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On certain families of Drinfeld quasi-modular forms

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ABSTRACT

This paper deals with two problems arising in the study of Drinfeld quasi-modular forms. The first problem is to find the maximal order of vanishing at infinity of a non-zero Drinfeld quasi-modular form and leads to the notion of “extremal” quasi-modular form (highest possible order of vanishing for fixed weight and depth). The second problem is determining differential properties of extremal forms, leading to the notion of “differentially extremal” form. From our investigations, we will obtain an upper bound for the order of vanishing at infinity of non-zero Drinfeld quasi-modular forms of small depths. The paper ends with a collection of tools used in the previous parts. The notion of “extremal” form is similar to one introduced by Kaneko and Koike in [M. Kaneko, M. Koike, On extremal quasimodular forms, Kyushu J. Math. 60 (2006) 457–470].

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1. Introduction

The primary aim of this paper is to describe some partial advances in the solution of the following two problems.

1. Find the maximal order of vanishing at infinity of a non-zero Drinfeld quasi-modular form of given weight.
2. Determine differential properties of Drinfeld quasi-modular forms of given weight and depth with maximal order of vanishing at infinity (these forms will be called *extremal*).

Our results are obtained in a constructive way, studying families of forms with peculiar properties. For our purposes, general tools need to be developed. Some will appear of independent interest.

Before going deeper in these topics and rigorously defining the entities above, we present an overview of the more familiar framework of quasi-modular forms on the complex upper-half plane, for the group $\mathbf{SL}_2(\mathbb{Z})$.

1.1. The classical framework

Let $z = x + iy \in \mathbb{C}$ with $y > 0$ and $u \in \mathbb{C}$ be complex numbers related by $u = e^{2\pi iz}$, so that $0 < |u| < 1$. For $i = 1, 2, 3$ the series:

$$E_{2i}(z) = 1 + b_i \sum_{n=1}^{\infty} n^{2i-1} \frac{u^n}{1-u^n},$$

with $b_1 = -24$, $b_2 = 240$, $b_3 = -504$, normally converge in any compact subset of the domain determined by $|u| < 1$ and represent algebraically independent functions. The \mathbb{C} -algebra of quasi-modular forms is the polynomial algebra $\tilde{M} := \mathbb{C}[E_2, E_4, E_6]$, which is graded by the weights (where the weight of E_{2i} is $2i$ for $i = 1, 2, 3$) and filtered by the depths (the depth of a polynomial in \tilde{M} is its degree in E_2):

$$\tilde{M} = \bigoplus_{w \geq 0} \bigcup_{l \geq 0} \tilde{M}_w^{\leq l},$$

where $\tilde{M}_w^{\leq l}$ is the \mathbb{C} -vector space spanned by the forms of weight w and *depth* $\leq l$.

Any element f of $\tilde{M}_w^{\leq l} \setminus \{0\}$ has a non-vanishing u -expansion

$$f(z) = \sum_{i=0}^{\infty} c_i u^i, \quad c_i \in \mathbb{C} \quad (1)$$

and the natural problem of determining the image of the function

$$\begin{aligned} v_{\infty} : \tilde{M}_w^{\leq l} \setminus \{0\} &\rightarrow \mathbb{Z}_{\geq 0}, \\ f &\mapsto \inf\{i, c_i \neq 0\}, \quad \text{with } f \text{ as in (1)} \end{aligned} \quad (2)$$

arises, for given l, w .

In [9, p. 459], Kaneko and Koike ask whether the image of v_{∞} on the set $\tilde{M}_w^{\leq l} \setminus \{0\}$ is precisely the interval $[0, 1, \dots, \dim_{\mathbb{C}}(\tilde{M}_w^{\leq l}) - 1]$, as numerical investigations suggest, for w small. This property, if true, would imply that for any $f \in \tilde{M}_w^{\leq l} \setminus \{0\}$,

$$v_{\infty}(f) \leq \dim_{\mathbb{C}}(\tilde{M}_w^{\leq l}) - 1 \leq \frac{1}{12}(w + 11l + l(w - l)). \quad (3)$$

We recall that \tilde{M} is a D -differential algebra, with $D := u \frac{d}{du} = (2\pi i)^{-1} \frac{d}{dz}$. Looking at the resultant $\text{Res}_{E_2}(f, Df)$ of f and Df , seen as polynomials in E_2 , it is not difficult to prove that for f as above, irreducible,¹

$$v_{\infty}(f) \leq \frac{1}{12}(w + 2l(w - l)). \quad (4)$$

Similar inequalities have already been used to describe diophantine properties of certain complex numbers, see e.g. [6, 16]. In fact, in order to prove (4), the above resultant can only be used if it does not vanish, that is, if Df is not divisible by f . To prove the estimate for the remaining forms, we need to characterise those forms f such that f divides Df . The key point is here to remark that the only principal prime ideal of \tilde{M} which is stable by differentiation is the ideal (Δ) , where Δ is the discriminant function (see [12, Chapter 10, Lemma 5.2]). The remaining case $f = \Delta$ in (4) can then be checked directly.

Apart from some choices (l, w) with $1 \leq l \leq 10$ and $w \leq 20$ and the case $l = 0$, the upper bound of (4) is weaker than that of (3). The truth of the sharper estimate (3) remains unknown for general l, w .

Let l, w be integers such that $\tilde{M}_w^{\leq l} \neq (0)$, and let $f_{l,w}$ be the unique non-vanishing normalised² quasi-modular form of the space $\tilde{M}_w^{\leq l}$ with the property that the function (2) attains its maximal value on it.

In [9, Theorem 2.1], Kaneko and Koike constructed a family of quasi-modular forms which turns out to be, up to a non-zero scalar factor and by means of elementary arguments, the family $(f_{1,2i})$.³

¹ The inequality (4) also holds for f not necessarily irreducible.

² A formal series $f = \sum_{i \geq i_0} c_i u^i$ with $c_{i_0} \neq 0$ is said to be *normalised* if $c_{i_0} = 1$. A quasi-modular form is normalised if its u -expansion is.

³ In [9], Kaneko and Koike call any non-vanishing form $f \in \tilde{M}_k^{\leq l} \setminus \tilde{M}_k^{\leq l-1}$ for which the equality $v_{\infty}(f) = \dim_{\mathbb{C}}(\tilde{M}_k^{\leq l}) - 1$ holds, *extremal*. We warn the reader that in this paper, we will use this terminology in a different way.

They prove that $v_\infty(f_{1,2i}) = \lfloor \frac{i}{3} \rfloor = \dim(\tilde{M}_{2i}^{\leq 1}) - 1$ ($i \geq 0$), where $\lfloor \cdot \rfloor$ denotes the lower integer part of a real number. Their construction is performed with an inductive process, in which the differential operators (“Serre’s differential operators”, cf. p. 467 of [9]) defined by:

$$\theta_d^{(n-1)} f = D^n f - \frac{n+d-1}{12} [E_2, f]_{n-1}^{(2,d)}, \quad (5)$$

play a crucial role, $[f, g]_{n-1}^{(2,d)}$ being a suitable normalisation of the Rankin–Cohen bracket of f and g , of order $n-1$ and weights 2 and d (its definition is available, for example, on p. 466 of [9]). It turns out that for all i , the form $f_{1,2i}$ is essentially unique satisfying:

$$\theta_{2i-1}^{(1)} f_{1,2i} = \lambda_{1,2i} \Delta f_{1,2i-8}, \quad i \geq 1, \quad (6)$$

where $\Delta = (E_4^3 - E_6^2)/1728$ and $\lambda_{1,2i} \in \mathbb{Q}$ with $\lambda_{1,2i} = 0$ if and only if $3 \nmid i$.

For $l=2$, Kaneko and Koike develop similar constructions in [9]. We omit to describe their results referring to [8,9] for further details and references; however, we did not find any other result in the direction of inequalities (3) for $l \geq 2$.

1.2. The drinfeldian framework: our results

From now on, the symbols $u, \tilde{M}, \Delta, v_\infty, D$ will be used with a new meaning which we now describe. Occasionally, the older meanings related to the classical framework will reappear, but the reader should not encounter any trouble with these occurrences.

In the following, $q = p^e$ is a positive power of a given prime number p fixed once and for all, and θ will be an indeterminate over \mathbb{F}_q . Certain results of this text do not hold for certain choices of q ; this will be highlighted on a case to case basis.

Let C be the completion of an algebraic closure $\overline{K_\infty}$ of the field $K_\infty := \mathbb{F}_q((1/\theta))$ for the unique extension of the valuation $-\deg_\theta$ to $\overline{K_\infty}$ (the θ^{-1} -adic valuation). For this valuation, K_∞ is the completion of its subfield $K = \mathbb{F}_q(\theta)$. We will also denote by A the \mathbb{F}_q -algebra $\mathbb{F}_q[\theta]$. On C, K_∞, K, A , we will denote by $|\cdot|$ the ultrametric absolute value $q^{\deg_\theta(\cdot)}$.

The story of Drinfeld modular forms⁴ begins with the pioneering work of Goss [4,5]. Later, in the important paper [3], Gekeler considered the two functions E, g and discovered the function h , allowing us, later in [1], to investigate some properties of the C -algebra of Drinfeld quasi-modular forms $\tilde{M} := C[E, g, h]$.⁵ As “drinfeldian framework” (title of this section) we mean a natural counterpart of the theory sketched in Section 1.1, for these Drinfeld quasi-modular forms.

We refer to the first part of our paper [1] for all the basic properties of Drinfeld quasi-modular forms, noticing that the indeterminate T there becomes θ here.⁶ Gekeler’s functions E, g, h are algebraically independent quasi-modular forms for the (homographical) action of the group $\mathbf{GL}_2(A)$ on $\Omega := C \setminus K_\infty$. For the three functions E, g, h , the triples $(w, m, l) \in \mathbb{Z}_{\geq 0} \times (\mathbb{Z}/(q-1)\mathbb{Z}) \times \mathbb{Z}_{\geq 0}$, where w is the weight, m the type and l the depth, are $(2, 1, 1)$, $(q-1, 0, 0)$ and $(q+1, 1, 0)$, respectively.

If we denote by $\tilde{M}_{w,m}^{\leq l}$ the C -vector space of Drinfeld quasi-modular forms of weight w , type m and depth $\leq l$ (which is by definition the space (0) if $l < 0$), we have [1, Proposition 2.2]:

$$\tilde{M} = \bigoplus_{\substack{w \in \mathbb{Z}_{\geq 0} \\ m \in \mathbb{Z}/(q-1)\mathbb{Z}}} \bigcup_{l \geq 0} \tilde{M}_{w,m}^{\leq l}.$$

⁴ Rigid analytic modular forms, where the “base fields” are global, of positive characteristic.

⁵ Notice that in the introduction of [1], the sentence attributing to Gekeler’s paper the first occurrence of modular forms for $\mathbf{GL}_2(\mathbb{F}_q[\theta])$ is obviously incorrect.

⁶ Several notations of [1] change in the present work. We adopt at the same time notations issued from Gekeler as in [3] and notations that will be compatible with Papanikolas [13] since we believe that the use of t -motives will eventually intervene in the theory of Drinfeld modular forms, and we want to keep certain symbols, such as t , free for that occasion.

1.2.1. First part: extremality

We recall [1, Section 2 and Lemma 4.2 (i)–(iii)] that E, g, h have u -expansions convergent in a neighbourhood of $u = 0$, with $u(z) = 1/e_C(\tilde{\pi}z)$, where e_C is Carlitz's exponential function and $\tilde{\pi} \in C$ is one of its fundamental periods (chosen once and for all)⁷. There is a C -algebra homomorphism $\tilde{M} \hookrightarrow C[[u]]$. We will identify $f \in \tilde{M}$ with its image $\sum_{i \geq 0} c_i u^i$ in $C[[u]]$. We write $\nu_\infty(0) = \infty$ and, for $f \neq 0$, $\nu_\infty(f) = \min\{i \text{ such that } c_i \neq 0\}$.

Definition 1.1. Let w, l be fixed non-negative integers and m be a class in $\mathbb{Z}/(q-1)\mathbb{Z}$. A non-zero quasi-modular form $f \in \tilde{M}_{w,m}^{\leq l}$ is said to be *extremal of depth $\leq l$* if for all $g \in \tilde{M}_{w,m}^{\leq l} \setminus \{0\}$, $\nu_\infty(g) \leq \nu_\infty(f)$.

A quasi-modular form which is extremal of depth $\leq l$ needs not to be extremal of depth $\leq l'$ for $l' > l$. On the other hand, a straightforward argument shows that, for w, m, l fixed, a quasi-modular form of weight w and type m which is extremal of depth $\leq l$ is uniquely determined, up to multiplication by an element of C^\times .

When a quasi-modular form $f \in \tilde{M}_{w,m}^{\leq l} \setminus \tilde{M}_{w,m}^{\leq l-1}$ is extremal of depth $\leq l$, we will often say that f is *extremal*. We will adopt this simplified terminology when the context allows the complete determination of l .

For all l, w, m such that $\tilde{M}_{w,m}^{\leq l} \neq (0)$, let $f_{l,w,m} \in \tilde{M}_{w,m}^{\leq l}$ be the unique normalised extremal quasi-modular form of depth $\leq l$. Contrary to the classical framework, u -expansions of Drinfeld quasi-modular forms are difficult to compute, and Gekeler's algorithms developed in [3] are required. Thanks to them, we did experiments that, after observation of the cases $q = 2, 3, 5$, $w \leq q^3 + 1$, $l \leq q^2 + 1$ and any value of m , suggest the existence of a (conjectural) estimate as follows. For all $\epsilon > 0$ and for all l big enough depending on ϵ :

$$\nu_\infty(f_{l,w,m}) \leq (1 + \epsilon)l(w - l) \quad (7)$$

(notice that if $l > 0$, $w > l$).

Just like inequality (3), this inequality seems to be rather difficult to prove. Even weaker estimates like an analogue of (4) are presently unavailable (see discussion in Section 1 of [1]). This is essentially because in our case there are infinitely many irreducible quasi-modular forms f such that Df is divisible by f , and in this last case there is no obvious candidate to replace the modular form $\text{Res}_E(f, Df)$.

It would be interesting to find an explicit function $c(w, l)$ of the weight and the depth such that for all $l \gg 0$, $\nu_\infty(f_{l,w,m}) \leq c(w, l)$. Showing the existence of a constant $c(q) > 0$, depending on q only, such that if $l > 0$,

$$\nu_\infty(f_{l,w,m}) \leq c(q)l(w - l),$$

would also have interesting arithmetic consequences. For instance, the results of the present paper show that if $c(q)$ exists, then $c(q) \geq 1$ (cf. Proposition 2.12). In the first part of this paper, we discuss partial advances toward these estimates for small depths.

Although analogues of “higher Serre's operators” can be constructed (this paper, Section 4.1), ideas of proof of Kaneko and Koike as in [8] cannot extend to our case because these operators have too big kernels due to the positive characteristic (but see Section 3.3 for some condition analogous to (6)).

In Section 2 we study the sequence of Drinfeld quasi-modular forms $(x_k)_{k \geq 0}$, with $x_k \in \tilde{M}_{q^{k+1},1}^{\leq 1} \setminus \tilde{M}_{q^{k+1},1}^{\leq 0}$, defined by $x_0 = -E$, $x_1 = -Eg - h$ and by the recursion formula

$$x_k = x_{k-1}g^{q^{k-1}} - [k-1]x_{k-2}\Delta^{q^{k-2}}, \quad k \geq 2,$$

⁷ In [1], we wrote about t -expansions instead of u -expansions, and the period of Carlitz's exponential was denoted by $\bar{\pi}$ instead of $\tilde{\pi}$, according to Gekeler [3].

where $\Delta := -h^{q-1}$. After having dealt with some basic properties of this sequence, we show in Proposition 2.3 that, for all $k \geq 0$, $\nu_\infty(x_k) = q^k$. This shows that in general,

$$\nu_\infty(f_{l,w,m}) > \dim_{\mathbb{C}}(\tilde{M}_{w,m}^{\leq l}) - 1,$$

in apparent contradiction with Kaneko and Koike's prediction (3) in the classical framework.

We can show that for all k , x_k is an irreducible polynomial in E, g, h and a resultant argument (see Section 2.1.4) yields:

Theorem 1.2. *For all $k \geq 0$, x_k is an extremal quasi-modular form.*

Looking back at subsequences of Kaneko and Koike's sequence $(f_{1,2i})_{i \geq 0}$, there does not seem to be similar recursion formulas, with weights varying as sequences like $(\alpha q^k + \beta)_{k \geq 0}$ rather than as arithmetic progressions. But experimentally, congruences between u -expansions of certain forms $f_{1,2i}$'s seem to occur. They could be consequence of Clausen–von Staudt congruences for Bernoulli numbers.

Our investigation was pushed a step further, with the sequence $(\xi_k)_{k \geq 0}$ defined, for $k \geq 0$ by:

$$\xi_k = [k]^q x_{k+1} x_{k-1}^q - [k+1] x_k^{q+1} \in \tilde{M}_{(q+1)(q^k+1),2}^{\leq q+1} \setminus \tilde{M}_{(q+1)(q^k+1),2}^{\leq q}, \quad (8)$$

where $[k] := \theta^{q^k} - \theta$ ($k \geq 1$), $[0] := 1$, and we have set $x_{-1} := -h^{1/q}$. Again, we could compute $\nu_\infty(\xi_k)$ for all k and prove that ξ_k is always irreducible, implying the following (Section 2.2):

Theorem 1.3. *Assuming that $q \geq 3$, the form ξ_k is extremal for all $k \geq 0$.*

As a product of these investigations, we obtain the following multiplicity estimate:

Theorem 1.4. *Let w and m be integers such that $\tilde{M}_{w,m}^{\leq q^2} \neq \{0\}$, and let f be a non-vanishing form in $\tilde{M}_{w,m}^{\leq q^2}$. Then*

$$\nu_\infty(f) \leq (q^3 + 1)(w - l).$$

1.2.2. Second part: differential properties

We were surprised to remark that the forms x_k and ξ_k also enjoy a rich differential structure, and the second part of this text, Section 3, is devoted to reporting our knowledge on this topic.

In all the following, we write $D = (D_n)_{n \geq 0}$ for the collection of higher derivatives on the \mathbb{C} -algebra of holomorphic functions on $\Omega = \mathbb{C} \setminus K_\infty$ introduced in [1, Section 1]. Therefore, $D_1 = (-\tilde{\pi})^{-1} d/dz = u^2 d/du$. By Theorem 2 of [1], D_n induces a \mathbb{C} -linear map

$$\tilde{M}_{w,m}^{\leq l} \xrightarrow{D_n} \tilde{M}_{w+2n,m+n}^{\leq l+n},$$

so that the \mathbb{C} -algebra \tilde{M} is D -stable (or hyperdifferential).

Already in [1], we have remarked that the problem of estimating the quantity $\nu_\infty(f)$ for a Drinfeld quasi-modular form f is intimately related to its differential properties (this point of view was inherited by Nesterenko, and finds its foundations in Siegel and Shidlovski's work).

In the papers [4,5], u -expansions were already considered, and their behaviour immediately appeared to be surprisingly erratic. Later, in [3], Gekeler described algorithms to compute their u -expansions. However, the unpredictable character of the coefficients of the u -expansions of all these Drinfeldian forms remains nowadays one of the typical aspects making this theory independent from the classical one. Similar observations can be made concerning the problem of Hecke's eigenforms.

As a strange resurgence of older problems, we remarked in [1] that also the operators D_n behave erratically over the polynomial algebra $C[E, g, h]$.

Here, we introduce the following:

Definition 1.5. Let f be a non-zero element of \tilde{M} . We define the *differential exponent* $\epsilon_D(f)$ of f as follows: it is ∞ if $\frac{D_n f}{f} \in \tilde{M}$ for all $n \geq 1$, and otherwise it is the smallest integer $k \geq 0$ satisfying $\frac{D_{E^k} f}{f} \notin \tilde{M}$ (thus $\epsilon(f) = 0$ means $\frac{D_1 f}{f} \notin \tilde{M}$).

Let l, w, m be such that $\tilde{M}_{w,m}^{\leq l} \setminus \tilde{M}_{w,m}^{\leq l-1} \neq \emptyset$ (with the convention that if $l < 0$, $\tilde{M}_{w,m}^{\leq l} := (0)$), let f be a quasi-modular form in this set. We say that f is *differentially extremal* of weight w , type m , depth l , if it attains the biggest *finite* differential exponent within this set.

By [1, Proposition 3.6], a differentially extremal quasi-modular form cannot be proportional to a power of h and if $q \neq 2, 3$ we obtained, in [1, Theorem 3], that if f is not proportional to a power of h , then f has a finite differential exponent. In Section 3, by using a result of Cornelissen in [2] on the factorisation in $K[g, h]$ of certain normalised Eisenstein series, we prove:

Theorem 1.6. For all $k \geq 0$, x_k is differentially extremal of differential exponent $(k+1)e$.

We are presently unable to show the differential extremality of the forms ξ_k , but in Section 3.2, we describe numerical computations which seem to confirm this hypothesis.

Sections 3.3 and 3.4 develop two questions, partly independent but not completely disjoint to the problem of finding differentially extremal quasi-modular forms. The content of these sections will be presented at their respective beginning; the reader can skip them in a first reading of the paper.

1.2.3. Third part: differential tools

The proofs of the statements above require several technical tools which appear in Section 4 of this paper. In this section, the reader can find several results, some of which are of independent interest, described in the summary below.

Let n, d be non-negative integers. We define the n -th Serre's operator of degree d , $\partial_n^{(d)} : \tilde{M} \rightarrow \tilde{M}$, by the formula

$$\partial_n^{(d)} f = D_n f + \sum_{i=1}^n (-1)^i \binom{d+n-1}{i} (D_{n-i} f) (D_{i-1} E). \quad (9)$$

These operators can be considered as analogues of higher Serre's C -linear differential operators (5) in the drinfeldian framework. In Theorem 4.1 of Section 4.1 we show the (not obvious at all) property that

$$\partial_n^{(w-l)} : \tilde{M}_{w,m}^{\leq l} \rightarrow \tilde{M}_{w+2n,m+n}^{\leq l}$$

(compare with [9, Proposition 3.3]). The properties of these operators are essential in the proof of Theorem 1.6. A further application of the operators $\partial_n^{(d)}$ is contained in Section 4.1.1, where we indicate a new technique to determine modular eigenforms of all the Hecke operators.

In Section 4.2 we furnish algorithms to compute the polynomials

$$D_n E, D_n g, D_n h \in C[E, g, h].$$

These algorithms can be viewed as variants of Gekeler's algorithms in [3]. Proposition 4.9 is crucial, for example, in the computations of the polynomials $D_n \xi_k$ we made, as well as in the proof of Theorem 1.6.

1.2.4. Final remarks

It is strongly possible that the forms ξ_k are all differentially extremal. This can be checked, in principle, by using the tools of Section 4.2, up to tremendous calculations we had not the courage to do, but that can be done. In fact, we got interested in these forms ξ_k in an attempt of finding differentially extremal forms by solving linear equations; later, we found that these forms are extremal, yielding the actual presentation of this paper. This convinced us to follow a constructive approach to produce multiplicity estimates.

The reader may remark that the problem of finding such families is essentially of a diophantine nature. It can be proved, just as in [8, p. 153], that the sequence $(x_k)_{k \geq 0}$ is related to the convergents of the continued fraction expansion of the function h/E as a formal series in $j := g^{q+1}/\Delta$. On another side, in [7], Kaneko reveals a connection between certain modular solutions of the differential equations $\theta_k^{(1)} f = 0$ and Apéry's approximations of $\zeta(2) = \pi^2/6$. Hence, it could reveal difficulty to explicitly construct new interesting families in higher depth. Nevertheless, we think that the connection between extremality and differential extremality of certain families of Drinfeld quasi-modular forms is such a topic that will deserve further surprises.

2. Order of vanishing and extremality

As already mentioned in the introduction, the main objective of this section is to introduce and study two families of extremal Drinfeld quasi-modular forms: one in depth ≤ 1 (the forms x_k , see Section 2.1) and one in depth $\leq q+1$ (the forms ξ_k , see Section 2.2). We use these forms to prove multiplicity estimates for quasi-modular forms of depth $\leq q$ or $\leq q^2$ (Sections 2.1.3 and 2.2.2). The tools developed in this section allow to compute, in Proposition 2.12, certain extremal forms of depth $< \frac{q+1}{2}$.

2.1. The family $(x_k)_{k \geq 0}$

We begin by defining three sequences of Drinfeld quasi-modular forms:

$$(g_k)_{k \geq 0}, \quad (h_k)_{k \geq 0}, \quad (x_k)_{k \geq 0}.$$

Eisenstein's series for the group $\mathbf{GL}_2(A)$ are defined on p. 681 of [3]:

$$E^{(w)}(z) = \sum'_{a,b \in A} (az+b)^{-w}, \quad (10)$$

where the dash indicates that the sum is restricted to a, b not simultaneously zero. It is easy to prove that the series $E^{(w)}$ converges uniformly on every compact subset of Ω , for all integers $w > 0$, to a Drinfeld modular form of weight w and type 0 which is non-zero if and only if $w \equiv 0 \pmod{q-1}$ (see [3] p. 682).

Following [3] p. 684 (and the notations of this reference), let us write $g_0 = 1$ and, for $k \geq 1$,

$$g_k = (-1)^{k+1} \pi^{1-q^k} L_k E^{(q^k-1)}, \quad (11)$$

where $L_k := [k][k-1] \cdots [1]$.

For $k \geq 0$, g_k is a non-vanishing normalised modular form of weight $q^k - 1$ and type 0, whose expansion at infinity belongs to $A[[u]]$.

We have [3, Proposition 6.9]: $g_0 = 1$, $g_1 = g$, and

$$g_k = g_{k-1} g^{q^{k-1}} - [k-1] g_{k-2} \Delta^{q^{k-2}}, \quad k \geq 2. \quad (12)$$

In order to define the forms h_k 's ($k \geq 0$) we introduce, for fixed $w \in \mathbb{N}$ and $m \in \mathbb{Z}/(q-1)\mathbb{Z}$, Serre's operator $\partial_1^{(w)} : M_{w,m} \rightarrow \tilde{M}$, which is defined by

$$\partial_1^{(w)} f = D_1 f - w E f. \quad (13)$$

It is well known (see [3, Section (8.5)]) that

$$\partial_1^{(w)}(M_{w,m}) \subset M_{w+2,m+1},$$

so we have in fact an operator $\partial_1^{(w)} : M_{w,m} \rightarrow M_{w+2,m+1}$ (note that in [3] Serre's operator is denoted by ∂_w and defined by the formula $\partial_w f = w E f - D_1 f = -\partial_1^{(w)} f$).

We now define, for $k \geq 0$:

$$h_k = -\partial_1^{(q^k-1)} g_k, \quad k \geq 0.$$

For all k , h_k is a modular form of weight $q^k + 1$, type 1. Moreover, $h_0 = 0$ and $h_1 = h$ [3, Theorem (9.1)]. Finally, we define the sequence $(x_k)_{k \geq 0}$ by:

$$x_0 = -E \quad \text{and} \quad x_k = D_1 g_k, \quad k \geq 1.$$

We will see in a little while that this definition is compatible with that of the introduction.

Since by definition we have, for $k \geq 1$, $\partial_1^{(q^k-1)} f = D_1 f + E f$, we find $h_k = -D_1 g_k - E g_k = -x_k - E g_k$. Hence the following identity holds (one immediately checks that it is also true for $k=0$):

$$x_k = -E g_k - h_k, \quad k \geq 0. \quad (14)$$

Therefore, the form x_k is, for $k \geq 0$, a non-modular quasi-modular form of weight $q^k + 1$, type 1 and depth 1.

It turns out that the three families $(g_k)_k$, $(h_k)_k$ and $(x_k)_k$ satisfy the same recursion formula.

Proposition 2.1. *The sequence $(h_k)_{k \geq 0}$ is determined by the initial conditions*

$$h_0 = 0, \quad h_1 = h$$

and the recursion formula

$$h_k = h_{k-1} g^{q^{k-1}} - [k-1] h_{k-2} \Delta^{q^{k-2}}, \quad k \geq 2.$$

Similarly, the sequence $(x_k)_{k \geq 0}$ is determined by the initial conditions

$$x_0 = -E, \quad x_1 = -E g - h$$

and the recursion formula

$$x_k = x_{k-1} g^{q^{k-1}} - [k-1] x_{k-2} \Delta^{q^{k-2}}, \quad k \geq 2.$$

Proof. We begin with the recursion relation for x_k . The formulas $x_0 = -E$ (definition) and $x_1 = D_1 g = -E g_1 - h_1 = -E g - h$ have been already remarked. If $k=2$, then $g_2 = [1] h^{q-1} + g^{q+1}$, so that by formulas (2) of [1]:

$$\begin{aligned}x_2 &= D_1 g_2 = -[1]h^{q-1}E - (Eg + h)g^q \\ &= x_1 g^q - [1]x_0 \Delta,\end{aligned}$$

which is the expected relation. If now $k > 2$, then $D_1 g^{q^{k-1}} = 0$ and $D_1 \Delta^{q^{k-2}} = 0$, so that, by (12):

$$\begin{aligned}x_k &= D_1 g_k = D_1(g_{k-1}g^{q^{k-1}}) + D_1(-[k-1]g_{k-2}\Delta^{q^{k-2}}) \\ &= (D_1 g_{k-1})g^{q^{k-1}} - [k-1](D_1 g_{k-2})\Delta^{q^{k-2}} \\ &= x_{k-1}g^{q^{k-1}} - [k-1]x_{k-2}\Delta^{q^{k-2}}.\end{aligned}$$

The proof of the statement about the sequence $(h_k)_{k \geq 0}$ is now clear. Indeed, by (14) and the result on the sequence $(x_k)_{k \geq 0}$ we have just proved, we have, for $k \geq 2$:

$$-(Eg_k + h_k) = (-Eg_{k-1} - h_{k-1})g^{q^{k-1}} + [k-1](Eg_{k-2} + h_{k-2})\Delta^{q^{k-2}}.$$

Using now the recursion formula (12) for the sequence $(g_k)_{k \geq 0}$, we get the same recursion formula for $(h_k)_{k \geq 0}$. \square

2.1.1. Order of vanishing of the form x_k

In this section, we determine the order of vanishing at infinity of the form x_k (for all k).

First of all, we recall our conventions for binomial coefficients [1, Section 3]. For $n \in \mathbb{Z}$ and $i \in \mathbb{Z}$ with $i \geq 0$:

$$\binom{n}{i} := \frac{\prod_{k=1}^i (n - k + 1)}{i!}.$$

We begin with useful, although elementary observations on derivatives of g_k and x_k for $k \geq 0$. The following formula is easy to check, for $a, b \in C$ not both vanishing:

$$\mathcal{D}_n((az + b)^{-w}) = \binom{w + n - 1}{n} (-1)^n \frac{a^n}{(az + b)^{n+w}}, \quad (15)$$

where the operators $\mathcal{D}_n = (-\tilde{\pi})^n D_n$ have been introduced in [1, Section 1].

Lemma 2.2. For $q \neq 2$, $k \geq 1$ or for $q = 2$, $k \geq 2$, we have:

$$D_2 g_k = \cdots = D_{q^k-1} g_k = 0.$$

Moreover, for all q and for all $k \geq 1$, we have:

$$D_1 x_k = \cdots = D_{q^k-1} x_k = 0.$$

Proof. Assuming that $k \geq 2$ for $q = 2$ or $k \geq 1$ otherwise, the integer $w = q^k - 1$ is ≥ 2 and we have the following congruences:

$$\begin{aligned}n = 2, \quad \binom{w + n - 1}{2} &= \binom{q^k}{2} \equiv 0 \pmod{p}, \\ n = p^s, \quad s = 1, \dots, ek - 1, \quad \binom{w + n - 1}{n} &= \binom{p^s + q^k - 2}{p^s} \equiv 0 \pmod{p}.\end{aligned}$$

Therefore, from the computations above, the uniform convergence of Eisenstein series (10) and formula (15):

$$\mathcal{D}_2 E^{(q^k-1)} = \mathcal{D}_p E^{(q^k-1)} = \mathcal{D}_{p^2} E^{(q^k-1)} = \dots = \mathcal{D}_{p^{ek-1}} E^{(q^k-1)} = 0. \quad (16)$$

By (11) and the fact that D is iterative (use formulas (18) and (19) of [1]), we obtain the property concerning the g_k 's.

The property of the derivatives of the x_k 's then follows from the definition $x_k = D_1 g_k$. Indeed, $D_1 x_k = D_1(D_1 g_k) = 2D_2 g_k = 0$ (this holds when $q = 2$ by the congruence $2 \equiv 0$ and when $q \neq 2$ by the equality $D_2 g_k = 0$ we have just proved). Furthermore, for $2 \leq i \leq q^k - 1$, $D_i x_k = D_i(D_1 g_k) = D_1(D_i g_k) = 0$. \square

We recall, in the next proposition and for the rest of the paper, that $L_k := [k] \cdots [1]$ for $k > 0$. We also set $L_0 := 1$.

Proposition 2.3. *For all $k \geq 0$, we have*

$$x_k = (-1)^{k+1} L_k u^{q^k} + \dots \quad (17)$$

In particular, $v_\infty(x_k) = q^k$.

Proof. By Lemma 2.2 and [1, Lemma 5.2] we have, for all $k \geq 0$, $x_k \in C[[u^{q^k}]]$. Since x_k vanishes at infinity, we may write:

$$x_k = \sum_{i \geq 1} c_{k,i} u^{iq^k}, \quad c_{k,i} \in C, \quad k \geq 0.$$

From Corollaries (10.5), (10.11) and (10.4) of [3], collected in the first part of Lemma 4.2 of [1], we find the following u -expansions:

$$\begin{aligned} g &= 1 - [1]u^{q-1} + \dots \in C[[u^{q-1}]], \\ h &= -u(1 + u^{(q-1)^2} + \dots) \in uC[[u^{q-1}]], \\ E &= u(1 + u^{(q-1)^2} + \dots) \in uC[[u^{q-1}]]. \end{aligned}$$

The third u -expansion tells that the result is true for $k = 0$, since $x_0 = -E = -u + \dots$. We also verify the result for $k = 1$ because the three u -expansions yield $x_1 = -Eg - h = [1]u^q + \dots$.

We finish the proof by induction on $s = k - 2 \geq 0$ with the help of Proposition 2.1. From the recursion formula for x_{s+2} we see that the coefficient of $u^{q^{s+1}}$ in the u -expansion of x_{s+2} is $c_{s+1,1} + [s+1]c_{s,1}$. But $x_{s+2} \in C[[u^{q^{s+2}}]]$ and there cannot be a non-trivial contribution by a term proportional to $u^{q^{s+1}}$; we deduce that this coefficient is zero. Therefore, $c_{s+1,1} = -[s+1]c_{s,1}$. \square

Remark 2.4. *It can be proved that for all k , the normalisation of x_k lies in $A[[u^{q^k}]]$.*

2.1.2. Tables

Table 1 collects several useful data checked above. The order of vanishing of h_k easily follows from (14), the fact that $v_\infty(g_k) = 0$ and Proposition 2.3. The index k is supposed to be ≥ 1 .

Table 2 describes the first values of x_k , from which one easily deduces the corresponding values of g_k, h_k thanks to (14):

Table 1

form f	weight	depth	type	$v_\infty(f)$
g_k	$q^k - 1$	0	0	0
h_k	$q^k + 1$	0	1	1
x_k	$q^k + 1$	1	1	q^k

Table 2

$x_0 = -E$ $x_1 = -Eg - h$ $x_2 = -E(g^{q+1} + [1]h^{q-1}) - g^q h$

2.1.3. A multiplicity estimate for forms of depth $\leq q$

The next simple lemma will be used quite often.

Lemma 2.5. Let f, g be quasi-modular forms, with $f \in \tilde{M}_{w,m}^{\leq l}$ and $g \in \tilde{M}_{w',m'}^{\leq l'}$, considered as polynomials in $C[E, g, h]$. Then, their resultant $\rho := \mathbf{Res}_E(f, g)$ with respect to E is a Drinfeld modular form of weight $w(\rho) = lw' + wl' - 2ll'$ and type $m(\rho) = lm' + l'm - ll'$.

Proof. This is elementary and follows by a suitable adaptation of, for example, [14, Lemma 6.1] (see also [17, Theorem 6.1]). The information on the type will not be used in this paper but is given for the sake of completeness. \square

The degree $d(f)$ of a quasi-modular form f is by definition the positive integer $d(f) := w(f) - l(f)$, difference between its weight and its depth.

As an application of the previous results, we prove here a multiplicity estimate for quasi-modular forms of depth $\leq q$ that will be used later in this paper.

Proposition 2.6. Let w and m be integers such that $\tilde{M}_{w,m}^{\leq q} \neq \{0\}$, and let f be a non-vanishing form in $\tilde{M}_{w,m}^{\leq q}$. Then

$$v_\infty(f) \leq \frac{q^2 + 1}{q + 1} d(f).$$

Proof. If the bound of the proposition holds for two forms f_1 and f_2 with $f_1 f_2 \in \tilde{M}_{w,m}^{\leq q}$, then the bound clearly holds for $f_1 f_2$ too, by adding the inequalities. So we may suppose that f is irreducible in the ring $C[E, g, h]$. Let k be the smallest integer ≥ 0 such that

$$w(f) < q^k + q. \quad (18)$$

If there is $\lambda \in C$ such that $f = \lambda x_k$, then $v_\infty(f) = q^k$ and $d(f) = q^k$, so the bound of Proposition 2.6 holds. If it is not the case, then consider the resultant (with respect to the indeterminate E)

$$\rho := \mathbf{Res}_E(f, x_k).$$

The function ρ is a non-zero modular form of weight $w(f) + l(f)(q^k - 1)$ according to Lemma 2.5, so we have, by [3, formula (5.14)],

$$v_\infty(\rho) \leq w(\rho)/(q + 1) \leq (w(f) + q(q^k - 1))/(q + 1). \quad (19)$$

On the other hand, since there exist $U, V \in \tilde{M}$ such that $\rho = Uf + Vx_k$, we have

$$v_\infty(\rho) \geq \min\{v_\infty(f), v_\infty(x_k)\} = \min\{v_\infty(f), q^k\}. \quad (20)$$

By (18) we have $q^k > (w(f) + q(q^k - 1))/(q + 1)$, so the compatibility of (19) and (20) implies $\min\{v_\infty(f), q^k\} = v_\infty(f)$, hence

$$v_\infty(f) \leq (w(f) + q(q^k - 1))/(q + 1). \quad (21)$$

We distinguish now two cases. If $k = 0$, then we get (note that $d(f) > 0$)

$$(q + 1)v_\infty(f) \leq w(f) \leq d(f) + q \leq (q + 1)d(f)$$

and the result follows in this case. If $k \geq 1$, then by minimality of k satisfying (18) we have $q^{k-1} + q \leq w(f)$, hence $q^k \leq q(w(f) - q) \leq qd(f)$. Replacing in (21) and using the estimate $w(f) \leq d(f) + q$, we get the result. \square

Remark 2.7. Observing the end of the proof of this Proposition, we notice that if $l(f) < q$, then we have the strict inequality $w(f) < d(f) + q$, and hence we get the strict inequality $v_\infty(f) < (q^2 + 1)/(q + 1)d(f)$. This remark will be crucial in the proof of Lemma 2.14.

2.1.4. Proof of Theorem 1.2

Table 1 of Section 2.1.2 shows that x_k has a high vanishing order compared to its weight. Here we prove that this vanishing order is the highest possible among forms in $\tilde{M}_{q^{k+1}, 1}^{\leq 1}$, that is, that the form x_k is extremal in $\tilde{M}_{q^{k+1}, 1}^{\leq 1}$. In fact we will even get a slightly more general result (Proposition 2.12).

Lemma 2.8. For all $k \geq 0$ we have:

$$\rho_k := \det \begin{pmatrix} g_k & h_k \\ g_{k+1} & h_{k+1} \end{pmatrix} = (-1)^k L_k h^{q^k}.$$

Proof. To compute ρ_k it suffices to substitute g_{k+1}, h_{k+1} by their expressions as linear combinations of g_k, g_{k-1} and h_k, h_{k-1} with coefficients in M (cf. (12) and Proposition 2.1). We obtain the formula $\rho_k = -[k]h^{q^{k-1}(q-1)}\rho_{k-1}$; since $\rho_0 = \det \begin{pmatrix} 1 & 0 \\ g & h \end{pmatrix}$, the lemma follows. \square

Lemma 2.9. For all $k \geq 0$, the form x_k is irreducible as a polynomial of $C[E, g, h]$.

Proof. Assume by contradiction that δ is a non-trivial divisor of the polynomial x_k . Since $x_k = -g_k E - h_k$ is of depth 1, we can assume without loss of generality that δ is a modular form, common divisor of g_k and h_k . But then, δ divides the form ρ_k of Lemma 2.8, which tells that δ is a multiple by an element of C^\times of a power of h . Hence, h divides g_k , which does not vanish at infinity; contradiction, because h does. \square

The next lemma gives a sufficient condition for an irreducible quasi-modular form to be extremal.

Lemma 2.10. Let $\varphi \in \tilde{M}_{w, m}^{\leq l(\varphi)}$ be a quasi-modular form satisfying

$$(q + 1)v_\infty(\varphi) > 2l(\varphi)d(\varphi). \quad (22)$$

Then, for all non-zero quasi-modular forms $f \in \tilde{M}_{w, m}^{\leq l(\varphi)}$ without non-constant common factor with φ , we have $v_\infty(f) \leq v_\infty(\varphi)$. In particular, if φ is irreducible in $C[E, g, h]$ then it is extremal.

Proof. Suppose that there exists a non-zero form $f \in \tilde{M}_{w,m}^{\leq l(\varphi)}$ such that $v_\infty(f) > v_\infty(\varphi)$, and such that f and φ have no common factor. Then the resultant $\rho := \text{Res}_E(f, \varphi)$ is non-zero. Note that ρ is a modular form of weight $w(\rho) = wl(\varphi) + wl(f) - 2l(\varphi)l(f) \leq 2l(\varphi)d(f)$ (Lemma 2.5). Now, we have on one side $v_\infty(\rho) \leq w(\rho)/(q+1)$ (since ρ is modular), and on the other side $v_\infty(\rho) \geq \min\{v_\infty(\varphi), v_\infty(f)\} = v_\infty(\varphi)$. Thus we find

$$v_\infty(\varphi) \leq w(\rho)/(q+1) \leq 2l(\varphi)d(f)/(q+1).$$

But this contradicts the hypothesis (22). This shows that $v_\infty(f) \leq v_\infty(\varphi)$ and the first part of the lemma is proved. The second one is clear, since when φ is irreducible, then any $f \in \tilde{M}_{w,m}^{\leq l(\varphi)}$ either has no common factor with φ or has the form $\lambda\varphi$ with $\lambda \in C$. \square

Proof of Theorem 1.2. It follows at once from Lemma 2.10 applied with $\varphi = x_k$, since x_k is irreducible by Lemma 2.9 and since the condition (22) is clearly satisfied (see Table 1 of Section 2.1.2). \square

Remark 2.11. We can generalise Theorem 1.2 a little bit as follows.

Proposition 2.12. Let r_0, \dots, r_s be non-negative integers not all of which are zero, let us write $l = r_0 + \dots + r_s$, $w = r_0 + r_1q + \dots + r_sq^s + l$ and let m be the class of reduction modulo $q-1$ of l . If $l < \frac{q+1}{2}$, then the quasi-modular form

$$x = x_0^{r_0} \dots x_s^{r_s} \in \tilde{M}_{w,m}^{\leq l} \setminus \tilde{M}_{w,m}^{\leq l-1}$$

is extremal.

Proof. Let f be a non-zero element of $\tilde{M}_{w,m}^{\leq l}$. Write $f = \varphi^* \delta$ and $x = \varphi \delta$, where $\delta, \varphi, \varphi^*$ are elements of $C[E, g, h]$ such that φ and φ^* are coprime. If φ is constant, then f is a multiple of x in $C[E, g, h]$, so $f = \lambda x$ for some $\lambda \in C^*$ (since $w(f) = w(x) = w$ by hypothesis). Thus $v_\infty(f) = v_\infty(x)$ in this case. If now φ is not constant, then φ has the form (up to an element of C^*)

$$\varphi = x_0^{\alpha_0} \dots x_s^{\alpha_s}$$

with $\alpha_0, \dots, \alpha_s$ not all zero such that $0 \leq \alpha_i \leq r_i$ for all i . One readily checks that the condition $(q+1)v_\infty(\varphi) > 2l(\varphi)d(\varphi)$ of Lemma 2.10 is equivalent to $q+1 > 2l$, so it is satisfied. Applying now this lemma yields $v_\infty(\varphi^*) \leq v_\infty(\varphi)$, or equivalently $v_\infty(f) \leq v_\infty(x)$. Thus x is extremal. \square

2.2. The family $(\xi_k)_{k \geq 0}$

We recall that the definition of the forms ξ_k occurs in (8). In this section we study the forms ξ_k and give proofs of Theorems 1.3 and 1.4.

Proposition 2.13. For all $k \geq 0$, the quasi-modular form ξ_k satisfies

$$w(\xi_k) = (q^k + 1)(q + 1), \quad l(\xi_k) = q + 1, \quad v_\infty(\xi_k) = q^{k+2} + q^k.$$

Proof. The fact that $w(\xi_k) = (q^k + 1)(q + 1)$ immediately follows from the definition of ξ_k . Let us prove that the depth of ξ_k is $q + 1$. If $k = 0$, a straightforward computation yields $\xi_0 = -[1]E^{q+1} + ghE + h^2$, so the result is clear. Suppose now that $k \geq 1$. By (14), $x_k = -g_kE - h_k$ for all k . We get, by definition of ξ_k :

$$\xi_k = [k]^q (g_{k-1}^q E^q + h_{k-1}^q) (g_{k+1}E + h_{k+1}) - [k+1] (g_kE + h_k) (g_k^q E^q + h_k^q). \quad (23)$$

If we consider ξ_k as a polynomial in E , the coefficient of E^{q+1} is therefore equal to $\alpha := [k]^q g_{k+1} g_{k-1}^q - [k+1] g_k^{q+1}$. This last form is non-zero since the constant term of its u -expansion is (using the fact that $g_i = 1 + \dots$ for all i) $[k]^q - [k+1] = -[1] \neq 0$. It follows that the degree in E of the form ξ_k is exactly $q+1$, hence $l(\xi_k) = q+1$.

It remains to prove that $v_\infty(\xi_k) = q^{k+2} + q^k$. To do this, we first notice that the following relation holds:

$$-\Delta^{q^k} \xi_k = x_{k+1}^{q+1} - x_k^q x_{k+2}.$$

Indeed, using the recursion formula of the sequence (x_k) (Proposition 2.1), we have:

$$\begin{aligned} x_{k+1}^{q+1} - x_k^q x_{k+2} &= (g^{q^{k+1}} x_k^q - [k]^q \Delta^{q^k} x_{k-1}^q) x_{k+1} - x_k^q (g^{q^{k+1}} x_{k+1} - [k+1] \Delta^{q^k} x_k) \\ &= -\Delta^{q^k} \xi_k. \end{aligned}$$

Thus, it suffices to show that

$$v_\infty(x_{k+1}^{q+1} - x_k^q x_{k+2}) = q^{k+2} + q^{k+1}. \quad (24)$$

But by Proposition 2.3 we have:

$$\begin{aligned} x_{k+1}^{q+1} - x_k^q x_{k+2} &= ((-1)^{k+2} L_{k+1} u^{q^{k+1}} + \dots)^{q+1} - ((-1)^{k+1} (-1)^{k+3} L_k^q L_{k+2} u^{q^{k+1}+q^{k+2}} + \dots) \\ &= ([k+1]^q - [k+2])[k+1] L_k^{q+1} u^{q^{k+1}+q^{k+2}} + \dots \\ &= -[k+1][1] L_k^{q+1} u^{q^{k+1}+q^{k+2}} + \dots \end{aligned}$$

(we have used the fact that $[k+1]^q - [k+2] = -[1]$). Hence (24) holds and the proposition is proved. \square

2.2.1. Proof of Theorem 1.3

Lemma 2.14. For every $k \geq 0$ the form ξ_k is irreducible in $C[E, g, h]$.

Proof. Suppose that ξ_k is reducible. Write $\xi_k = ab$, where a, b are non-constant quasi-modular forms. Suppose first that $l(a) \geq 1$ and $l(b) \geq 1$, or, equivalently, that $l(a) \leq q$ and $l(b) \leq q$. Since $l(\xi_k) = q+1$, a and b cannot both have a depth equal to q , so we may suppose $l(a) < q$. By Proposition 2.6 and Remark 2.7, we have

$$v_\infty(a) < \frac{q^2+1}{q+1} d(a) \quad \text{and} \quad v_\infty(b) \leq \frac{q^2+1}{q+1} d(b).$$

Hence, since $d(a) + d(b) = d(\xi_k) = q^k(q+1)$,

$$v_\infty(\xi_k) = v_\infty(a) + v_\infty(b) < \frac{q^2+1}{q+1} (d(a) + d(b)) = q^k(q^2+1).$$

But this contradicts the fact that $v_\infty(\xi_k) = q^k(q^2+1)$ (Proposition 2.13).

Thus we have $l(a) = 0$ or $l(b) = 0$. We will suppose in what follows that $l(a) = 0$, i.e. a is a modular form. We will even assume, without loss of generality, that a is irreducible. Returning to the expression (23) of ξ_k , we see that $\xi_k = \alpha E^{q+1} + \beta E^q + \gamma E + \delta$ with

$$\begin{aligned}\alpha &= [k]^q g_{k-1}^q g_{k+1} - [k+1] g_k^{q+1}, \\ \beta &= [k]^q g_{k-1}^q h_{k+1} - [k+1] g_k^q h_k, \\ \gamma &= [k]^q h_{k-1}^q g_{k+1} - [k+1] h_k^q g_k, \\ \delta &= [k]^q h_{k-1}^q h_{k+1} - [k+1] h_k^{q+1}.\end{aligned}$$

As in Lemma 2.8, we define

$$\rho_k := \det \begin{pmatrix} g_k & h_k \\ g_{k+1} & h_{k+1} \end{pmatrix}.$$

We have (Lemma 2.8) $\rho_k = (-1)^k L_k h^{q^k}$. Since a is a modular form dividing ξ_k , it divides α , β , γ and δ , and thus also the two forms

$$h_{k-1}^q \alpha - g_{k-1}^q \gamma = [k+1] g_k \rho_{k-1}^q = (-1)^{k-1} L_{k-1}^q [k+1] h^{q^k} g_k$$

and

$$h_{k-1}^q \beta - g_{k-1}^q \delta = [k+1] h_k \rho_{k-1}^q = (-1)^{k-1} L_{k-1}^q [k+1] h^{q^k} h_k.$$

But h does not divide α as this form does not vanish at infinity (see e.g. the proof of Proposition 2.13). So a must divide both g_k and h_k . But then a divides the form ρ_k , hence is equal to h (up to a constant factor). But this is impossible since h does not divide α . Finally, the contradiction obtained shows that ξ_k is irreducible, as announced. \square

Proof of Theorem 1.3. By Proposition 2.13 we have $v_\infty(\xi_k) = q^k(q^2 + 1)$, $l(\xi_k) = q + 1$ and $d(\xi_k) = q^k(q + 1)$, hence the condition $(q + 1)v_\infty(\xi_k) > 2l(\xi_k)d(\xi_k)$ is satisfied for $q > 2$. Since ξ_k is irreducible by Lemma 2.14, we may therefore apply Lemma 2.10. We get that ξ_k is extremal. \square

Remark 2.15. For $q = 2$, numerical computations show that ξ_k is also extremal for $k = 0, 1$ (cf. Section 3.2).

2.2.2. Proof of Theorem 1.4

We argue as in the proof of Proposition 2.6. We may suppose that f is irreducible in the ring $C[E, g, h]$. Let k be the smallest integer ≥ 0 such that

$$w(f) < q^k + q^2. \quad (25)$$

If there is $\lambda \in C$ such that $f = \lambda \xi_k$, then $v_\infty(f) = q^k(q^2 + 1)$ and $d(f) = q^k(q + 1)$, so the bound of the theorem holds. If it is not the case, then consider the resultant $\rho := \text{Res}_E(f, \xi_k)$. The function ρ is a non-zero modular form of weight $(w(f) + l(f)(q^k - 1))(q + 1)$ by Lemma 2.5, so we have

$$v_\infty(\rho) \leq w(\rho)/(q + 1) \leq w(f) + q^2(q^k - 1). \quad (26)$$

On the other hand, we have

$$v_\infty(\rho) \geq \min\{v_\infty(f), v_\infty(\xi_k)\} = \min\{v_\infty(f), q^k(q^2 + 1)\}. \quad (27)$$

By (25) we have $q^k(q^2 + 1) > w(f) + q^2(q^k - 1)$, so the compatibility of (26) and (27) implies $\min\{v_\infty(f), q^k(q^2 + 1)\} = v_\infty(f)$, hence

$$v_\infty(f) \leq w(f) + q^2(q^k - 1). \quad (28)$$

We now distinguish two cases. If $k = 0$, then we get (notice that $d(f) > 0$)

$$v_\infty(f) \leq w(f) \leq d(f) + q^2 \leq (q^2 + 1)d(f)$$

and the result follows in this case. If $k \geq 1$, then by minimality of k satisfying (25) we have $q^{k-1} + q^2 \leq w(f)$, hence $q^k \leq q(w(f) - q^2) \leq qd(f)$. Replacing in (28) and using the estimate $w(f) \leq d(f) + q^2$, we get the result.

3. Differential extremality

In Section 5 of [1], we have introduced the following subgroups of F^\times where $F = C(E, g, h)$:

$$\Psi_k = \{f \in F^\times; (D_{p^j} f)/f \in \tilde{M} \text{ for all } 0 \leq j \leq k\},$$

with the additional notation $\Psi_{-1} := F^\times$. The differential exponent (the map we have mentioned in the introduction) can then be defined in the following alternative way:

$$\epsilon_D : F^\times \rightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

such that $\epsilon_D(f) = k + 1$ if $f \in \Psi_k \setminus \Psi_{k+1}$ and $\epsilon_D(f) = \infty$ if $f \in \Psi_\infty := \bigcap_{i=-1}^\infty \Psi_i$. This makes sense because for all $k \geq -1$, $\Psi_k \supsetneq \Psi_{k+1}$.

Lemma 3.1. *The following properties of the differential exponent hold, with $f, g \in F^\times$.*

1. If $f \in C^\times h^{\mathbb{Z}}$, then $\epsilon_D(f) = \infty$.
2. If $q \neq 2, 3$, then $\epsilon_D(f) = \infty$ implies that $f \in C^\times h^{\mathbb{Z}}$.
3. We have $\epsilon_D(f^p) = \epsilon_D(f) + 1$. Moreover, if $p \nmid m$, $\epsilon_D(f^m) = \epsilon_D(f)$.
4. We have $\epsilon_D(fg) \geq \inf\{\epsilon_D(f), \epsilon_D(g)\}$, and equality holds when $\epsilon_D(f) \neq \epsilon_D(g)$.

Sketch of proof. The first property follows from [1, Proposition 3.6]. The second property is a paraphrase of [1, Theorem 3]. In the third property, the first part is clear by using the formula $D_{p^{k+s}} f^{p^k} = (D_{p^s} f)^{p^k}$, which follows easily from Leibniz rule [1, Eq. (16)]. As for the second part, we observe that $\epsilon_D(f^m) \geq \epsilon_D(f)$ because the sets Ψ_k are, as already observed, multiplicative subgroups of F^\times . Let us write $k = \epsilon_D(f)$. Then, by Leibniz's formula (15) of [1],

$$\frac{D_{p^k}(f^m)}{f^m} = m \frac{D_{p^k} f}{f} + \sum_{\substack{i_1 + \dots + i_m = p^k \\ 0 \leq i_1, \dots, i_m < p^k}} \frac{D_{i_1} f}{f} \dots \frac{D_{i_m} f}{f}.$$

The sum on the right-hand side is an element of \tilde{M} while $(D_{p^k} f)/f \notin \tilde{M}$, which implies the inequality $\epsilon_D(f^m) \leq \epsilon_D(f)$. The fourth property can be proved in a similar way; its proof is then left to the reader. \square

As in [1, Section 5] we denote by F_r the following subset of F , which turns out to be a subfield:

$$F_r = \bigcap_{0 \leq i \leq r} \ker D_{p^i}.$$

For $r = -1$ we define $F_{-1} = F$. We have $F_{r+1} \subset F_r$ and $F_r^p \subset F_{r+1}$ for all $r \geq 0$.

Proposition 3.2. Let f be an element of Ψ_k ($k \geq 0$). Then, $h^{-v_\infty(f)} f \in F_k^\times$. More precisely,

$$\frac{D_n f}{f} = \frac{D_n(h^{v_\infty(f)})}{h^{v_\infty(f)}}, \quad n = 0, \dots, p^{k+1} - 1.$$

Proof. Let us write $v_\infty(f) = n_0 + \dots + n_k p^k + m p^{k+1}$ with $n_0, \dots, n_k \in \{0, \dots, p-1\}$ and $m \geq 0$. Let us also define inductively:

$$f_{-1} = f, \quad f_0 = f_{-1}/h^{n_0}, \quad \dots, \quad f_s = f_{s-1}/h^{n_s p^s}, \quad \dots$$

By [1, Proposition 3.6], we know that $f_{-1}, f_0, \dots, f_k \in \Psi_k$.

We now prove by induction on s that $f_s \in F_s^\times$ for $0 \leq s \leq k$. From elementary weight considerations, there exists $\alpha \in C$ such that

$$D_1 f_{-1} = \alpha E f_{-1}.$$

Writing $f_{-1} = cu^i + \dots$ (the dots \dots are understood as a series of higher powers of u and $c \neq 0$) and comparing $D_1 f_{-1} = c i u^{i+1} + \dots$ with $\alpha E f_{-1} = \alpha(u + \dots)(cu^i + \dots)$ yields $\alpha \equiv i \equiv n_0 \pmod{p}$. Hence $\alpha \in \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ and we can choose a representative α of this class, with $\alpha = n_0$ (allowing an abuse of notation). Now,

$$D_1 h^{n_0} = n_0 E h^{n_0},$$

which implies $D_1 f_0 = 0$, hence $f_0 \in F_0$.

Assuming now that $f_s \in F_s^\times$ for $s < k$, we proceed to prove that $f_{s+1} \in F_{s+1}^\times$. Since $f_s \in \Psi_s$, we can apply Lemma 5.9 of [1] to check that:

$$D_{p^{s+1}} f_s = \alpha_s E^{p^{s+1}} f_s, \quad \alpha_s \in \{0, \dots, p-1\}.$$

Since $f_s \in F_s^\times$ by hypothesis, Lemma 5.2 of [1] implies that

$$f_s = c_s u^{i_s p^{s+1}} + \dots \in C((u^{p^{s+1}})), \quad c_s \neq 0.$$

Comparing $D_{p^{s+1}} f_s = c_s i_s u^{(i_s+1)p^{s+1}} + \dots$ with $\alpha_s E^{p^{s+1}} f_s = \alpha_s (u^{p^{s+1}} + \dots)(c_s u^{i_s p^{s+1}} + \dots)$ yields $\alpha_s \equiv i_s \equiv n_{s+1} \pmod{p}$ which implies $\alpha_s = n_{s+1}$. Now,

$$D_{p^{s+1}} h^{n_{s+1} p^{s+1}} = n_{s+1} E^{p^{s+1}} h^{n_{s+1} p^{s+1}}$$

so that $f_{s+1} \in F_{s+1}^\times$.

Finally,

$$\frac{f}{h^{v_\infty(f)}} = \frac{f}{h^{n_0 + n_1 p + \dots + n_k p^k}} \frac{1}{h^{m p^{k+1}}} = \frac{f_k}{h^{m p^{k+1}}} \in F_k^\times,$$

and for all $0 \leq n \leq p^{k+1} - 1$,

$$\frac{D_n f}{f} = \frac{D_n(h^{v_\infty(f)})}{h^{v_\infty(f)}}. \quad \square$$

3.1. Differential extremality of the forms x_k

Here we prove Theorem 1.6. The proof makes use at once of the main result of [2], where the factorisations in $K[g, h]$ of Gekeler's forms g_k 's are determined, and of differential tools that will be developed in Section 4. We begin with a subsection devoted to the description of the needed result from [2], and to some preliminary lemmas.

3.1.1. Preliminaries to the proof of Theorem 1.6

Since the C -algebra \tilde{M} is equal to the polynomial ring $C[E, g, h]$, the group $\mathbf{Aut}(C/K)$ naturally acts on \tilde{M} , the action on an element $f \in \tilde{M}$ being given by the action on the coefficients of f seen as a polynomial in E, g, h . Since the functions E, g, h have their u -expansions with coefficients in K , if σ is a K -automorphism and if $f = \sum_{i \geq 0} c_i u^i$, then we have

$$f^\sigma = \sum_{i \geq 0} c_i^\sigma u^i.$$

If L is a subfield of C , we will say that $f \in \tilde{M}$ is *defined over L* if $f \in L[E, g, h]$.

Lemma 3.3. *For $k \geq 0$, let f be normalised and differentially extremal in $\tilde{M}_{q^{k+1}, 1}^{\leq 1}$. Then, f is defined over the inseparable closure of K in C .*

Proof. Let f be as in the statement of the lemma. We know that $\epsilon_D(f) \geq e(k+1) = \epsilon_D(x_k)$ by Proposition 4.9. Hence, for all n such that $1 \leq n \leq q^{k+1} - 1$, f divides $D_n f$. Let $A_n \in \tilde{M}$ be the polynomial such that $D_n f = A_n f$, for n as above. By Proposition 3.2, there exists $\nu \in \mathbb{Z}$ such that $A_n = (D_n h^\nu)/h^\nu$, so A_n is defined over K by [1, formula (3)]. Let σ be any element of $\mathbf{Aut}(C/K)$. For all n with $1 \leq n \leq q^{k+1} - 1$, we have:

$$D_n f^\sigma = (D_n f)^\sigma = (A_n f)^\sigma = A_n f^\sigma,$$

where the first identity comes again from [1, formula (3)]. Define $\nu := \nu_\infty(f) = \nu_\infty(f^\sigma)$ and suppose that $f \neq f^\sigma$. By Proposition 3.2 and [1, Lemma 5.2], we have $fh^{-\nu}, f^\sigma h^{-\nu} \in C((u^{q^{k+1}}))$, so there exists $\alpha \in \mathbb{Z}$ such that $\nu_\infty((f - f^\sigma)h^{-\nu}) = \alpha q^{k+1}$. Now, since f and f^σ are normalised, the form $\phi := f - f^\sigma$ satisfies $\nu_\infty(\phi) > \nu$. Hence we get $\alpha > 0$, which implies $\nu_\infty(\phi) = \nu + \alpha q^{k+1} > q^k$. But since $\nu_\infty(x_k) = q^k$, this contradicts the extremality of x_k (Theorem 1.2). Hence we have $f = f^\sigma$. \square

The following conjecture seems plausible, although pretty difficult to reach with the tools at our disposal.

Conjecture 3.4. *If f is differentially extremal, then f is defined over K .*

We rewrite the needed result from [2] in the following lemma.

Lemma 3.5. *Let $k \geq 2$ be even. Then the polynomial $g_k \in K[g, h]$ is irreducible and totally decomposable over a separable extension of K .*

Proof. We know from [2, Section 2] that if we set $j = g^{q+1}/\Delta$, there exist a non-zero element $c_k \in K$ and a polynomial $P_k \in K[X]$ such that

$$g_k = c_k \Delta^{\deg(P_k)} P_k(j), \quad (29)$$

where the degree of P_k is

$$\frac{q^k - 1}{q^2 - 1}.$$

Hence $p \nmid \deg(P_k)$, so that P_k is separable over K . Now, formula (29) establishes a bijective correspondence between the irreducible factors in $\bar{K}[X]$ of P_k (\bar{K} being an algebraic closure of K in C), and the irreducible factors in $\bar{K}[g, h]$ of g_k . This shows the second statement of the lemma. The statement on the irreducibility of g_k follows from [2, Theorem E2]. \square

Remark 3.6. If $k \geq 3$ is odd, it is easy to show from the results in [2] that g_k is totally decomposable over a separable extension of K , and equal to a product of g and an irreducible polynomial of $K[g, h]$. However, we will only use the case k even.

Lemma 3.7. Let \mathcal{K} be any field. Let a, b, c, d be polynomials in $\mathcal{K}[X_1, \dots, X_n]$, where $n \geq 1$, and write $F = aX_0 + b$, $G = cX_0 + d$. Let $\mathcal{I} = (F, G)$ be the ideal generated by F, G in $\mathcal{K}[X_0, X_1, \dots, X_n]$, and define $\mathcal{J} := \mathcal{I} \cap \mathcal{K}[X_1, \dots, X_n]$. Then, if a, c are coprime in $\mathcal{K}[X_1, \dots, X_n]$, the ideal $\mathcal{J} \subset \mathcal{K}[X_1, \dots, X_n]$ is principal, generated by the resultant $\rho := \text{Res}_{X_0}(F, G) = ad - bc$.

Proof. Let $AF + BG \in \mathcal{K}[X_1, \dots, X_n]$ be any polynomial of \mathcal{J} , where $A, B \in \mathcal{K}[X_0, \dots, X_n]$. Let us write

$$A = \sum_{i=0}^m \alpha_i X_0^i, \quad B = \sum_{i=0}^{m'} \beta_i X_0^i$$

with $\alpha_i, \beta_j \in \mathcal{K}[X_1, \dots, X_n]$ and $\alpha_m \beta_{m'} \neq 0$. Considering the leading term of the expansion of $AF + BG$ in powers of X_0 , we see that $m = m'$. Moreover, assuming $m > 0$, we have

$$\alpha_m a + \beta_m c = 0 \tag{30}$$

and

$$\alpha_i b + \alpha_{i-1} a + \beta_i d + \beta_{i-1} c = 0, \quad 1 \leq i \leq m. \tag{31}$$

Since a, c are coprime, we deduce from (30) that there exists $u_m \in \mathcal{K}[X_1, \dots, X_n]$ such that $\alpha_m = cu_m$, $\beta_m = -au_m$. Substituting in (31) for $i = m$, we get similarly the existence of $u_{m-1} \in \mathcal{K}[X_1, \dots, X_n]$ such that

$$\alpha_{m-1} = du_m + cu_{m-1}, \quad \beta_{m-1} = -bu_m - au_{m-1}.$$

Arguing by induction, one sees that there exist elements $u_{m-2}, \dots, u_0 \in \mathcal{K}[X_1, \dots, X_n]$ such that

$$\alpha_i = du_{i+1} + cu_i, \quad \beta_i = -bu_{i+1} - au_i, \quad i = 0, \dots, m-1.$$

Now,

$$AF + BG = \alpha_0 b + \beta_0 d = (du_1 + cu_0)b + (-bu_1 - au_0)d = (bc - ad)u_0 = -\rho u_0.$$

Thus, $AF + BG \in (\rho)$ and hence $\mathcal{J} \subset (\rho)$. Since conversely $\rho = aG - cF \in \mathcal{J}$, we find $\mathcal{J} = (\rho)$. \square

We recall the definition (9) of the operators $\partial_n^{(d)}$:

$$\partial_n^{(d)} f = D_n f + \sum_{i=1}^n (-1)^i \binom{d+n-1}{i} (D_{n-i} f) (D_{i-1} E). \quad (32)$$

If $f \in \tilde{M}_{w,m}^{\leq l} \setminus \tilde{M}_{w,m}^{\leq l-1}$, we write $\partial_n f$ instead of $\partial_n^{(w-l)} f$.

Lemma 3.8. *Let $k \geq 1$. Let $f = aE + b$ be in $\tilde{M}_{q^{k+1},1}^{\leq 1} \setminus M$, with $a, b \in M$, and such that f divides $D_1 f$. Then $f = -D_1 a$ and $D_1 f = 0$.*

Proof. We have $D_1 f = Af$ with $A \in \tilde{M}_{2,1}^{\leq 1}$, so that $D_1 f = \mu Ef$ with $\mu \in C$. By definition (32) we have $D_1 f = d(f)Ef + \partial_1 f = \partial_1 f$, hence $\partial_1 f = \mu Ef$. Since by Theorem 4.1 $\partial_1 f$ has depth ≤ 1 , we deduce $\mu = 0$. Definition (32) gives $\partial_1 a = D_1 a + aE$ and $\partial_1 b = D_1 b - bE$. Therefore,

$$\begin{aligned} 0 = D_1 f &= (D_1 a)E + a(D_1 E) + D_1 b = (\partial_1 a - aE)E + aE^2 + \partial_1 b + bE \\ &= (\partial_1 a)E + \partial_1 b + bE. \end{aligned}$$

Since $\partial_1 a, \partial_1 b \in M$ by Theorem 4.1, we get $\partial_1 a = -b$ and $\partial_1 b = 0$. This yields $f = aE - \partial_1 a = -D_1 a$. \square

3.1.2. Proof of Theorem 1.6

The fact that $\epsilon_D(x_k) = e(k+1)$ follows from Proposition 4.9.

The theorem is clear for $k=0$ because $\tilde{M}_{2,1}^{\leq 1}$ has dimension 1 and is generated by $E = -x_0$.

For $k > 0$, let $f = aE + b$, with $a, b \in M$, be normalised in $\tilde{M}_{q^{k+1},1}^{\leq 1} \setminus M$ and differentially extremal, so that $\epsilon_D(f) \geq e(k+1)$. We want to prove that f is proportional to x_k . Let us assume by contradiction that this is not the case. We will use Lemma 3.5, which forces us to distinguish the even and the odd k cases.

Case of k odd. We claim that a, g_{k+1} are coprime in M . Let us assume the contrary. We first note that f , thus a , is defined over the inseparable closure of K by Lemma 3.3. Now, since $k+1$ is even, we can apply Lemma 3.5 to g_{k+1} . We deduce that if a, g_{k+1} have a non-constant common factor, then it must be g_{k+1} (up to a non-zero constant). So g_{k+1} must divide a which is obviously false, by elementary weight consideration.

The ideal $\mathcal{I}_k \cap M$ of $M = C[g, h]$ is principal, generated by the resultant $\rho = \text{Res}_E(f, x_{k+1})$ by Lemma 3.7. This is a modular form of weight $w = q^k(q+1)$, by Lemma 2.5, and it is non-zero since x_{k+1} and f are coprime by the irreducibility of x_{k+1} (Lemma 2.9).

We have $D_j f / f \in \tilde{M}$ and $D_j x_{k+1} = 0$ for $1 \leq j \leq q^{k+1} - 1$. Hence, the ideal $\mathcal{I}_k = (f, x_{k+1})$ of \tilde{M} contains the sets $D_1 \mathcal{I}_k, \dots, D_{q^{k+1}-1} \mathcal{I}_k$.

By Theorem 4.1, $\partial_j \rho \in \mathcal{I}_k \cap M$, $j = 1, \dots, q^{k+1} - 1$. Since $\mathcal{I}_k \cap M$ is principal, we get:

$$(\partial_j \rho) / \rho \in M, \quad j = 1, \dots, q^{k+1} - 1. \quad (33)$$

It is then straightforward to see, from identity (32), that

$$(D_1 \rho) / \rho, \dots, (D_{q^{k+1}-1} \rho) / \rho \in \tilde{M}. \quad (34)$$

By [1, Lemma 5.7] and Proposition 3.2, the property (34) implies that

$$\rho = h^{v_\infty(\rho)} \phi^{q^{k+1}}, \quad \text{for some } \phi \in M \text{ with } v_\infty(\phi) = 0.$$

This equality of modular forms yields the following equality of weights:

$$q^k(q+1) = v_\infty(\rho)(q+1) + w(\phi)q^{k+1}. \quad (35)$$

If $q > 2$ (resp. $q = 2$), the equality above can hold only if $w(\phi) = 0$ (resp. $w(\phi) = 0, 1$). Indeed, when $w(\phi) \neq 0$, we have $w(\phi) \geq q - 1$. But the non-negativity of $v_\infty(\rho)$ in (35) is contradictory with the conditions $q > 2$ and $w(\phi) \geq q - 1$, or $q = 2$ and $w(\phi) \geq 2$.

Let us assume, for $q \geq 2$, that $w(\phi) = 0$, that is, that ϕ is a non-zero constant. There exists c in C^\times , such that

$$\rho = -g_{k+1}f + ax_{k+1} = ch^{q^k}.$$

We have proved in Proposition 2.3 that $v_\infty(x_{k+1}) = q^{k+1}$. It then follows that $v_\infty(f) = v_\infty(g_{k+1}f) = q^k$ (remember that $v_\infty(g_{k+1}) = 0$). By the extremality of x_k , f is proportional to x_k , a contradiction.

It remains to treat the case $q = 2$ and $w(\phi) = 1$ in (35). This yields $3 \cdot 2^k = 3v_\infty(\rho) + 2^{k+1}$, that is, $3v_\infty(\rho) = 2^k$, which is impossible by the integrality of $v_\infty(\rho)$. This concludes the proof of the theorem in the case k odd.

Case of k even. In this case, with the same arguments that we have used for the case k odd, we see that two subcases hold. The first is the case $f = aE + b$ with a, g_k coprime, the second is the case of a proportional to g_k .

First subcase. This case can be handled in about the same way as the case of k odd, again with additional nested subcases corresponding to $q > 2$ and $q = 2$. Regardless to the value of q , the resultant $\rho = \text{Res}_E(f, x_k)$ has weight $2q^k$ and satisfies:

$$\rho = h^{v_\infty(\rho)}\phi^{q^{k+1}},$$

by means of arguments very similar to that we have used in the case k odd. This yields

$$2q^k = v_\infty(\rho)(q+1) + w(\phi)q^{k+1}. \quad (36)$$

From this we see that $2q^k \geq w(\phi)q^{k+1}$, so that identity (36) can hold only when $w(\phi) = 0$ if $q > 2$ or when $w(\phi) = 0, 1$ if $q = 2$. But $w(\phi) = 0$ is impossible in (36) since $q + 1$ never divides $2q^k$. It remains to treat the case $q = 2$ and $w(\phi) = 1$. In this case, $v_\infty(\rho) = 0$, ϕ is proportional to g , and there exists c in C^\times such that $\rho = -g_k f + ax_k = cg^{q^{k+1}}$. By Proposition 3.2, $D_1 f = \dots = D_{q^{k+1}-1} f = 0$. However, by [1, Proposition 5.4, identity (37)], the depth of $D_{q^k} f$ is equal to $q^k + 1$ and $D_{q^k} f$ cannot vanish; a contradiction.

Second subcase. It remains to treat the case of $f = aE + b$ with a proportional to g_k . But in this case, by Lemma 3.8, f is proportional to $D_1 g_k$, thus proportional to x_k by definition of the sequence $(x_k)_{k \geq 0}$. The proof of the theorem is now complete. \square

3.2. Numerical observations

We made several numerical computations, essentially with $q = 2, 3, 5$, thanks to the algorithms of Section 4.2 and with the help of a computer. Let us introduce the following family of quasi-modular forms (where we recall that $L_k := [k] \cdots [1]$ if $k > 0$ and $L_0 := 1$):

$$\eta_k := L_{k+1}x_k^q + L_k^q g x_{k+1} \in \tilde{M}_{q(q^k+1),1}^{\leq q} \setminus \tilde{M}_{q(q^k+1),1}^{\leq q-1}, \quad k \geq 0.$$

It is easy to prove that $v_\infty(\eta_k) = q^{k+1} + q - 1$ and $e_D(\eta_k) = 0$, for all $k \geq 0$ (use Lemma 3.1 for the latter identity).

We present Table 3 describing both the results of this paper and the analysis of the results of the numerical experiments we made. We look at extremal and differentially extremal quasi-modular

Table 3

l	f	$\nu_\infty(f)$	$\epsilon_D(f)$	f'
1	x_k	q^k	$(k+1)e$	x_k
2	x_k^2	$2q^k$	$(k+1)e$	x_k^2 *
\vdots	\vdots	\vdots	\vdots	\vdots
s	x_k^s	sq^k	$(k+1)e$	x_k^s *
s'	$x_k^{s'}$ *	$s'q^k$	$(k+1)e$	$x_k^{s'}$ *
\vdots	\vdots	\vdots	\vdots	\vdots
$q-1$	x_k^{q-1} *	$q^{k+1} - q^k$	$(k+1)e$	x_k^{q-1} *
q	η_k *	$q^k + q - 1$	0	x_k^q *
$q+1$	ξ_k *	$q^{k+2} + q^k$	$(k+2)e$ *	ξ_k *

forms in the vector spaces $\tilde{M}_{l(q^k+1),l}^{\leq l}$ with $l = 1, \dots, q+1$ and $k \geq 0$. The integer s is supposed to be $< \frac{q+1}{2}$ while the integer s' satisfies $\frac{q+1}{2} \leq s' \leq q-1$.

In the first column we enter the integer l which determines the vector space where we look for an extremal quasi-modular form. In the second column, there is an extremal quasi-modular form f (unique up to multiplication by a scalar in C^\times). If an asterisk (*) appears, the resulting form is certified by numerical computations solely in the case of $q = 2, 3, 5$ and $k = 0, 1$. If no asterisk figures, the validity of the entry is understood for all q and for all $k \geq 0$.

The third column contains $\nu_\infty(f)$. In the fourth column, the differential exponent $\epsilon_D(f)$ of the corresponding form f is computed or estimated. Again, in absence of an asterisk, the result is unconditional. Otherwise, the value of the entry represents a lower bound for $\epsilon_D(f)$ and its exact value for $q = 2, 3, 5$ and $k = 0, 1$.

In the fifth column we enter a differentially extremal quasi-modular form f' in $\tilde{M}_{l(q^k+1),l}^{\leq l} \setminus \tilde{M}_{l(q^k+1),l}^{\leq l-1}$. In this case, there is no need to write down all the orders of vanishing and differential exponents, since they can be easily computed applying the results of this text and in particular Lemma 3.1. Of course, the presence of the asterisk tells that the corresponding quantity is checked only for $q = 2, 3, 5$ and $k = 0, 1$. It seems that these differentially extremal forms are unique up to multiplication by a scalar in C^\times .

Table 3 indicates that extremality and differential extremality are inequivalent conditions. However, it seems that certain forms, notably the x_k 's and ξ_k 's, have the interesting property of being at once irreducible, extremal and differentially extremal.

These *primitive* forms will become, in the opinion of the authors, of particular importance and will deserve a crucial role in the forthcoming researches in this topic. In the case $q = 2$, computations have been pushed forward to higher values of l and k disclosing the existence of other primitive forms, that will be studied elsewhere.

3.3. The forms x_k 's as solutions of certain differential systems

In this section, we give yet another property that characterises the collection of forms x_k up to scalars in C^\times . The theorem below could be understood as a reasonable substitute of Theorem 2 of [8]. We recall that the operators $\partial_n^{(d)}$ have been defined in (9).

Theorem 3.9. *Let f be a non-constant element of $\tilde{M}_{q^k+1,1}^{\leq 1}$. We have*

$$\partial_1^{(q^k)} f = \dots = \partial_{q^{k+1}-1}^{(q^k)} f = 0$$

if and only if f is proportional to x_k .

The proof of this theorem requires the two following lemmas:

Lemma 3.10. *We have:*

$$D_{iq^k-1}E = E^{iq^k}, \quad k \geq 0, \quad 1 \leq i < q. \quad (37)$$

Proof. It follows by the same arguments of the proof of [1, Lemma 3.8]. Since we have [3, p. 686]

$$E = \frac{1}{\pi} \sum_{a \in A} \sum_{\text{monic } b \in A} \frac{a}{az + b},$$

we deduce

$$\begin{aligned} D_{iq^k-1}E &= \frac{1}{\pi^{iq^k}} \sum_{a \in A} \sum_{\text{monic } b \in A} \left(\frac{a^i}{(az + b)^i} \right)^{q^k} \\ &= (D_{i-1}E)^{q^k} \\ &= E^{iq^k}. \quad \square \end{aligned}$$

Lemma 3.11. *Let f be a non-constant element of $\Psi_{e(k+1)-1}$ and n be an integer $\leq q^{k+1} - 1$. The following conditions are equivalent:*

1. $D_n f = 0$ if q^k does not divide n and $D_n f = E^{iq^k} f$ if $n = iq^k$ with $0 \leq i < q$.
2. $\partial_1^{(q^k)} f = \dots = \partial_{q^{k+1}-1}^{(q^k)} f = 0$.

We first need to focus on certain structures which appear in the operator $\partial_n^{(q^k)}$. With $d = q^k$ and $1 \leq n \leq q^{k+1} - 1$, the definition of the operator $\partial_n^{(d)}$ reads:

$$\partial_n^{(q^k)} f = D_n f + R_1 + R_2,$$

where

$$\begin{aligned} R_1 &= (-1)^n \sum_{0 \leq i \leq n-1, q^k \nmid i} (-1)^i \binom{q^k + n - 1}{n - i} (D_i f)(D_{n-i-1} E), \\ R_2 &= (-1)^n \sum_{0 \leq i \leq n-1, q^k \mid i} (-1)^i \binom{q^k + n - 1}{n - i} (D_i f)(D_{n-i-1} E). \end{aligned}$$

We have the congruences modulo p :

$$\binom{q^k + n - 1}{n - jq^k} \equiv \begin{cases} \binom{i}{i-j} & \text{if } n = iq^k, \\ 0 & \text{if } q^k \nmid n, \end{cases} \quad \text{for } 0 \leq j \leq q-1, \quad jq^k \leq n \leq q^{k+1} - 1. \quad (38)$$

The binomial in the left-hand side of the congruence is, up to multiplication by a power of -1 , the coefficient of $(D_{jq^k} f)(D_{n-jq^k-1} E)$ in the expression defining $\partial_n^{(q^k)} f$. Hence, after (38) and (37),

$$\begin{aligned}
 R_2 &= (-1)^i \sum_{j=0}^{i-1} (-1)^j \binom{i}{i-j} (D_{jq^k} f) (D_{(i-j)q^k-1} E) \\
 &= (-1)^i \sum_{j=0}^{i-1} (-1)^j \binom{i}{i-j} (D_{jq^k} f) E^{(i-j)q^k}, \quad \text{if } n = iq^k,
 \end{aligned} \tag{39}$$

and

$$R_2 = 0, \quad \text{otherwise.} \tag{40}$$

Proof of Lemma 3.11. $1 \Rightarrow 2$. We prove that $\partial_n^{(q^k)} f = 0$ for $n = 1, \dots, q^{k+1} - 1$. We have $R_1 = 0$ by hypothesis. If $q^k \nmid n$, $D_n f = 0$ and $\partial_n^{(q^k)} f = 0$ because of (40). If $n = iq^k$, $D_n f = E^{iq^k} f$ by hypothesis and by (39)

$$R_2 = (-1)^i E^{iq^k} f \sum_{j=0}^{i-1} (-1)^j \binom{i}{i-j} \tag{41}$$

$$= -E^{iq^k} f. \tag{42}$$

Hence, in this case too, $\partial_n^{(q^k)} f = D_n f + R_2 = 0$.

$2 \Rightarrow 1$. For $n = 1$, the statement is true because $\partial_1^{(q^k)} f = D_1 f$. Let us assume that we have already proved that, for all $n \leq m-1$ (m being an integer ≥ 2) $D_n f = 0$ if q^k does not divide n and $D_n f = E^{iq^k} f$ if $n = iq^k$. We know that $\partial_m^{(q^k)} f = 0$. Hence $D_m f = -R_1 - R_2 = -R_2$ (because $R_1 = 0$ by induction hypothesis). If $q^k \nmid m$, $R_2 = 0$ by (40) and $D_m f = 0$. Otherwise, the induction hypothesis implies equality (42) and $D_m f = E^m f$. \square

Proof of Theorem 3.9. If $f = cx_k$, $c \in C^\times$, then identity (69) of Proposition 4.9 and Lemma 3.11 imply that the identities involving the operators $\partial_n^{(q^k)}$ hold. For the other implication, we first apply Lemma 3.11 and then, Theorem 1.6. \square

Remark 3.12. We could not find an analogue of Theorem 3.9 for the family $(\xi_k)_{k \geq 0}$. Similarly, we did not find a reasonable substitute of Theorem 1 of [8].

3.4. Structure of the subfields F_k

The content of this section is independent on our quest of finding differentially extremal quasi-modular forms and can be skipped in a first reading of the paper.

Proposition 3.2 says that every differentially extremal quasi-modular form is multiple by a power of the form h of a (necessarily isobaric) element of F_k . By [11, Proposition 2.2], there exists, for all $k \geq 0$, an element $z_k \in F \setminus F_k$ such that F is a F_k -vector space of dimension p^k of basis $(1, z_k, \dots, z_k^{p^k-1})$ and such that for all $n, m \in \mathbb{N}$, $D_n z_k^m = \binom{m}{n} z_k^{m-n}$. However, this does not clarify much the structure of the fields F_k themselves.

We now define the sequence $(y_k)_{k \geq 1}$ by

$$y_k = \Delta^{q^{k-1}} x_{k-1}.$$

Here, we prove:

Theorem 3.13. *Let $r \geq 0$ be a non-negative integer. For every integer k such that $q^{k+1} > p^r$, we have*

$$F_r = F_{r-1}^p(x_{k+1}, y_{k+1}).$$

For instance, if we take $k = r$, we find $F_k = F_{k-1}^p(x_{k+1}, y_{k+1})$ for all $k \geq 0$. To prove Theorem 3.13 we will need two lemmas.

Lemma 3.14. *For all $r \geq 0$ we have*

$$[F_r : F_{r-1}^p] = p^2.$$

Proof. We know that $[F_{r-1} : F_r] = p$ for all r . Let us prove by induction that $[F_r : F_r^p] = p^3$. This is clear for $r = -1$. If now this property holds for the integer $r - 1$, then

$$[F_r : F_r^p] = \frac{[F_{r-1} : F_{r-1}^p][F_{r-1}^p : F_r^p]}{[F_{r-1} : F_r]} = \frac{p^3 \cdot p}{p} = p^3,$$

hence the property also holds for r . Thus it holds for all $r \geq 0$, from which we deduce

$$[F_r : F_{r-1}^p] = \frac{[F_r : F_r^p]}{[F_{r-1}^p : F_r^p]} = \frac{p^3}{p} = p^2. \quad \square$$

Lemma 3.15. *For all $k \geq 0$ the elements $x_{k+1}^i y_{k+1}^j$ ($1 \leq i, j \leq p - 1$) are linearly independent over F^p .*

Proof. Let us first consider the case $k = 0$. Suppose that there exists a relation

$$\sum_{\substack{0 \leq i \leq p-1 \\ 0 \leq j \leq p-1}} \lambda_{ij}^p x_1^i y_1^j = 0 \quad (43)$$

where $\lambda_{ij} \in F$. Since $x_1 = -(Eg + h)$ and $y_1 = Eh^{q-1}$ this rewrites, after multiplying (43) by h^{p-1} :

$$\sum_{\substack{0 \leq i \leq p-1 \\ 0 \leq j \leq p-1}} ((-1)^i \lambda_{ij} h^{jp^{e-1}})^p (Eg + h)^i E^j h^{p-1-j} = 0. \quad (44)$$

Since the forms $Eg + h$, E and h are algebraically independent over C by [1, Lemma 2.4], the forms $(Eg + h)^i E^j h^k$ ($0 \leq i, j, k \leq p - 1$) are obviously linearly independent over F^p . In particular, it follows from (44) that $(-1)^i \lambda_{ij} h^{jp^{e-1}} = 0$ for all i, j , hence $\lambda_{ij} = 0$ for all i, j . This proves the result for $k = 0$.

Suppose now that $k \geq 1$. Since $x_{k+1} = g^{q^k} x_k - [k]y_k$ and $y_{k+1} = \Delta^{q^k} x_k$, we have:

$$F^p(x_{k+1}, y_{k+1}) = F^p(x_{k+1}, x_k) = F^p(g^{q^k} x_k - [k]y_k, x_k) = F^p(x_k, y_k).$$

By induction, it follows that $F^p(x_{k+1}, y_{k+1}) = F^p(x_1, y_1)$, hence

$$[F^p(x_{k+1}, y_{k+1}) : F^p] = [F^p(x_1, y_1) : F^p] = p^2$$

by the case $k = 0$ we have just proved. But this means that the p^2 generators $x_{k+1}^i y_{k+1}^j$ ($1 \leq i, j \leq p-1$) of the F^p -algebra $F^p(x_{k+1}, y_{k+1})$ are F^p -linearly independent. \square

Proof of Theorem 3.13. Let $r \geq 0$ and $k \geq 0$ be as in the theorem. First of all, we note that by Lemma 2.2 we have $x_{k+1} \in F_r$. Next, the relation $[k+1]y_{k+1} = x_{k+1}g^{q^{k+1}} - x_{k+2}$ together with the fact that $g^{q^{k+1}}, x_{k+1}, x_{k+2}$ all belong to F_r shows that $y_{k+1} \in F_r$. Thus we have the inclusions

$$F_{r-1}^p \subset F_{r-1}^p(x_{k+1}, y_{k+1}) \subset F_r.$$

Now, we have $[F_r : F_{r-1}^p] = [F_{r-1}^p(x_{k+1}, y_{k+1}) : F_{r-1}^p] = p^2$ by Lemmas 3.14 and 3.15. It follows that $F_r = F_{r-1}^p(x_{k+1}, y_{k+1})$. \square

Remark 3.16. Let us define, for $s \in \mathbb{Z}$ and $k \geq 0$:

$$\mathcal{A}_k^{(s)} := \{f \in \Psi_k \text{ such that } v_\infty(f) \equiv s \pmod{p^{k+1}}\} \cup \{0\}.$$

It is an F_k -vector space by Proposition 3.2 and [1, Lemma 5.2]. Then it is easy to see that we have the following direct sum:

$$\mathcal{A}_k := C[\Psi_k] = \bigoplus_{s \in \mathbb{Z}/p^{k+1}\mathbb{Z}} \mathcal{A}_k^{(s)}.$$

\mathcal{A}_k is a $\mathbb{Z}/p^{k+1}\mathbb{Z}$ -graded F_k -algebra. Proposition 3.2 implies that a basis of this algebra is $(1, h, \dots, h^{p^{k+1}-1})$. The difficulty of constructing differentially extremal quasi-modular forms comes from the difficulty of computing the intersections $\mathcal{A}_k^{(s)} \cap \tilde{M}_{w,m}^{\leq 1}$ for given s, k, l, w, m . This seems to explain why we did not really take advantage of Theorem 3.13.

It is easy to deduce from Proposition 3.2 that $\mathcal{A}_k^{(s)} \cap M = M^{p^{k+1}} h^s$ with $M = C[g, h]$, that is, Lemma 5.7 of [1] (this result also follows from [15, Theorem 2.6]).

4. Differential tools

This section, divided in two distinct subsections, contains two contributions to the study of differential properties of Drinfeld quasi-modular forms, the proof of Theorem 4.1 for the operators $\partial_n^{(d)}$ and the description of an algorithm which allows to compute higher derivatives of Drinfeld quasi-modular forms. Although these tools have been already used, namely in Section 3, we decided to collect them in a separate section as they can be of interest independent on the study of differential extemality.

4.1. Higher Serre's operators

Here we study the higher Serre's operators (9) and prove Theorem 4.1.

Let n, d be non-negative integers. We have defined the n -th Serre's operator of degree d , $\partial_n^{(d)} : \tilde{M} \rightarrow \tilde{M}$, by the following formula, that we quote again to ease the reading of this section:

$$\partial_n^{(d)} f = D_n f + \sum_{i=1}^n (-1)^i \binom{d+n-1}{i} (D_{n-i} f)(D_{i-1} E). \quad (45)$$

Notice that in this definition the integer d is arbitrary: In particular, it is not necessarily the weight or the degree of f (at this stage f is not supposed to be a quasi-modular form, anyway). If $n = 0$, then $\partial_n^{(d)} f = \partial_0^{(d)} f = f$. If $n = 1$, we have $\partial_n^{(d)} f = D_1 f - d E f$, which coincides with the formula (13) when $d = w(f)$.

In [9], the authors use the family of operators $\theta_k^{(r)}$, described in the introduction, which act on vector spaces of quasi-modular forms for $\mathbf{SL}_2(\mathbb{Z})$. When $n \geq 1$, there is a strong similarity between our operator $\partial_n^{(d)}$ and the operator $\frac{1}{n!}\theta_d^{(n-1)}$.

In [18], the authors introduce a class of operators on Drinfeld modular forms which could play the role of Rankin–Cohen operators in the drinfeldian framework. Their definition appears in formula (3.14) of their Theorem 3.7, and the notation they adopt for their operator is $[\cdot, \cdot]_{k,l,n}$. It is easy to prove that for all $n, d \geq 1$ there exists $\lambda_{n,d} \in \mathbb{F}_q$ such that:

$$\partial_n^{(d)} f = D_n f - \lambda_{n,d} (-\tilde{\pi})^{1-n} (n+d-1) [E, f]_{2,d,n-1}.$$

However, we do not have a general receipt to compute $\lambda_{n,d}$; for instance, we notice that it can vanish for certain choices of n, d .

The remarkable feature of the operator $\partial_n^{(d)}$ is that it does not increase the depth of quasi-modular forms of degree d . In the classical case, a proof of this fact appears, for example, in [9, Proposition 3.3].

Theorem 4.1. *Let w, l be non-negative integers with $w \geq 2l$ and let m be a class in $\mathbb{Z}/(q-1)\mathbb{Z}$. Define $d := w - l$. For every $n \in \mathbb{N}$, we have*

$$\partial_n^{(d)} (\tilde{M}_{w,m}^{\leq l}) \subset \tilde{M}_{w+2n,m+n}^{\leq l}. \quad (46)$$

In particular, $\partial_n^{(w)}$ sends modular forms of weight w and type m on modular forms of weight $w + 2n$ and type $m + n$.

To prove this theorem we will use the notion of polynomial associated with a Drinfeld quasi-modular form introduced in [1]. We recall for convenience the definitions and properties we will need here. If f is a Drinfeld quasi-modular form of weight w and type m , then there exists a unique polynomial $P_f = \sum_{i=0}^l f_i X^i \in \tilde{M}[X]$ such that

$$f(\gamma(z)) = \frac{(cz+d)^w}{(\det \gamma)^m} \sum_{i=0}^l f_i(z) \left(\frac{c}{cz+d} \right)^i$$

for every $z \in \Omega$ and every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(A)$. This polynomial is the *polynomial associated with f* , its coefficients are in fact Drinfeld quasi-modular forms. If $f = 0$, then $P_f = 0$ and if $f \neq 0$, then the degree of P_f is equal to the depth of f . When f is a modular form, then we have $P_f = f$ and for $f = E$ we have $P_E = E - \tilde{\pi}^{-1}X$. Finally, if f_1, f_2 are two Drinfeld quasi-modular forms, then $P_{f_1 f_2} = P_{f_1} P_{f_2}$ and, if f_1, f_2 have the same weight and the same type, then $P_{f_1 + f_2} = P_{f_1} + P_{f_2}$.

It will be convenient to introduce the following notation, where X is an indeterminate over the ring \tilde{M} , which is equal to $C[E, g, h]$ in virtue of [1, Theorem 1].

Definition 4.2. Let $f = f(E, g, h)$ be an element of \tilde{M} . For every $n \in \mathbb{N}$ we define $\partial_n^{(E)} f \in \tilde{M}$ by the formula

$$f(E + X, g, h) = \sum_{n \geq 0} (\partial_n^{(E)} f) X^n \in C[E, g, h, X], \quad (47)$$

The family $\partial^{(E)} = (\partial_n^{(E)})_{n \geq 0}$ is obviously an iterative higher derivation on \tilde{M} . We have $\partial_1^{(E)} = \frac{\partial}{\partial E}$ and $(\frac{\partial}{\partial E})^n = n! \partial_n^{(E)}$. The interest of introducing $\partial^{(E)}$ is that the following property clearly holds (for all Drinfeld quasi-modular forms $f \neq 0$ and all integers $l \geq 0$):

$$l(f) \leq l \iff \partial_n^{(E)} f = 0 \text{ for all } n \geq l+1. \quad (48)$$

In fact, it turns out that the $\partial_n^{(E)} f$'s are, up to powers of $-\tilde{\pi}$, the coefficients of the associated polynomial P_f .

Lemma 4.3. *Let f be a Drinfeld quasi-modular form. Then we have*

$$P_f(X) = \sum_{n \geq 0} \frac{\partial_n^{(E)} f}{(-\tilde{\pi})^n} X^n.$$

Proof. We may assume that f is non-zero. Let w , m and l denote respectively the weight, the type and the depth of f . Let us write $f = \sum_{i=0}^l f_i E^i$, where $f_i = f_i(g, h) \in M_{w-2i, m-i}$. We have

$$P_f = \sum_{i=0}^l P_{f_i E^i} = \sum_{i=0}^l f_i P_E^i = \sum_{i=0}^l f_i \left(E - \frac{1}{\tilde{\pi}} X \right)^i = f \left(E - \frac{1}{\tilde{\pi}} X, g, h \right),$$

so the result immediately follows from (47). \square

This very simple result allows to translate any property about the coefficients of the polynomial P_f into a property about the higher derivatives $\partial_n^{(E)} f$ and conversely. For example, the reader will easily check that the fact that $\partial^{(E)}$ is iterative is equivalent to Lemma 2.5 of [1]. For the proof of Theorem 4.1, we will use the reformulation of [1, Proposition 3.1] in terms of the higher derivation $\partial^{(E)}$. This yields the following commutation rule between $\partial_j^{(E)}$ and D_n . Note that in the complex case and for $j = 1$, the analogous formula is established in [9] (during the proof of Proposition 3.3).

Lemma 4.4. *Let $j \geq 0$ and $n \geq 0$ be non-negative integers, and let f be a Drinfeld quasi-modular form of weight w . Then we have*

$$\partial_j^{(E)} D_n f = \sum_{r=0}^n \binom{w+n-j+r-1}{r} D_{n-r} \partial_{j-r}^{(E)} f,$$

where we set $\partial_i^{(E)} = 0$ if $i < 0$.

Proof. This is exactly the formula for $P_{D_n f}$ given in Proposition 3.1 of [1], expressed in terms of the higher derivation $\partial^{(E)}$ with help of Lemma 4.3, and taking into account the fact that $D_i = \frac{1}{(-\tilde{\pi})^i} \mathcal{D}_i$. \square

The next lemma is analogous to a similar formula appearing in the proof of [9, Proposition 3.3]. However, the proof in our case cannot be done by induction as in [9] and thus requires more care.

Lemma 4.5. *Let n , d , w and k be non-negative integers, and let f be a Drinfeld quasi-modular form of weight w . Then we have*

$$\partial_k^{(E)} \partial_n^{(d)} f = \sum_{i=0}^k (-1)^i \binom{d+k-w-1}{i} \partial_{n-i}^{(d+i)} \partial_{k-i}^{(E)} f, \quad (49)$$

where we define $\partial_j^{(d)} = 0$ if $j < 0$.

Proof. The formula (49) being plainly true if $n = 0$, we will assume in the following that $n \geq 1$. Applying the operator $\partial_k^{(E)}$ to Eq. (45) and using Leibniz rule for the product $\partial_k^{(E)}((D_{n-i}f)(D_{i-1}E))$, we get

$$\partial_k^{(E)} \partial_n^{(d)} f = \partial_k^{(E)} D_n f + \sum_{\substack{1 \leq i \leq n \\ 0 \leq j \leq k}} (-1)^i \binom{d+n-1}{i} (\partial_j^{(E)} D_{n-i} f) (\partial_{k-j}^{(E)} D_{i-1} E). \quad (50)$$

Applying Lemma 4.4 with $f = E$ and noting that $\partial_{k-j-r}^{(E)} E = 0$ if $r > k - j$ or $r < k - j - 1$, we have (we use the convention $D_i = 0$ if $i < 0$)

$$\begin{aligned} \partial_{k-j}^{(E)} D_{i-1} E &= \sum_{r \geq 0} \binom{i-k+j+r}{r} D_{i-1-r} \partial_{k-j-r}^{(E)} E \\ &= \binom{i}{k-j} D_{i+j-k-1} E + \binom{i-1}{k-j-1} D_{i+j-k}(1) \\ &= \begin{cases} \binom{i}{k-j} D_{i+j-k-1} E & \text{if } j > k-i, \\ 1 & \text{if } j = k-i, \\ 0 & \text{if } j < k-i. \end{cases} \end{aligned}$$

Substituting in (50), we obtain

$$\partial_k^{(E)} \partial_n^{(d)} f = A + B, \quad (51)$$

where

$$A = \sum_{0 \leq i \leq n} (-1)^i \binom{d+n-1}{i} \partial_{k-i}^{(E)} D_{n-i} f$$

and

$$B = \sum_{\substack{1 \leq i \leq n \\ k-i+1 \leq j \leq k}} (-1)^i \binom{d+n-1}{i} \binom{i}{k-j} (\partial_j^{(E)} D_{n-i} f) (D_{i+j-k-1} E).$$

Applying Lemma 4.4 again and then making the change of variable $l = i + r$, we first find the following expression for A :

$$\begin{aligned} A &= \sum_{0 \leq i \leq n} \sum_{0 \leq r \leq n-i} (-1)^i \binom{d+n-1}{i} \binom{w+n-k+r-1}{r} D_{n-i-r} \partial_{k-i-r}^{(E)} f \\ &= \sum_{0 \leq l \leq n} \left[\sum_{0 \leq i \leq l} (-1)^i \binom{d+n-1}{i} \binom{w+n-k+l-i-1}{l-i} \right] D_{n-l} \partial_{k-l}^{(E)} f \\ &= \sum_{0 \leq l \leq n} (-1)^l \binom{d+k-w-1}{l} D_{n-l} \partial_{k-l}^{(E)} f. \end{aligned} \quad (52)$$

Similarly for B , we apply Lemma 4.4 to $\partial_j^{(E)} D_{n-i} f$ and then make the changes of variable $J = i + j - k$, $I = i + r - J$. This yields

$$B = \sum_{\substack{0 \leq I \leq n-1 \\ 1 \leq J \leq n-I}} (-1)^{I+J} S_{IJ} (D_{n-I-J} \partial_{k-I}^{(E)} f) (D_{J-1} E),$$

where

$$S_{IJ} = \sum_{r=0}^I (-1)^r \binom{d+n-1}{I+J-r} \binom{I+J-r}{I-r} \binom{w+n-k-J+r-1}{r}.$$

Since

$$\binom{d+n-1}{I+J-r} \binom{I+J-r}{I-r} = \binom{d+n-1}{d+n-1-J} \binom{d+n-J-1}{I-r},$$

the sum S_{IJ} is equal to

$$\binom{d+n-1}{d+n-1-J} T_{IJ}$$

with

$$\begin{aligned} T_{IJ} &= \sum_{r=0}^I (-1)^r \binom{d+n-J-1}{I-r} \binom{w+n-k-J+r-1}{r} \\ &= \binom{d+k-w-1}{I} \end{aligned}$$

by [1, Lemma 3.2]. Hence we obtain the following expression for B :

$$B = \sum_{\substack{0 \leq I \leq n-1 \\ 1 \leq J \leq n-I}} (-1)^{I+J} \binom{d+k-w-1}{I} \binom{d+n-1}{d+n-1-J} (D_{n-I-J} \partial_{k-I}^{(E)} f) (D_{J-1} E). \quad (53)$$

Now, we note that in the formula (52) the summands vanish if $I > n$ or $I > k$, so we may assume that I runs from 0 to k . Similarly, in the formula (53) we can let I vary from 0 to k . Using this remark, and substituting (52) and (53) in (51), we get:

$$\begin{aligned} \partial_k^{(E)} \partial_n^{(d)} f &= \sum_{l=0}^k (-1)^l \binom{d+k-w-1}{l} \\ &\quad \times \left[D_{n-l} \partial_{k-l}^{(E)} f + \sum_{J=1}^{n-l} (-1)^J \binom{d+n-1}{d+n-1-J} (D_{n-l-J} \partial_{k-l}^{(E)} f) (D_{J-1} E) \right]. \end{aligned}$$

This is nothing else than the formula (49), so Lemma 4.5 is proved. \square

We can now prove Theorem 4.1.

Proof of Theorem 4.1. Let $f \in \tilde{M}_{w,m}^{\leq l}$, and let k be any integer such that $k \geq l + 1$. We want to show that $\partial_k^{(E)} \partial_n^{(d)} f = 0$, which will prove the proposition by property (48). We use for this Lemma 4.5. Since here $d = w - l$, we have

$$\partial_k^{(E)} \partial_n^{(d)} f = \sum_{i=0}^k (-1)^i \binom{k-l-1}{i} \partial_{n-i}^{(d+i)} \partial_{k-i}^{(E)} f. \quad (54)$$

In this sum, if $i \leq k - l - 1$, then $k - i \geq l + 1$ and hence $\partial_{k-i}^{(E)} f = 0$ (property (48)). If now $i \geq k - l$, then $\binom{k-l-1}{i} = 0$. So all the summands in the right-hand side of (54) vanish and $\partial_k^{(E)} \partial_n^{(d)} f = 0$. \square

4.1.1. Digression: an application to eigenforms of Hecke operators

Theorem 4.1 has already been used in the proof of Theorem 1.6 (in Section 3). Another interesting application of this theorem is that it can be used to construct *a priori* new eigenforms for Hecke operators, from given ones.

Let $\mathfrak{p} = (P)$ be a non-zero prime ideal of A , where P is a monic polynomial. Following [5, § 1.8] or [3, § 7], we define, for any quasi-modular form $f \in \tilde{M}_{w,m}^{\leq l}$, $T_{\mathfrak{p}} f$ by the formula

$$(T_{\mathfrak{p}} f)(z) = P^w f(Pz) + \sum_{\substack{b \in A \\ \deg_{\theta} b < \deg_{\theta} P}} f\left(\frac{z+b}{P}\right), \quad (55)$$

where we remark the dependence of this operator on the weight w . We also notice that there is no reason for $T_{\mathfrak{p}} f$ to lie in \tilde{M} , except when we already know that $f \in \tilde{M}$.

Lemma 4.6. If $f \in \tilde{M}_{w,m}^{\leq l}$ is a quasi-modular form which is an eigenform for $T_{\mathfrak{p}}$ with eigenvalue $\lambda \in C$ (that is, such that $T_{\mathfrak{p}} f = \lambda f$), then for all $n \geq 1$, $D_n f \in \tilde{M}_{w+2n,m+n}^{\leq l+n}$ also is an eigenform with eigenvalue λP^n .

Proof. The function $\varphi : z \mapsto f(Pz)$ satisfies $(D_n \varphi)(z) = P^n (D_n f)(Pz)$ for all n , and the functions $f_b : z \mapsto f((z+b)/P)$ satisfy $(D_n f_b)(z) = P^{-n} (D_n f)((z+b)/P)$. Since the weight of $D_n f$ is $w + 2n$, we then see that

$$T_{\mathfrak{p}}(D_n f) = P^n D_n(T_{\mathfrak{p}} f) \quad \text{for all } n \geq 0,$$

where this time, the operator $T_{\mathfrak{p}}$ in the left-hand side is defined as in (55) but with w replaced by $w + 2n$. It immediately follows that if $T_{\mathfrak{p}} f = \lambda f$ for some $\lambda \in C$, then $T_{\mathfrak{p}}(D_n f) = \lambda P^n (D_n f)$. \square

Example. The normalised Eisenstein series g_k defined in (11) (for $k \geq 1$) are modular eigenforms of weight $q^k - 1$ and type 0 of all the operators $T_{\mathfrak{p}}$ with corresponding eigenvalue P^{q^k-1} , by [3, Proposition 7.2]. Therefore, Lemma 4.6 says that $x_k = D_1 g_k$ is eigenform of all the operators $T_{\mathfrak{p}}$ with corresponding eigenvalue P^{q^k} (if $k \geq 1$).

Question. Are the forms ξ_k of Section 2.2 eigenforms for all the Hecke operators?

More interesting is the particular case in which f is a modular form which is known to be an eigenform for all the Hecke operators, and n is an integer such that $D_n f$ is again a modular form. The next lemma implies that when $d \geq 2$, there are infinitely many n 's for which $D_n = \partial_n^{(d)}$, and thus every eigenform f of $T_{\mathfrak{p}}$ of degree ≥ 2 potentially yields other eigenforms (these forms may be identically zero, but in many examples they are not).

Lemma 4.7. Let d, k be non-negative integers with $d \geq 2$ and $p^k(p-1) \geq d-1$, and write $n = 1 - d + p^{k+1}$. Then $\partial_n^{(d)} = D_n$.

Proof. By using e.g. the formula (14) of [1], one easily checks that

$$\binom{d+n-1}{i} \equiv 0 \pmod{p} \quad \text{for all } i = 1, \dots, n.$$

The conclusion follows by applying these congruences to the formula defining the operators $\partial_n^{(d)}$. \square

If f is a modular form of weight $d \geq 2$ and if n is the integer of Lemma 4.7, $D_n f$ is a modular form by Theorem 4.1. Note that for a given f , there might be other choices of n for which $\partial_n^{(d)} f = D_n f$, but it is not difficult to show that there is at most one more choice for n than the one in the lemma, for which $\binom{d+n-1}{i} \equiv 0 \pmod{p}$ for all $i = 1, \dots, n$.

Example. We know [3, Corollary (7.6)] that h is an eigenform of all the Hecke operators. Here $d = q + 1$, and we can take $n = p^{k+1} - q$ for every k such that $p^k(p-1) \geq q$. Then we get an infinite family of eigenforms $(D_{p^{k+1}-q} h)$. We remark that we already knew from [1, Lemma 3.10] that these functions are modular. We do not know yet how to characterise the integers n such that $D_n h \neq 0$.

By using the tools developed in Section 4.2, some of these forms can be computed explicitly. Then, it can be checked that not all of them are zero, and they do not belong to the families known by the work of Gekeler and Goss [3,5]. For example, one computes easily:

$$D_{q^2-q} h = \frac{g^{q-1} h^q}{[1]^{q-1}}, \quad D_q(\Delta) = \frac{gh^q}{[1]},$$

so that $g^{q-1} h^q$ and gh^q are Hecke eigenforms.

4.2. An algorithm to compute higher derivatives

Let $\delta = (\delta_n)_{n \geq 0}$ be a higher derivation on a C -algebra F . Then, X being an indeterminate over F , the map (Taylor's homomorphism)

$$\mathcal{T}_X^\delta : F \rightarrow F[[X]]$$

defined by

$$\mathcal{T}_X^\delta(x) = \sum_{n=0}^{\infty} (\delta_n x) X^n$$

is a C -algebra homomorphism [10, Section 27].

Over $F[[X]]$ there also is the iterative higher derivation $\delta' = (\delta'_n)_{n \geq 0}$ uniquely determined by

$$\delta'_n(fX^i) = \binom{i}{n} fX^{i-n} \quad (56)$$

for $f \in F$ and $n, i \geq 0$. One checks that $(\delta_n)_{n \geq 0}$ is iterative if and only if, over F ,

$$\mathcal{T}_X^\delta \circ \delta_n = \delta'_n \circ \mathcal{T}_X^\delta, \quad n \geq 0. \quad (57)$$

Indeed, this condition is equivalent to the commutativity of the diagram on p. 209 of [10].

From now on we work with $F = C(E, g, h)$ and $\delta = D = (D_n)_{n \geq 0}$. We also write, to ease notations, $\mathcal{T}_X = \mathcal{T}_X^D$. We will also look at the fraction field $F(X)$ of $F[X]$ as embedded in $F((X))$. For example, the expression $1/(1 - EX)$ represents the formal series $1 + EX + E^2X^2 + \dots \in F[[X]]$.

Let f be an element of F . In the following, we will make use of the polynomials

$$\mathcal{T}_{X,k}(f) := \sum_{i=0}^{q^k-1} (D_i f) X^i \in F[X],$$

so that, in $F[[X]]$, we have the following congruence modulo the ideal (X^{q^k}) :

$$\mathcal{T}_X(f) \equiv \mathcal{T}_{X,k}(f) \pmod{(X^{q^k})}.$$

These polynomials provide approximations to arbitrary order for the formal series $\mathcal{T}_X(f)$ as $\mathcal{T}_X(f) = \lim_{n \rightarrow \infty} \mathcal{T}_{X,n}(f)$ (limit for the X -adic metric).

The map $\mathcal{T}_{X,k} : F \rightarrow F[X]$ is not a C -algebra homomorphism itself, but induces a C -algebra homomorphism:

$$\mathcal{T}_{X,k} : F \rightarrow \frac{F[[X]]}{(X^{q^k})} = \frac{F[X]}{(X^{q^k})}.$$

The following identities and congruences will prove to be useful (the product being equal to 1 when the indexing set is empty):

$$\mathcal{T}_{X,r+s}(f^{q^s}) = \mathcal{T}_{X,r}(f)^{q^s}, \quad (58)$$

$$\mathcal{T}_{X,r}(f^{-1}) \equiv f^{-q^r} \prod_{i=0}^{r-1} \mathcal{T}_{X,r-i}(f^{q^{-1}})^{q^i} \pmod{(X^{q^r})}. \quad (59)$$

Equality (58) holds for $f \in F$ and $r, s \geq 0$ and its validity is easy to check. Congruence (59) holds for $f \in F^\times$ and $r \geq 0$ and can be proved as follows. Since $\mathcal{T}_X(f)^{q^r} \equiv f^{q^r} \pmod{(X^{q^r})}$, we have

$$\mathcal{T}_{X,r}(f^{q^{-1}})^{1+q+\dots+q^{r-1}} \mathcal{T}_{X,r}(f) \equiv f^{q^r} \pmod{(X^{q^r})},$$

yielding the desired congruence.

Proposition 4.8. *Let $r, s \geq 0$ be integers. The following congruence holds, modulo the ideal $(X^{q^{r+s+1}})$ of $F[[X]]$:*

$$\mathcal{T}_{X,r+s+1}(g_s) \equiv [s+1]^{-1} \mathcal{T}_{X,r+1}(\Delta^{-1})^{q^s} (\mathcal{T}_{X,r}(g)^{q^{s+1}} \mathcal{T}_{X,r+s+1}(g_{s+1}) - \mathcal{T}_{X,r+s+1}(g_{s+2})). \quad (60)$$

Proof. We appeal to the formula (12), which is equivalent to

$$g_s = [s+1]^{-1} \Delta^{-q^s} (g^{q^{s+1}} g_{s+1} - g_{s+2}), \quad s \geq 0.$$

Congruence (60) is obtained applying the C -algebra homomorphism \mathcal{T}_X to both sides of the latter identity (formulas (58) and (59) can help):

$$\mathcal{T}_X(g_s) = [s+1]^{-1} \mathcal{T}_X(\Delta)^{-q^s} (\mathcal{T}_X(g)^{q^{s+1}} \mathcal{T}_X(g_{s+1}) - \mathcal{T}_X(g_{s+2})), \quad s \geq 0, \quad (61)$$

and then reducing modulo the ideal $(X^{q^{r+s+1}})$ of $F[[X]]$. \square

We now give an algorithm for the explicit computation of $\mathcal{T}_X(g_1)$, $\mathcal{T}_X(g_2)$, ... and $\mathcal{T}_X(\Delta)$, for which Proposition 4.8 and Lemma 2.2 are the key tools.

The best way to describe our algorithm, naturally presented as an induction process, is to begin by giving the detail of the explicit computation of its first steps. We start with the explicit computation of the polynomials $\mathcal{T}_{X,1}(\Delta)$ and $\mathcal{T}_{X,s+1}(g_s)$ ($s \geq 1$) (first approximation). Then, we proceed to the computation of $\mathcal{T}_{X,2}(\Delta)$ and of a representative of $\mathcal{T}_X(g_s)$ modulo the ideal $(X^{q^{s+1}+2})$ (second approximation). Unfortunately, the full computation of the polynomials $\mathcal{T}_{X,s+2}(g_s)$ would be too long to present in this article.

These explicit computations will prepare the reader for the general process which generates explicit expressions for the polynomials $\mathcal{T}_{X,r+s+1}(g_s)$ ($s \geq 1$) and $\mathcal{T}_{X,r+1}(\Delta)$ for all $r \geq 1$; he or she will then be ready to understand the algorithm. At the same time, the accomplished explicit computations are used for different purposes in several parts of this paper.

In [1, Theorem 4.1] we have computed $D_1 f, D_q f, D_{q^2} f \in C[E, g, h]$ with $f \in \{E, g, h\}$, applying the classical technique consisting in solving linear equations in C -vector spaces of modular forms with prescribed order of vanishing at infinity. This method of computation can be pushed beyond to compute also $D_{q^3} f, D_{q^4} f, \dots$ but then it requires that one first computes the coefficients of the u -expansions of E, g, h with Gekeler's algorithm of [3], before entering the linear algebra part. However, the computation of the u -expansions of E, g, h is not an easy matter, since it also needs computation of the so-called Goss polynomials, a task that usually generates large computations.

The algorithm we give here is of a different nature and can be considered as a variant of Gekeler's techniques of computing Goss polynomials (cf. Proof of Lemma 3.3), and using the recurrence relations (12) to yield u -expansions of modular forms. Our algorithm is easier to use, compared to the methods introduced in [1] because it does not need any preliminary computation of u -expansions.

4.2.1. First approximation

We know that:

$$\mathcal{T}_{X,s}(g_s) = g_s + x_s X, \quad s \geq 1, \quad (62)$$

$$\mathcal{T}_{X,1}(\Delta) = \Delta(1 - EX). \quad (63)$$

The first formula follows easily from Lemma 2.2, while the second can be obtained by using [1, Theorem 4.1(iii)] or applying Lemma 2.2 to g_1, g_2 and then use the formula

$$\Delta = [1]^{-1}(g_1^{q+1} - g_2), \quad (64)$$

easily deduced from (12).

From (63) we obtain:

$$\begin{aligned} \mathcal{T}_{X,1}(\Delta^{-1}) &\equiv \Delta^{-1} \frac{1}{1 - EX} \pmod{(X^q)} \\ &= \Delta^{-1}(1 + EX + E^2 X^2 + \dots + E^{q-1} X^{q-1}). \end{aligned}$$

Substituting (62) in (60) we obtain the following congruence modulo the ideal $(X^{q^{s+1}})$ for $\mathcal{T}_{X,s+1}(g_s)$, where we suppose that $s \geq 1$:

$$\begin{aligned} \mathcal{T}_{X,s+1}(g_s) &\equiv [s+1]^{-1} \Delta^{-q^s} \frac{1}{1 - E^{q^s} X^{q^s}} ((g^{q^{s+1}} + x_1^{q^{s+1}} X^{q^{s+1}})(g_{s+1} + x_{s+1} X) - (g_{s+2} + x_{s+2} X)) \\ &\equiv \frac{g_s + x_s X}{1 - E^{q^s} X^{q^s}}. \end{aligned} \quad (65)$$

We thus obtain the following formula for $\mathcal{T}_{X,s+1}(g_s)$:

$$\mathcal{T}_{X,s+1}(g_s) = g_s + x_s X + E^{q^s} X^{q^s} (g_s + x_s X) + \cdots + E^{(q-1)q^s} X^{(q-1)q^s} (g_s + x_s X).$$

This in turn allows, inserting the congruence (65) for $s = 1, 2$ in (64), to compute $\mathcal{T}_{X,2}(\Delta)$. Here is the formula we find (congruences modulo the ideal (X^{q^2})):

$$\begin{aligned} \mathcal{T}_{X,2}(\Delta) &\equiv [1]^{-1} (\mathcal{T}_{X,2}(g_1)^q \mathcal{T}_{X,2}(g_1) - \mathcal{T}_{X,2}(g_2)) \\ &\equiv [1]^{-1} \left(\frac{(g + x_1 X)^{q+1}}{(1 - E^q X^q)^{q+1}} - (g_2 + x_2 X) \right) \\ &\equiv \Delta \left(1 - EX + \left(\frac{gh}{[1]} - E^q \right) X^q + \left(E^{q+1} - \frac{Egh}{[1]} - \frac{h^2}{[1]} \right) X^{q+1} \right) (1 - E^q X^q)^{-1}. \end{aligned} \quad (66)$$

To check the last congruence, the reader can make use of Tables 1 and 2 of Section 2.1.2 and identity (64).

4.2.2. Second approximation

By using (60) and (65), we get, for all $s \geq 1$, the following congruences modulo the ideal $(X^{q^{s+2}})$:

$$\begin{aligned} [s+1]\mathcal{T}_{X,s+2}(g_s) &\equiv \mathcal{T}_{X,2}(\Delta^{-1})^{q^s} (\mathcal{T}_{X,1}(g)^{q^{s+1}} \mathcal{T}_{X,s+2}(g_{s+1}) - \mathcal{T}_{X,s+2}(g_{s+2})) \\ &\equiv \mathcal{T}_{X,2}(\Delta^{-1})^{q^s} \left((g^{q^{s+1}} + x_1^{q^{s+1}} X^{q^{s+1}}) \frac{g_{s+1} + x_{s+1} X}{1 - E^{q^{s+1}} X^{q^{s+1}}} - (g_{s+2} + x_{s+2} X) \right). \end{aligned}$$

The explicit computation of $\mathcal{T}_{X,2}(\Delta^{-1})$ is possible with Formula (59). Unfortunately, it is rather complicated to handle, so we limit ourselves to its determination modulo the ideal (X^{q+1}) . This is why we do not fully compute $\mathcal{T}_{X,s+2}(g_s)$ in this text.

Since $\mathcal{T}_{X,2}(\Delta) \equiv \Delta(1 - EX + ([1]^{-1}gh - E^q)X^q) \pmod{(X^{q+1})}$ we compute easily:

$$\mathcal{T}_{X,2}(\Delta^{-1}) \equiv \Delta^{-1} \left(\sum_{i=0}^{q-1} E^i X^i + \left(E^q - \frac{gh}{[1]} \right) X^q \right) \pmod{(X^{q+1})}. \quad (67)$$

We reduce the q^s -th power of the polynomial (67) modulo the ideal $(X^{q^{s+1}+2})$. Looking at the formula (12) in the form

$$\Delta^{q^s} g_s = [s+1]^{-1} (g^{q^{s+1}} g_{s+1} - g_{s+2})$$

and $x_1 = -Eg - h$, we find the congruences (modulo $(X^{q^{s+1}+2})$):

$$\begin{aligned} [s+1]\mathcal{T}_X(g_s) &\equiv \mathcal{T}_{X,2}(\Delta^{-1})^{q^s} ((g^{q^{s+1}} + x_1^{q^{s+1}} X^{q^{s+1}})(g_{s+1} + x_{s+1} X)(1 + E^{q^{s+1}} X^{q^{s+1}}) - (g_{s+2} + x_{s+2} X)) \\ &\equiv \mathcal{T}_{X,2}(\Delta^{-1})^{q^s} (g^{q^{s+1}} g_{s+1} - g_{s+2} + (g^{q^{s+1}} x_{s+1} - x_{s+2})X + (x_1^{q^{s+1}} g_{s+1} + E^{q^{s+1}} g_{s+1} g^{q^{s+1}})X^{q^{s+1}} \\ &\quad - h^{q^{s+1}} x_{s+1} X^{q^{s+1}+1}) \\ &\equiv \mathcal{T}_{X,2}(\Delta^{-1})^{q^s} (\Delta^{q^s} [s+1]g_s + \Delta^{q^s} [s+1]x_s X - h^{q^{s+1}} g_{s+1} X^{q^{s+1}} - h^{q^{s+1}} x_{s+1} X^{q^{s+1}+1}) \end{aligned}$$

$$\equiv \left(\sum_{i=0}^{q-1} E^{iq^s} X^{iq^s} + \left(E^{q^{s+1}} - \frac{g^{q^s} h^{q^s}}{[1]^{q^s}} \right) X^{q^{s+1}} \right) \\ \times ([s+1]g_s + [s+1]x_s X + h^{q^s} g_{s+1} X^{q^{s+1}} + h^{q^s} x_{s+1} X^{q^{s+1}+1}) \pmod{(X^{q^{s+1}+2})}.$$

Therefore, we obtain:

Proposition 4.9. For $s \geq 1$, we have the congruences:

$$\mathcal{T}_X(g_s) \equiv \frac{g_s + x_s X}{1 - E^{q^s} X^{q^s}} - h^{q^s} \left(-\frac{g_{s+1}}{[s+1]} + \frac{g^{q^s} g_s}{[1]^{q^s}} \right) X^{q^{s+1}} - h^{q^s} \left(-\frac{x_{s+1}}{[s+1]} + \frac{g^{q^s} x_s}{[1]^{q^s}} \right) X^{q^{s+1}+1} \\ \pmod{(X^{q^{s+1}+2})}, \quad (68)$$

$$\mathcal{T}_X(x_s) \equiv \frac{x_s}{1 - E^{q^s} X^{q^s}} - h^{q^s} \left(-\frac{x_{s+1}}{[s+1]} + \frac{g^{q^s} x_s}{[1]^{q^s}} \right) X^{q^{s+1}} \pmod{(X^{q^{s+1}+1})}. \quad (69)$$

In particular, $\epsilon_D(x_k) = (k+1)e$ for all $k \geq 0$.

The second congruence of the proposition follows from the first and (57) because we have, for $s \geq 1$:

$$\mathcal{T}_X(x_s) = \mathcal{T}_X(D_1 g_s) = \frac{\partial}{\partial X} \mathcal{T}_X(g_s).$$

The value of $\epsilon_D(x_k)$ is now easy to determine thanks to (69). Indeed, the first term of the right-hand side of this identity tell that $\epsilon_D(x_k) \geq (k+1)e$. Equality follows by checking that the coefficient of X^{k+1} is a polynomial of $C[E, g, h]$ which is coprime with x_k by Lemma 2.9.

Remark 4.10. Notice that, substituting $s = 0$, the second formula agrees with [1, (iv) and (vii) of Theorem 4.1]. There is no simple explanation of this fact.

4.2.3. End of description of the algorithm

We shall now describe, in its generality, the algorithm, which uses induction on $r \geq 1$. Taking into account Proposition 4.9 and formula (67), we can assume that for an integer $r \geq 1$ we have already computed $\mathcal{T}_{X,r+s}(g_s)$ (for all $s \geq 1$) and $\mathcal{T}_{X,r+1}(\Delta)$, explicitly as polynomials of $C(E, g, h)[X]$.

Therefore, in (60), the polynomials

$$\mathcal{T}_{X,r+s+1}(g_{s+2}), \quad \mathcal{T}_{X,r}(g), \quad \mathcal{T}_{X,r+s+1}(g_{s+1})$$

are all known. By using (59), we compute $\mathcal{T}_{X,r+1}(\Delta^{-1})$, then we use (58) to raise the expression to the q^s -th power; we obtain the polynomials $\mathcal{T}_{X,r+s+1}(g_s)$, $s \geq 1$.

In particular, for $s = 1, 2$ we have an explicit expression of $\mathcal{T}_{X,r+2}(g_1)$ and $\mathcal{T}_{X,r+3}(g_2)$ and the degree $< q^{r+2}$ representative of the class of reduction modulo $(X^{q^{r+2}})$ of the polynomial

$$[1]^{-1}(\mathcal{T}_{X,r+2}(g_1)^{q+1} - \mathcal{T}_{X,r+3}(g_2))$$

is the polynomial $\mathcal{T}_{X,r+2}(\Delta)$ by (64).

This ends the description of the algorithm.

4.2.4. An algorithm which computes $D_n f$ for $f = E, g, h, n \geq 1$

It remains to explain how to compute the polynomials $\mathcal{T}_{X,r}(f)$ with $f \in \{E, g, h\}$ for $r \geq 0$. First of all, $\mathcal{T}_{X,r}(g)$ and $\mathcal{T}_{X,r}(\Delta)$ are computed by the algorithm of Section 4.2.3 (case $s = 1$).

We have $\Delta = -h^{q-1}$. Hence, $\mathcal{T}_X(\Delta) = -\mathcal{T}_X(h)^{q-1}$. By [1, Proposition 3.6], there exist two formal series

$$f_h = \sum_{m=0}^{\infty} c_m X^m, \quad f_{\Delta} = \sum_{m=0}^{\infty} d_m X^m \in F[[X]]$$

with $c_0 = d_0 = 1$, such that $\mathcal{T}_X(h) = hf_h$ and $\mathcal{T}_X(\Delta) = \Delta f_{\Delta}$. We have

$$f_h^{q-1} = f_{\Delta},$$

so that the coefficients c_n are uniquely determined by the following relations:

$$c_n = \ell_n - \sum_{i+j=n, i \neq n} c_i d_j, \quad \text{where } \ell_n = \begin{cases} 0 & \text{if } q \nmid n, \\ c_{n/q}^q & \text{if } q \mid n, \end{cases} \quad n \geq 0.$$

The coefficients c_n can be computed by induction on $n \geq 0$ and this allows to compute the polynomials $\mathcal{T}_{X,r}(h)$.

Now, $D_1 h = Eh$, so that $\mathcal{T}_X(D_1 h) = \mathcal{T}_X(E)\mathcal{T}_X(h)$. But $\mathcal{T}_X(D_1 h) = \delta'_1 \mathcal{T}_X(h)$ because D is iterative and we have (57), so that

$$\mathcal{T}_X(E) = (\delta'_1 \mathcal{T}_X(h))\mathcal{T}_X(h)^{-1}$$

where $(\delta'_n)_{n \in \mathbb{N}}$ is the iterative derivation defined in (56). Therefore,

$$\mathcal{T}_{X,r}(E) \equiv \left(\frac{\partial}{\partial X} \mathcal{T}_{X,r}(h) \right) \mathcal{T}_{X,r}(h^{-1}) \pmod{(X^{q^r})}, \quad r \geq 0. \quad (70)$$

The computation of the polynomials $\mathcal{T}_{X,r}(h^{-1})$ can be made with (59).

In fact, since $D_1(\Delta) = -E\Delta$, the computation of the sequence $\mathcal{T}_{X,r}(E)$ can be achieved avoiding the use of h , by using (57).

Remark 4.11. By Theorem 1 and Proposition 3.1 of [1], we also know that all the fractions $D_n E, D_n g, D_n h \in F$ belong to $\tilde{M} = C[E, g, h]$. In fact, we have $\mathcal{T}_X(f) \in \tilde{M}[[X]]$ for all $f \in \tilde{M}$. This property seems not to follow from Proposition 4.8.

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