

Keywords: Nonlinear systems, time–delay systems, differential geometry, linearization, discontinuous control

1. INTRODUCTION

Time–delay systems have now become a topic of major importance in the control system community due to the large number of applications which can be described by such mathematical models Borri et al. (2017), Driver (1977), Gu et al. (2003), Hespanha et al. (2007), Marshall (1979). Many theoretical advances have been obtained either in the comprehension of the structural properties of this class of systems or in the stabilization techniques (see for example Hovelaque et al. (1997), Ito el al. (2013), Krstic (2009), Sename (2014), Sename et al. (2010), Zheng et al. (2010), Zheng et al. (2010b), Bartosiewicz et al. (2020), Zheng et al. (2011) and the references therein).

More in detail with reference to the class of systems affected by constant commensurate delays several results have been obtained by using a differential representation of the system in order to better understand to what extent some classical problems such as the feedback linearization problem or the equivalence to the input–output injection linear observable canonical form, could be addressed and solved, Califano et al. (2011), Califano et al. (2013), Germani et al. (1996), Kaldmae et al. (2015), Marquez-Martinez (2000).

In the present paper we show how some results can be achieved, at least on a time window, by weakening the structure of the control law, and using the delay structure of the system. While the proposed approach can be applied to several control problems, we illustrate the idea and the result to the linear feedback equivalence one.

Example 1. For instance consider the dynamics with commensurate delays, so that it can be written as involving one single delay τ:

$$\begin{aligned}\dot{x}_1(t) &= (2+\sin x_2(t-\tau))u(t)\\ \dot{x}_2(t) &= u(t-\tau)\end{aligned} \tag{1}$$

It is immediately clear that such a system cannot be linearized for all time t via a continuous static state feedback law of the form

$$u(t) = \alpha(x(t), \cdots x(t-j\tau)) + \beta((t), \cdots x(t-j\tau))v(t)$$

with j an appropriate finite integer. However, consider now the following discontinuous static feedback

$$\begin{aligned}u(t) &= v(t), & t &\in [(2k-1)\tau, 2k\tau)\\ u(t) &= \frac{1}{2+\sin x_2(t-\tau)}v(t), & t &\in [2k\tau, (2k+1)\tau)\end{aligned} \tag{2}$$

with $k = 0, 1, 2, \cdots$ Then one immediately verifies that the previous feedback yields a linear time-delay closed loop system for $t \in [2k\tau, (2k+1)\tau)$, $k = 0, 1, \cdots$, since on such intervals the dynamics reads

$$\begin{aligned}\dot{x}_1(t) &= v(t)\\ \dot{x}_2(t) &= v(t-\tau)\end{aligned} \quad t \in [2k\tau, (2k+1)\tau), k = 0, 1, \cdots \tag{3}$$

Although it is not possible to achieve linearization for all t, the problem is solved on time intervals $[2k\tau, (2k+1)\tau)$ as the inputs $u(t)$ and $u(t-\tau)$ are independent, whereas time intervals $[(2k-1)\tau, 2k\tau)$ redefine the "initial condition" for the next time interval.

Towards a high-gain piecewise stabilization

An obvious application of this piecewise feedback linearization is stabilization, using a linear time-delay stabilization methodology. This is tractable only when the stabilization is fast enough during the "linearized dynamics windows" so that future uncontrolled windows do not destabilize in a significant manner. A general result in that perspective is out of the scope of this paper and requires further assumptions on the class of systems under consideration. Nevertheless, this stabilization process is easily illustrated on Example 1.

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2021.10.375

(1) For $t \in [-2\tau, -\tau)$, set $u(t) = 0$.
(2) For $t \in [-\tau, 0)$, set $u(t) = -k_2 x_2(t)$ where $k_2 = \frac{1}{\tau}$. The closed loop system then yields a nonlinear dynamics for x_1 and $\dot{x}_2 = 0$.
(3) For $t \in [0, \tau)$, set $u(t) = -\frac{k_1 x_1(t)}{2+\sin x_2(t-\tau)}$ so that the closed loop system becomes

$$\begin{cases} \dot{x}_1(t) = -k_1 x_1(t) \\ \dot{x}_2(t) = -k_2 x_2(t-\tau) = -k_2 x_2(0) \end{cases}$$

whose solution is

$$\begin{cases} x_1(t) = x_1(0)\exp(-k_1 t) \\ x_2(t) = x_2(0)(1-k_2 t) \end{cases}$$

Since $k_2 = \tau$ one gets $x_2(\tau) = 0$ and k_1 can be chosen to get $x_1(\tau)$ arbitrarily close to the origin.

The paper is organized as follows. In Section 2 the mathematical setting and tools are recalled, while in Section 3 the feedback linearization problem is addressed and solved by weakening the class of feedback laws and coordinates change considered and generalizing the conditions recovered in Califano and Moog (2011). In Section 4 the main results are stated. Some conclusions are given in Section 5.

2. RECALLS AND NOTATIONS

Let us now consider a multi-input nonlinear time delay system affected by constant commensurate delays. Without loss of generality such a system is represented by the differential equation

$$\dot{x}(t) = F(x(t), x(t-\tau), ..., .x(t-s\tau)) + \\ + \sum_{j=1}^{m}\sum_{i=0}^{l} G_{ij}(x(t), x(t-\tau), ..., .x(t-s\tau))u_j(t-i\tau) \quad (4)$$

where τ is a constant delay, $s, l \geq 0$ are integers and the functions $G_{ij}(\cdot)$, $i \in [0, l]$, $j \in [1, m]$ and $F(\cdot)$ are analytic in their arguments. The notations used throughout the paper, issued from Califano and Moog (2017), are recalled hereafter.

- $x_{[s]} = (x^T(t), ..., x^T(t-s\tau)) \in R^{(s+1)n}$, denotes the vector consisting of the first $(s+1)n$ components of the state of the infinite dimensional system (4). $x_{[0]} = [x_{1,[0]}, ..., x_{n,[0]}]^T \in R^n$ will denote the values of the state variable at time t.
- $x_{[s]}(-i) = (x^T(t-i\tau), ..., x^T(t-s\tau-i\tau)) \in R^{(s+1)n}$. Accordingly, $x_{j,[0]}(-i) := x_j(t-i\tau)$ denotes the j-th component of the instantaneous values of the state variable delayed by $D = i\tau$. When no confusion is possible the subindex can be dropped so that $\mathbf{x}$ will stand for $x_{[s]}$.
- $\mathcal{K}$ denotes the field of casual meromorphic functions $f(x_{[s]}, u_{[j]})$, with $s, j \in N$ (meromorphic functions are single-valued, that is analytic in all but possibly a discrete subset of their domain).
- Given a function $f(x_{[s]}, u_{[j]})$, we will denote by $f(-l) = f(x_{[s]}(-l), u_{[j]}(-l))$.
- d is the standard differential operator.
- δ represents the backward time-shift operator: for $a(.), f(.) \in \mathcal{K} : \delta[a\, df] = a(-1)\delta df = a(-1)df(-1)$. Denoting by ε the vector space spanned by the differentials $dx(t-i); i \in N$ over the field $\mathcal{K}$, the elements of ε are called one-forms. The shift operator is applied to the vector space ε in this way: if ω is the one-form $\omega = \sum_{i=1}^{n}\sum_{j=0}^{k} a_i dx_i(t-j)$ then $\delta\omega$ is given by $\delta\omega = \sum_{i=1}^{n}\sum_{j=0}^{k} \delta(a_i)dx_i(t-j-1)$.
- $\mathcal{K}(\delta]$ is the (left) ring of polynomials in δ with coefficients in $\mathcal{K}$. Every element of $\mathcal{K}(\delta]$ may be rewritten as $\alpha(\delta] = \sum_{j=0}^{r_\alpha} \alpha_j(.)\delta^j$ with $\alpha_j(.) \in K$ and $r_\alpha = deg(\alpha(\delta])$ the polynomial degree in δ.
- $\mathbf{u}^{[i]}$ will stand for $(u, \dot{u}, \cdots, u^{(i)})$, with $\mathbf{u}^{[-1]}$ the empty set.
- Given a right submodule
$$\Delta(\delta] = \text{span}_{\mathcal{K}(\delta]}\{\tau_1(\mathbf{x}, \delta), \cdots \tau_j(\mathbf{x}, \delta)\}$$
of rank j, $\Delta_c(\delta]$ is the right closure of $\Delta(\delta]$ that is the largest right submodule of rank j containing $\Delta(\delta]$.

With the notations introduced so far, system (4) can be rewritten as

$$\dot{x}_{[0]} = F(x_{[s]}) + \sum_{k=1}^{m}\sum_{i=0}^{l} G_{ik}(x_{[s]})u_{k,[0]}(-i). \quad (5)$$

As in Bartosiewicz et al. (2020), the approach in this paper starts by considering the differential representation of the given dynamics. Thus one gets that, using the notation just introduced, such an infinitesimal representation is given by

$$d\dot{x}_{[0]} = f(x_{[s]}, u_{[s]}, \delta)dx_{[0]} + \sum_{k=0}^{m} g_{1,k}(x_{[s]}, \delta)du_{k,[0]} \quad (6)$$

where $f(x_{[s]}, u_{[s]}, \delta)$ is a $n \times n$ matrix representing the differential with respect to the state variable and is given by

$$f(x_{[s]}, u_{[s]}, \delta) = \sum_{j=0}^{s} \frac{\partial F(x_{[s]})}{\partial x_{[s]}(-j)}\delta^j \\ + \sum_{j=0}^{s}\sum_{k=1}^{m}\sum_{i=0}^{l} u_{k,[0]}(-i)\frac{\partial G_{ik}(x_{[s]})}{\partial x_{[s]}(-j)}\delta^j, \quad (7)$$

while $g_{1,k}(x_{[s]}, \delta) = \sum_{i=0}^{l} g_{1,k}^i(x)\delta^i$, for $k \in [1, m]$, is a $n \times 1$ vector representing the differential of the dynamics with respect to the control u_k, and given by

$$g_{1,k}(x_{[s]}, \delta) = \sum_{i=0}^{l} G_{i,k}(x_{[s]})\delta^i = \sum_{i=0}^{l} g_{1,k}^i(\mathbf{x})\delta^i. \quad (8)$$

Let us now recall that in the delay–free case the feedback linearization problem can be easily solved by using geometric arguments and tools. More precisely the necessary an sufficient conditions are stated in terms of the existence of appropriate functions which can be taken as linearizing outputs. To this end the conditions reduce to requiring that the system is controllable and that the distributions defined by the controllability directions are locally of constant dimension and involutive.

In order to solve the problem in the delay context these conditions need to be generalized. We thus end this section by recalling the notions of accessibility matrix and the concept of polynomial Lie bracket and involutivity introduced in Califano and Moog (2017) and which are necessary to face the problem in this context.

First of all starting from (8), one defines the accessibility directions iteratively, for $i \geq 1$ as

$$g_{i+1,k}(\mathbf{x},\mathbf{u}^{[i-1]}\delta) = \dot{g}_{i,k} - f(\mathbf{x},\mathbf{u},\delta)g_{i,k}(\mathbf{x},\mathbf{u}^{[i-2]}\delta), \quad (9)$$

and consistently, setting

$$g_i(\mathbf{x},\mathbf{u}^{[i-2]}\delta) = g_{i,1}(\mathbf{x},\mathbf{u}^{[i-2]}\delta), \cdots, g_{i,m}(\mathbf{x},\mathbf{u}^{[i-2]}\delta),$$

the accessibility matrix

$$\mathcal{R}_n = (g_1(\mathbf{x},\delta), g_2(\mathbf{x},\mathbf{u},\delta), \cdots, g_n(\mathbf{x},\mathbf{u}^{[n-2]},\delta)).$$

Given $r(\mathbf{x},\delta) = \sum_{j=0}^{s} r^j(\mathbf{x})\delta^j$, we consider its action on a function $\epsilon(t)$, and we denote its image by $\mathbf{R}(\mathbf{x},\epsilon) := \sum_{j=0}^{s} r_1^j(\mathbf{x})\epsilon(-j)$. Then the Polynomial Lie Bracket for causal terms is defined as follows:

Definition 1. Given $r_i(\mathbf{x}_{[s]},\delta) \in \mathcal{K}^n(\delta]$, $i = 1,2$, the Polynomial Lie Bracket for causal terms $[\mathbf{R}_1(\mathbf{x},\epsilon), r_2(\mathbf{x},\delta)]$ is defined as

$$[\mathbf{R}_1(\mathbf{x},\epsilon), r_2(\mathbf{x},\delta)] := ad_{\mathbf{R}_1(\mathbf{x}_{[s]},\epsilon)} r_2(\mathbf{x}_{[s]},\delta) =$$

$$\dot{r}_2(\mathbf{x},\delta)|_{\dot{x}_{[0]}=\mathbf{R}_1(\mathbf{x},\epsilon)} - \sum_{k=0}^{s} \frac{\partial \mathbf{R}_1(\mathbf{x}_{[s]},\epsilon)}{\partial \mathbf{x}_{[0]}(-k)} \delta^k r_2(\mathbf{x},\delta).$$

Accordingly we have the following definition of involutivity of a submodule of causal terms

Definition 2. The submodule of causal elements

$$\mathcal{M} = \text{span}_{\mathcal{K}(\delta]}\{r_1(\mathbf{x},\delta), \cdots r_j(\mathbf{x},\delta)\}$$

is involutive, if for any two indices $i,k \in [1,j]$ the polynomial Lie bracket

$$[\mathbf{R}_i(\mathbf{x},\epsilon), r_k(\mathbf{x},\delta)] \in \mathcal{M}_c \quad \forall \epsilon$$

where $\mathcal{M}_c$ is the right closure of $\mathcal{M}$.

The first result was stated in Califano and Moog (2017) and concerns accessibility, a fundamental property in this context.

Theorem 1. System (4) is locally accessible if and only if the accessibility matrix

$$\mathcal{R}_n = (g_1(\mathbf{x},\delta), g_2(\mathbf{x},\mathbf{u},\delta), \cdots, g_n(\mathbf{x},\mathbf{u}^{[n-2]},\delta)\}))$$

has rank n over $\mathcal{K}(\delta]$ for some $\mathbf{u}^{[n-2]}$.

The second one concerns feedback linearization under regular bicausal static state feedback for single input systems taken form Califano and Moog (2011) and which is here restated using the framework in Califano and Moog (2021).

Theorem 2. The single input system described by (4) with $m = 1$ is locally feedback equivalent to a linear accessible system with delays if and only if the following conditions are satisfied

i) The accessibility matrix $\mathcal{R}_n$ has rank n over $\mathcal{K}(\delta]$
ii) The submodule

$$\mathcal{M} = \text{span}_{\mathcal{K}(\delta]}\{g_1(\mathbf{x},\delta), \cdots g_{n-1}(\mathbf{x},\delta)\}$$

is involutive

iii) There exist matrices $Q_1(\delta)$ unimodular and lower triangular, $Q_2(\delta) = diag(1, c_2(\delta),, c_2(\delta)c_n(\delta))$, $T(\mathbf{x},\delta)$ unimodular and $\Phi_1(\mathbf{x})$ lower triangular, such that denoting by $\lambda(\mathbf{x})$ a function with relative degree n and closed, the following conditions are satisfied

a)

$$\begin{pmatrix} d\lambda(\mathbf{x}) \\ d\lambda(\mathbf{x}) \\ \cdots \\ d\lambda^{(n-1)}(\mathbf{x}) \end{pmatrix} = \Phi_1(\mathbf{x})^{-1} Q_1(\delta) Q_2(\delta) T(\mathbf{x},\delta) dx_{[0]}$$

where

$$\Phi_1(\mathbf{x}) = \begin{pmatrix} \varphi_1(\lambda) & 0 & \cdots & 0 \\ \dot{\varphi}_1(\lambda) & \varphi_1(\lambda) & \cdots & 0 \\ \vdots & \ddots & & \\ \varphi_1^{(n-1)} & \binom{n-1}{1}\varphi_1^{(n-2)}(\lambda) & \cdots & \varphi_1(\lambda) \end{pmatrix}$$

with $\varphi_1(\lambda) = \frac{\partial \varphi(\lambda)}{\partial \lambda}$

b) Setting $d\lambda^{(n)} = a(\mathbf{x},\mathbf{u},\delta)dx_{[0]} + b(\mathbf{x},\mathbf{u},\delta)du_{[0]}$, one must have

$$b(\mathbf{x},\mathbf{u},\delta) = b(\mathbf{x},0,\delta) = \varphi_1^{-1}(\lambda)\tilde{b}(\delta)\beta(\mathbf{x})$$

and

$$\varphi_1(\lambda)a(\mathbf{x},0,\delta) + \Phi_{n+1}(\mathbf{x})\Phi_1(\mathbf{x})^{-1}Q_1(\delta)Q_2(\delta)T(\mathbf{x},\delta)$$
$$= (\tilde{a}(\delta) + \tilde{b}(\delta)\Gamma(\mathbf{x},0,\delta))T(\mathbf{x},\delta)$$

with

$$\Phi_{n+1}(\mathbf{x}) = \left(\binom{n}{0}\varphi_1^{(n)}(\lambda) \cdots \binom{n}{n-1}\dot{\varphi}_1(\lambda)\right)$$

3. MIMO FEEDBACK LINEARIZATION

In the present section necessary and sufficient conditions for achieving feedback linearization in the case on multi–input time delay systems are given, by weakening some requirements on the feedback law and on the change of coordinates. More precisely

- In the delay context it is common to ensure that the change of coordinates considered is bicausal, that is: it is causal and has a causal polynomial inverse in δ of finite degree. This limitation leads in the single input case to the strong condition represented by *iii) a)* in Theorem 2, since the fictitious output function λ, which has actually relative degree n, defines with its derivatives a matrix which has rank n over $\mathcal{K}(\delta]$ but not in general unimodular.

  In the present context following also some new trends in the literature (see for example Haidar et al. (2020)), such a requirement is not necessarily fulfilled. It is in fact required that the change of coordinates is defined by a full rank matrix on $\mathcal{K}(\delta]$, which ensures the existences of an inverse map not necessarily associated to a causal polynomial of finite degree
- Similarly for the feedback law used, instead of ensuring the existence of a polynomial inverse, which in the single input case leads to condition *iii) b)* in Theorem 2, an hybrid control scheme proposed first in Marquez-Martinez et al. (2002) and independently in the sampled-data context in Monaco et al. (2017) is considered, which allows its implementation. More precisely, the control

$$v(t) = \lambda(x(t), \cdots, x(t-s), u(t), u(t-s)), \frac{\partial \lambda}{\partial u(t)} \neq 0$$

is implemented through the hybrid scheme

$$\begin{aligned} \xi_1(t+1) &= u(t) \\ \xi_2(t+1) &= \xi_1(t) \\ &\vdots \\ \xi_s(t+1) &= \xi_{s-1}(t) \\ u(t) &= \varphi(\mathbf{x}, \xi(t), v(t)) \end{aligned} \quad (10)$$

where $\varphi(\mathbf{x}, \xi(t), v(t))$ satisfies

$$\lambda(x(t), \cdots, x(t-s), \varphi(\mathbf{x}, \xi(t), v(t)), \xi) = v(t). \quad (11)$$

We are now ready to state our first result.

Theorem 3. The nonlinear time delay system (4) is accessible and feedback linearizable through the hybrid control scheme (10) if and only if the following conditions are satisfied

i) $\mathcal{R}_n$ has constant dimension n locally around x_0 for $\delta = 0$

ii) $\text{span}_{\mathcal{K}(\delta]}\{g_1(\mathbf{x}, \delta), \cdots, g_i(\mathbf{x}, \delta)\}$ is involutive for $i = 1, \cdots, n-1$ and of constant dimension k_i locally around x_0 for $\delta = 0$.

Proof. Assume that condition i) and ii) are satisfied. Then condition i) guarantees the accessibility of the system for $\delta = 0$. Condition ii) instead guarantees the existence of m independent functions $\lambda_1, \cdots \lambda_m$ such that

$$(d\lambda_1^T, \cdots, d(\lambda_1^{(r_1-1)})^T, \cdots, d\lambda_m^T, \cdots, d(\lambda_m^{(r_m-1)})^T)^T$$

are independent differentials with $r_1 + \cdots + r_m = n$, and

$$\begin{pmatrix} d\lambda_1^{(r_1)} \\ \vdots \\ d\lambda_m^{(r_m)} \end{pmatrix} = \Gamma(\mathbf{x}, \mathbf{u}, \delta)dx_{[0]} + U(\mathbf{x}, \delta)du_{[0]} \quad (12)$$

with $\text{rank}(U(\mathbf{x}, 0)) = m$. Let now for $i = 1, \cdots, m$, s_i be the maximum degree of δ in the i-th column of $U(\mathbf{x}, \delta)$. Let s_{i_1}, s_{i_2}, s_{i_j} be all the indices such that $s_{i_l} \geq 1$ for $l \in [1, j]$. Then set

$$\begin{aligned} \xi_{l1}(t+1) &= u_l(t) \\ \xi_{l2}(t+1) &= \xi_{l1}(t) \\ &\vdots \qquad\qquad l \in [1, j] \quad (13) \\ \xi_{ls_{i_l}}(t+1) &= \xi_{l,s_{i_l}-1}(t) \end{aligned}$$

and accordingly, one can chose

$$u(t) = \varphi(\mathbf{x}, \xi(t), v(t))$$

such that $\lambda_i^{(r_i)} = v_i(t)$, for $i \in [1, m]$. In fact, with (13), one immediately gets from (12) that

$$\begin{pmatrix} \lambda_1^{(r_1)} \\ \vdots \\ \lambda_m^{(r_m)} \end{pmatrix} = \gamma_0(\mathbf{x}, \xi) + U(\mathbf{x}, 0)u(t) \quad (14)$$

and since by assumption $U(\mathbf{x}, 0)$ has full rank one can compute a feedback law $u(t) = \varphi(\mathbf{x}, \xi(t), v(t))$ such that (14) holds true.

In the coordinates

$$\begin{aligned} z_{11} &= \lambda_1, \cdots, z_{1,r_1} = \lambda_1^{(r_1-1)} \\ &\vdots \\ z_{m1} &= \lambda_m, \cdots, z_{m,r_m} = \lambda_m^{(r_m-1)} \end{aligned}$$

the closed loop system is then linear and delay free.

Conversely. Assume that there exists a change of coordinates $z = \phi(x)$, not necessarily bicausal, and a feedback law $u = \alpha(x, v, u(-1), \cdots, u(-s))$ such that in the new coordinates the closed loop system reads

$$\dot{z}(t) = \sum_{l=0}^{j} A_j z(t-j) + \sum_{i=1}^{m}\sum_{k=0}^{p} B_{ki} v_i(t-k) \quad (15)$$

with

$$dim(B(\delta), A(\delta)B(\delta) \cdots A^{n-1}(\delta)B(\delta)) = n,$$

then there exist m output functions $\lambda_i = C_i z$ with relative degree r_i and such that $r_1 + r_2 + \cdots + r_m = n$. In the x coordinates such functions will continue to have relative degree r_i for $i \in [1, m]$ and will have the expression $\lambda_i(x) = C_i\phi(x)$. Being exact differentials they imply the involutivity of the distributions

$$(g_1(\mathbf{x}, \delta), \cdots, g_i(\mathbf{x}, \delta)) \qquad i \in [1, n]$$

thus proving the result. ◁

4. FEEDBACK LINEARIZATION OVER A TIME WINDOW

When feedback linearization cannot be achieved via a regular static state feedback, one may argue if it is possible to use the delay which affects the system to obtain that at least on a time window such a result can be obtained. This is the case of system (1) in Example 1 where implementing the control (2) allows to achieve feedback linearization on the time window $[2k\tau, (2k+1)\tau)$.

The main idea behind next result is that the presence of a delay allows the appearence of the control and its delays, which at least on some time interval are considered as independent inputs. Once they are fixed on a time window in order to achieve a given goal, in this case feedback linearization, one has to wait that they become independent again to ensure that the linearization goal can be fulfilled again. Note that Example 1 does not fulfil the conditions of Theorem 3.

For simplicity consider the single input dynamics

$$\begin{aligned} \dot{x}(t) &= F(x(t), ..., .x(t-s\tau)) + \\ &+ \sum_{i=0}^{l} G_i(x(t), x(t-\tau), ..., .x(t-s\tau))u(t-i\tau) \quad (16) \end{aligned}$$

and suppose that it does not satisfy the conditions of Theorem 2 nor of Theorem 3, that is it is not static state feedback linearizable neither with a bicausal change of coordinates and a bicausal feedback, nor with the hybrid controller of the form (10) which requires weaker conditions.

4.1 Feedback linearization over the $[2k\tau, (2k+1)\tau)$ time windows

Let us now consider $u(t)$ and $u(t-\tau)$ independent over a window of width τ. Setting $u(t) = u_1(t)$ and $u(t-\tau) = u_2(t)$, and $\bar{l} = \lfloor (l+1)/2 \rfloor$, where $\lfloor * \rfloor$ denotes the integer part of $*$. One gets that $u(t-2k\tau) = u_1(t-2k\tau)$ while $u(t-(2k+1)\tau) = u_2(t-2k\tau)$ the dynamics (16) will read

$$\dot{x}(t) = F(x(t), x(t-\tau), ..., .x(t-s\tau)) + \\ + \sum_{i=0}^{\bar{l}} G_{2i,1}(x(t), x(t-\tau), ..., .x(t-s\tau))u_1(t-2i\tau) \\ + \sum_{i=0}^{\bar{l}} G_{2i+1,2}(x(t), x(t-\tau), ..., .x(t-s\tau))u_2(t-2i\tau) \tag{17}$$

Accordingly, one gets the differential representation

$$d\dot{x}(t) = f(\mathbf{x}, \mathbf{u}, \delta)d\mathbf{x} + g_1(\mathbf{x}, \delta)du_{1,0} + g_2(\mathbf{x}, \delta)du_{2,0} \tag{18}$$

Since $u_{1,0}$ and $u_{2,0}$ are not independent the result of Theorem 3 cannot be applied, even if the conditions are satisfied. In fact one has to consider the fact that $u_{1,0}$ and $u_{2,0}$ are linked through the relation

$$u_{2,0}(t) = u_{1,0}(t-\tau).$$

However by using a discontinuous control which switches every 2τ, the following result can be stated.

Theorem 4. The nonlinear time delay system (16) is accessible and feedback linearizable on each time window $[2k\tau, (2k+1)\tau)$ if and only if the dynamics (18) satisfies the conditions of Theorem 3.

Proof. Let the controller

$$\begin{aligned}
\xi_{11}(t+1) &= u_1(t) \\
\xi_{21}(t+1) &= \xi_{11}(t) \\
&\vdots \\
\xi_{s_1,1}(t+1) &= \xi_{s_1-1,1}(t) \\
\xi_{12}(t+1) &= u_2(t) \\
\xi_{22}(t+1) &= \xi_{12}(t) \\
&\vdots \\
\xi_{s_2,2}(t+1) &= \xi_{s_2-1,2}(t) \\
\begin{pmatrix} u_{1,[0]} \\ u_{2,[0]} \end{pmatrix} &= \alpha(\mathbf{x}, \xi) + \beta(\mathbf{x})v_{[0]}
\end{aligned}$$

together with the change of coordinates

$$z = \phi(x)$$

transform the dynamics (17) into a linear two input system. Then the controller

$$\begin{aligned}
\chi_{11}(t+2) &= u(t) = u_1(t) \\
\chi_{21}(t+2) &= \chi_{11}(t) = u(t-2) \\
&\vdots \\
\chi_{\bar{s}_1,1}(t+2) &= \chi_{\bar{s}_1-1,1}(t) = u(t-2(s_1-1)) \\
\chi_{12}(t+2) &= u(t-1) = u_2(t) \\
\chi_{22}(t+2) &= \chi_{12}(t) = u(t-3) \\
&\vdots \\
\chi_{\bar{s}_2,2}(t+2) &= \chi_{\bar{s}_2-1,2}(t) = u(t-2(s_2-1)-1)
\end{aligned}$$

with the feedback

$$\begin{pmatrix} u(\mu) \\ u(\mu-1) \end{pmatrix} = \begin{pmatrix} u_{1,[0](\mu)} \\ u_{2,[0](\mu)} \end{pmatrix} = \alpha(\mathbf{x}, \chi) + \beta(\mathbf{x})v_{[0](\mu)}$$

where $\mu \in [t+2k\tau, t+(2k+1)\tau)$, $k = 0, 1, 2, \cdots$, together with the change of coordinates

$$z = \phi(x)$$

transforms the single input dynamics (16) into a two input linear system over the time interval $[t+2i\tau, t+(2i+1)\tau)$. $\triangleleft$

4.2 Feedback linearization over the $[3k\tau, (3k+1)\tau)$ time windows

Whenever the given dynamics is affected by $u(t)$, $u(t-\tau)$ and $u(t-2\tau)$, and the conditions of Theorem 4 are not fulfilled, then the process can be further investigated over three successive time intervals considering $u(t)$, $u(t-\tau)$ and $u(t-2\tau)$ independent.

Set $u(t) = u_1(t)$, $u(t-\tau) = u_2(t)$, $u(t-2\tau) = u_3(t)$ and $\hat{l} = \lfloor (l+1)/3 \rfloor$, so that $u(t-3k\tau) = u_1(t-3k\tau)$, $u(t-(3k+1)\tau) = u_2(t-3k\tau)$ and $u(t-(3k+2)\tau) = u_3(t-3k\tau)$. The dynamics (16) will read

$$\dot{x}(t) = F(x(t), x(t-\tau), ..., .x(t-s\tau)) + \\ + \sum_{i=0}^{\hat{l}} G_{3i,1}(x(t), x(t-\tau), ..., .x(t-s\tau))u_1(t-3i\tau) \\ + \sum_{i=0}^{\hat{l}} G_{3i+1,2}(x(t), x(t-\tau), ..., .x(t-s\tau))u_2(t-3i\tau) \\ + \sum_{i=0}^{\hat{l}} G_{3i+2,3}(x(t), x(t-\tau), ..., .x(t-s\tau))u_3(t-3i\tau) \tag{19}$$

Accordingly, one gets the differential representation

$$d\dot{x}(t) = f(\mathbf{x}, \mathbf{u}, \delta)d\mathbf{x} + g_1(\mathbf{x}, \delta)du_{1,0} + g_2(\mathbf{x}, \delta)du_{2,0} + \\ + g_3(\mathbf{x}, \delta)du_{3,0} \tag{20}$$

Theorem 4 is then naturally extended as follows.

Theorem 5. The nonlinear time delay system (16) is accessible and feedback linearizable on each time window $[3k\tau, (3k+1)\tau)$ if and only if the dynamics (20) satisfies the conditions of Theorem 3.

4.3 Feedback linearization over some time window

When the previous conditions are not fulfilled, then the process can be continued up to the consideration of $(l+1)$ successive time intervals. The ultimate result which incorporates all previous ones is stated next.

First, define

$$\dot{x}(t) = F(x(t), x(t-\tau), ..., x(t-s\tau)) + \\ + \sum_{i=0}^{1} G_{(l+1)i,1}(x(t), ..., x(t-s\tau))u_1(t-(l+1)i\tau) \\ \vdots \\ + \sum_{i=0}^{1} G_{(l+1)i+l,l+1}(x(t), ..., x(t-s\tau))u_{l+1}(t-(l+1)i\tau) \tag{21}$$

Accordingly, one gets the differential representation

$$d\dot{x}(t) = f(\mathbf{x}, \mathbf{u}, \delta)d\mathbf{x} + g_1(\mathbf{x}, \delta)du_{1,0} + g_2(\mathbf{x}, \delta)du_{2,0} + \\ +... + g_{l+1}(\mathbf{x}, \delta)du_{l+1,0} \quad (22)$$

Theorem 6. The nonlinear time delay system (16) is accessible and feedback linearizable on some time window if and only if the dynamics (22) satisfies the conditions of Theorem 3.

The conditions of Theorem 6 are the weakest possible, but the linearization is effective eventually on one time interval over $l + 1$, where l denotes the number of delays. When the conditions of Theorem 6 are not fulfilled there is no static state feedback able to solve the linearization problem on any time window. The broader class of dynamic state feedbacks deserves to be investigated in this case.

5. CONCLUSIONS

Static state feedback linearization has been considered for non linear time delays systems in a very general problem statement. A full solution has been provided for single input systems. The generalization to multi input systems is essentially a matter of notations which become a bit more involved. If the weakest conditions in Theorem 6 are not fulfilled, then it is interesting to check whether a single input time delay system may be linearized on some time window by a dynamic state feedback. This would be a significant difference with respect to delay free nonlinear systems.

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