Polarizability of dielectric prolate half ellipse

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***Abstract*—This article presents a method for solving the polarizability of a dielectric prolate half ellipse as a function of its relative electric permittivity. The considered geometry consists of two conjoined half ellipses with different permittivities. The polarizability depends on the excitation field direction, therefore can be presented in the form of dyadic consisting of two components that are series and parallel polarizabilities. The method is based on analytical series expansions with coefficients obtained as a numerical solution of a matrix equation.**

***Index Terms*—Elliptic coordinates, Polarizability, Matrix representation**

I. Introduction

Study of electrostatic responses of different shaped particles is crucial especially for designing artificial materials [1], [2]. Single-particle response is informative to know the response of composite medium made up of these particles embedded in free space [3]. The quantity that measures the electrical response of any object is its polarizability α that characterizes the magnitude of polarization in presence of a static field. It is defined as the ratio of the dipole moment and the magnitude of the incident field. Several articles have been published that study the polarizabilities of many different geometries like spheres [4], [5], circular cylinders or 2D disk [6], [7], hemisphere [8], half-disk [9] and anisotropic elliptic inclusions [11] as well. However, polarizability of dielectric half ellipse, which is also the topic of present study has not been considered yet. Due to their manageable mathematics, elliptic coordinates/geometries have been analyzed widely in different areas of electromagnetics [11], [12], [13], [14], [15], [16].

In this paper, we focus on the computation of the polarizability of a dielectric prolate half ellipse placed in a homogeneous background medium. For this geometry normalized polarizability can be expressed as a dyadic [8]

$$\mathbf{p} = \overline{\overline{\alpha}}\mathbf{E_e} \tag{1}$$

where

$$\overline{\overline{\alpha}} = \alpha_{\|}\mathbf{u}_x\mathbf{u}_x + \alpha_{\perp}\mathbf{u}_y\mathbf{u}_y \tag{2}$$

where the $\alpha_{\|}$ and $\alpha_{\perp}$ are are parallel and orthogonal components, respectively of the normalized polarizability.

The elliptic system of coordinates has been shown in Fig. 1. For different chosen values of η, the coordinate curves in Fig. 1 take the shape of confocal ellipses. The two foci $(-a, 0)$ and $(0, a)$ are generally taken to be fixed at $-a$ and $+a$, respectively on the x-axis of the cartesian coordinate system.

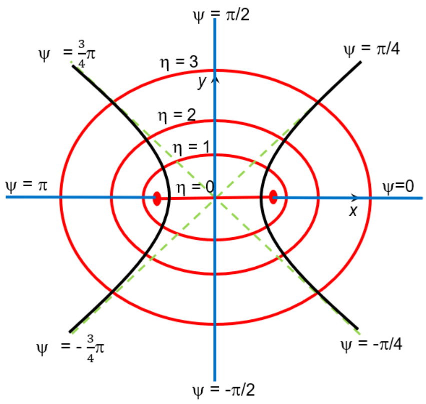

Fig. 1. System of elliptic coordinates. The ψ coordinates are confocal hyperbolae symmetrical about the x-axis with focal points at $x = a$ and $x = -a$. The η coordinates are confocal ellipses centered on the origin.

In this paper, we introduce a semianalytical method for solving the polarizability components of a prolate half ellipse. We begin with writing the potential function as a series expansion, and by applying the boundary conditions, we are able to construct a matrix equation, whose solution gives the unknown coefficients.

II. Methodology

A. Series solution for the electrostatic potential in a semi-elliptical region

Let us first consider a general situation where the object consists of two half ellipses with different permittivities. Thus, the 2D space must now be divided into three regions and we can write the electrostatic potential as a series expansion in each region. Depending on the orientation of the external electric field, a pair of two conjoined half ellipses is referred as a series and a parallel [9]. For the series and parallel cases, the external electric field $\mathbf{E}_e$ is x and y-directed respectively, as shown in Fig. 2. If the permittivity of one half of the ellipse is the same as of the surrounding environment, we are left with a single half ellipse.

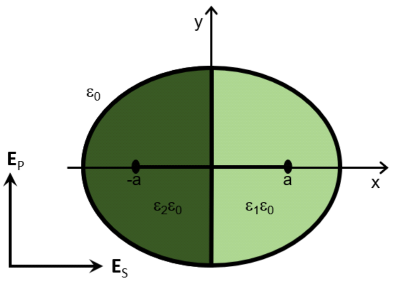

Fig. 2. A double half-ellipse with elliptic coordinate curve η=η_o in external electric field. The series configuration (left) and the parallel configuration (right) are presented.

1) Half-ellipses in series: First we consider series case. The excitation field is $\mathbf{E}_e = E_e \mathbf{u}_x$ and the corresponding potential will be $\phi_e = -V_o \cosh \eta \cos \psi$, where $V_o = E_e a$. The required potential as series expansion can be written as

$$\phi_o \approx \sum_{n=1}^{N} B_n e^{-n\eta} \cos(n\psi) - V_o \cosh(\eta) \cos(\psi), \quad \eta \geq \eta_o \tag{3}$$

$$\phi_r \approx \sum_{n=0}^{N} C_n \cosh(n\eta) \cos(n\psi), \quad \eta \leq \eta_o \quad \frac{-\pi}{2} \leq \psi \leq \frac{\pi}{2} \tag{4}$$

$$\phi_l \approx \sum_{n=0}^{N} D_n \cosh(n\eta) \cos(n\psi), \quad \eta \leq \eta_o \quad \frac{\pi}{2} \leq \psi \leq \frac{3\pi}{2} \tag{5}$$

where subscripts r and l refer to the right and the left half of double half-ellipse as presented in Fig. 1 and subscript o refers to the space outside the elliptic region. The unknown coefficients B_n, C_n, and D_n are solved by applying the boundary conditions. The continuity of the potential and the continuity of its normal component are required. Considering these conditions on the boundary between the half-ellipses, we obtain

$$C_n = \eta_n D_n, \quad \eta_n = \begin{cases} 1, & n = even \\ \frac{\varepsilon_2}{\varepsilon_1}, & n = odd \end{cases} \tag{6}$$

where ε_1 and ε_2 are the dimensionless relative permittivity values of the right and left half-ellipses, respectively.

Since it is not possible to have closed form of solution to solve the coefficients, we have constructed a set of N linearly independent equations that can be written as a $N \times N$ matrix equation. Each matrix element would then have a closed-form analytic expression but it has to be solved numerically. Following procedure has been adopted. By considering the boundary conditions on the outer contours, at $\eta = \eta_o$, we obtain four equations. These equations are then multiplied by $\cos m\psi$, $0 \leq m \leq N$, and integrated w.r.t ψ. Equations related to the right half are integrated over the interval $-\pi/2 \leq \psi \leq \pi/2$ and the equations related to the left half over the interval $\pi/2 \leq \psi \leq 3\pi/2$. This means we have to encounter following integrals

$$U_{m,n} = \int_{-\pi/2}^{\pi/2} \cos m\psi \cos n\psi d\psi \tag{7}$$

and

$$I_{m,n} = \int_{\pi/2}^{3\pi/2} \cos m\psi \cos n\psi d\psi \tag{8}$$

These integrals can be computed analytically as shown

$$U_{m,n} = \begin{cases} \pi, \quad m = n = 0 \\ \frac{\pi}{2}, \quad m = n \neq 0 \\ 0, \quad m + n = even, \quad m, n \neq 0, m \neq n \\ (-1)^{\frac{1}{2}(m+n-1)}(\frac{2n}{n^2-m^2}), m = even, n = odd \\ (-1)^{\frac{1}{2}(m+n-1)}(\frac{2m}{n^2-m^2}), n = even, m = odd \end{cases} \tag{9}$$

and

$$I_{m,n} = (-1)^{m+n} U_{m,n} \tag{10}$$

After applying condition in Eq. (4) and some algebra, we have obtain following set of equations to solve coefficients B_n.

$$\begin{aligned} &\sum_{n=1}^{N} [\eta_m \{m\varepsilon_1 \tanh(m\eta_o) + n\} + \\ &\quad (-1)^{m+n} \{m\varepsilon_2 \tanh(m\eta_o) + n\}] e^{-n\eta_o} B_n U_{m,n} \\ &\quad = [\eta_m \{m\varepsilon_1 \cosh \eta_o \tanh(m\eta_o) - \sinh \eta_o\} + \\ &\quad (-1)^{m+1} \{m\varepsilon_2 \cosh \eta_o \tanh(m\eta_o) - \sinh \eta_o\}] V_o U_{m,1} \end{aligned} \tag{11}$$

and by considering the values of η_n and $U_{m,n}$ we can also write

$$\begin{aligned} &\sum_{n=1,3..}^{N} (\varepsilon_1 - \varepsilon_2) e^{-n\eta_o} U_{m,n} B_n + \\ &(\varepsilon_1 + \varepsilon_2 + 2\coth(m\eta_o)) e^{-m\eta_o} \frac{\pi}{2} B_m \\ &= (\varepsilon_1 - \varepsilon_2) \cosh \eta_o U_{m,1} V_o \qquad m = 2, 4, ...N \end{aligned} \tag{12}$$

and

$$\begin{aligned} &\sum_{n=2,4..}^{N} n(\varepsilon_2 - \varepsilon_1) e^{-n\eta_o} U_{m,n} B_n + \\ &m(2\varepsilon_1\varepsilon_2 \tanh(m\eta_o) + \varepsilon_1 + \varepsilon_2) e^{-m\eta_o} \frac{\pi}{2} B_m \\ &= (2\varepsilon_2\varepsilon_1 \tanh(m\eta_o) \cosh \eta_o - \varepsilon_2 \sinh \eta_o - \\ &\varepsilon_1 \sinh \eta_o) \frac{\pi}{2} \delta_{m,1} V_o \qquad m = 1, 3, ...N \end{aligned} \tag{13}$$

The above equation system can be written as $N \times N$ matrix equation and solved numerically.

2) Half-ellipses in parallel: For parallel case, the excitation field is y-directed such that $\mathbf{E}_e = E_e \mathbf{u}_y$ and the corresponding potential will be $\phi_e = -V_o \sinh\eta \sin\psi$, where $V_o = E_e a$. The required potential as series expansion can be written as

$$\phi_o \approx \sum_{n=1}^{N} F_n e^{-n\eta} \sin(n\psi) - V_o \sinh\eta \sin\psi, \quad \eta \geq \eta_o \quad (14)$$

$$\phi_r \approx \sum_{n=1}^{N} G_n \sinh(n\eta) \sin(n\psi), \quad \eta \leq \eta_o \quad \frac{-\pi}{2} \leq \psi \leq \frac{\pi}{2} \quad (15)$$

$$\phi_l \approx \sum_{n=1}^{N} H_n \sinh(n\eta) \sin(n\psi), \quad \eta \leq \eta_o \quad \frac{\pi}{2} \leq \psi \leq \frac{3\pi}{2} \quad (16)$$

The coefficients have been obtained here as in case of series, but the relation between G_n and H_n becomes

$$G_n = \eta_n H_n, \quad \eta_n = \begin{cases} 1, & n = odd \\ \frac{\varepsilon_2}{\varepsilon_1}, & n = even \end{cases} \quad (17)$$

Here again four equations have been obtained corresponding to the boundary conditions at outer contours. These equations are multiplied by $\sin m\psi$ for this case and integrated with respect to ψ. We need to evaluate these following integrals

$$V_{m,n} = \int_{-\pi/2}^{\pi/2} \sin m\psi \sin n\psi d\psi \quad (18)$$

and

$$W_{m,n} = \int_{\pi/2}^{3\pi/2} \sin m\psi \sin n\psi d\psi \quad (19)$$

Performing few calculations we get

$$V_{m,n} = \begin{cases} 0, & m = n = 0 \\ U_{m,n}, & m = n \neq 0 \\ 0, & m + n = even, \quad m, n \neq 0, \quad m \neq n \\ \frac{m}{n} U_{m,n}, & m = even, \quad n = odd \\ \frac{n}{m} U_{m,n}, & n = even, \quad m = odd \end{cases} \quad (20)$$

and

$$W_{m,n} = (-1)^{m+n} V_{m,n} \quad (21)$$

We must have

$$\sum_{n=1,3..}^{N} (\varepsilon_2 - \varepsilon_1) e^{-n\eta_o} U_{m,n} F_n + (\varepsilon_1 + \varepsilon_2 + 2\varepsilon_1\varepsilon_2 \tanh(m\eta_o)) e^{-m\eta_o} \frac{\pi}{2} F_m = (\varepsilon_1 - \varepsilon_2) cosh\eta_o U_{m,1} V_o \qquad m = 2, 4, ...N \quad (22)$$

and

$$\sum_{n=2,4..}^{N} n(\varepsilon_1 - \varepsilon_2) e^{-n\eta_o} \tanh(m\eta_o) U_{m,n} F_n + m[(\varepsilon_1 + \varepsilon_2) \tanh(m\eta_o) + 2] e^{-m\eta_o} \frac{\pi}{2} F_m = [(\varepsilon_1 + \varepsilon_2) \tanh(m\eta_o) \sinh\eta_o - 2\cosh\eta_o] \delta_{m,1} V_o \quad m = 1, 3, ...N \quad (23)$$

B. Polarizability of dielectric half ellipse

If the observation point is far enough, or with large values of η, ellipse starts to resemble a circle. For that, the polarized ellipse must be approximated by a 2D dipole. If $\varepsilon_2 = 1$, only one half ellipse is left. The normalized polarizability dyadic in Eq. 2 for the this case becomes

$$\overline{\overline{\alpha}} = \alpha_s \mathbf{u}_x \mathbf{u}_x + \alpha_p \mathbf{u}_y \mathbf{u}_y \quad (24)$$

where α_s and α_p are series and parallel polarizabilities respectively.

The potential produced by the dipole with dipole moment $\mathbf{p}$ along x-axis is

$$\phi_d(\rho, \varphi) = \frac{p \cos\varphi}{2\pi\varepsilon_e \rho} \quad (25)$$

ϕ_d has been expressed in polar coordinates. The connection between polar and elliptic coordinates is

$$\rho \approx \frac{ae^\eta}{2} \quad (26)$$

$$\varphi \approx \psi \quad (27)$$

Using above connection Eq. 25 can be transformed into elliptic coordinates. The corresponding term in Eq. 3 is

$$\phi_d(\eta, \psi) = B_1 e^{-\eta} \cos\psi \quad (28)$$

When we combine Eqs. 1, 25-28 , we are left with

$$\alpha_s = 2\frac{B_1}{V_o} \quad (29)$$

Similarly for y-directed dipole, we end up finding parallel polarizability that is

$$\alpha_p = 2\frac{F_1}{V_o} \quad (30)$$

C. Numerical simulations and discussion

Fig. 3 presents the normalized polarizability component α_s of half ellipse and normalized polarizability component of a homogeneous ellipse with x-directed external excitation computed analytically using results of [11] and then substituting $\varepsilon_t = \varepsilon_n = \varepsilon_r$ as a function of relative permittivity ε_r computed with matrix size $N = 100$. Similarly Fig. 4 presents the normalized polarizability component α_p of half ellipse and normalized polarizability component of a homogeneous ellipse of when external excitation is y-directed as a function of ε_r computed with matrix size $N = 100$.

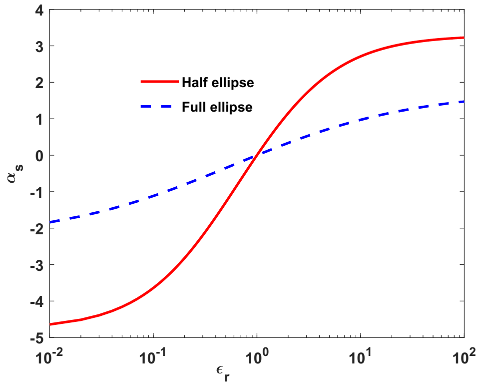

Fig. 3. The normalized polarizability component α_s of a half ellipse with N = 100 compared with corresponding normalized polarizability component of a homogeneous ellipse as a function of ε_r.

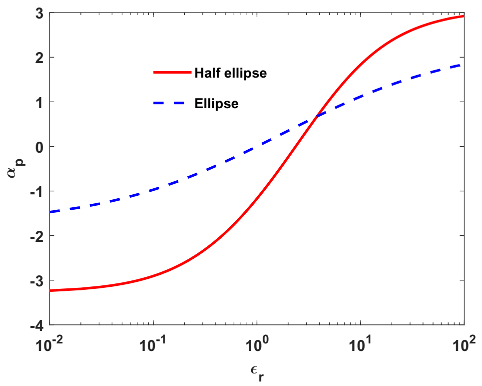

Fig. 4. The normalized polarizability component α_p of a half ellipse with N = 100 compared with corresponding normalized polarizability component of a homogeneous ellipse as a function of ε_r.

It can be seen that the magnitude of the series and parallel polarizabilities of half ellipse is greater than the corresponding polarizabilities of whole ellipse, in both cases. Fig. 5 presents the comparison between circular disk, average normalized polarizability of a homogeneous ellipse and the average polarizability of half ellipse. These observations seems reasonable and in agreement with the results in [18], [19]. As sphere is the object with minimum polarizability and any other geometrically different object from sphere has increased polarizability depending on its deviation from sphere. Therefore, in 2D any deviation from circular geometry results in an increase in magnitude of polarizability.

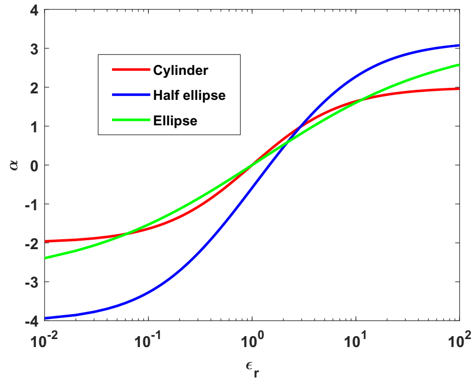

Fig. 5. The normalized polarizability of a homogeneous disk, the average polarizability of a homogeneous ellipse and the average polarizability of a half ellipse as a function of ε_r.

III. Conclusion

In this article, we considered the polarizability of dielectric prolate half ellipse. The polarizability consisted of two components, the series polarizability α_s and the parallel polarizability α_p. Based on analytical approach, we presented a method in which the electrostatic potential function has been written as a series expansion. We have splitted series into even and odd values of indices, as they have shown different results corresponding to even and odd indices. However, it is not possible to solve the coefficient of expansions separately. We must have to consider an equation system of N equations and write it in matrix form. With this method, we calculated approximate formulas for normalized polarizability of a half ellipse as a function of its relative permittivity. Normalized polarizability consists of two orthogonal components, that are series and parallel one. This work still needs numerical treatment for complete validation, so we will be working on it and soon results will be published.

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