SAPIENZA
UNIVERSITÀ DI ROMA

# Conformal Embeddings in Basic Lie Superalgebras

Cristoforo Caffi

*Advisor:* Prof. Paolo Papi

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# Introduction

Let $V$ and $W$ be vertex operator superalgebras equipped with Virasoro vectors $\omega_V$, $\omega_W$ and assume that $W$ is a vertex subalgebra of $V$.

**Definition 1** *We say that $W$ is conformally embedded in $V$ if $\omega_V = \omega_W$.*

In this work we deal with following problems.

1. Classify conformal embeddings when $V, W$ are affine vertex superalgebras endowed with $\omega_V$, $\omega_W$ given by Sugawara construction.

2. Under the assumptions in (1), analyze the decomposition of $V$ as a $W$-module.

We will give a complete solution to problem (1) when $V$ is the affine vertex superalgebra associated to a basic Lie superalgebra (BSA) and $W$ is the subalgebra generated by $\{x_{(-1)}\mathbf{1} \mid x \in \mathfrak{g}^0\}$, $\mathfrak{g}^0$ being any regular equal rank basic Lie subsuperalgebra of $\mathfrak{g}$. Furthermore we will answer the following question

(0) Give an explicit criterion to locate regular maximal equal rank basic Lie subsuperalgebras.

Finally, we discuss some examples related to problem (2).

We start briefly discussing the origin of Definition 1 and its relevance.

## Some history

The general definition of conformal embedding introduced above is a natural generalization of the following notion, which has been popular in physics literature in the mid 80's, due to its relevance for string compactifications.

Let $\mathfrak{g}$ be simple finite-dimensional complex Lie algebra and $\mathfrak{g}^0$ a reductive quadratic subalgebra of $\mathfrak{g}$ (i.e. such that the restriction to $\mathfrak{g}^0$ of an invariant form on $\mathfrak{g}$ is non-degenerate). The embedding $\mathfrak{g}^0 \hookrightarrow \mathfrak{g}$ is called conformal if the central charge of the Sugawara construction for the affinization $\hat{\mathfrak{g}}$, acting on a level 1 integrable module, equals that for the natural embedding of $\widehat{\mathfrak{g}^0}$ in $\hat{\mathfrak{g}}$. Such an embedding is called maximal if no reductive subalgebra $\mathfrak{a}$ with $\mathfrak{g}^0 \subsetneq \mathfrak{a} \subsetneq \mathfrak{g}$ embeds conformally in $\mathfrak{g}$.

Maximal conformal embeddings were classified in [30], [11], and the corresponding decompositions are described in [25], [23], [13]. In the vertex algebra framework the definition can be rephrased as follows: the simple affine vertex algebras $V_{\mathbf{k}}(\mathfrak{g}^0)$ and $V_1(\mathfrak{g})$ have the same Sugawara conformal vector for some multiindex $\mathbf{k}$ of levels. One may wonder whether the embedding $V_{\mathbf{k}}(\mathfrak{g}^0) \subset V_k(\mathfrak{g})$ is conformal according to Definition 1 for some $k$, not necessarily 1. Such a $k$ will be called a *conformal level*. This question was answered in [4], where it was also proved that equality of central charges still detects conformality of maximal equal rank subalgebras and that the same criterion holds also we dealing with non maximal equal rank embeddings. Question (2) has appeared many times in literature. Complete answers when $\mathfrak{g}^0$ is semisimple (and an almost complete answer when $\mathfrak{g}^0$ has a nonzero center) have been given in [4]. Explicit decompositions were also given in *loc. cit.*.

Next, the non-equal rank case was treated in [7]. The further step was to analyze the super case, i.e. when $\mathfrak{g}$ is a basic Lie superalgebra. This has been done in [9] in the special case when $\mathfrak{g}^0 = \mathfrak{g}_{\bar{0}}$, i.e. it is the even part of $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$. In particular conformal levels have been computed.

## Relevance of the definition

Beyond the motivations coming from Physics, conformal embeddings share interesting properties from the point of view of modularity of characters of affine algebras [25]. They provide very interesting connections with the combinatorics of abelian ideals of Borel subalgebras [13].

In the realm of VOA theory, conformal embeddings have been thoroughly investigated by Adamović, Perše, Adamović-Perše (see e.g. [1], [29], [2], [3]) and the relations with simple current extensions have been discussed in [22]. It is worthwhile to notice that a technical tool provided in [3] to detect conformal embeddings (and further generalized in [4]) will be of fundamental importance for our treatment: it will be called the *AP-criterion*.

Quite recently, Adamović, Kac, Möseneder Frajria, Papi, Perše, have studied conformal embeddings of affine vertex algebras in minimal $W$-algebras [5], [6]. The minimal $W$-algebras are vertex algebras obtained by quantum Hamiltonian reduction starting from a basic Lie superalgebra and an even minimal nilpotent element: see [26]; they are quite relevant in Physics, since they include the superconformal algebras. It turns out that the notion of conformal embedding, together with that of *collapsing level*, are crucial in providing, through the theory of $W$-

algebras, semisimplicity results for certain categories of representations of affine Lie algebras [8].

## Overview of the results

In this subsection we shall display the main results of our work in rough terms. We shall concentrate on pairs $(\mathfrak{g}^0, \mathfrak{g})$ where $\mathfrak{g}$ is a basic Lie superalgebras and $\mathfrak{g}^0$ is a semisimple (see def 2.1.2) regular equal rank basic Lie subalgebra of $\mathfrak{g}$. A regular subalgebra $\mathfrak{g}^0$ is one that admits a basis consisting of elements of some Cartan Subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and root vectors of $\mathfrak{g}$ relative to $\mathfrak{h}$.

• **Problem (0):** Give an explicit criterion to locate regular maximal equal rank basic Lie subsuperalgebras.

In the Lie algebra case maximal subalgebras are detected by the Borel-De Siebenthal theorem via an explicit procedure [12].

In the super case, we can follow the method described in [31] to find all possible regular subalgebras of $\mathfrak{g}$: it consists in taking all possible affine Dinkin diagrams of $\mathfrak{g}$ and then deleting one or more dots from it. In this way, we obtain the Dynkin diagram corresponding to a regular subalgebra. Then, repeating the same operation on the obtained Dynkin diagram, that is adding a dot associated to the lowest root of a simple ideal and cancelling arbitrary dots as many times as necessary, we will obtain all the Dynkin diagrams associated with regular semisimple BSA. To obtain equal rank subalgebras, only the first step will be considered.

We formulate a super analogue of the Borel-De Siebenthal algorithm.

**Theorem 1** *Let $\mathfrak{g}$ be a BSA and $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. A Lie subsuperalgebra $\mathfrak{g}' \subset \mathfrak{g}$ is an equal rank maximal regular subalgebra if and only if there exists an affine simple root system (cf 1.5.1) $\hat{\Pi} = \{\alpha_0, \dots, \alpha_l\}$ in $\mathfrak{h}^*$ such that if $\sum_{i=0}^{l} c_i\alpha_i = 0$ is the linear dependence with coprime non-negative integer coefficients, then*

$$\mathfrak{g}' = \mathfrak{h}' \oplus \bigoplus_{\alpha \in \Delta'} \mathfrak{g}_\alpha,$$

*where $\Delta'$ is the root system generated by $\Pi = \hat{\Pi} \setminus \{\alpha_i\}$ with $c_i$ prime and $\mathfrak{h}' \subset \mathfrak{h}$ is generated by $h_\alpha$, $\alpha \in \Pi$.*

Unfortunately, we cannot offer a conceptual proof of this result, which has to be proved by direct verification.

• **Problem (1)** Let $\mathscr{V}_k(\mathfrak{g}^0)$ be the affine vertex subalgebra of $V_k(\mathfrak{g})$ generated by $\{x_{(-1)}\mathbf{1} \mid x \in \mathfrak{g}^0\}$. Classify levels $k$ such that the $\mathscr{V}_k(\mathfrak{g}^0)$ is conformally embedded into $V_k(\mathfrak{g})$ when $\mathfrak{g}^0$ is a regular equal rank subalgebra.

As noted above, the main technical tool to detect a conformal embedding is the AP criterion, that can be stated as follows. We assume that $\mathfrak{g}^0 = \mathfrak{g}^0_0 \oplus \mathfrak{g}^0_1 \oplus$

$\cdots \oplus \mathfrak{g}_s^0$, where $\mathfrak{g}_0^0$ is even abelian and $\mathfrak{g}_i^0$ are basic classical ideals for $i > 0$. The algebras $\mathfrak{g}$ and $\mathfrak{g}_i^0$ are equipped with bilinear invariant forms $(\cdot,\cdot)$ and $(\cdot,\cdot)_j$, with the normalizations given in 1.2. Let $\mathfrak{g}^1$ be the orthogonal complement of $\mathfrak{g}^0$ and assume that is completely reducible as $\mathfrak{g}^0$-module:

$$\mathfrak{g}^1 = V_{\mathfrak{g}^0}(\mu_1) \oplus \cdots V_{\mathfrak{g}^0}(\mu_s).$$

Then $\mathscr{V}_k(\mathfrak{g}^0)$ is conformally embedded in $V_k(\mathfrak{g})$ if and only if

$$\sum_{j=0}^{s} \frac{(\mu_i^j, \mu_i^j + 2\rho^j)_j}{2(u_j(k) + h_j^\vee)} = 1$$

for all $i > 0$, where

- $\mu_i^j$ is the restriction of $\mu_i$ on $\mathfrak{g}_j^0 \cap \mathfrak{h}^*$
- $h_j^\vee$ are the dual Coxeter numbers
- $u_j(k) = \frac{(\cdot,\cdot)_j}{(\cdot,\cdot)} k$

We apply this criterion to all regular equal rank subalgebras to obtain all conformal levels $k$ and the results are detailed from Chapter 3 to Chapter 9.

**Theorem 2** *Up to an explicit list of exceptions, the conformal levels for the pair* $(\mathfrak{g}^0, \mathfrak{g})$ *are the same of the pair* $(\mathfrak{g}_{\bar{0}}, \mathfrak{g})$.

Recall that conformal levels for the pair $(\mathfrak{g}_{\bar{0}}, \mathfrak{g})$ have been found in [9].

• **Problem (2)** Describe the structure of $V_k(\mathfrak{g})$ as $\mathscr{V}_k(\mathfrak{g}^0)$-module.

An approach to these problems comes from the *fusion rule argument*, that has been introduced by Adamović-Perše in [2] for Lie algebras and it has been generalized to the Lie superalgebras in [9]. Using the same notation already adopted, assume that $\mathfrak{g}_0^0 = \{0\}$ and $\mathfrak{g}^0$ is the fixed space of an automorphism of $\mathfrak{g}$ of finite order. Decompose

$$\mathfrak{g}^1 = \bigoplus_{\mu} V_{\mathfrak{g}^0}(\mu). \tag{1}$$

**Theorem 3 (Fusion rule argument)** *Assume that, if $\nu$ is the weight of a $\mathfrak{g}^0$-primitive vector occurring in $\mathfrak{g}^1 \otimes \mathfrak{g}^1$ then there is a $\mathfrak{g}^0$-primitive vector in $V_k(\mathfrak{g})$ of weight $\nu$ if and only if $\nu = \mu$ for some $\mu$ in (1). Then $V_k(\mathfrak{g}_0)$ is simple and*

$$V_k(\mathfrak{g}) = V_k(\mathfrak{g}^0) \oplus \bigoplus_{\mu} L_{\mathfrak{g}^0}(\mu),$$

*where $L_{\mathfrak{g}^0}(\mu)$ is the irreducible representation of $V^k(\mathfrak{g}^0)$ with top component $V_{\mathfrak{g}^0}(\mu)$.*

The numerical criterion of this method is the following:

**Corollary 1** *The hypothesis of the previous theorem hold whenever for all primitive vectors of weight $\nu$ occurring in $\mathfrak{g}^1 \otimes \mathfrak{g}^1$, one has either $\nu = \mu$ for some $\mu$ in 1 or*

$$\sum_{r=1}^{s} \frac{(\mu^r, \mu^r + 2\rho^r)}{2(u_r(k) + h_r^\vee)} \notin \mathbb{Z}_+$$

By using the fusion rules argument, we provide decompositions when $\mathfrak{g} = B(m,n)$ (for both conformal levels $k = 1$ and $k = (3-2m+2n)/2$) and $\mathfrak{g} = D(m,n)$ (only for $k = 1$).

## Structure of the thesis

The thesis in organized as follows:

- In Chapter 1, we display the setup of our work: the general property of the Basic Lie Superalgebras and we describe the process to obtain regular Lie subalgebras by means of affine diagrams. Then, we present the concept of conformal embedding and the AP criterion.

- In Chapter 2, we give the notion of regular subalgebra, the method to find them and the result analogous to the Borel-De Siebenthal theorem for the Lie Superalgebras.

- From Chapter 3 to 9, we detail the computation of all regular equal rank subalgebras of BSAs and we compute their conformal levels.

- In the last chapter, we report the fusion rule argument and we verify that it can be applied to equal rank subalgebras of $B(m,n)$ (for both conformal levels $k = 1$ and $k = (3-2m+2n)/2$) and $D(m,n)$ (only for $k = 1$).

# Chapter 1

# Basic definitions and notation

## 1.1 Basic Lie Superalgebras

Let us start with a brief reminder about basic notions of basic Lie superalgebras (BSA).

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be an associative superalgebra; the Lie bracket on the homogeneous elements is defined by the equality:

$$[a,b] = ab - (-1)^{p(a)p(b)}ba \qquad \text{for all} \quad a,b \in \mathfrak{g}_0 \cup \mathfrak{g}_1,$$

the parity $p$ being 0 for elements in $\mathfrak{g}_0$ and 1 for elements in $\mathfrak{g}_1$. With this structure, $\mathfrak{g}_0$ is a Lie algebra and $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module under the adjoint action.

Recall the a Lie superalgebra is simple if it has not trivial graded ideals. Finite dimensional Lie superalgebras have been classified by V. Kac [19].

We will be interested in the so called basic Lie superalgebras (BSA), which are defined by the following conditions:

- $\mathfrak{g}$ is simple;
- $\mathfrak{g}_0$ is a reductive Lie algebra;
- $\mathfrak{g}$ has a non-degenerate invariant even (that is $(\mathfrak{g}_0, \mathfrak{g}_1) = 0$) supersymmetric bilinear form. Supersymmetric means that it is symmetric on the even part and skew-symmetric on the odd part.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_0$. A root $\alpha$ of $\mathfrak{g}$ ($\alpha \neq 0$) is an element $\alpha \in \mathfrak{h}^*$, the dual of $\mathfrak{h}$, such that

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h,x] = \alpha(h)x, h \in \mathfrak{h}\} \neq 0.$$

A root is called *even* (resp. *odd*) if $\mathfrak{g}_0 \cap \mathfrak{g}_\alpha \neq 0$ (resp. $\mathfrak{g}_1 \cap \mathfrak{g}_\alpha \neq 0$); we will denote by $\Delta_0$ (resp. $\Delta_1$) the set of even (resp. odd) roots.

Let $E$ be the real vector space spanned by $\Delta$, a total order on $E$ is always assumed to be compatible with the real vector space structure; that is, $v > w$ and $v' > w'$ imply that $v + v' > w + w'$, $-v < -w$ and $cv > cw$ for $c \in \mathbb{R}$ and $c > 0$.

A positive system $\Delta^+$ is a subset of $D$ consisting precisely of all those roots $\alpha \in \Delta$ satisfying $\alpha > 0$ for some total ordering of $E$. Given a positive system $\Delta^+$, we define the *fundamental system* $\Pi \subset D^+$ to be the set of $\alpha \in \Delta^+$ which cannot be written as a sum of two roots in $\Delta^+$. We refer to elements in $D^+$ as *positive roots* and elements in $\Pi$ as *simple roots*. Similarly, we denote by $\Delta^-$ the corresponding set of negative roots.

The classification of BSA is part of the classification performed in [19]. In the following we give an explicit presentation of the BSAs and of their root systems.

## The Lie superalgebra $A(m,n)$ or $\mathfrak{sl}(m+1,n+1)$

These superalgebras are defined by

$$\mathfrak{sl}(m+1,n+1) = \left\{ x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \mathrm{str}(x) = \mathrm{tr}(a) - \mathrm{tr}(b) = 0 \right\},$$

where $a, b, c$ and $d$ are $(m+1)\times(m+1)$, $(m+1)\times(n+1)$, $(n+1)\times(m+1)$ and $(n+1)\times(n+1)$ matrices, respectively. The even elements are of the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ and the odd elements of the form $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$. The Cartan subalgebra $H$ is the subspace of diagonal matrices. For a diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_{m+n+2})$ we define

$$\begin{aligned} \varepsilon_i(D) &= d_i \qquad (i = 1, \cdots m+1) \\ \delta_j(D) &= d_{m+1+j} \qquad (j = 1, \cdots n+1) \end{aligned}$$

Then the even and the odd roots are

$$\begin{aligned} \Delta_0 &= \{\varepsilon_i - \varepsilon_j (i,j = 1, \cdots, m+1; i \neq j), \delta_i - \delta_j (i,j = 1, \cdots, n+1; i \neq j)\} \\ \Delta_1 &= \{\pm(\varepsilon_i - \delta_j)(i = 1, \cdots, m+1; j = 1, \cdots, n+1)\} \end{aligned}$$

This superalgebra is simple if $m \neq n$. If they are equal, we can consider the simple quotient

$$\mathfrak{psl}(m+1) = \mathfrak{sl}(m+1, m+1)/\mathbb{C}I$$

## The Orthosymplectic superalgebra $B(m,n)$ or $\mathfrak{osp}(2m+1,2n)$

$B(m,n)$ is a subalgebra of $\mathfrak{sl}(2m+1,2n)$ consisting on those $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for which the even and odd components $x_\zeta (\zeta = 0,1)$ satisfy

$$x_\zeta^T B + (-1)^\zeta B x_\zeta = 0,$$

where $x^T = \begin{pmatrix} a^t & -c^t \\ b^t & d^t \end{pmatrix}$ is the supertranspose of $x$ and

$$B = \left(\begin{array}{ccc|cc} 0 & I_m & 0 & & \\ I_m & 0 & 0 & & \\ 0 & 0 & 1 & & \\ \hline & & & 0 & I_n \\ & & & -I_n & 0 \end{array}\right)$$

The Cartan subalgebra consists of diagonal matrices $D$ and we put

$$\begin{aligned} \varepsilon_i(D) &= d_i \qquad (i = 1, \ldots, m) \\ \delta_j(D) &= d_{2m+1+j} \qquad (j = 1 \cdots, n) \end{aligned}$$

Then the even and the odd roots are

$$\begin{aligned} &\Delta_0 = \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i (i, j = 1, \cdots, m; i \neq j), \pm\delta_i \pm \delta_j, \pm 2\delta_i (i, j = 1, \cdots, n; i \neq j)\} \\ &\Delta_1 = \{\pm\varepsilon_i \pm \delta_j, \pm\delta_j (i = 1, \cdots, m; j = 1, \cdots, n)\} \end{aligned}$$

## The Orthosymplectic superalgebras $D(m,n)$ and $C(n)$

$D(m,n) = \mathfrak{osp}(2m,2n)$ is a subalgebra of $\mathfrak{sl}(2m,2n)$ consisting on those $x$ for which

$$x_\zeta^T B + (-1)^\zeta B x_\zeta = 0,$$

where

$$B = \left(\begin{array}{cc|cc} 0 & I_m & & \\ I_m & 0 & & \\ \hline & & 0 & I_n \\ & & -I_n & 0 \end{array}\right)$$

As usual, the Cartan subalgebra consists of diagonal matrices $D$, we put

$$\begin{aligned} \varepsilon_i(D) &= d_i \qquad (i = 1, \ldots, m) \\ \delta_j(D) &= d_{2m+j} \qquad (j = 1 \cdots, n) \end{aligned}$$

Then the even and the odd roots are given by

$$\begin{aligned} &\Delta_0 = \{\pm\varepsilon_i \pm \varepsilon_j (i, j = 1, \cdots, m; i \neq j), \pm\delta_i \pm \delta_j, \pm 2\delta_i (i, j = 1, \cdots, n; i \neq j)\} \\ &\Delta_1 = \{\pm\varepsilon_i \pm \delta_j (i = 1, \cdots, m; j = 1, \cdots, n)\} \end{aligned}$$

The Lie superalgebras $C(n+1) = \mathfrak{osp}(2,2n)$ form a special case. The roots of $C(n+1)$ are given by

$$\begin{aligned} &\Delta_0 = \{\pm\delta_i \pm \delta_j, \pm 2\delta_i (i, j = 1, \cdots, n; i \neq j)\} \\ &\Delta_1 = \{\pm\varepsilon \pm \delta_j (j = 1, \cdots, n)\} \end{aligned}$$

### The exceptional superalgebra $D(2,1;\alpha)$, $\alpha \in \mathbb{C}$, $(\alpha \neq 0, -1)$

The even part of this superalgebra is $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ and the odd part is the tensor product of three 2-dimensional representations, one for each $\mathfrak{sl}(2)$. In terms of $2\varepsilon_1, 2\varepsilon_2, 2\varepsilon_3$ (the roots of the three $\mathfrak{sl}(2)$), the even and the odd roots of $D(2,1;\alpha)$ are given by

$$\Delta_0 = \{\pm 2\varepsilon_i (i = 1, \cdots, 3\}$$
$$\Delta_1 = \{\pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3\}$$

### The exceptional superalgebra $G(3)$

The even part of $G(3)$ is $\mathfrak{sl}(2) \oplus G_2$ and the odd part is the tensor product of the 2-dimensional representation of $\mathfrak{sl}(2)$ and the fundamental 7-dimensional $G_2$-module. In terms of $2\delta$ (the root of $\mathfrak{sl}(2)$) and $\varepsilon_1, \varepsilon_2, \varepsilon_3$ with $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$ the roots are

$$\Delta_0 = \{\pm 2\delta, \varepsilon_i - \varepsilon_j, \pm \varepsilon_i\}$$
$$\Delta_1 = \{\pm \delta, \pm \varepsilon_i \pm \delta)\}$$

### The exceptional superalgebra $F(4)$

The even part of $F(4)$ is $\mathfrak{sl}(2) \oplus \mathfrak{so}(7)$ and the odd part is the tensor product of the 2-dimensional representation of $\mathfrak{sl}(2)$ and the simple $\mathfrak{so}(7)$-spin module. In terms of $\delta$ (the root of $\mathfrak{sl}(2)$) and $\varepsilon_1, \varepsilon_2, \varepsilon_3$ the roots are

$$\Delta_0 = \{\pm \delta, \pm \varepsilon_i \pm \varepsilon_j, \pm \varepsilon_i\}$$
$$\Delta_1 = \{1/2(\pm \delta \pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3)\}$$

## 1.2 Normalized form

Every BSA has a non-degenerate bilinear supersymmetric invariant form. Throughout our discussion we will use the normalizations of the form given in the following table (these normalizations are taken form [24], Table 6.1):

| Lie superalgebra | Normalization |
|---|---|
| $A(m,n), \mathfrak{psl}(n), B(m,n)$ $C(n+1), D(m,n)$ | $(\varepsilon_i, \varepsilon_j) = \delta_{ij} \quad (\delta_i, \delta_j) = -\delta_{ij} \quad (\varepsilon_i, \delta_j) = 0$ |
| $D(2,1;\alpha)$ | $(\varepsilon_1, \varepsilon_1) = -(1+\alpha)/2, \quad (\varepsilon_2, \varepsilon_2) = 1/2,$ $(\varepsilon_3, \varepsilon_3) = \alpha/2, \quad (\varepsilon_i, \varepsilon_j) = 0 (i \neq j)$ |
| $G(3)$ | $(\varepsilon_i, \varepsilon_i) = 2/3, \quad (\varepsilon_i, \varepsilon_j) = -2/3, \quad (i \neq j)$ $(\delta, \delta) = -2/3, \quad (\varepsilon_i, \delta) = 0$ |
| $F(4)$ | $(\varepsilon_i, \varepsilon_j) = \delta_{ij}, \quad (\delta, \delta) = -3, \quad (\delta, \varepsilon_i) = 0$ |

We also give a normalization for a Lie algebra: we require that $(\theta,\theta)=2$, where $\theta$ is the highest root.

## 1.3 Dynkin diagrams

For every choice of simple root system $\Pi\subset\Delta$ we can draw a diagram, called *Dynkin Diagram*, using these rules (cf [16]):

- to each simple even root we associate an empty circle $\bigcirc$, to each simple odd isotropic root we associate the symbol $\bigotimes$ and to each simple odd non isotropic root we associate a black circle, $\bullet$
- the $i$ and the $j$ circles are joined by $\eta_{ij}$ lines with

$$\eta_{ij}=\frac{2|a_{ij}|}{\min\left(|a_{ii}|,|a_{jj}|\right)} \qquad \text{if } a_{ii},a_{jj}\neq 0 \tag{1.1}$$

$$\eta_{ij}=\frac{2|a_{ij}|}{\min_{a_{kk}\neq 0}\left(|a_{kk}|\right)} \qquad \text{if } a_{ii}=0 \text{ or } a_{jj}=0 \tag{1.2}$$

- we add an arrow on the lines connecting the $i$-th and $j$-th dot when $\eta_{ij}>1$ pointing from $i$ to $j$ if $a_{ii}a_{jj}\neq 0$ and $|a_{ii}|>|a_{jj}|$ or if $a_{ii}=0,a_{jj}\neq 0,|a_{jj}|<2$ and pointing from $j$ to $i$ if $a_{ii}=0,a_{jj}\neq 0,|a_{jj}|>2$

## 1.4 Real and Odd reflections

One of the main difference between semisimple Lie algebras and BSA stands in the fact that in Lie algebras simple roots systems are conjugated under the action of Weyl Group and they have an unique Dynkin diagram, up to isomorphism. Instead, in Lie superalgebras, there are different conjugacy classes of simple systems, each one with its own Dynkin Diagram.

In order to move from one diagram to another *odd reflections* are used: given a simple system $\Pi$ and an isotropic odd root $\alpha\in\Pi$ the reflection $r_\alpha$ is a bijection between simple sets of roots and sends $\Pi$ to

$$r_\alpha(\Pi)=\Pi^\alpha=\{\beta\in\Pi\mid(\beta,\alpha)=0,\beta\neq\alpha\}\cup\{\beta+\alpha\mid(\beta,\alpha)\neq 0\}\cup\{-\alpha\}$$

The positive system of $\Pi^\alpha$ is

$$\Delta^\alpha=\{-\alpha\}\cup\Delta\setminus\{\alpha\}.$$

As in the case of Lie algebras, for a non isotropic root $\alpha$ we can associate the (real) reflection

$$r_\alpha(x) = x - 2\frac{(x,\alpha)}{(\alpha,\alpha)}\alpha.$$

For an odd non isotropic root $\alpha$, we have $2\alpha \in \Delta^+$ and we define the reflection $r_\alpha$ as the real reflection $r_{2\alpha}$ associated to $2\alpha$.

**Proposition 1.4.1** *Given two fundamental systems $\Pi$ and $\Pi'$ there exists a sequence consisting of real and odd reflections $r_1 \ldots, r_k$ such that, up to the action of the Weyl Group of $\mathfrak{g}$,*

$$r_1 \cdots r_k(\Pi) = \Pi'.$$

**Remark 1.4.2** *Real reflections do not change the Dynkin Diagram, then if we want to obtain all possible Dynkin Diagrams we can use only odd reflections.*

# 1.5 The Kac-Moody affinization

The *Kac-Moody affinization* of a superalgebra $\mathfrak{g}$ is the Lie superalgebra

$$\hat{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}K,$$

where $\mathfrak{g}[t,t^{-1}]$ stands for the algebra of Laurent polynomials. The Lie bracket is defined on the generators of $\hat{\mathfrak{g}}$ by

$$[xt^m, xt^n] = [x,y]t^{m+n} + m(x,y)\delta_{m-n}K, \qquad [K,\mathfrak{g}] = 0$$

Let $\hat{\Pi}$ be obtained from a simple root system $\Pi$ of $\mathfrak{g}$ by adding the corresponding lowest root.

**Definition 1.5.1** *We call an affine simple root system a set of root obtained from $\hat{\Pi}$ by applying even and odd reflections.*

Given an affine simple root system we can draw its (affine) Dynkin Diagram using the same conventions as in section 1.3.

# 1.6 Conformal superalgebras

In this section, we provide the notions of conformal embeddings between vertex operator superalgebras and we describe the AP criterion from [9] to find them in a numerical way.

We use the definition of *Vertex operator algebra* given in [20] and we recall here the notion of *conformal algebra* (cf [21]).

**Definition 1.6.1** *A* Lie conformal superalgebra *is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}[T]$-module $R = R_0 \oplus R_1$ endowed with a parity-preserving $\mathbb{C}$-bilinear map $R \otimes R \to \mathbb{C}[\lambda] \otimes R$ denoted by $[a_\lambda b]$, such that the following axioms holds:*

| | |
|---|---|
| *sesquilinearity* | $[Ta_\lambda b] = -\lambda [a_\lambda b] \qquad T[a_\lambda b] = [Ta_\lambda b] + [a_\lambda Tb]$ |
| *skewsymmetry* | $[b_\lambda a] = -p(a,b)[a_{-\lambda-T} b]$ |
| *Jacobi identity* | $[a_\lambda [b_\mu c]] + p(a,b)[b_\mu [a_\lambda c]] = [[a_\lambda c]_{\lambda+\mu} c]$ |

**Remark 1.6.2** *A vertex algebra can be endowed with the structure of a Lie conformal superalgebra by introducing the $\lambda$-product via the formula*

$$[a_\lambda b] = \sum_{n\geq 0} \frac{\lambda^n}{n!}(a_{(n)} b).$$

Given a Lie conformal superalgebra $R$ one can construct its *universal enveloping vertex algebra* $V(R)$. This vertex algebra is characterized by the following properties:

- There is an embedding $R \to V(R)$ of Lie conformal algebras
- Given an ordered basis $a_i$ of $R$, the monomials $:a_{i_1} a_{i_2} \cdots a_{i_n}:$ with $i_j \leq i_{j+1}$ and $i_j < i_{j+1}$ if $p(a_{i_j}) = 1$ form a basis of $V(R)$

The normal order product of more than two elements is defined from right to left.

## The Affine Vertex Algebra

Given a BSA $\mathfrak{g}$ with a non-degenerate invariant bilinear form $(\cdot,\cdot)$, one defines the *Current Lie conformal algebra* $Cur(\mathfrak{g})$ as

$$Cur(\mathfrak{g}) = \mathbb{C}[T] \otimes \mathfrak{g} \oplus \mathbb{C}K,$$

with $TK = 0$ and the $\lambda$-bracket defined for $a, b \in \mathfrak{g}$ by

$$[a_\lambda b] = [a,b] + \lambda(a,b)K \qquad [\mathfrak{g}_\lambda K] = [K_\lambda K] = 0$$

Let $V(\mathfrak{g})$ be its universal enveloping vertex algebra. The vertex algebra

$$V^k(\mathfrak{g}) := V(\mathfrak{g})/(K-k)V(\mathfrak{g})$$

is called the *level k universal affine vertex algebra*. Its unique simple quotient is denoted by $V_k(\mathfrak{g})$ and called *Simple affine vertex algebra*.

## 1.7 Conformal Vector and Conformal Embeddings

Let $\mathfrak{g}$ be a BSA and let $k \neq -h^\vee$, where $h^\vee$ is the *dual Coxeter number*, that is the half of the eigenvalue of the action of Casimir element on $\mathfrak{g}$. Let $\{x_i\}$ be a basis of $\mathfrak{g}$ and let $\{y_i\}$ its dual basis, that is $(x_i, y_j) = \delta_{ij}$. We can define the *conformal vector*, inside the algebras $V^k(\mathfrak{g})$ and $V_k(\mathfrak{g})$, as

$$\omega_{\mathfrak{g}} = \frac{1}{2(h^\vee + k)} \sum_{i=0}^{\dim \mathfrak{g}} :y_i x_i:$$

Let $\mathfrak{g}^0$ be an equal rank subalgebra contained in $\mathfrak{g}$ such that the restriction of $(\cdot,\cdot)$ is nondegenerate. We further assume that $\mathfrak{g}^0$ decomposes as $\mathfrak{g}^0 = \mathfrak{g}^0_0 \oplus \cdots \oplus \mathfrak{g}^0_s$ with $\mathfrak{g}^0_0$ even abelian and $\mathfrak{g}^0_i$ BSA ideals for $i > 0$.

Let $(\cdot,\cdot)_j$ be normalized invariant form on $\mathfrak{g}^0_j$ and set $\{x^j_i\}, \{y^j_i\}$ to be dual bases of $\mathfrak{g}^0_j$ with respect to $(\cdot,\cdot)_j$. Let $h^\vee_j$ be the dual Coxeter number of $\mathfrak{g}^0_j$. For $\mathfrak{g}^0_0$, let $\{x^0_i\}, \{y^0_i\}$ be dual bases of $\mathfrak{g}^0_0$ with respect to $(\cdot|\cdot)_0 = (\cdot|\cdot)_{|\mathfrak{g}^0_0 \times \mathfrak{g}^0_0}$ and set $h^\vee_0 = 0$.

If $\mathbf{k} = (k_0, \dots, k_s)$ is a multi-index of levels we set

$$V^{\mathbf{k}}(\mathfrak{g}^0) = V^{k_0}(\mathfrak{g}^0_0) \otimes \cdots \otimes V^{k_s}(\mathfrak{g}^0_s),$$

and, assuming $k_j + h^\vee_j \neq 0$ for all $j$, we let

$$V_{\mathbf{k}}(\mathfrak{g}^0) = V_{k_0}(\mathfrak{g}^0_0) \otimes \cdots \otimes V_{k_s}(\mathfrak{g}^0_s).$$

We consider also $V^{\mathbf{k}}(\mathfrak{g}^0)$ and $V_{\mathbf{k}}(\mathfrak{g}^0)$, as conformal vertex algebras with conformal vector $\omega_{\mathfrak{g}^0}$ :

$$\omega_{\mathfrak{g}^0} = \sum_{j=0}^{s} \frac{1}{2(k_j + h^\vee_j)} \sum_{i=1}^{\dim \mathfrak{g}^0_j} :y^j_i x^j_i: .$$

We let $\mathscr{V}_k(\mathfrak{g}^0)$ denote the vertex subalgebra of $V_k(\mathfrak{g})$ generated by $x(-1)\mathbf{1}$, $x \in \mathfrak{g}^0$. Note that, given $k \in \mathbb{C}$, there is a uniquely determined multi-index $\mathbf{u}(k)$ such that $\mathscr{V}_k(\mathfrak{g}_{\bar{0}})$ is a quotient of $V^{\mathbf{u}(k)}(\mathfrak{g}^0)$ hence, if $u_j(k) + h^\vee_j \neq 0$ for each $j$, $\omega_{\mathfrak{g}^0}$ is a conformal vector in $\mathscr{V}_k(\mathfrak{g}^0)$. We will say that $\mathscr{V}_k(\mathfrak{g}^0)$ is conformally embedded in $V_k(\mathfrak{g})$ if $\omega_{\mathfrak{g}} = \omega_{\mathfrak{g}^0}$.

Our aim is the study of conformal embeddings of $\mathscr{V}_k(\mathfrak{g}^0)$ in $V_k(\mathfrak{g})$. The basis of our investigation is the following result [9, Theorem 2.1]. Let $\mathfrak{g}^1$ be the orthocomplement of $\mathfrak{g}^0$ in $\mathfrak{g}$.

**Theorem 1.7.1** *In the above setting, $\mathscr{V}_k(\mathfrak{g}^0)$ is conformally embedded in $V_k(\mathfrak{g})$ if and only for any $x \in \mathfrak{g}^1$ we have*

$$(\omega_{\mathfrak{g}^0})_0 x(-1)\mathbf{1} = x(-1)\mathbf{1}. \tag{1.3}$$

Assume that $\mathfrak{g}^1$ is completely reducible as a $\mathfrak{g}^0$-module, and let

$$\mathfrak{g}^1 = \bigoplus_{i=1}^{t} V_{\mathfrak{g}^0}(\mu_i)$$

be its decomposition. Set $\mu_0 = 0$.

**Corollary 1.7.2** *$\mathscr{V}_k(\mathfrak{g}^0)$ is conformally embedded in $V_k(\mathfrak{g})$ if and only if*

$$\sum_{j=0}^{s} \frac{(\mu_i^j, \mu_i^j + 2\rho^j)_j}{2(u_j(k) + h_j^\vee)} = 1 \tag{1.4}$$

*for all $i > 0$.*

**Definition 1.7.3** *We say that $k \in \mathbb{C}$ is a* conformal level *for the embedding $\mathfrak{g}^0 \in \mathfrak{g}$ if*

- $h^\vee + k \neq 0$
- $u_j k + k \neq 0$ *for all $j$*
- *$\mathscr{V}_k(\mathfrak{g}^0)$ is conformally embedded in $V_k(\mathfrak{g})$*

We will refer to the condition of Corollary 1.4 as AP criterion. From chapter 3 to chapter 9, for each BSA $\mathfrak{g}$, we detect regular equal rank subalgebras using the method described in 2.1 and we compute the conformal levels by using the AP criterion.

## 1.8 Decomposition in the vertex algebras

Another two interesting problems concerning the embedding $\mathscr{V}_k(\mathfrak{g}^0) \subset V_k(\mathfrak{g})$ are the following:

- *Simplicity problem*: determine the structure of $\mathscr{V}_k(\mathfrak{g}^0)$, in particular determine when it is simple.
- *Decomposition problem*: describe the structure of $V_k(\mathfrak{g})$ as $\mathscr{V}_k(\mathfrak{g}^0)$-module.

Both problems can be dealt with a statement called *fusion rule argument*. It was applied firstly in the Lie case in [2] and then generalized to the superlie case in [9]. We briefly describe here this result.

Assume $\mathfrak{g}_0^0 = 0$ and that $\mathfrak{g}^0$ is the set of fixed points of an automorphism $\sigma$ of order $s$ and let

$$\mathfrak{g} = \bigoplus_{i=0}^{s-1} \mathfrak{g}^{(i)}$$

be the corresponding eigenvalue decomposition. Note that $\mathfrak{g}^{(0)} = \mathfrak{g}^0$ and $\mathfrak{g}^1 = \oplus_{i=1}^{s-1} \mathfrak{g}^{(i)}$. Since $\mathfrak{g}^1$ is assumed to be completely reducible as $\mathfrak{g}^0$-module, we have

$$\mathfrak{g}^1 = \oplus_\mu V_{\mathfrak{g}^0}(\mu) \tag{1.5}$$

**Theorem 1.8.1** *Assume that, if $\nu$ is the weight of a $\mathfrak{g}^0$-primitive vector occurring in $\mathfrak{g}^1 \otimes \mathfrak{g}^1$ then there is a $\mathfrak{g}^0$-primitive vector in $V_k(\mathfrak{g})$ of weight $\nu$ if and only if $\nu = \mu$ for some $\mu$ in (1.5). Then $V_k(\mathfrak{g}_0)$ is simple and*

$$V_k(\mathfrak{g}) = V_k(\mathfrak{g}^0) \oplus \bigoplus_\mu L_{\mathfrak{g}^0}(\mu)$$

Here $L_{\mathfrak{g}^0}(\mu)$ is the irreducible representation of $V^k(\mathfrak{g}^0)$ with $V_{\mathfrak{g}^0}(\mu)$ as top component ($V_{\mathfrak{g}^0}(\mu)$ the irreducible representation of $\mathfrak{g}^0$ of maximal weight $\mu$).

The numerical criterion to verify the condition of the previous Theorem is described in the following remark:

**Remark 1.8.2** *The hypothesis of the Theorem 1.8.1 hold whenever for all primitive vectors of weight $\nu$ occurring in $\mathfrak{g}^1 \otimes \mathfrak{g}^1$, one has either $\nu = \mu$ for some $\mu$ in 1.5 or*

$$\sum_{r=1}^{s} \frac{(\mu^r, \mu^r + 2\rho^r)}{2(u_r(k) + h_r^\vee)} \notin \mathbb{Z}_+$$

In Chapter 10 we will apply this argument to find the decomposition of maximal regular (cf def 2.1.1) equal rank subalgebras of $B(m,n)$ (for both conformal levels $k = 1$ and $k = (3 - 2m + 2n)/2$) and $\mathfrak{g} = D(m,n)$ (only for $k = 1$).

# Chapter 2

# Regular subalgebras

## 2.1 Semisimple regular Lie subalgebras

In this section, we describe the subalgebras we are interested in and we detail a way to find them.

**Definition 2.1.1** *Let $\mathfrak{g}$ be a BSA. A subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ is* regular *if there exists a basis of $\mathfrak{g}'$ consisting of elements of some Cartan subalgebra $\mathfrak{h}$ and root vectors of $\mathfrak{g}$ relative to $\mathfrak{h}$.*

Then $\mathfrak{g}'$ can be written in the following form:

$$\mathfrak{g}' = \mathfrak{h}' \oplus \bigoplus_{\alpha \in \Delta'} \mathfrak{g}_\alpha,$$

with $\mathfrak{h}' \subset \mathfrak{h}$ and $\Delta' \subset \Delta$.

**Definition 2.1.2** *Let $\mathfrak{g}$ be a BSA. A subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ is* semisimple *if it is the direct sum of components which are either a BSA or simple Lie algebras.*

For the whole discussion by "regular" we will mean "semisimple regular". The method for finding the (semisimple) regular subalgebras of a BSA is completely analogous to the usual one for Lie algebras by means of extended Dynkin diagrams [15]. The main difference is that for the superalgebras case one has to consider all the Dynkin diagrams associated to the nonequivalent simple root systems.

A first classification of the regular semi-simple subalgebras of basic BSA and the proof of the method has been given in [31]. Therefore we will limit ourselves to summarize the used techniques.

Given for a BSA a simple root system and the associated Dynkin diagram, we draw the affine Dynkin diagram by adding to it a dot corresponding to the lowest root. Now, deleting arbitrarily one or more than one dot of this extended diagram, will yield one Dynkin diagram or a set of disjoint Dynkin diagrams corresponding to a regular subalgebra of $\mathfrak{g}$. Indeed, taking away one or more roots, one is left with a set of linearly independent roots which constitute the simple root system of a regular subalgebra of $\mathfrak{g}$.

Then repeating the same operation on the obtained Dynkin diagrams, that is adding a dot associated to the lowest root of a simple part and cancelling one arbitrary dot (or two dots in the case $A(m,n)$) as many times as necessary, we will obtain in this way all the Dynkin diagrams associated with regular semisimple BSA. One can easily notice that in order to get the maximal regular semisimple BSA of the same rank as $\mathfrak{g}$, only the first step has to be achieved (which does not mean that all the so-obtained subalgebras are maximal).

The other possible maximal regular subalgebras of $\mathfrak{g}$ if they exist, will be obtained by deleting one dot in the (non-extended) Dynkin diagram of $\mathfrak{g}$ and will be therefore of rank $r-1$.

In chapters 3 to 9, we will use this method to find all possible equal rank regular subalgebras of BSA.

## 2.2 Maximal semisimple subalgebras of Lie algebras

In this section we briefly recall some fact concerning semisimple subalgebras in Lie simple algebras.

Let $\mathfrak{g}$ be a semisimple Lie algebra: the method of finding semisimple subalgebras is by means of removing dots to the affine Dynkin diagram, as described above for regular subsuperalgebras (cf. [28], Chapter 12).

There is also a characterization of subalgebras that are maximal among semisimple subalgebras in terms of the coefficients of simple roots with respect to the maximal root. It comes from the Borel- De Siebenthal theorem, in the version for Lie algebras (again in [28], Chapter 12)

**Theorem 2.2.1 (Borel-de Siebenthal)** *Let $\mathfrak{g}$ be a simple Lie algebra and $\Delta$ be an irreducible root system. Let $\{\alpha_1,\cdots,\alpha_l\}\subset\Delta$ be a fundamental system. Denote with $\alpha_0$ be the highest root of $\Delta$ with respect to $\{\alpha_1,\cdots,\alpha_l\}$ and expand*

$$\alpha_0=\sum_{i=1}^{l}c_i\alpha_i.$$

*Then the subroot systems of $\Delta$ that generate a maximal semisimple subalgebra are those with fundamental systems*

*i)* $\{\alpha_1,\cdots,\alpha_l\}\setminus\{\alpha_i\}$, *where* $c_i=1$

*ii)* $\{-\alpha_0,\cdots,\alpha_l\}\setminus\{\alpha_i\}$, *where* $c_i=p$ *is a prime number*

## 2.3 Maximal regular subsuperalgebras

A natural question that can arise is whether a result similar to theorem 2.2.1 can be stated When $\mathfrak{g}$ is a BSA.

We show a partial positive answer when dealing with regular subalgebras, as stated in the following result:

**Theorem 2.3.1** *Let $\mathfrak{g}$ be a BSA and $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. A Lie subsuperalgebra $\mathfrak{g}' \subset \mathfrak{g}$ is an equal rank maximal regular subalgebra if and only if there exists an affine simple root system (cf 1.5.1) $\hat{\Pi} = \{\alpha_0, \dots, \alpha_l\}$ in $\mathfrak{h}^*$ such that if $\sum_{i=0}^{l} c_i \alpha_i = 0$ is the linear dependence with coprime non-negative integer coefficients, then*

$$\mathfrak{g}' = \mathfrak{h}' \oplus \bigoplus_{\alpha \in \Delta'} \mathfrak{g}_\alpha,$$

*where $\Delta'$ is the root system generated by $\Pi = \hat{\Pi} \setminus \{\alpha_i\}$ with $c_i$ prime and $\mathfrak{h}' \subset \mathfrak{h}$ is generated by $h_\alpha$, $\alpha \in \Pi$.*

The proof of the Theorem 2.2.1 relies on the geometrical description of root systems that cannot be extended to Lie superalgebras; unfortunately we cannot offer a conceptual proof of this result.

However, we see that the statement holds by direct verification: from chapters 3 to 9 we will detect all possible equal rank regular subalgebras and we will remark that the criterion holds.

# Chapter 3

# Embeddings of Regular Subalgebras in $A(m,n)$

## 3.1 Affine Dynkin Diagrams of $A(m,n)$

Let $\mathfrak{g} = A(m,n) = \mathfrak{sl}(m+1,n+1)$, using the notation of Chapter 1 the roots are

$$\Delta_0 = \{\pm\varepsilon_i \pm \varepsilon_j, (1 \leq i \neq j \leq m+1) \pm \delta_i \pm \delta_j (1 \leq i \neq j \leq n+1)\}$$
$$\Delta_1 = \{\pm\varepsilon_i \pm \delta_j (1 \leq i \leq m+1, 1 \leq j \leq n+1)\},$$

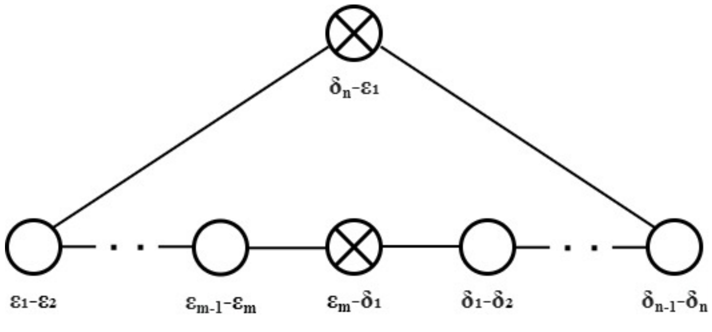
with the relation $\sum \varepsilon_i - \sum \delta_j = 0$ The invariant bilinear form on $\mathfrak{h}^*$ is:

$$(\varepsilon_i, \varepsilon_j)_{A(m,n)} = \delta_{ij}$$
$$(\delta_i, \delta_j)_{A(m,n)} = -\delta_{ij}$$
$$(\varepsilon, \delta_i)_{A(m,n)} = 0$$

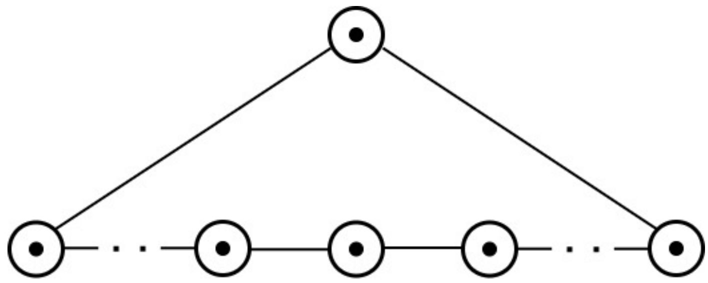
Taking as simple system of roots

$$\Pi = \{\varepsilon_1 - \varepsilon_2, \cdots, \varepsilon_m - \varepsilon_{m-1}, \varepsilon_m - \delta_1, \delta_1 - \delta_2, \cdots, \delta_n - \delta_{n+1}\},$$

the minimum root is $\delta_{n+1} - \varepsilon_1$. The corresponding affine Dynkin Diagram is



Applying repeatedly odd reflections on the diagram we obtain, up to permutation of $\varepsilon_i$ and $\delta_i$, all the possible affine diagrams of $A(m,n)$: they are in bijection with the sequences of $m$ $\varepsilon$ and $n$ $\delta$. They are of the form



(here $\odot$ can be either $\bigcirc$ or $\otimes$)

## 3.2 Regular subalgebras

Deleting one root from the previous diagram we obtain the whole algebra $A(m,n)$. Then there are no maximal equal rank regular subalgebras.

We also notice that for every choice of a simple root system $\{\alpha_i\}_{i=1\ldots,m+n-1}$of $A(m,n)$, the maximal root is $\sum \alpha_i$, then the statement of the Theorem 2.3.1 is verified.

In order to find all regular subalgebras we must delete a root from the diagrams of $A(m,n)$. In this way the subalgebras obtained are all of the type

$$\mathfrak{sl}(p,q) \oplus \mathfrak{sl}(m+1-p,n+1-q),$$

and they are maximal among the regular subalgebras.

## 3.3 Equal rank subalgebras

We can obtain equal rank subalgebras if we take a maximal regular subalgebra and add the 1-dimensional centralizer. The result is a subalgebra of type

$$\mathfrak{g}^0 = \mathfrak{sl}(p,q) \oplus \mathfrak{sl}(m+1-p,n+1-q) \oplus \mathbb{C}\omega$$

We define $\overline{\varepsilon}$ as the dual of $\omega$.

We split the analysis of the conformal embedding of these subalgebras into three cases:

- $p \neq 0, m+1$ and $q \neq 0, n+1$
- $\mathfrak{g}^0 = \mathfrak{sl}(q) \oplus \mathfrak{sl}(m+1,n+1-q) \oplus \mathbb{C}\omega$.
- $\mathfrak{g}^0 = \mathfrak{g}_0 = \mathfrak{sl}(m+1) \oplus \mathfrak{sl}(n+1) \oplus \mathbb{C}\omega$

## Case $p \neq 0, m+1$ and $q \neq 0, n+1$

We can suppose without loss of generality, that the root systems of the subalgebras $\mathfrak{sl}(p,q)$ and $\mathfrak{sl}(m+1-p, n+1-q)$ are given by

$$\begin{aligned}
\Delta_0^{\mathfrak{sl}(p,q)} &= \{\pm(\varepsilon_i - \varepsilon_j)(1 \le i \neq j \le p), \pm(\delta_i - \delta_j), (1 \le i \neq j \le q)\} \\
\Delta_1^{\mathfrak{sl}(p,q)} &= \{\pm(\varepsilon_i - \delta_j), (1 \le i \le p, 1 \le j \le q)\} \\
\Delta_0^{\mathfrak{sl}(m+1-p,n+1-q)} &= \{\pm(\varepsilon_i - \varepsilon_j), (p+1 \le i \neq j \le m), \pm(\delta_i - \delta_j), (q+1 \le i \neq j \le n)\} \\
\Delta_1^{\mathfrak{sl}(m+1-p,n+1-q)} &= \{\pm(\varepsilon_i - \delta_j), (p+1 \le i \le m, q+1 \le j \le n)\}
\end{aligned}$$

As simple root systems we take

$$\begin{aligned}
\Pi^{\mathfrak{sl}(p,q)} &= \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \cdots, \varepsilon_{p-1} - \varepsilon_p, \varepsilon_p - \delta_1, \delta_1 - \delta_2, \cdots, \delta_{q-1} - \delta_q\} \\
\Pi^{\mathfrak{sl}(m+1-p,n+1-q)} &= \{\varepsilon_{p+1} - \varepsilon_{p+2}, \varepsilon_{p+2} - \varepsilon_{p+3}, \cdots, \varepsilon_m - \varepsilon_{m+1}, \varepsilon_{m+1} - \delta_{q+1}, \cdots, \delta_n - \delta_{n+1}\}
\end{aligned}$$

The centralizer of $\mathfrak{g}^0$ is a one dimensional space $\mathbb{C}\omega$ and we choose

$$\omega = \frac{1}{n-m} \left( \begin{array}{cc|cc}
(n-m+p-q)I_p & 0 & 0 & 0 \\
0 & (p-q)I_{m+1-p} & 0 & 0 \\
\hline
0 & 0 & (n-m+p-q)I_q & \\
0 & 0 & 0 & (p-q)I_{n+1-q}
\end{array} \right)$$

The orthogonal complement $\mathfrak{g}^1$ of is generated by $\mathfrak{g}_\alpha$, where

$$\begin{aligned}
\alpha = &\pm(\varepsilon_i - \varepsilon_j) \quad (1 \le i \le p, p+1 \le j \le m+1), \\
&\pm(\delta_i - \delta_j) \quad (1 \le i \le q, q+1 \le j \le n+1), \\
&\pm(\varepsilon_i - \delta_j) \quad (1 \le i \le p, q+1 \le j \le n+1 \text{ or } p+1 \le i \le m+1, 1 \le j \le q),
\end{aligned}$$

The maximal weights of $\mathfrak{g}^1$ wrt $\mathfrak{g}^0$ are $\varepsilon_1-\delta_{n+1}$ and $\varepsilon_{p+1}-\delta_1$. We have to split these weights into a linear combination of roots of the subalgebra:

$$\begin{aligned}\varepsilon_1-\delta_{n+1}=&\underbrace{\varepsilon_1-\frac{1}{p-q}\left(\sum_{i=1}^{p}\varepsilon_i-\sum_{j=1}^{q}\delta_j\right)}_{\mathfrak{sl}(p,q)}\underbrace{-\delta_{n+1}+\frac{1}{m-n-p+q}\left(\sum_{i=p+1}^{m}\varepsilon_i-\sum_{j=q+1}^{n}\delta_j\right)}_{\mathfrak{sl}(m+1-p,n+1-q)}+\\&+\underbrace{\frac{1}{p-q}\left(\sum_{i=1}^{p}\varepsilon_1-\sum_{j=1}^{q}\delta_j\right)-\frac{1}{m-n-p+q}\left(\sum_{i=p+1}^{m}\varepsilon_i-\sum_{j=q+1}^{n}\delta_j\right)}_{\mathbb{C}\omega}\end{aligned}$$

$$\begin{aligned}\varepsilon_{p+1}-\delta_1=&\underbrace{-\delta_1+\frac{1}{p-q}\left(\sum_{i=1}^{p}\varepsilon_i-\sum_{j=1}^{q}\delta_j\right)}_{\mathfrak{sl}(p,q)}+\underbrace{\varepsilon_{p+1}-\frac{1}{m-n-p+q}\left(\sum_{i=p+1}^{m}\varepsilon_i-\sum_{j=q+1}^{n}\delta_j\right)}_{\mathfrak{sl}(m+1-p,n+1-q)}+\\&\underbrace{-\frac{1}{p-q}\left(\sum_{i=1}^{p}\varepsilon_1-\sum_{j=1}^{q}\delta_j\right)+\frac{1}{m-n-p+q}\left(\sum_{i=p+1}^{m}\varepsilon_i-\sum_{j=q+1}^{n}\delta_j\right)}_{\mathbb{C}\omega}\end{aligned}$$

Then

$$\begin{aligned}\mathfrak{g}^1=&V_{\mathfrak{sl}(p,q)}\left(\varepsilon_1-\frac{1}{p-q}\left(\sum_{i=1}^{p}\varepsilon_i-\sum_{j=1}^{q}\delta_j\right)\right)\otimes\\&\otimes V_{\mathfrak{sl}(m+1-p,n+1-q)}\left(-\delta_{n+1}+\frac{1}{m-n-p+q}\left(\sum_{i=p+1}^{m}\varepsilon_i-\sum_{j=q+1}^{n}\delta_j\right)\right)\otimes\\&\otimes V_{\mathbb{C}\omega}\left(\frac{1}{p-q}\left(\sum_{i=1}^{p}\varepsilon_i-\sum_{j=1}^{q}\delta_j\right)-\frac{1}{m-n-p+q}\left(\sum_{i=p+1}^{m}\varepsilon_i-\sum_{j=q+1}^{n}\delta_j\right)\right)\oplus\\&\oplus V_{\mathfrak{sl}(p,q)}\left(-\delta_1+\frac{1}{p-q}\left(\sum_{i=1}^{p}\varepsilon_i-\sum_{j=1}^{q}\delta_j\right)\right)\otimes\\&\otimes V_{\mathfrak{sl}(m+1-p,n+1-q)}\left(\varepsilon_{p+1}-\frac{1}{m-n-p+q}\left(\sum_{i=p+1}^{m}\varepsilon_i-\sum_{j=q+1}^{n}\delta_j\right)\right)\otimes\\&\otimes V_{\mathbb{C}\omega}\left(-\frac{1}{p-q}\left(\sum_{i=1}^{p}\varepsilon_i-\sum_{j=1}^{q}\delta_j\right)+\frac{1}{m-n-p+q}\left(\sum_{i=p+1}^{m}\varepsilon_i-\sum_{j=q+1}^{n}\delta_j\right)\right)\\&=V_{\mathfrak{sl}(p,q)}(\omega_1)\otimes V_{\mathfrak{sl}(m+1-p,n+1-q)}(\omega_{m+n-p-q})\otimes V_{\mathbb{C}\omega}(\overline{\varepsilon})\oplus\\&\oplus V_{\mathfrak{sl}(p,q)}(\omega_{p+q-2})\otimes V_{\mathfrak{sl}(m+1-p,n+1-q)}(\omega_1)\otimes V_{\mathbb{C}\omega}(-\overline{\varepsilon}),\end{aligned}$$

where $\omega_i$ are the fundamental weights and $\overline{\varepsilon} = \omega^*$.

## $\mathfrak{sl}(p,q)$

- $u(k) = k$
- $2\rho = \sum_{i=1}^{p}(p-q+1-2i)\varepsilon_i + \sum_{j=1}^{q}(q+p+1-2j)\delta_j$
- $\omega_1 = \varepsilon_1 - \frac{1}{p-q}\left(\sum_{i=1}^{p}\varepsilon_i - \sum_{j=1}^{q}\delta_j\right)$
- $\omega_{p+q-2} = -\delta_1 + \frac{1}{p-q}\left(\sum_{i=1}^{p}\varepsilon_i - \sum_{j=1}^{q}\delta_j\right)$
- $(2\rho+\omega_1,\omega_1) = \dfrac{(p-q+1)(p-q-1)}{p-q}$
- $(2\rho+\omega_{p+q-2},\omega_{p+q-2}) = \dfrac{(p-q+1)(p-q-1)}{p-q}$
- $h^\vee = p-q$

## $\mathfrak{sl}(m+1-p,n+1-q)$

- $u(k) = k$
- $2\rho = \sum_{i=1}^{m+1-p}(p-q+1-2i)\varepsilon_{p+i} + \sum_{j=1}^{n+1-q}(q+p+1-2j)\delta_{q+j}$
- $\omega_1 = \varepsilon_{p+1} - \frac{1}{m-n-p+q}\left(\sum_{i=p+1}^{m}\varepsilon_i - \sum_{j=q+1}^{n}\delta_j\right)$
- $\omega_{m+n-p-q} = -\delta_{n+1} + \frac{1}{m-n-p+q}\left(\sum_{i=p+1}^{m}\varepsilon_i - \sum_{j=q+1}^{n}\delta_j\right)$
- $(2\rho+\omega_1,\omega_1) = \dfrac{(m-p-n+q+1)(m-p-n+q-1)}{m-p-n+q}$
- $(2\rho+\omega_{m+n-p-q},\omega_{m+n-p-q}) = \dfrac{(m-p-n+q+1)(m-p-n+q-1)}{m-p-n+q}$
- $h^\vee = m-p-n+q$

## $C\omega$

- $u(k) = k$
- $2\rho = 0$
- $\overline{\varepsilon} = \frac{1}{p-q}\left(\sum_{i=1}^{p}\varepsilon_1 - \sum_{j=1}^{q}\delta_j\right) - \frac{1}{m-n-p+q}\left(\sum_{i=p+1}^{m}\varepsilon_i - \sum_{j=q+1}^{n}\delta_j\right)$
- $(\overline{\varepsilon},\overline{\varepsilon}) = \dfrac{m-n}{(p-q)(m-p-n+q)}$
- $h^\vee = 0$

The equation from the AP criterion is the same for the two components of $\mathfrak{g}^1$ and it is

$$\frac{m-n}{2k(p-q)(m-p-n+q)} + \frac{(p-q+1)(p-q-1)}{2(k+p-q)(p-q)} + \frac{(m-p-n+q+1)(m-p-n+q-1)}{2(k+m-n-p+q)(m-p-n+q)} = 1$$

The solutions are $k=1$, $k=-1$ and $k=\dfrac{n-m}{2}$.

**Exceptional cases**

If $\hat{h} = -k$, we cannot accept the conformal level: it happens when $p-q$ or $m-p-n+q$ equals one of the numbers $1,-1,-(n-m)/2$.

## Case $\mathfrak{g}^0 = \mathfrak{sl}(q) \oplus \mathfrak{sl}(m+1,n+1-q) \oplus \mathbb{C}\omega$

We can suppose without loss of generality, that the root systems of the subalgebras $\mathfrak{sl}(q)$ and $\mathfrak{sl}(m+1,n+1-q)$ are given by

$$\begin{aligned}
\Delta^{\mathfrak{sl}(q)} &= \{\pm(\delta_i - \delta_j), (1 \le i \ne j \le q)\} \\
\Delta_0^{m+1,n+1-q} &= \{\pm(\varepsilon_i - \varepsilon_j), (1 \le i \ne j \le m+1), \pm(\delta_i - \delta_j), (q+1 \le i \ne j \le n+1)\} \\
\Delta_1^{m+1,n+1-q} &= \{\pm(\varepsilon_i - \delta_j), (1 \le i \le m, q+1 \le j \le n)\}
\end{aligned}$$

As simple root systems we take

$$\begin{aligned}
\Pi^{\mathfrak{sl}(q)} &= \{\delta_1 - \delta_2, \cdots, \delta_{q-1} - \delta_q\} \\
\Pi^{\mathfrak{sl}(m+1.n+1-q)} &= \{\varepsilon_1 - \varepsilon_2, \cdots, \varepsilon_m - \varepsilon_{m+1}, \varepsilon_{m+1} - \delta_{q+1}, \cdots, \delta_n - \delta_{n+1}\}
\end{aligned}$$

The centralizer of $\mathfrak{g}^0$ is a one dimensional space $\mathbb{C}\omega$ and we choose

$$\omega = \frac{1}{m-n}\left(\begin{array}{c|cc} qI_{m+1} & 0 & 0 \\ \hline 0 & (m-n+q)I_q & 0 \\ 0 & 0 & qI_{n+1-q} \end{array}\right)$$

The orthogonal complement $\mathfrak{g}^1$ of is generated by $\mathfrak{g}_\alpha$, where

$$\begin{aligned} \alpha = &\pm(\delta_i - \delta_j) \quad (1 \le i \le q, q+1 \le j \le n+1), \\ &\pm(\varepsilon_i - \delta_j) \quad (1 \le i \le m+1, 1 \le j \le q), \end{aligned}$$

The maximal weights of $\mathfrak{g}^1$ wrt $\mathfrak{g}^0$ are $\delta_1 - \delta_{n+1}$ and $\varepsilon_1 - \delta_q$. We have to split these weights in a linear combination of roots of the subalgebra:

$$\begin{aligned} \delta_1 - \delta_{n+1} = &\underbrace{\delta_1 - \frac{1}{q}\sum_{j=1}^{q}\delta_j}_{\mathfrak{sl}(q)} \underbrace{- \delta_{n+1} + \frac{1}{m-n+q}\left(\sum_{i=1}^{m}\varepsilon_i - \sum_{j=q+1}^{n}\delta_j\right)}_{\mathfrak{sl}(m+1,n+1-q)} + \\ &\underbrace{+\frac{1}{q}\sum_{j=1}^{q}\delta_j - \frac{1}{m-n+q}\left(\sum_{i=1}^{m}\varepsilon_i - \sum_{j=q+1}^{n}\delta_j\right)}_{\mathbb{C}\omega} \\ \varepsilon_1 - \delta_q = &\underbrace{-\delta_q + \frac{1}{q}\sum_{j=1}^{q}\delta_j}_{\mathfrak{sl}(q)} + \underbrace{\varepsilon_1 - \frac{1}{m-n+q}\left(\sum_{i=1}^{m}\varepsilon_i - \sum_{j=q+1}^{n}\delta_j\right)}_{\mathfrak{sl}(m+1,n+1-q)} + \\ &\underbrace{-\frac{1}{q}\sum_{j=1}^{q}\delta_j + \frac{1}{m-n+q}\left(\sum_{i=1}^{m}\varepsilon_i - \sum_{j=q+1}^{n}\delta_j\right)}_{\mathbb{C}\omega} \end{aligned}$$

Then

$$\begin{aligned}
\mathfrak{g}^1 &= V_{\mathfrak{sl}(q)}\left(\delta_1 - \frac{1}{q}\sum_{j=1}^{q}\delta_j\right) \otimes V_{\mathfrak{sl}(m+1,n+1-q)}\left(-\delta_{n+1} + \frac{1}{m-n+q}\left(\sum_{i=1}^{m}\varepsilon_i - \sum_{j=q+1}^{n}\delta_j\right)\right) \otimes \\
&\otimes V_{\mathbb{C}\omega}\left(\frac{1}{q}\sum_{j=1}^{q}\delta_j - \frac{1}{m-n+q}\left(\sum_{i=1}^{m}\varepsilon_i - \sum_{j=q+1}^{n}\delta_j\right)\right) \oplus \\
&\oplus V_{\mathfrak{sl}(q)}\left(-\delta_q + \sum_{j=1}^{q}\delta_j\right) \otimes V_{\mathfrak{sl}(m+1,n+1-q)}\left(\varepsilon_1 - \frac{1}{m-n+q}\left(\sum_{i=1}^{m}\varepsilon_i - \sum_{j=q+1}^{n}\delta_j\right)\right) \otimes \\
&\otimes V_{\mathbb{C}\omega}\left(-\frac{1}{q}\sum_{j=1}^{q}\delta_j + \frac{1}{m-n+q}\left(\sum_{i=1}^{m}\varepsilon_i - \sum_{j=q+1}^{n}\delta_j\right)\right) \\
&= V_{\mathfrak{sl}(q)}(\omega_1) \otimes V_{\mathfrak{sl}(m+1,n+1-q)}(\omega_{m+n-q}) \otimes V_{\mathbb{C}\omega}(\overline{\varepsilon}) \oplus V_{\mathfrak{sl}(q)}(\omega_{q-1}) \otimes V_{\mathfrak{sl}(m+1,n+1-q)}(\omega_1) \otimes V_{\mathbb{C}\omega}(-\overline{\varepsilon}
\end{aligned}$$

where $\omega_i$ are the fundamental weights and $\overline{\varepsilon} = \omega^*$.

$\mathfrak{sl}(q)$

- $u(k) = -k$
- $2\rho = \sum_{j=1}^{q}(q+1-2j)\delta_j$
- $\omega_1 = \delta_1 - \frac{1}{q}\sum_{j=1}^{q}\delta_j$
- $\omega_{q-1} = -\delta_q + \sum_{j=1}^{q}\delta_j$
- $(2\rho+\omega_1,\omega_1) = \dfrac{(q+1)(q-1)}{q}$
- $(2\rho+\omega_{q-1},\omega_{q-1}) = \dfrac{(q+1)(q-1)}{q}$
- $h^\vee = q$

$\mathfrak{sl}(m+1,n+1-q)$

- $u(k) = k$
- $2\rho = \sum_{i=1}^{m+1}(-q+1-2i)\varepsilon_i + \sum_{j=1}^{n+1-q}(q+1-2j)\delta_{q+j}$
- $\omega_1 = \varepsilon_1 + \frac{1}{m-n+q}\left(\sum_{i=1}^{m}\varepsilon_i - \sum_{j=q+1}^{n}\delta_j\right)$

- $\omega_{m+n-q} = -\delta_{n+1} + \frac{1}{m-n+q}\left(\sum_{i=1}^{m} \varepsilon_i - \sum_{j=q+1}^{n} \delta_j\right)$
- $(2\rho + \omega_1, \omega_1) = \dfrac{(m-n+q+1)(m-n+q-1)}{m-n+q}$
- $(2\rho + \omega_{m+n-q}, \omega_{m+n-q}) = \dfrac{(m-n+q+1)(m-n+q-1)}{m-n+q}$
- $h^\vee = m-n+q$

$C\omega$

- $u(k) = k$
- $2\rho = 0$
- $\overline{\varepsilon} = \frac{1}{q}\sum_{j=1}^{q} \delta_j - \frac{1}{m-n+q}\left(\sum_{i=1}^{m} \varepsilon_i - \sum_{j=q+1}^{n} \delta_j\right)$
- $(\overline{\varepsilon}, \overline{\varepsilon}) = \dfrac{n-m}{q(m-n+q)}$
- $h^\vee = 0$

The equation given by the AP criterion is the same for the two components of $\mathfrak{g}^1$ and it is

$$\frac{n-m}{2kq(m-n+q)} + \frac{(q+1)(q-1)}{2q(-k+q)} + \frac{(m-n+q+1)(m-n+q-1)}{2(k+m-n+q)(m-n+q)} = 1$$

The solutions are $k = 1$, $k = -1$ and $k = (n-m)/2$.

### Exceptional cases

If $\hat{h} = -k$, we cannot accept the conformal level: it happens when $q$ or $m-n+q$ equals one of the numbers $1, -1, -(n-m)/2$.

## Case $\mathfrak{g}^0 = \mathfrak{g}_0 = \mathfrak{sl}(m+1) \oplus \mathfrak{sl}(n+1) \oplus \mathbb{C}\omega$

The root systems of the subalgebras $\mathfrak{sl}(m+1)$ and $\mathfrak{sl}(n+1)$ are given by

$$\Delta^{\mathfrak{sl}(m+1)} = \{\pm(\varepsilon_i - \varepsilon_j), (1 \leq i \neq j \leq m+1)\}$$
$$\Delta^{\mathfrak{sl}(n+1)} = \{\pm(\delta_i - \delta_j), (1 \leq i \neq j \leq n+1)\}$$

As simple root systems we take

$$\Pi^{\mathfrak{sl}(m+1)} = \{\varepsilon_1 - \varepsilon_2, \cdots, \varepsilon_m - \varepsilon_{m+1}\}$$
$$\Pi^{\mathfrak{sl}(n+1)} = \{\delta_1 - \delta_2, \cdots, \delta_n - \delta_{n+1}\}$$

The centralizer of $\mathfrak{g}^0$ is a one dimensional space $\mathbb{C}\omega$ and we choose

$$\omega = \frac{1}{n-m}\left(\begin{array}{c|c} (n+1)I_{m+1} & 0 \\ \hline 0 & (m+1)I_{n+1} \end{array}\right)$$

The orthogonal complement $\mathfrak{g}^1$ of is generated by $\mathfrak{g}_\alpha$, where

$$\alpha = \pm(\varepsilon_i - \delta_j) \quad (1 \le i \le m+1, 1 \le j \le n+1),$$

The maximal weights of $\mathfrak{g}^1$ wrt $\mathfrak{g}^0$ are $\varepsilon_1 - \delta_{n+1}$ and $\delta_1 - \varepsilon_{m+1}$. We have to split these weights in a linear combination of roots of the subalgebra:

$$\varepsilon_1 - \delta_{n+1} = \underbrace{\varepsilon_1 - \frac{1}{m+1}\sum_{i=1}^{m+1}\varepsilon_i}_{\mathfrak{sl}(m+1)} \underbrace{- \delta_{n+1} + \frac{1}{n+1}\sum_{j=1}^{n+1}\delta_j}_{\mathfrak{sl}(n+1)} + \underbrace{\frac{1}{m+1}\sum_{i=1}^{m+1}\varepsilon_i - \frac{1}{n+1}\sum_{j=1}^{n+1}\delta_j}_{\mathbb{C}\omega}$$

$$\delta_1 - \varepsilon_{m+1} = \underbrace{-\varepsilon_{m+1} + \frac{1}{m+1}\sum_{i=1}^{m+1}\varepsilon_i}_{\mathfrak{sl}(m+1)} + \underbrace{\delta_1 - \frac{1}{n+1}\sum_{j=1}^{n+1}\delta_j}_{\mathfrak{sl}(n+1)} + \underbrace{-\frac{1}{m+1}\sum_{i=1}^{m+1}\varepsilon_i + \frac{1}{n+1}\sum_{j=1}^{n+1}\delta_j}_{\mathbb{C}\omega}$$

Then

$$\begin{aligned}
\mathfrak{g}^1 &= V_{\mathfrak{sl}(m+1)}\left(\varepsilon_1 - \frac{1}{m+1}\sum_{i=1}^{m+1}\varepsilon_i\right) \otimes V_{\mathfrak{sl}(n+1)}\left(-\delta_{n+1} + \frac{1}{n+1}\sum_{j=1}^{n+1}\delta_j\right) \otimes \\
&\otimes V_{\mathbb{C}\omega}\left(\frac{1}{m+1}\sum_{i=1}^{m+1} - \frac{1}{n+1}\sum_{j=1}^{n+1}\delta_j\right) \oplus \\
&\oplus V_{\mathfrak{sl}(m+1)}\left(-\varepsilon_{m+1} + \frac{1}{m+1}\sum_{i=1}^{m+1}\varepsilon_i\right) \otimes V_{\mathfrak{sl}(n+1)}\left(\delta_1 - \frac{1}{n+1}\sum_{j=1}^{n+1}\delta_j\right) \otimes \\
&\otimes V_{\mathbb{C}\omega}\left(-\frac{1}{m+1}\sum_{i=1}^{m+1}\varepsilon_i + \frac{1}{n+1}\sum_{j=1}^{n+1}\delta_j\right) \\
&= V_{\mathfrak{sl}(m+1)}(\omega_1) \otimes V_{\mathfrak{sl}(n+1)}(\omega_n) \otimes V_{\mathbb{C}\omega}(\overline{\varepsilon}) \oplus V_{\mathfrak{sl}(m+1)}(\omega_m) \otimes V_{\mathfrak{sl}(n+1)}(\omega_1) \otimes V_{\mathbb{C}\omega}(-\overline{\varepsilon}),
\end{aligned}$$

where $\omega_i$ are the fundamental weights and $\overline{\varepsilon} = \omega^*$.

$\mathfrak{sl}(m+1)$

- $u(k) = k$
- $2\rho = \sum_{i=1}^{m+1}(m+2-2i)\varepsilon_i$
- $\omega_1 = \varepsilon_1 - \frac{1}{m+1}\sum_{i=1}^{m+1}\varepsilon_i$
- $\omega_m = -\varepsilon_{m+1} + \frac{1}{m+1}\sum_{i=1}^{m+1}\varepsilon_i$
- $(2\rho+\omega_1,\omega_1) = \dfrac{m(m+2)}{m+1}$
- $(2\rho+\omega_m,\omega_m) = \dfrac{m(m+2)}{m+1}$
- $h^\vee = m+1$

$\mathfrak{sl}(n+1)$

- $u(k) = -k$
- $2\rho = \sum_{j=1}^{n+1}(2n+2-2j)\delta_j$
- $\omega_1 = \delta_1 - \frac{1}{n+1}\sum_{j=1}^{n+1}\delta_j$
- $\omega_n = -\delta_{n+1} + \frac{1}{n+1}\sum_{j=1}^{n+1}\delta_j$
- $(2\rho+\omega_1,\omega_1) = \dfrac{n(n+2)}{n+1}$
- $(2\rho+\omega_{m+n-q},\omega_{m+n-q}) = \dfrac{n(n+2)}{n+1}$
- $h^\vee = n+1$

$C\omega$

- $u(k) = k$
- $2\rho = 0$
- $\overline{\varepsilon} = -\frac{1}{m+1}\sum_{i=1}^{m+1}\varepsilon_i + \frac{1}{n+1}\sum_{j=1}^{n+1}\delta_j$
- $(\overline{\varepsilon},\overline{\varepsilon}) = \dfrac{n-m}{(m+1)(n+1)}$

- $h^\vee = 0$

The equation from the AP criterion is the same for the two components of $\mathfrak{g}^1$ and is

$$\frac{n-m}{2k(m+1)(n+1)} + \frac{m(m+2)}{2(m+1)(k+m+1)} + \frac{n(n+2)}{2(n+1)(-k+n+1)} = 1$$

The solutions are $k = 1$, $k = -1$ and $k = (n-m)/2$.

**Exceptional cases**

If $\hat{h} = -k$, we cannot accept the conformal level: it happens when $m+1$ or $n+1$ equals one of the numbers $-1, -(n-m)/2$.

### 3.3.1 Conclusions

When $\mathfrak{g} = \mathfrak{sl}(m+1,n+1)$, the subalgebras of type

$$\mathfrak{g}^0 = \mathfrak{sl}(p,q) \oplus \mathfrak{sl}(m+1-p,n+1-q) \oplus \mathbb{C}\omega$$

where $\omega$ centralize $\mathfrak{g}^0$, have the same conformal levels: they are $k = \pm 1$ and $k = (n-m)/2$.

## 3.4 Conformal Embeddings in $\mathfrak{psl}(m,m)$

When $\mathfrak{g} = \mathfrak{psl}(m,m)$, we can repeat the same arguments used for $A(m,n)$ removing the 1-dimensional space $\mathbb{C}\omega$. Then, the regular equal rank subalgebras of $\mathfrak{g}$ are of the form

- $\mathfrak{g}^0 = \mathfrak{sl}(p,q) \oplus \mathfrak{sl}(m-p,m-q)$
- $\mathfrak{g}^0 = \mathfrak{sl}(q) \oplus \mathfrak{sl}(m,m,-q)$
- $\mathfrak{g}^0 = \mathfrak{sl}(m) \oplus \mathfrak{sl}(m)$

The three equations given by the AP criterion are

$$\frac{(p-q+1)(p-q-1)}{2(k+p-q)(p-q)} + \frac{(p-q+1)(p-q-1)}{2(k-p+q)(q-p)} = 1$$
$$\frac{(q-1)(q+1)}{2q(-k+q)} + \frac{(q-1)(q+1)}{2q(k+q)} = 1$$
$$\frac{(m-1)(m+1)}{2m(k+m)} + \frac{(m-1)(m+1)}{2m(-k+m)} = 1$$

They are solved by $k = 1$ and $k = -1$

# Chapter 4

# Embeddings of Regular Subalgebras in $B(m,n)$

## 4.1 Affine Dynkin Diagrams of $B(m,n)$

Let $\mathfrak{g} = B(m,n) = \mathfrak{osp}(2m+1 \mid 2n)$, with the notation of Chapter 1 the roots are

$$\Delta_0 = \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i (1 \le i \ne j \le m), \pm\delta_i \pm \delta_j, \pm 2\delta_i (1 \le i \ne j \le n)\}$$
$$\Delta_1 = \{\pm\varepsilon_i \pm \delta_j, \pm\delta_j \mid i = 1, \ldots, m, j = 1, \ldots, n\}$$

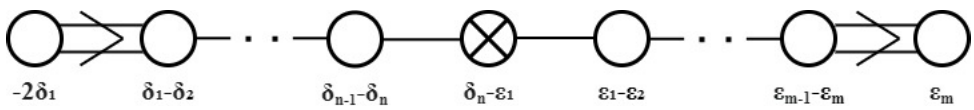
The bilinear invariant form is:

$$(\varepsilon_i, \varepsilon_j)_{B(m,n)} = \delta_{ij}$$
$$(\delta_i, \delta_j)_{B(m,n)} = -\delta_{ij}$$
$$(\varepsilon, \delta_i)_{B(m,n)} = 0$$

Taking the following as simple system of roots :

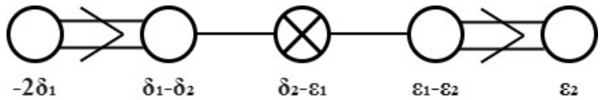
$$\{\delta_1 - \delta_2, \delta_2 - \delta_3, \cdots, \delta_n - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \cdots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m\},$$

the minimal root is $-2\delta_1$. The corresponding affine diagram is

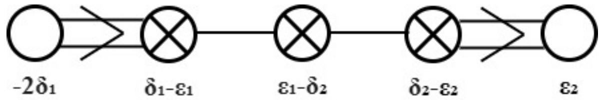



Applying repeatedly odd reflections on the diagram we obtain, up to the action of the Weyl group, all the possible affine diagrams of $B(m,n)$. The number of different diagrams this affine diagrams are in bijection with the sequences of $m$ $\varepsilon$ and $n$ $\delta$ (then they are $\binom{n+m}{m}$). For instance, we show these diagrams for $B(2,2)$.

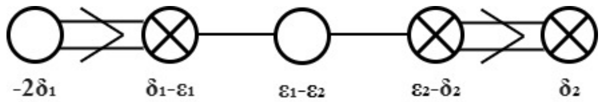
$\delta\delta\varepsilon\varepsilon$



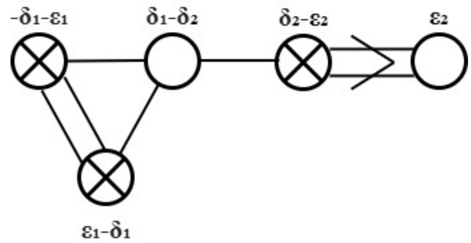

$\delta\varepsilon\delta\varepsilon$



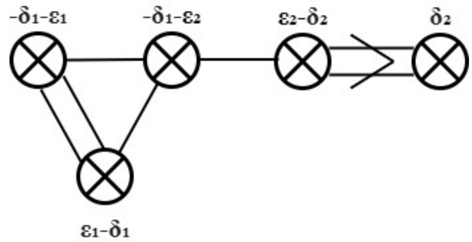

$\delta\varepsilon\varepsilon\delta$



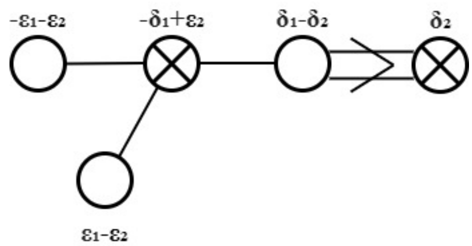

$\varepsilon\delta\delta\varepsilon$




$\varepsilon\delta\varepsilon\delta$

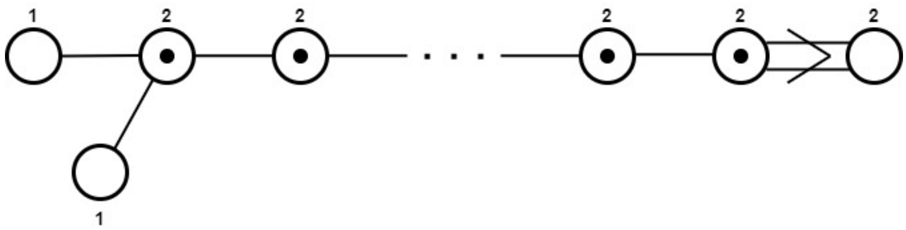


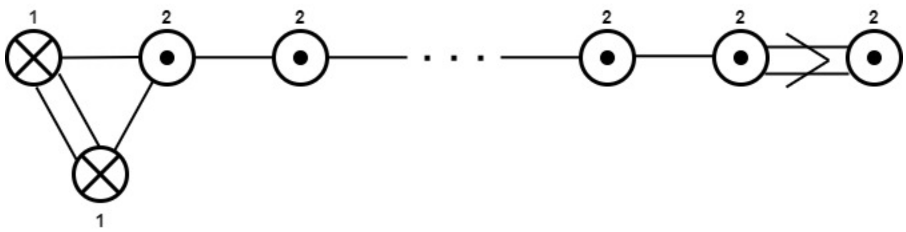

$\varepsilon\varepsilon\delta\delta$



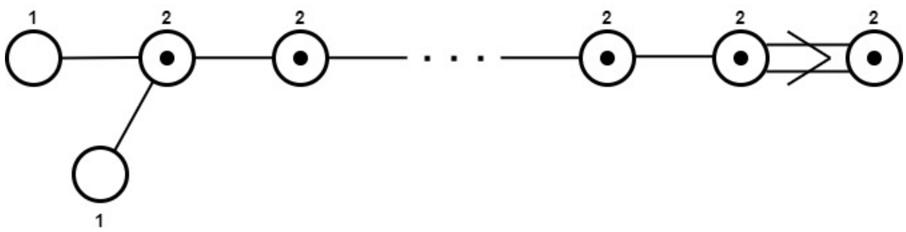
In general, for $B(m,n)$, the possible affine diagrams are the following (here $\odot$ can be either $\bigcirc$ or $\otimes$). The labels associated to the roots are the coefficient of the maximal root.

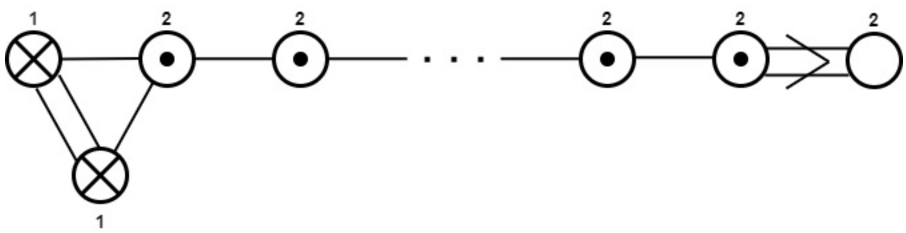
- If the number of odd simple roots is even


- If the number of odd simple roots is odd

## 4.2 Regular maximal subalgebras

If we delete one of the external left root from the previous diagrams we obtain the Dynkin diagram of $B(m,n)$.

Instead, deleting one internal node from one of these diagrams we obtain two diagrams: the left one of type $D(p,q)$ and the right one is of type $B(m-p,n-q)$.

Eventually, if we delete the last root we obtain the diagram of the algebra $D(m,n)$.

The proper subalgebras obtained in this way are all maximal among regular subalgebras because there are not strict inclusions between two of them.

Notice that Theorem 2.3.1 is verified.

## 4.3 Conformal embeddings of subalgebras

In this section we compute the conformal levels of the (equal rank) maximal regular subalgebras found previously. We split those subalgebras in the following groups, and study them separately:

- $D(p,q) \oplus B(m-p,n-q)$, with $p \neq 0$ and $q \neq 0,n$
- $C_q \oplus B(m,n-q)$, with $q \neq 0,n$
- $D_p \oplus B(m-p,n)$, with $p \neq 0$
- $C_n \oplus B_m$
- $D(m,n)$

### 4.3.1 Case $\mathfrak{g}^0 = D(p,q) \oplus B(m-p,n-q)$, with $p \neq 0$ and $q \neq 0,n$

In order to study the conformal embeddings, we can suppose without loss of generality, that the root systems of the subalgebras $D(p,q)$ and $B(m-p,n-q)$ are given by

$$\Delta_0^D = \{\pm\varepsilon_i \pm \varepsilon_j (1 \leq i \neq j \leq p), \pm\delta_i \pm \delta_j, \pm 2\delta_i (1 \leq i \neq j \leq q)\}$$
$$\Delta_1^D = \{\pm\varepsilon_i \pm \delta_j \ (1 \leq i \leq p, 1 \leq j \leq q)\}$$
$$\Delta_0^B = \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i (p+1 \leq i \neq j \leq m), \pm\delta_i \pm \delta_j, \pm 2\delta_i (q+1 \leq i \neq j \leq n)\}$$
$$\Delta_1^B = \{\pm\varepsilon_i \pm \delta_j, \pm\delta_j (p+1 \leq i \leq m, q+1 \leq j \leq n)\}$$

As simple root systems we take

$$\Pi^D = \{\delta_1 - \delta_2, \delta_2 - \delta_3, \cdots, \delta_q - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \cdots, \varepsilon_{p-1} - \varepsilon_p, \varepsilon_{p-1} + \varepsilon_p\}$$
$$\Pi^B = \{\delta_{q+1} - \delta_{q+2}, \delta_{q+2} - \delta_{q+3}, \cdots, \delta_n - \varepsilon_{p+1}, \varepsilon_{p+1} - \varepsilon_{p+2}, \cdots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m\}$$

The orthogonal complement $\mathfrak{g}^1$ of $\mathfrak{g}^0 = D(p,q) \oplus B(m-p,n-q)$ is generated by $\mathfrak{g}_\alpha$, where

$$\begin{aligned}
\alpha = &\pm\varepsilon_i \pm \varepsilon_j && (1 \le i \le p, p+1 \le j \le m),\\
&\pm\varepsilon_i && (1 \le i \le p),\\
&\pm\delta_i \pm \delta_j && (1 \le i \le q, q+1 \le j \le n),\\
&\pm\varepsilon_i \pm \delta_j && (1 \le i \le p, q+1 \le j \le n \text{ or } p+1 \le i \le m, 1 \le j \le q),\\
&\pm\delta_i && (1 \le i \le q)
\end{aligned}$$

The maximal weight of $\mathfrak{g}^1$ wrt $\mathfrak{g}^0$ is $\delta_1 + \delta_{q+1}$, then

$$\mathfrak{g}^1 = V_{(D(p,q))}(\delta_1) \otimes V_{B(m-p,n-q)}(\delta_{q+1})$$

$D(p,q)$

- $u(k) = k$
- $2\rho = 2\sum_{j=1}^{p}(p-j)\varepsilon_j + 2\sum_{i=1}^{q}(q-i-p+1)\delta_i$
- $\omega = \delta_1$
- $(2\rho+\omega,\omega) = 2p-2q-1$
- $h^\vee = 2p-2q-2$

$B(m-p,n-q)$

- $u(k) = k$
- $2\rho = \sum_{j=p+1}^{m}(2m-2j+1)\varepsilon_j + \sum_{i=q+1}^{n}(2n-2m+2p-2i+1)\delta_i$
- $\omega = \delta_{q+1}$
- $(2\rho+\omega,\omega) = 2(m-p)-2(n-q)$
- $h^\vee = 2(m-p)-2(n-q)-1$

The AP criterion reads

$$\frac{2p-2q-1}{2(k+2p-2q-2)} + \frac{2(m-p)-2(n-q)}{2(k+2(m-p)-2(n-q)-1)} = 1$$

The solutions are $k=1$ and $k = \frac{3-2m+2n}{2}$.

**Exceptional cases**

If $h^\vee = -k$, we cannot accept the conformal level: it happens when $k = 1$ and $m - p = n - q$.

### 4.3.2 Case $\mathfrak{g}^0 = C_q \oplus B(m, n-q)$, with $q \neq 0, n$

We can suppose without loss of generality, that the root systems of the subalgebras $C_q$ and $B(m, n-q)$ are given by

$$\begin{aligned}
\Delta^C &= \{\pm\delta_i \pm \delta_j, \pm 2\delta_i (1 \le i \neq j \le q)\} \\
\Delta_0^B &= \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i (1 \le i \neq j \le m), \pm\delta_i \pm \delta_j, \pm 2\delta_i (q+1 \le i \neq j \le n)\} \\
\Delta_1^B &= \{\pm\varepsilon_i \pm \delta_j, \pm\delta_j (p+1 \le i \le m, q+1 \le j \le n)\}
\end{aligned}$$

As simple root systems we take

$$\begin{aligned}
\Pi^C &= \{\delta_1 - \delta_2, \delta_2 - \delta_3, \cdots, \delta_{q-1} - \delta_q, 2\delta_q\} \\
\Pi^B &= \{\delta_{q+1} - \delta_{q+2}, \delta_{q+2} - \delta_{q+3}, \cdots, \delta_n - \varepsilon_{p+1}, \varepsilon_{p+1} - \varepsilon_{p+2}, \cdots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m\}
\end{aligned}$$

The orthogonal complement $\mathfrak{g}^1$ of $\mathfrak{g}^0 = D(p,q) \oplus B(m-p, n-q)$ is generated by $\mathfrak{g}_\alpha$, where

$$\begin{aligned}
\alpha = &\pm \delta_i \pm \delta_j \quad (1 \le i \le q, q+1 \le j \le n), \\
&\pm \varepsilon_i \pm \delta_j \quad (1 \le i \neq j \le m, 1 \le j \le q), \\
&\pm \delta_i \qquad\quad (1 \le i \le q)
\end{aligned}$$

The maximal weight of $\mathfrak{g}^1$ wrt $\mathfrak{g}^0$ is $\delta_1 + \delta_{q+1}$, then

$$\mathfrak{g}^1 = V_{C_q}(\delta_1) \otimes V_{B(m,n-q)}(\delta_{q+1})$$

$C_q$

- $u(k) = -1/2k$
- $2\rho = 2\sum_{i=1}^{q}(q-i+1)\delta_i$
- $\omega = \delta_1$
- $(2\rho + \omega, \omega) = q + 1/2$
- $h^\vee = q + 1$

$B(m,n-q)$

- $u(k)=k$
- $2\rho=\sum_{j=1}^{m}(2m-2j+1)\varepsilon_j+\sum_{i=q+1}^{n}(2n-2m-2i+1)\delta_i$
- $\omega=\delta_{q+1}$
- $(2\rho+\omega,\omega)=2m-2(n-q)$
- $h^\vee=2m-2(n-q)-1$

The AP criterion reads

$$\frac{q+1/2}{2(-k/2+q+1)}+\frac{2m-2(n-q)}{2(k+2m-2(n-q)-1)}=1$$

The solutions are $k=1$ and $k=\frac{3-2m+2n}{2}$.

**Exceptional cases**

If $h^\vee=-k$, we cannot accept the conformal level: it happens when $k=1$ and $m=n-q=$.

## Case $\mathfrak{g}^0=D_p\oplus B(m-p,n)$, with $p\neq 0$

We can suppose, without loss of generality, that the root systems of the subalgebras $D_p$ and $B(m-p,n)$ are given by

$$\begin{aligned}
&\Delta^D=\{\pm\varepsilon_i\pm\varepsilon_j(1\leq i\neq j\leq p)\\
&\Delta_0^B=\{\pm\varepsilon_i\pm\varepsilon_j,\pm\varepsilon_i(p+1\leq i\neq j\leq m),\pm\delta_i\pm\delta_j,\pm 2\delta_i(1\leq i\neq j\leq n)\}\\
&\Delta_1^B=\{\pm\varepsilon_i\pm\delta_j,\pm\delta_j(p+1\leq i\leq m,1\leq j\leq n)\}
\end{aligned}$$

As simple root systems we take

$$\begin{aligned}
&\Pi^D=\{\varepsilon_1-\varepsilon_2,\cdots,\varepsilon_{p-1}-\varepsilon_p,\varepsilon_{p-1}+\varepsilon_p\}\\
&\Pi^B=\{\delta_1-\delta_2,\delta_2-\delta_3,\cdots,\delta_n-\varepsilon_{p+1},\varepsilon_{p+1}-\varepsilon_{p+2},\cdots,\varepsilon_{m-1}-\varepsilon_m,\varepsilon_m\}
\end{aligned}$$

The orthogonal complement $\mathfrak{g}^1$ of $\mathfrak{g}^0=D_q\oplus B(m-p,n)$ is generated by $\mathfrak{g}_\alpha$, where

$$\begin{aligned}
\alpha=&\pm\varepsilon_i\pm\varepsilon_j && (1\leq i\leq p,p+1\leq j\leq m),\\
&\pm\varepsilon_i && (1\leq i\leq p),\\
&\pm\varepsilon_i\pm\delta_j && (1\leq i\leq p,1\leq j\leq n.
\end{aligned}$$

The maximal weight of $\mathfrak{g}^1$ wrt $\mathfrak{g}^0$ is $\varepsilon_1 + \delta_1$, then

$$\mathfrak{g}^1 = V_{D_q}(\varepsilon_1) \otimes V_{B(m-p,n)}(\delta_1)$$

$D_q$

- $u(k) = k$
- $2\rho = 2\sum_{j=1}^{p}(p-j)\varepsilon_j$
- $\omega = \varepsilon_1$
- $(2\rho + \omega, \omega) = 2p - 1$
- $h^\vee = 2p - 2$

$B(m-p,n)$

- $u(k) = k$
- $2\rho = \sum_{j=p+1}^{m}(2m-2j+1)\varepsilon_j + \sum_{i=q+1}^{n}(2n-2m+2p-2i+1)\delta_i$
- $\omega = \delta_1$
- $(2\rho + \omega, \omega) = 2(m-p) - 2n$
- $h^\vee = 2(m-p) - 2n - 1$

The AP criterion reads

$$\frac{2p-1}{2(k+2p-2)} + \frac{2(m-p)-2n}{2(k+2(m-p)-2n-1)} = 1$$

The solutions are $k = 1$ and $k = \frac{3-2m+2n}{2}$.

### Exceptional cases

If $h^\vee = -k$, we cannot accept the conformal level: it happens when $k = 1$ and $m = p + n =$.

## Case $\mathfrak{g}^0 = \mathfrak{g}_0 = C_n \oplus B_m$

The root systems of the subalgebras $C_n$ and $B_m$ are given by

$$\Delta^C = \{\pm\delta_i \pm \delta_j, \pm 2\delta_i (1 \le i \ne j \le n)\}$$
$$\Delta^B = \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i (1 \le i \ne j \le m)\}$$

As simple root systems we take

$$\Pi^C = \{\delta_1 - \delta_2, \delta_2 - \delta_3, \cdots, \delta_{n-1} - \delta_n, 2\delta_n\}$$
$$\Pi^B = \{\varepsilon_1 - \varepsilon_2, \cdots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m\}$$

The orthogonal complement $\mathfrak{g}_1$ is generated by $\mathfrak{g}_\alpha$, where

$$\begin{aligned} \alpha = &\pm \varepsilon_i \pm \delta_j \quad (1 \le i \ne j \le m, 1 \le j \le n), \\ &\pm \delta_i \qquad\quad (1 \le i \le n) \end{aligned}$$

The maximal weight of $\mathfrak{g}^1$ wrt $\mathfrak{g}^0$ is $\varepsilon_1 + \delta_1$, then

$$\mathfrak{g}^1 = V_{C_n}(\delta_1) \otimes V_{B_m}(\varepsilon_1)$$

$C_n$

- $u(k) = -1/2k$
- $2\rho = 2\sum_{i=1}^{n}(n-i+1)\delta_i$
- $\omega = \delta_1$
- $(2\rho + \omega, \omega) = n + 1/2$
- $h^\vee = n+1$

$B_m$

- $u(k) = k$
- $\rho = \sum_{j=1}^{m}(2m-2j+1)\varepsilon_j$
- $\omega = \varepsilon_1$
- $(2\rho + \omega, \omega) = 2m$
- $h^\vee = 2m-1$

The AP criterion reads

$$\frac{n+1/2}{2(-k/2+n+1)} + \frac{2m}{2(k+2m-1)} = 1$$

The solutions are $k = 1$ and $k = \frac{3-2m+2n}{2}$.

**Exceptional cases**

If $h^\vee = -k$, we cannot accept the conformal level: it happens when $k = 1$ and $m = n$.

### 4.3.3 Case $\mathfrak{g}^0 = D(m,n)$

The root system of $\mathfrak{g}^0$ is given by

$$\Delta_0^D = \{\pm\varepsilon_i \pm \varepsilon_j (1 \le i \ne j \le m), \pm\delta_i \pm \delta_j, \pm 2\delta_i (1 \le i \ne j \le n)\}$$
$$\Delta_1^D = \{\pm\varepsilon_i \pm \delta_j \ (1 \le i \le m, 1 \le j \le n)\}$$

As simple root systems we take

$$\Pi^D = \{\delta_1 - \delta_2, \delta_2 - \delta_3, \cdots, \delta_n - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \cdots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_{m-1} + \varepsilon_m\}$$

The orthogonal complement is generated by $\mathfrak{g}_\alpha$, where

$$\begin{aligned} \alpha = &\pm \varepsilon_i && (1 \le i \le m) \\ &\pm \delta_i && (1 \le i \le n) \end{aligned}$$

The maximal weight of $\mathfrak{g}^1$ wrt $\mathfrak{g}^0$ is $\delta_1$, then

$$\mathfrak{g}^1 = V_{D(m,n)}(\delta_1)$$

- $u(k) = k$
- $2\rho = 2\sum_{j=1}^m (m-j)\varepsilon_j + 2\sum_{i=1}^n (n-i-m+1)\delta_i$
- $\omega = \delta_1$
- $(2\rho + \omega, \omega) = 2m - 2n - 1$
- $h^\vee = 2m - 2n - 2$

The AP criterion reads

$$\frac{2m-2n-1}{2(k+2m-2n-2)} = 1$$

The solution is $k = \frac{3-2m+2n}{2}$.

### 4.3.4 Conclusions

When $\mathfrak{g} = B(m,n)$, every maximal regular equal rank subalgebra except $D(m,n)$ has the conformal levels $k = 1, \frac{3-2m+2n}{2}$ (unless $k + h^\vee = 0$). The subalgebra $D(m,n)$ has only the conformal level $k = \frac{3-2m+2n}{2}$.

# Chapter 5

# Embeddings of Regular Subalgebras in $D(m,n)$

Let $\mathfrak{g} = D(m,n) = \mathfrak{osp}(2m \mid 2n)$, with the notation of Chapter 1 the roots are

$$\begin{aligned}\Delta_0 &= \{\pm\varepsilon_i \pm \varepsilon_j (1 \le i \ne j \le m), \pm\delta_i \pm \delta_j, \pm 2\delta_i (1 \le i \ne j \le n)\}\\ \Delta_1 &= \{\pm\varepsilon_i \pm \delta_j (1 \le i \le m, 1 \le j \le n)\}\end{aligned}$$

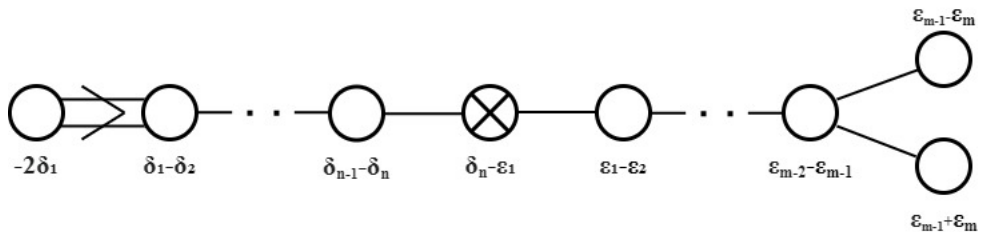
The invariant bilinear form is:

$$\begin{aligned}(\varepsilon_i, \varepsilon_j)_{D(m,n)} &= \delta_{ij}\\ (\delta_i, \delta_j)_{D(m,n)} &= -\delta_{ij}\\ (\varepsilon, \delta_i)_{D(m,n)} &= 0\end{aligned}$$

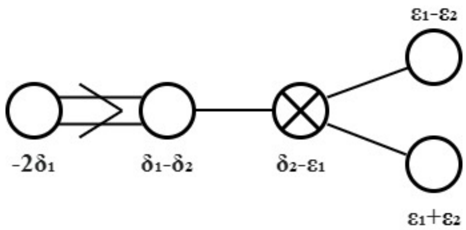
Taking the following as simple system of roots :

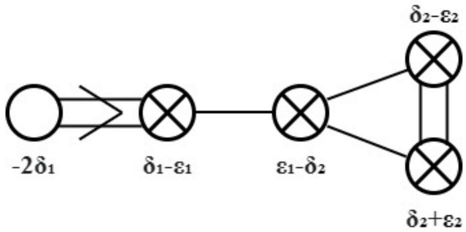
$$\{\delta_1 - \delta_2, \delta_2 - \delta_3, \cdots, \delta_n - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \cdots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_{m-1} + \varepsilon_m\},$$

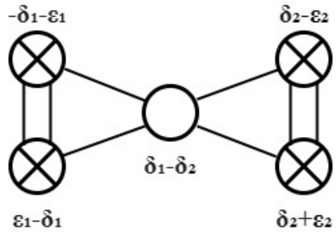
the minimal root is $-2\delta_1$. The corresponding affine diagram is

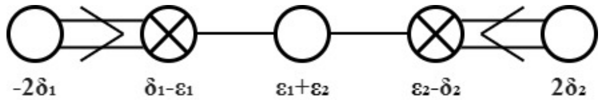



Applying repeatedly odd reflections on the diagram we obtain, up to the action of the Weyl group, all the possible affine diagrams of $D(m,n)$. For instance, we show these diagrams for $D(2,2)$.



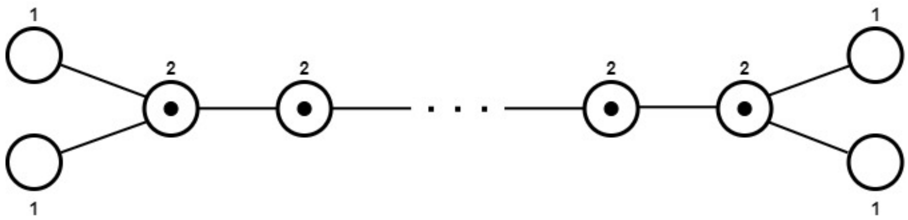



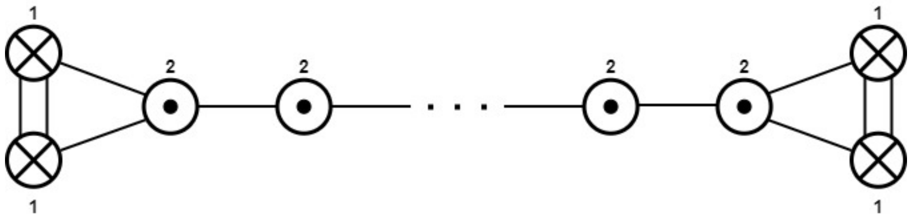



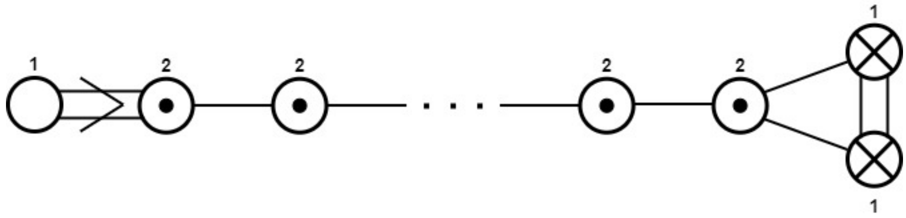




In general, for $D(m,n)$, the possible affine diagrams are the following (here $\odot$ can be either $\bigcirc$ or $\otimes$)

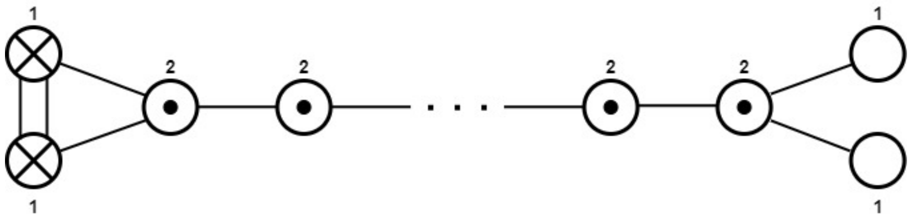
- If the number of odd simple roots is even

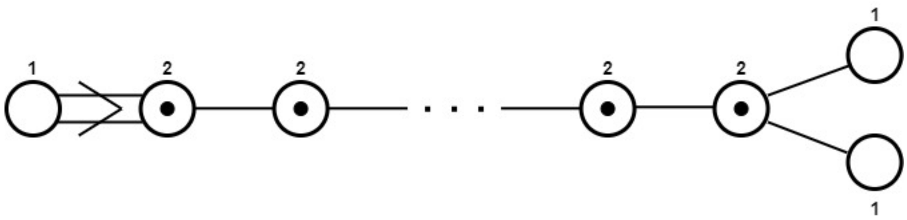




- If the number of odd simple roots is odd





## 5.1 Regular equal rank subalgebras

Deleting one node from one of these diagrams we obtain equal rank regular subalgebras: if we choose a root at the edge of the diagram we obtain the diagram of the whole algebra $D(m,n)$. If we choose a root of the internal part of the diagram we obtain two diagrams of type $D(p,q)$ and $D(m-p,n-q)$, where $p$ and $q$ are not both equal to zero (here $D(p,0)=D_p$ and $D(0,q)=C_q$).

Since there are not strictly inclusions between tho of these subalgebras they are all maximal among regular subalgebras.

We split the study of the conformal levels in the following cases:

- $\mathfrak{g}^0 = D(p,q)\oplus D(m-p,n-q)$, with $p\neq 0,m$ and $q\neq 0,n$
- $\mathfrak{g}^0 = D(p,n)\oplus D_{m-p}$, with $p\neq 0,m$.
- $\mathfrak{g}^0 = D(m,q)\oplus C_{n-q}$, with $q\neq 0,n$
- $\mathfrak{g}^0 = \mathfrak{g}_0 = D_m\oplus C_n$

### 5.1.1 Case $\mathfrak{g}^0 = D(p,q) \oplus D(m-p,n-q)$, with $p \neq 0,m$ and $q \neq 0,n$

In order to study the conformal embeddings, we can suppose without loss of generality, that the root systems of the subalgebras $D(p,q)$ and $D(m-p,n-q)$ are given by

$$\begin{aligned}
\Delta_0^{D(p,q)} &= \{\pm\varepsilon_i \pm \varepsilon_j (1 \le i \neq j \le p), \pm\delta_i \pm \delta_j, \pm 2\delta_i (1 \le i \neq j \le q)\} \\
\Delta_1^{D(p,q)} &= \{\pm\varepsilon_i \pm \delta_j \ (1 \le i \le p, 1 \le j \le q)\} \\
\Delta_0^{D(m-p,n-q)} &= \{\pm\varepsilon_i \pm \varepsilon_j (p+1 \le i \neq j \le m), \pm\delta_i \pm \delta_j, \pm 2\delta_i (q+1 \le i \neq j \le n)\} \\
\Delta_1^{D(m-p,n-q)} &= \{\pm\varepsilon_i \pm \delta_j (p+1 \le i \le m, q+1 \le j \le n)\}
\end{aligned}$$

As simple root systems we take

$$\begin{aligned}
\Pi^{D(p,q)} &= \{\delta_1 - \delta_2, \delta_2 - \delta_3, \cdots, \delta_q - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \cdots, \varepsilon_{p-1} - \varepsilon_p, \varepsilon_{p-1} + \varepsilon_p\} \\
\Pi^{D(m-p,n-q)} &= \{\delta_{q+1} - \delta_{q+2}, \delta_{q+2} - \delta_{q+3}, \cdots, \delta_n - \varepsilon_{p+1}, \varepsilon_{p+1} - \varepsilon_{p+2}, \cdots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_{m-1} + \varepsilon_m\}
\end{aligned}$$

The orthogonal complement $\mathfrak{g}^1$ of $\mathfrak{g}^0 = D(p,q) \oplus B(m-p,n-q)$ is generated by $\mathfrak{g}_\alpha$, where

$$\begin{aligned}
\alpha = &\pm \varepsilon_i \pm \varepsilon_j \quad (1 \le i \le p, p+1 \le j \le m), \\
&\pm \delta_i \pm \delta_j \quad (1 \le i \le q, q+1 \le j \le n), \\
&\pm \varepsilon_i \pm \delta_j \quad (1 \le i \le p, q+1 \le j \le n \text{ or } p+1 \le i \le m, 1 \le j \le q)
\end{aligned}$$

The maximal weight of $\mathfrak{g}^1$ wrt $\mathfrak{g}^0$ is $\delta_1 + \delta_{q+1}$, then

$$\mathfrak{g}^1 = V_{D(p,q)}(\delta_1) \otimes V_{D(m-p,n-q)}(\delta_{q+1})$$

$D(p,q)$

- $u(k) = k$
- $2\rho = 2\sum_{i=1}^{p}(p-i)\varepsilon_i + 2\sum_{j=1}^{q}(q-j-p+1)\delta_j$
- $\omega = \delta_1$
- $(2\rho + \omega, \omega) = 2p - 2q - 1$
- $h^\vee = 2p - 2q - 2$

$D(m-p,n-q)$

- $u(k)=k$
- $2\rho=-2\sum_{i=p+1}^{m} j\varepsilon_i+2\sum_{j=q+1}^{n}(-j-p+1)\delta_j$
- $\omega=\delta_{q+1}$
- $(2\rho+\omega,\omega)=2m-2n-2p+2q-1$
- $h^\vee=2m-2n-2p+2q-2$

The AP criterion reads

$$\frac{2p-2q-1}{2(k+2p-2q-2)}+\frac{2(m-p)-2(n-q)-1}{2(k+2(m-p)-2(n-q)-2)}=1$$

The solutions are $k=1$ and $k=2-m+n$.

**Exceptional cases**

If $h^\vee=-k$, we cannot accept the conformal level: it happens when $k=2-m+n$ and $2p-2q-2$ or $2m-2n-2p+2q-2$ equals $-k=m-n-2$.

### 5.1.2 Case $\mathfrak{g}^0=D(p,n)\oplus D_{m-p}$

We can suppose without loss of generality, that the root systems of the subalgebras $D(p,n)$ and $D_{m-p}$ are given by

$$\begin{aligned}
\Delta_0^{D(p,n)} &= \{\pm\varepsilon_i\pm\varepsilon_j(1\le i\ne j\le p),\pm\delta_i\pm\delta_j,\pm2\delta_i(1\le i\ne j\le n)\}\\
\Delta_1^{D(p,n)} &= \{\pm\varepsilon_i\pm\delta_j\ (1\le i\le p,1\le j\le n)\}\\
\Delta^{D_{m-p}} &= \{\pm\varepsilon_i\pm\varepsilon_j(p+1\le i\ne j\le m)\}
\end{aligned}$$

As simple root systems we take

$$\begin{aligned}
\Pi^{D(p,n)} &= \{\delta_1-\delta_2,\delta_2-\delta_3,\cdots,\delta_n-\varepsilon_1,\varepsilon_1-\varepsilon_2,\cdots,\varepsilon_{p-1}-\varepsilon_p,\varepsilon_{p-1}+\varepsilon_p\}\\
\Pi^{D_{m-p}} &= \{\varepsilon_{p+1}-\varepsilon_{p+2},\cdots,\varepsilon_{m-1}-\varepsilon_m,\varepsilon_{m-1}+\varepsilon_m\}
\end{aligned}$$

The orthogonal complement $\mathfrak{g}^1$ of $\mathfrak{g}^0$ is generated by $\mathfrak{g}_\alpha$, where

$$\begin{aligned}
\alpha=&\pm\varepsilon_i\pm\varepsilon_j \quad (1\le i\le p,p+1\le j\le m),\\
&\pm\varepsilon_i\pm\delta_j \quad (p+1\le i\le m,1\le j\le n)
\end{aligned}$$

The maximal weight of $\mathfrak{g}^1$ wrt $\mathfrak{g}^0$ is $\delta_1+\varepsilon_{p+1}$, then

$$\mathfrak{g}^1=V_{D(p,n)}(\delta_1)\otimes V_{D_{n-q}}(\varepsilon_{p+1})$$

$D(p,n)$

- $u(k) = k$
- $2\rho = 2\sum_{i=1}^{p}(p-i)\varepsilon_i + 2\sum_{j=1}^{n}(n-j-p+1)\delta_j$
- $\omega = \delta_1$
- $(2\rho+\omega,\omega) = 2p-2n-1$
- $h^\vee = 2p-2n-2$

$D_{m-p}$

- $u(k) = k$
- $2\rho = 2\sum_{i=p+1}^{m}(m-i)\varepsilon_i$
- $\omega = \varepsilon_{p+1}$
- $(2\rho+\omega,\omega) = 2m-2p-1$
- $h^\vee = 2m-2p-2$

The AP criterion reads

$$\frac{2p-2n-1}{2(k+2p-2n-2)} + \frac{2m-2p-1}{2(k+2(m-p)-2)} = 1$$

The solutions are $k = 1$ and $k = 2-m+n$.

### 5.1.3 Case $\mathfrak{g}^0 = D(m,q) \oplus C_{n-q}$

We can suppose, without loss of generality, that the root systems of the subalgebras $D(m,q)$ and $C_{n-q}$ are given by

$$\begin{aligned}
\Delta_0^{D(m,q)} &= \{\pm\varepsilon_i \pm \varepsilon_j (1 \le i \ne j \le m), \pm\delta_i \pm \delta_j, \pm 2\delta_i (1 \le i \ne j \le q)\} \\
\Delta_1^{D(m,q)} &= \{\pm\varepsilon_i \pm \delta_j \ (1 \le i \le m, 1 \le j \le q)\} \\
\Delta^{C_{n-q}} &= \{\pm\delta_i \pm \delta_j, \pm 2\delta_i (q+1 \le i \ne j \le n)\}
\end{aligned}$$

As simple root systems we take

$$\begin{aligned}
\Pi^{D(m,q)} &= \{\delta_1-\delta_2, \delta_2-\delta_3, \cdots, \delta_q-\varepsilon_1, \varepsilon_1-\varepsilon_2, \cdots, \varepsilon_{m-1}-\varepsilon_m, \varepsilon_{m-1}+\varepsilon_m\} \\
\Pi^{C_{n-q}} &= \{\delta_{q+1}-\delta_{q+2}, \cdots, \delta_{n-1}-\delta_n, 2\delta_n\}
\end{aligned}$$

The orthogonal complement $\mathfrak{g}^1$ of $\mathfrak{g}^0$ is generated by $\mathfrak{g}_\alpha$, where

$$\begin{aligned} \alpha = & \pm\delta_i \pm \delta_j \quad (1 \le i \le q, q+1 \le j \le m), \\ & \pm\varepsilon_i \pm \delta_j \quad (1 \le i \le m, q+1 \le j \le n) \end{aligned}$$

The maximal weight of $\mathfrak{g}^1$ wrt $\mathfrak{g}^0$ is $\delta_1 + \delta_{q+1}$, then

$$\mathfrak{g}^1 = V_{D(m,q)}(\delta_1) \otimes V_{C_{n-q}}(\delta_{q+1})$$

$D(m,q)$

- $u(k) = k$
- $2\rho = 2\sum_{i=1}^{m}(m-i)\varepsilon_i + 2\sum_{j=1}^{q}(q-j-m+1)\delta_j$
- $\omega = \delta_1$
- $\theta = 2\delta_1$
- $(2\rho+\omega,\omega) = 2m-2q-1$
- $h^\vee = 2m-2q-2$

$C_{n-q}$

- $u(k) = -k/2$
- $2\rho = 2\sum_{j=q+1}^{n}(n-j+1)\delta_j$
- $\omega = \delta_{q+1}$
- $\theta = 2\delta_{q+1}$
- $(2\rho+\omega,\omega) = n-q+1/2$
- $h^\vee = n-q+1$

The AP criterion reads

$$\frac{2m-2q-1}{2(k+2m-2q-2)} + \frac{2n-2q+1}{2(-k+2(n-q)+2)} = 1$$

The solutions are $k=1$ and $k = 2-m+n$.

**Exceptional cases**

If $h^\vee = -k$, we cannot accept the conformal level: it happens when $k = 2-m+n$ and $2m-2q-2$ or $n-q+1$ equals $-k = m-n-2$.

### 5.1.4 Case $\mathfrak{g}^0 = \mathfrak{g}_0 = D_m \oplus C_n$

Here $\mathfrak{g}^0 = C_n \oplus D_m$. The root systems of the subalgebras $C_n$ and $B_m$ are given by

$$\Delta^C = \{\pm\delta_i \pm \delta_j, \pm 2\delta_i (1 \le i \ne j \le n)\}$$
$$\Delta^D = \{\pm\varepsilon_i \pm \varepsilon_j, (1 \le i \ne j \le m)\}$$

As simple root systems we take

$$\Pi^C = \{\delta_1 - \delta_2, \delta_2 - \delta_3, \cdots, \delta_{n-1} - \delta_n, 2\delta_n\}$$
$$\Pi^D = \{\varepsilon_1 - \varepsilon_2, \cdots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_{m-1} + \varepsilon_m\}$$

The orthogonal complement $\mathfrak{g}_1$ is generated by $\mathfrak{g}_\alpha$, where

$$\alpha = \pm\varepsilon_i \pm \delta_j \quad (1 \le i \ne j \le m, 1 \le j \le n),$$

The maximal weight of $\mathfrak{g}^1$ wrt $\mathfrak{g}^0$ is $\varepsilon_1 + \delta_1$, then

$$\mathfrak{g}^1 = V_{C_n}(\delta_1) \otimes V_{D_m}(\varepsilon_1)$$

$C_n$

- $u(k) = -1/2k$
- $2\rho = 2\sum_{j=1}^{n}(n-j+1)\delta_j$
- $\omega = \delta_1$
- $(2\rho + \omega, \omega) = n + 1/2$
- $h^\vee = n+1$

$D_m$

- $u(k) = k$
- $2\rho = \sum_{i=1}^{m}(2m-2i)\varepsilon_i$

- $\omega = \varepsilon_1$
- $(2\rho + \omega, \omega) = 2m - 1$
- $h^\vee = 2m - 2$

The AP criterion reads

$$\frac{n+1/2}{2(-k/2+n+1)} + \frac{2m-1}{2(k+2m-2)} = 1$$

The solutions are $k = 1$ and $k = 2 - m + n$.

**Exceptional cases**

If $h^\vee = -k$, we cannot accept the conformal level: it happens when $k = 2 - m + n$ and $n + 1$ or $2m - 2$ equals $-k = m - n - 2$.

### 5.1.5 Conclusions

When $\mathfrak{g} = D(m,n)$,every maximal regular equal rank subalgebra has the conformal levels $k = 1$ and $k = 2 - m + n$ (unless $k + h^\vee = 0$).

# Chapter 6

# Embeddings of Regular Subalgebras in $C(n+1)$

## 6.1 Affine Diagrams of $C(n+1)$

Let $\mathfrak{g} = C(n+1) = \mathfrak{osp}(2|2n)$, with the notation of Chapter 1 the roots are

$$\Delta_0 = \{\pm\delta_i \pm \delta_j, \pm 2\delta_i \mid i,j = 1,\ldots,n\}$$
$$\Delta_1 = \{\pm\varepsilon \pm \delta_i \mid i = 1,\ldots,n\},$$

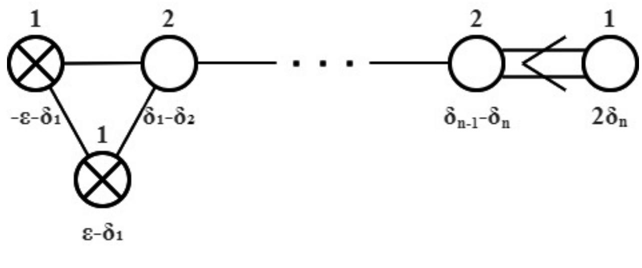
The bilinear invariant form is:

$$(\varepsilon,\varepsilon)_{C(n+1)} = 1$$
$$(\delta_i,\delta_j)_{C(n+1)} = -\delta_{ij}$$
$$(\varepsilon,\delta_i)_{C(n+1)} = 0$$

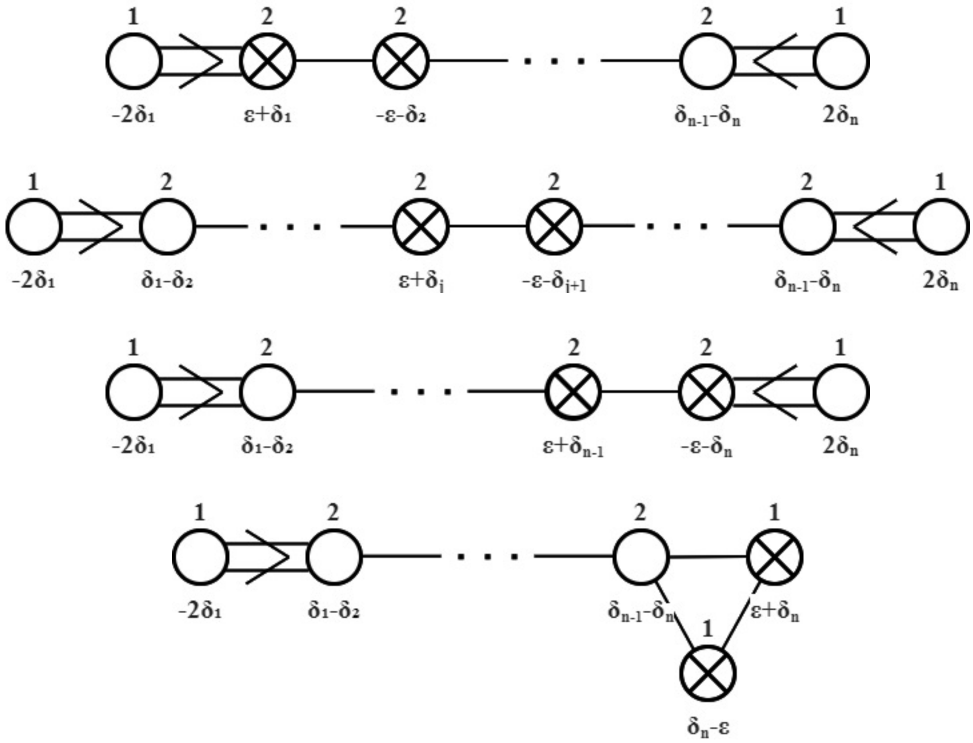
Taking the following as simple system of roots :

$$\{\varepsilon - \delta_1, \delta_1 - \delta_2, \cdots, \delta_{n-1} - \delta_n, 2\delta_n\},$$

the minimum root is $-\varepsilon - \delta_1$. The corresponding affine Dynkin Diagram is the following:



Applying repeatedly odd reflections on the diagram we obtain, up to permutation of $\delta_i$, all the possible affine diagrams of $C(n+1)$:




## 6.2 Equal rank regular Subalgebras

Deleting one root at a time we obtain the diagram of a equal rank subalgebra $\mathfrak{g}_0$.

If you delete a node with label equal to 1 we obtain the whole algebra $C(n+1)$.

Instead, deleting a node with coefficient equal to 2 this operation give us a subalgebra of the type

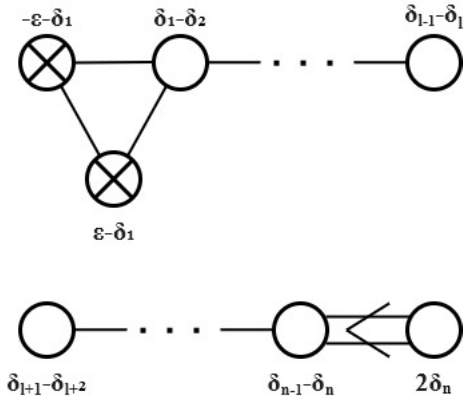
$$\mathfrak{g}^0 = C(l+1) \oplus C_{n-l},$$

with $0 < l < n$. They are all maximal among regular subalgebras because there are no strict inclusion between two of them.

Also in this case we see that the Theorem 2.3.1 is verified.

## 6.3 Conformal embedding

In order to study the conformal levels of these subalgebras, without loss of generality, we can take the subalgebra obtained form the first diagram without the root $\delta_l - \delta_{l+1}$.




The orthogonal complement $\mathfrak{g}^1$ is generated by

$$\{\mathfrak{g}_\alpha \mid \alpha = \pm\delta_i \pm \delta_j, \pm\varepsilon \pm \delta_j, i \leq l, j > l\}$$

The maximal weight with respect to the action of $\mathfrak{g}^0$ is $-\delta_l + \delta_{l+1}$

$$\mathfrak{g}^1 = V_{\mathfrak{g}^0}(-\delta_l + \delta_{l+1}) = V_{C(l+1)}(-\delta_l) \otimes V_{C_{n-l}}(\delta_{l+1})$$

| | $u(k)$ | $2\rho$ | $\omega$ | $(2\rho+\omega,\omega)$ | $h^\vee$ |
|---|---|---|---|---|---|
| $C(l+1)$ | $k$ | $-2\sum_{j=1}^{l}(j-1)\delta_j$ | $-\delta_l$ | $1-2l$ | $-2l$ |
| $C_{n-l}$ | $-k$ | $2\sum_{j=1}^{n-l}2(n-l-j+1)\delta_{l+j}$ | $\delta_{l+1}$ | $(2n-2l+1)/2$ | $n-l+1$ |

The equation given by the AP criterion is

$$\frac{1-2l}{2(k-2l)} + \frac{n-l+\frac{1}{2}}{-k+2n-2l+2} = 1$$

The solutions are $k=1$ and $k=1+n$.

### 6.3.1 Exceptional cases

If $h^\vee = -k$, we cannot accept the conformal level: it happens when $k=1+n$ and $2l = k = 1+n$.

# Chapter 7

# Embeddings of Regular Subalgebras in $G(3)$

## 7.1 Affine diagrams of $G(3)$

Let $\mathfrak{g} = G(3)$, with the notation of Chapter 1 the roots are

$$\Delta_0 = \{\varepsilon_i - \varepsilon_j, \pm\varepsilon_i, \pm 2\delta\} \qquad i,j = 1,2,3$$
$$\Delta_1 = \{\pm\varepsilon_i \pm \delta, \pm\delta\}$$

where $\varepsilon_j - \varepsilon_k, \pm\varepsilon_j$ are the roots of $G_2$, satisfying $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$ and $\pm 2\delta$ are the roots of $A_1$ in $G(3)_0 = G_2 \oplus A_1$.

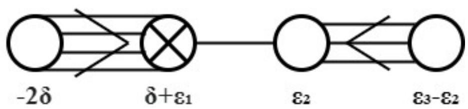
The bilinear invariant form is:

$$(\varepsilon_i, \varepsilon_i)_{G(3)} = \frac{2}{3}$$
$$(\varepsilon_i, \varepsilon_j)_{G(3)} = -\frac{1}{3} \qquad i \neq j$$
$$(\delta, \delta)_{G(3)} = -\frac{2}{3}$$
$$(\varepsilon_i, \delta)_{G(3)} = 0$$

Taking the following as simple root system:

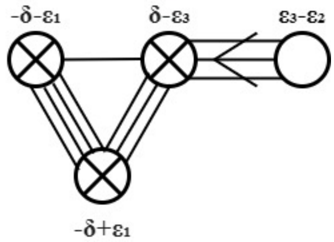
$$\{\delta + \varepsilon_1, \varepsilon_2, \varepsilon_3 - \varepsilon_2\},$$

the minimum root is $-2\delta$. The corresponding affine Dynkin Diagram is

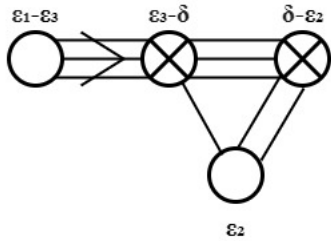
**I** 

Applying repeatedly odd reflections on the diagram we obtain, up to isomorphism, all the possible affine diagrams of $G(3)$.

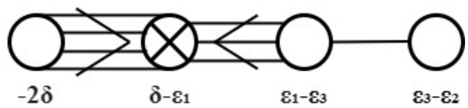
**II**



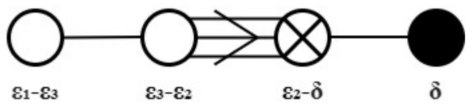

**III**




**IV**




**V**



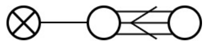
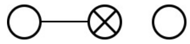
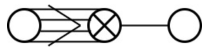

## 7.2 Equal rank subalgebras

Deleting one root at once we obtain equal rank subalgebras. In the following tables we will detail all possible subalgebras obtained in this way with a special attention to the coefficient of the root deleted.

### Diagram I

The relation between the roots is

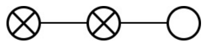
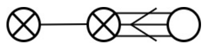
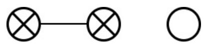
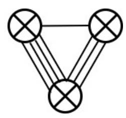
$$-2\delta + 2(\delta + \varepsilon_1) + 4\varepsilon_2 + 2(\varepsilon_3 - \varepsilon_2) = 0$$

| Root deleted | Coefficient | Diagram | Subalgebra | Maximality |
|---|---|---|---|---|
| $-2\delta$ | 1 |  | $G(3)$ | - |
| $\delta+\varepsilon_1$ | 2 |  | $A_1 \oplus G_2$ | Yes |
| $\varepsilon_2$ | 4 |  | $A(1,0) \oplus A_1$ | No |
| $\varepsilon_3-\varepsilon_2$ | 2 |  | $D(2,1;3)$ | Yes |

## Diagram II

The relation between the roots is

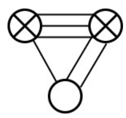
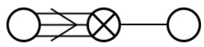
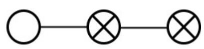
$$3(-\delta-\varepsilon_1)+(-\delta+\varepsilon_1)+4(\delta-\varepsilon_3)+2(\varepsilon_3-\varepsilon_2)=0$$

| Root deleted | Coefficient | Diagram | Subalgebra | Maximality |
|---|---|---|---|---|
| $-\delta-\varepsilon_1$ | 3 |  | $A(2,0)$ | Yes |
| $-\delta+\varepsilon_1$ | 1 |  | $G(3)$ | - |
| $\delta-\varepsilon_3$ | 4 |  | $A(1,0) \oplus A_1$ | No |
| $\varepsilon_3-\varepsilon_2$ | 2 |  | $D(2,1;3)$ | Yes |

## Diagram III

The relation between the roots is

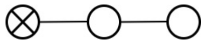
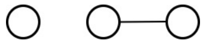
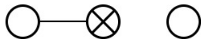
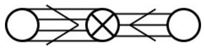
$$(\varepsilon_1-\varepsilon_3)+2(-d+\varepsilon_3)+2(\delta-\varepsilon_2)+3\varepsilon_2=0$$

| Root deleted | Coefficient | Diagram | Subalgebra | Maximality |
|---|---|---|---|---|
| $\varepsilon_1-\varepsilon_3$ | 1 |  | $G(3)$ | - |
| $\varepsilon_3-\delta$ | 2 |  | $B(1,1) \oplus A_1$ | Yes |
| $\delta-\varepsilon_2$ | 2 |  | $D(2,1;3)$ | Yes |
| $\varepsilon_2$ | 3 |  | $A(2,0)$ | Yes |

## Diagram IV

The relation between the roots is

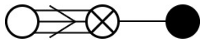
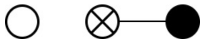
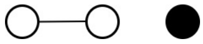
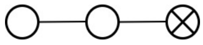
$$3(-2\delta)+6(\delta-\varepsilon_1)+4(\varepsilon_1-\varepsilon_3)+2(\varepsilon_3-\varepsilon_2)=0$$

| Root deleted | Coefficient | Diagram | Subalgebra | Maximality |
|---|---|---|---|---|
| $-2\delta$ | 3 |  | $A(2,0)$ | Yes |
| $\delta-\varepsilon_1$ | 6 |  | $A_1\oplus A_2$ | No |
| $\varepsilon_1-\varepsilon_3$ | 4 |  | $A(1,0)\oplus A_1$ | No |
| $\varepsilon_3-\varepsilon_2$ | 2 |  | $D(2,1;3)$ | Yes |

## Diagram V

The relation between the roots is

$$(\varepsilon_1-\varepsilon_3)+2(\varepsilon_3-\varepsilon_2)+3(\varepsilon_2-\delta)+3\delta=0$$

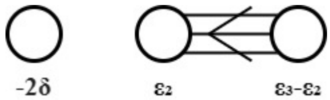
| Root deleted | Coefficient | Diagram | Subalgebra | Maximality |
|---|---|---|---|---|
| $\varepsilon_1-\varepsilon_3$ | 1 |  | $G(3)$ | - |
| $\varepsilon_3-\varepsilon_2$ | 2 |  | $B(1,1)\oplus A_1$ | Yes |
| $\varepsilon_2-\delta$ | 3 |  | $A_2\oplus B(0,1)$ | Yes |
| $\delta$ | 3 |  | $A(2,0)$ | Yes |

- A subalgebra of the form $A(1,0)\oplus A_1$ is contained in one $B(1,1)\oplus A_1$
- A subalgebra of the form $A_2\oplus A_1$ is contained in one $B(0,1)\oplus A_2$

# 7.3 Conformal Embeddings

For every subalgebra obtained, up to isomorphism, we search the conformal levels of the embedding in $G(3)$.

### 7.3.1 $\mathfrak{g}^0 = A_1 \oplus G_2$ (Diagram I)




The bilinear form of $G(3)$ restrict to the normalized form of $G_2$, while

$$\frac{(\cdot,\cdot)_{A_1}}{(\cdot,\cdot)_{G(3)}} = -\frac{3}{4}.$$

The subalgebra the even part of $G(3)$, then the orthogonal complement is $\mathfrak{g}_1$, generated by $\{\mathfrak{g}_\alpha \mid \alpha = \pm\varepsilon_i \pm \delta, \pm\delta\}$. The highest weight wrt $\mathfrak{g}^0$ is $-\delta - \varepsilon_1$. Then

$$\mathfrak{g}^1 = V_{\mathfrak{g}^0}(-\delta - \varepsilon_1) = V_{A_1}(-\delta) \otimes V_{G_2}(-\varepsilon_1)$$

In the following table we list the data to compute the levels of the conformal embeddings.

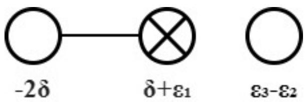
| | $u(k)$ | $2\rho$ | $\omega$ | $(2\rho+\omega,\omega)$ | $h^\vee$ |
|---|---|---|---|---|---|
| $A_1$ | $-3/4k$ | $-2\delta$ | $-\delta$ | 3/2 | 2 |
| $G_2$ | $k$ | $\varepsilon_2 + 3\varepsilon_3 - 3\varepsilon_1$ | $-\varepsilon_1$ | 4 | 4 |

The AP criterion reads

$$\frac{3/2}{2(-3/4k+2)} + \frac{4}{2(k+4)} = 1$$

Its solutions are $k = 1$ and $k = -4/3$.

### 7.3.2 $\mathfrak{g}^0 = A(1,0) \oplus A_1$ (Diagram I)




The bilinear form of $G(3)$ restrict to the normalized form of $A_1$, while

$$\frac{(\cdot,\cdot)_{A(1,0)}}{(\cdot,\cdot)_{G(3)}} = -\frac{3}{4}.$$

The orthogonal complement $\mathfrak{g}^1$ is generated by

$$\{\mathfrak{g}_\alpha \mid \alpha = \pm\delta \pm \varepsilon_2, \pm\delta \pm \varepsilon_3, \pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_1 - \varepsilon_3)\}.$$

In the following diagrams we describe the action of $\mathfrak{g}^0$ to $\mathfrak{g}^1$; $A_1$ acts by horizontal arrows and $A(1,0)$ acts by vertical arrows.

$$\begin{array}{ccc}
-\varepsilon_2 & \leftarrow & -\varepsilon_3 \\
\uparrow & & \uparrow \\
-\delta+\varepsilon_3 & \leftarrow & -\delta+\varepsilon_2 \\
\uparrow & & \uparrow \\
\delta+\varepsilon_3 & \leftarrow & \delta+\varepsilon_2 \\
\uparrow & & \uparrow \\
-\varepsilon_1+\varepsilon_3 & \leftarrow & -\varepsilon_1+\varepsilon_2
\end{array}
\qquad
\begin{array}{ccc}
\varepsilon_1-\varepsilon_2 & \leftarrow & \varepsilon_1-\varepsilon_3 \\
\uparrow & & \uparrow \\
-\delta-\varepsilon_2 & \leftarrow & -\delta-\varepsilon_3 \\
\uparrow & & \uparrow \\
\delta-\varepsilon_2 & \leftarrow & \delta-\varepsilon_3 \\
\uparrow & & \uparrow \\
\varepsilon_3 & \leftarrow & \varepsilon_2
\end{array}
\qquad
\begin{array}{c}
\varepsilon_1 \\
\uparrow \\
-\delta \\
\uparrow \\
\delta \\
\uparrow \\
-\varepsilon_1
\end{array}$$

Then

$$\begin{aligned}
\mathfrak{g}^1 =& V_{\mathfrak{g}^0}(-\varepsilon_2) \oplus V_{\mathfrak{g}^0}(\varepsilon_1-\varepsilon_2) \oplus V_{\mathfrak{g}^0}(\varepsilon_1) \\
=& V_{A(1,0)}(1/4(-2\delta)+1/2(\delta+\varepsilon_1)) \otimes V_{A_1}(1/2(\varepsilon_3-\varepsilon_2)) \oplus \\
& V_{A(1,0)}(3/4(-2\delta)+3/2(\delta+\varepsilon_1)) \otimes V_{A_1}(1/2(\varepsilon_3-\varepsilon_2)) \oplus \\
& V_{A(1,0)}(1/2(-2\delta)+\delta+\varepsilon_1),
\end{aligned}$$

In the following tables we list the data to compute the levels of the conformal embeddings.

- For $V_{\mathfrak{g}^0}(-\varepsilon_2)$ we have

| | $u(k)$ | $2\rho$ | $\omega$ | $(2\rho+\omega,\omega)$ | $h^\vee$ |
|---|---|---|---|---|---|
| $A(1,0)$ | $-3/4k$ | $-2\varepsilon_1-2\delta$ | $1/2\varepsilon_1$ | 3/8 | 1 |
| $A_1$ | $k$ | $\varepsilon_3-3\varepsilon_2$ | $1/2(\varepsilon_3-\varepsilon_2)$ | 3/2 | 2 |

- For $V_{\mathfrak{g}^0}(\varepsilon_1-\varepsilon_2)$ we have

| | $u(k)$ | $2\rho$ | $\omega$ | $(2\rho+\omega,\omega)$ | $h^\vee$ |
|---|---|---|---|---|---|
| $A(1,0)$ | $-3/4k$ | $-2\varepsilon_1-2\delta$ | $3/2\varepsilon_1$ | 3/8 | 1 |
| $A_1$ | $k$ | $\varepsilon_3-3\varepsilon_2$ | $1/2(\varepsilon_3-\varepsilon_2)$ | 3/2 | 2 |

- For $V_{\mathfrak{g}^0}(-\varepsilon_1)$ we have

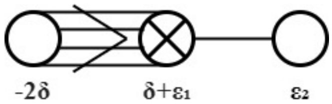
| | $u(k)$ | $2\rho$ | $\omega$ | $(2\rho+\omega,\omega)$ | $h^\vee$ |
|---|---|---|---|---|---|
| $A(1,0)$ | $-3/4k$ | $-2\varepsilon_1-2\delta$ | $\varepsilon_1$ | $1/2$ | 1 |

Then the conformal level $k$ must satisfies

$$\begin{cases}
\dfrac{3/8}{2(-3/4k+1)}+\dfrac{3/2}{2(k+2)}=1 \\
\dfrac{1/2}{2(-3/4k+1)}=1
\end{cases}$$

The solution is $k=1$.

### 7.3.3 $\mathfrak{g}^0 = D(2,1;3)$ (Diagram I)




We take as form of $D(2,1;3)$ the restriction of $(\cdot,\cdot)_{G(3)}$. (It turns out that every choice for the bilinear form brings at the same result, due to $h^\vee = 0$). The orthogonal complement is generated by

$$\{\mathfrak{g}_\alpha \mid \alpha = \pm\varepsilon_1, \pm\varepsilon_3, \pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_2 - \varepsilon_3), \pm\delta \pm \varepsilon_3, \pm\delta \pm \varepsilon_1\}$$

As $\mathfrak{g}^0$-module, $\mathfrak{g}^1$ has only one maximal root: $\varepsilon_2 - \varepsilon_3$.

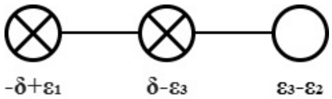
| | $u(k)$ | $2\rho$ | $\omega$ | $(2\rho+\omega,\omega)$ | $h^\vee$ |
|---|---|---|---|---|---|
| $D(2,1;3)$ | $k$ | $\varepsilon_2 - \varepsilon_1 + \varepsilon_3 - 2\delta$ | $\varepsilon_2 - \varepsilon_3$ | 2 | 0 |

The AP criterion reads

$$\frac{2}{2k} = 1,$$

then $k = 1$.

### 7.3.4 $\mathfrak{g}^0 = A(2,0)$ (Diagram II)




The bilinear form of $G(3)$ restricts to the normalized form of $A(2,0)$. The orthogonal complement $\mathfrak{g}^1$ is generated by

$$\{\mathfrak{g}_\alpha \mid \alpha = \pm\delta, \pm 2\delta, \pm\varepsilon_i, \pm(\delta + \varepsilon_i)\}.$$

.

The maximal weight in $\mathfrak{g}^1$ wrt $\mathfrak{g}^0$ are $\varepsilon_1$ and $\delta + \varepsilon_1$

$$\mathfrak{g}^1 = V_{A(2,0)}(\varepsilon_1) \oplus V_{A(2,0)}(\delta + \varepsilon_1).$$

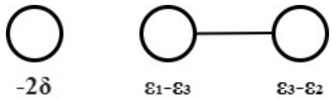
| | $u(k)$ | $2\rho$ | $\omega$ | $(2\rho+\omega,\omega)$ | $h^\vee$ |
|---|---|---|---|---|---|
| $A(2,0)$ | $k$ | $\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \delta$ | $\varepsilon_1$ | $4/3$ | 2 |
| $A(2,0)$ | $k$ | $\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \delta$ | $\delta + \varepsilon_1$ | $4/3$ | 2 |

The AP criterion reads

$$\frac{4/3}{2(k+2)} = 1,$$

solved by $k = -4/3$

### 7.3.5 $\mathfrak{g}^0 = A_1 \oplus A_2$ (Diagram IV)




The bilinear form of $G(3)$ restricts to the normalized form of $A_2$ and

$$\frac{(\cdot,\cdot)_{A_1}}{(\cdot,\cdot)_{G_3}} = -\frac{3}{4}$$

The orthogonal complement is generated by

$$\{\mathfrak{g}_\alpha \mid \alpha = \pm\varepsilon_i, \pm\varepsilon_j \pm \delta, \pm\delta\}$$

The action of $\mathfrak{g}^0$ on $\mathfrak{g}^1$ is described by the following diagrams:

$$\text{-}\delta \longleftarrow \delta \qquad \varepsilon_1 \longleftarrow \varepsilon_3 \longleftarrow \varepsilon_2$$

$$-\varepsilon_2 \longleftarrow -\varepsilon_3 \longleftarrow -\varepsilon_1$$

$$\begin{tikzcd}
-\delta+\varepsilon_1 & -\delta+\varepsilon_3 \arrow[l] & -\delta+\varepsilon_2 \arrow[l] & -\delta-\varepsilon_2 & -\delta-\varepsilon_3 \arrow[l] & -\delta-\varepsilon_1 \arrow[l] \\
\delta+\varepsilon_1 \arrow[u] & \delta+\varepsilon_3 \arrow[l] \arrow[u] & \delta+\varepsilon_2 \arrow[l] \arrow[u] & \delta-\varepsilon_2 \arrow[u] & \delta-\varepsilon_3 \arrow[l] \arrow[u] & \delta-\varepsilon_1 \arrow[l] \arrow[u]
\end{tikzcd}$$

Then the action splits as

$$\mathfrak{g}^1 = V_{A_1}(-\delta) \oplus V_{A_2}(\varepsilon_1) \oplus V_{A_2}(-\varepsilon_2) \oplus V_{A_1}(-\delta) \otimes V_{A_2}(\varepsilon_1) \oplus V_{A_1}(-\delta) \otimes V_{A_2}(-\varepsilon_2)$$

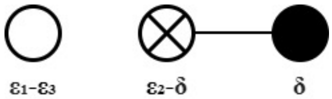
| | $u(k)$ | $2\rho$ | $\omega$ | $(2\rho+\omega,\omega)$ | $h^\vee$ |
|---|---|---|---|---|---|
| $A_1$ | $-3/4k$ | $-2\delta$ | $-\delta$ | $3/2$ | 2 |
| $A_2$ | $k$ | $2\varepsilon_1 - 2\varepsilon_2$ | $\varepsilon_1$ | $8/3$ | 3 |
| $A_2$ | $k$ | $2\varepsilon_1 - 2\varepsilon_2$ | $-\varepsilon_2$ | $8/3$ | 3 |

Then the conformal level $k$ must satisfies

$$\begin{cases} \dfrac{3/2}{2(-3/4k+2)} = 1 \\ \dfrac{8/3}{2(k+3)} = 1 \\ \dfrac{3/2}{2(-3/4k+2)} + \dfrac{8/3}{2(k+3)} = 1 \end{cases}$$

But there are not solutions.

### 7.3.6 $\mathfrak{g}^0 = A_1 \oplus B(1,1)$ (Diagram V)




The bilinear form restricts to the invariant form of $A_1$ and

$$\frac{(\cdot,\cdot)_{B(1,1)}}{(\cdot,\cdot)_{G_3}} = \frac{3}{2}$$

The orthogonal complement is generated by

$$\{\mathfrak{g}_\alpha \mid \alpha = \pm\varepsilon_1, \pm\varepsilon_3, \pm\varepsilon_1 \pm \delta, \pm\varepsilon_3 \pm \delta, \pm(\varepsilon_2 - \varepsilon_3), \pm(\varepsilon_1 - \varepsilon_3)\}$$

The only maximal weight of $g^1$ wrt $\mathfrak{g}^0$ is $\varepsilon_2 - \varepsilon_3$ Then

$$\mathfrak{g}^1 = V_{\mathfrak{g}^0}(\varepsilon_2 - \varepsilon_3) = V_{A_1}(1/2(\varepsilon_1 - \varepsilon_3)) \otimes V_{B(1,1)}(3/2(\varepsilon_2 - \delta) + 3/2\delta)$$

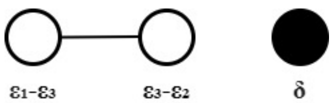
| | $u(k)$ | $2\rho$ | $\omega$ | $(2\rho+\omega,\omega)$ | $h^\vee$ |
|---|---|---|---|---|---|
| $A_1$ | $k$ | $\varepsilon_1 - \varepsilon_3$ | $1/2(\varepsilon_1 - \varepsilon_3)$ | $3/2$ | $2$ |
| $B(1,1)$ | $3/2k$ | $\delta - \varepsilon_2$ | $3/2\varepsilon_2$ | $3/4$ | $-1$ |

The AP criterion reads

$$\frac{3}{4(k+2)} + \frac{3/4}{2(3/2k-1)} = 1$$

Solved by $k = 1$ and $k = -4/3$

### 7.3.7 $\mathfrak{g}^0 = A_2 \oplus B(0,1)$ (Diagram V)




The bilinear form restricts to the invariant form of $A_2$ and

$$\frac{(\cdot,\cdot)_{B(0,1)}}{(\cdot,\cdot)_{G_3}} = \frac{2}{3}$$

The orthogonal complement is generated by

$$\{\mathfrak{g}_\alpha \mid \alpha = \pm\varepsilon_i, \pm\varepsilon_i \pm \delta\}$$

The two maximal weight of $\mathfrak{g}^1$ wrt $\mathfrak{g}^0$ are $\varepsilon_1+\delta$ and $-\varepsilon_2+\delta$, then

$$\begin{aligned}\mathfrak{g}^1 &= V_{\mathfrak{g}^0}(\varepsilon_1+\delta)\oplus V_{\mathfrak{g}^0}(-\varepsilon_2+\delta)\\ &= V_{A_2}(\varepsilon_1)\otimes V_{B(0,1)}(\delta)\oplus V_{A_2}(-\varepsilon_2)\otimes V_{B(0,1)}(\delta)\end{aligned}$$

| | $u(k)$ | $2\rho$ | $\omega$ | $(2\rho+\omega,\omega)$ | $h^\vee$ |
|---|---|---|---|---|---|
| $A_2$ | $k$ | $2\varepsilon_1-2\varepsilon_2$ | $\varepsilon_1$ | $8/3$ | $3$ |
| $A_2$ | $k$ | $2\varepsilon_1-2\varepsilon_2$ | $-\varepsilon_2$ | $8/3$ | $3$ |
| $B(0,1)$ | $3/2k$ | $\delta$ | $\delta$ | $-2$ | $-3$ |

The AP criterion reads

$$\frac{4}{3(k+3)}-\frac{2}{3k-6}=1$$

The solutions are $k=1$ and $k=-4/3$.

### 7.3.8 Conclusions

We summarize the results in the following table

| Subalgebra | Conformal levels | Maximality |
|---|---|---|
| $A_1\oplus G_2$ | $1,-4/3$ | Yes |
| $A(1,0)\oplus A_1$ | $1$ | No |
| $D(2,1;3)$ | $1$ | Yes |
| $A(2,0)$ | $-4/3$ | Yes |
| $A_1\oplus A_2$ | None | No |
| $A_1\oplus B(1,1)$ | $1,-4/3$ | Yes |
| $A_2\oplus B(0,1)$ | $1,-4/3$ | Yes |

# Chapter 8

# Embeddings of Regular Subalgebras in $F(4)$

## 8.1 Affine diagrams of $F(4)$

Let $\mathfrak{g} = F(4)$, with the notation of Chapter 1 the roots are

$$\Delta_0 = \{\pm\varepsilon_i, \pm\delta, \pm\varepsilon_i \pm \varepsilon_j\} \qquad i \neq j = 1,2,3$$
$$\Delta_1 = \{1/2(\pm\delta \pm \varepsilon_1 \pm \varepsilon_2, \pm\varepsilon_3)\}$$

The bilinear invariant form is:

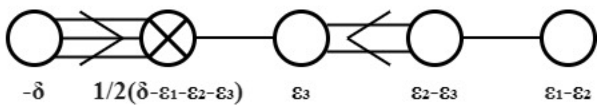
$$(\varepsilon_i, \varepsilon_j)_{F(4)} = \delta_{ij}$$
$$(\delta, \delta)_{F(4)} = -3$$
$$(\delta, \varepsilon_i)_{F(4)} = 0$$

Taking the following as simple root system:

$$\{1/2(\delta - \varepsilon_1 - \varepsilon_2 - \varepsilon_3), \varepsilon_3, \varepsilon_2 - \varepsilon_3, \varepsilon_1 - \varepsilon_2\},$$

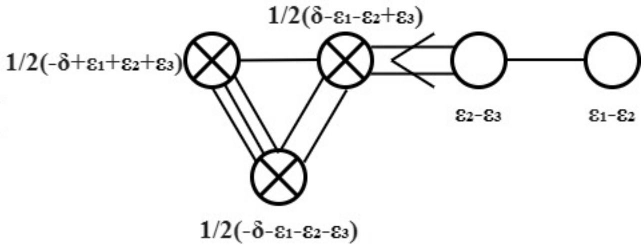
the minimum root is $-\delta$. The corresponding affine Dynkin Diagram is

**I**

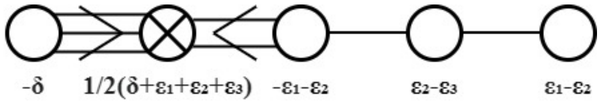



Applying repeatedly odd reflections on the diagram we obtain, up to isomorphism all the possible affine diagrams of $F(4)$:

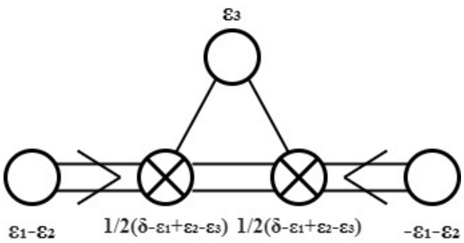
II




III




IV



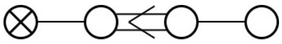
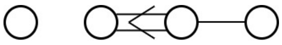
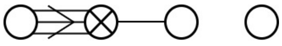
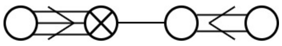

## 8.2 Equal rank subalgebras

Deleting one root at once we obtain equal rank subalgebras. In the following tables we will detail all possible subalgebras obtained in this way with a special attention to the coefficient of the root deleted.

### Diagram I

The relation between the roots is

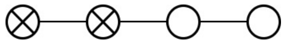
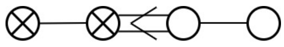
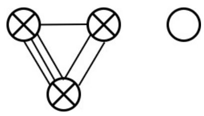
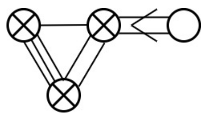
$$-\delta + 2(1/2(\delta - \varepsilon_1 - \varepsilon_2 - \varepsilon_3)) + 3\varepsilon_3 + 2(\varepsilon_2 - \varepsilon_3) + \varepsilon_1 - \varepsilon_2 = 0$$

| Root deleted | Coefficient | Diagram | Subalgebra | Maximality |
|---|---|---|---|---|
| $-\delta$ | 1 |  | $F(4)$ | - |
| $1/2(\delta - \varepsilon_1 - \varepsilon_2 - \varepsilon_3)$ | 2 |  | $A_1 \oplus B_3$ | Yes |
| $\varepsilon_3$ | 3 |  | $A(1,0) \oplus A_2$ | Yes |
| $\varepsilon_2 - \varepsilon_3$ | 2 |  | $D(2,1;2) \oplus A_1$ | Yes |
| $\varepsilon_1 - \varepsilon_2$ | 1 |  | $F(4)$ | - |

## Diagram II

The relation between the roots is

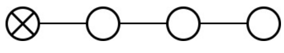
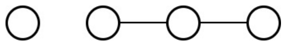
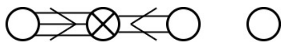
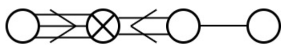
$$2(1/2(-\delta+\varepsilon_1+\varepsilon_2+\varepsilon_3))+1/2(-\delta-\varepsilon_1-\varepsilon_2-\varepsilon_3)+3(1/2(\delta-\varepsilon_1-\varepsilon_2+\varepsilon_3))+2(\varepsilon_2-\varepsilon_3)+\varepsilon_1-\varepsilon_2=0$$

| Root deleted | Coefficient | Diagram | Subalgebra | Maximality |
|---|---|---|---|---|
| $1/2(-\delta+\varepsilon_1+\varepsilon_2+\varepsilon_3)$ | 2 |  | $A(3,0)$ | Yes |
| $1/2(-\delta-\varepsilon_1-\varepsilon_2-\varepsilon_3)$ | 1 |  | $F(4)$ | - |
| $1/2(\delta-\varepsilon_1-\varepsilon_2+\varepsilon_3)$ | 3 |  | $A(1,0)\oplus A_2$ | Yes |
| $\varepsilon_2-\varepsilon_3$ | 2 |  | $D(2,1;2)\oplus A_1$ | Yes |
| $\varepsilon_1-\varepsilon_2$ | 1 |  | $F(4)$ | - |

## Diagram III

The relation between the roots is

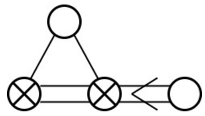
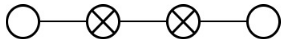
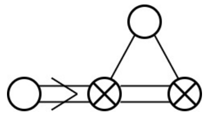
$$2(-\delta)+4(1/2(\delta+\varepsilon_1+\varepsilon_2+\varepsilon_3))+3(-\varepsilon_1-\varepsilon_2)+2(\varepsilon_2-\varepsilon_3)+\varepsilon_1-\varepsilon_2=0$$

| Root deleted | Coefficient | Diagram | Subalgebra | Maximality |
|---|---|---|---|---|
| $-\delta$ | 2 |  | $A(3,0)$ | Yes |
| $1/2(\delta+\varepsilon_1+\varepsilon_2+\varepsilon_3)$ | 4 |  | $A_1\oplus A_3$ | No |
| $-\varepsilon_1-\varepsilon_2$ | 3 |  | $A(1,0)\oplus A_2$ | Yes |
| $\varepsilon_2-\varepsilon_3$ | 2 |  | $D(2,1;2)\oplus A_1$ | Yes |
| $\varepsilon_1-\varepsilon_2$ | 1 |  | $F(4)$ | - |

## Diagram IV

The relation between the roots is

$$\varepsilon_1-\varepsilon_2+2(1/2(\delta-\varepsilon_1+\varepsilon_2-\varepsilon_3))+2\varepsilon_3+2(1/2(-\delta+\varepsilon_1+\varepsilon_2-\varepsilon_3))+(-\varepsilon_1-\varepsilon_2)=0$$

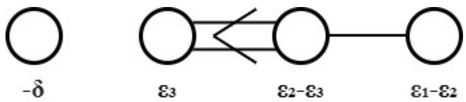
| Root deleted | Diagram | Subalgebra | Maximality | |
|---|---|---|---|---|
| $\varepsilon_1-\varepsilon_2$ | 1 |  | $F(4)$ | - |
| $1/2(\delta-\varepsilon_1+\varepsilon_2-\varepsilon_3)$ | 2 |  | $D(2,1;2)\oplus A_1$ | Yes |
| $\varepsilon_3$ | 2 |  | $A(3,0)$ | Yes |
| $1/2(-\delta+\varepsilon_1+\varepsilon_2-\varepsilon_3)$ | 2 |  | $D(2,1;2)\oplus A_1$ | Yes |
| $-\varepsilon_1-\varepsilon_2$ | 1 |  | $F(4)$ | - |

- A subalgebra of the form $A_1\oplus A_3$ is contained in one $A_1\oplus B_3$.

## 8.3 Conformal Embeddings

For every subalgebra obtained, up to isomorphism, we search the conformal levels of the embedding in $F(4)$.

### 8.3.1 $\mathfrak{g}^0=A_1\oplus B_3$ (Diagram I)




The bilinear form of $F(4)$ restrict to the normalized form of $B_3$, while

$$\frac{(\cdot,\cdot)_{A_1}}{(\cdot,\cdot)_{F(4)}}=-\frac{2}{3}.$$

The subalgebra is the even part of $F(4)$, then the orthogonal complement is $\mathfrak{g}_1$, generated by $\{\mathfrak{g}_\alpha \mid \alpha=1/2(\pm\delta\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3)\}$. The highest weight wrt $\mathfrak{g}^0$ is $1/2(-\delta+\varepsilon_1+\varepsilon_2+\varepsilon_3)$. Then

$$\mathfrak{g}^1=V_{\mathfrak{g}^0}(1/2(-\delta+\varepsilon_1+\varepsilon_2+\varepsilon_3))=V_{A_1}(-\delta/2)\otimes V_{B_3}(1/2(\varepsilon_1+\varepsilon_2+\varepsilon_3))$$

In the following table we list the data to compute the levels of the conformal embeddings.

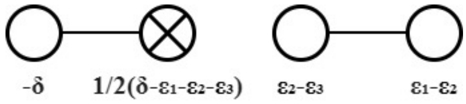
| | $u(k)$ | $2\rho$ | $\omega$ | $\theta$ | $(2\rho+\omega,\omega)$ | $h^\vee$ |
|---|---|---|---|---|---|---|
| $A_1$ | $-2/3k$ | $-\delta$ | $-\delta/2$ | $-\delta$ | $3/2$ | 2 |
| $B_3$ | $k$ | $5\varepsilon_1+3\varepsilon_2+\varepsilon_3$ | $1/2(\varepsilon_1+\varepsilon_2+\varepsilon_3)$ | $\varepsilon_1+\varepsilon_2$ | $21/4$ | 5 |

The AP criterion reads

$$\frac{3/2}{2(-2/3k+2)}+\frac{21/4}{2(k+5)}=1$$

Its solutions are $k=1$ and $k=-3/2$.

### 8.3.2 $\mathfrak{g}^0 = A(1,0)\oplus A_2$ (Diagram I)




The bilinear form of $F(4)$ restrict to the normalized form of $A_2$, while

$$\frac{(\cdot,\cdot)_{A(1,0)}}{(\cdot,\cdot)_{F(4)}}=-\frac{2}{3}.$$

The orthogonal complement $\mathfrak{g}^1$ is generated by

$$\{\mathfrak{g}_\alpha \mid \alpha=\pm\varepsilon_i,\pm(\varepsilon_i+\varepsilon_j),\pm 1/2(\delta\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3),\alpha\neq\pm 1/2(\pm\delta+\varepsilon_1+\varepsilon_2+\varepsilon_3)\}.$$

The maximal weight wrt $\mathfrak{g}^0$ are $-\varepsilon_3$ and $-\varepsilon_2-\varepsilon_3$, then

$$\begin{aligned}\mathfrak{g}^1 &= V_{\mathfrak{g}^0}(-\varepsilon_3)\oplus V_{\mathfrak{g}^0}(-\varepsilon_2-\varepsilon_3)\\ &= V_{A(1,0)}(-1/3(\varepsilon_1+\varepsilon_2+\varepsilon_3))\otimes V_{A_2}(1/3(\varepsilon_1+\varepsilon_2-2\varepsilon_3))\oplus\\ &\oplus V_{A(1,0)}(-2/3(\varepsilon_1+\varepsilon_2+\varepsilon_3))\otimes V_{A_2}(1/3(2\varepsilon_1-\varepsilon_2-\varepsilon_3)).\end{aligned}$$

In the following tables we list the data to compute the levels of the conformal embeddings.

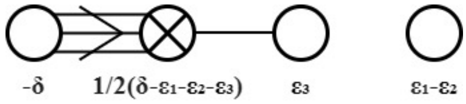
| | $u(k)$ | $2\rho$ | $\omega$ | $(2\rho+\omega,\omega)$ | $h^\vee$ |
|---|---|---|---|---|---|
| $A(1,0)$ | $-2/3k$ | $-\delta+\varepsilon_1+\varepsilon_2+\varepsilon_3$ | $-(\varepsilon_1+\varepsilon_2+\varepsilon_3)/3$ | $4/9$ | 1 |
| $A(1,0)$ | $-2/3k$ | $-\delta+\varepsilon_1+\varepsilon_2+\varepsilon_3$ | $-2/3(\varepsilon_1+\varepsilon_2+\varepsilon_3)$ | $4/9$ | 1 |
| $A_2$ | $k$ | $2\varepsilon_1-2\varepsilon_3$ | $1/3(\varepsilon_1+\varepsilon_2-2\varepsilon_3)$ | $8/3$ | 3 |
| $A_2$ | $k$ | $2\varepsilon_1-2\varepsilon_3$ | $1/3(2\varepsilon_1-\varepsilon_2-\varepsilon_3)$ | $8/3$ | 3 |

The AP criterion reads

$$\frac{4/9}{2(-2/3k+1)}+\frac{8/3}{2(k+3)}=1$$

Its solution are $k=1$ and $k=-3/2$.

### 8.3.3 $\mathfrak{g}^0 = D(2,1;2) \oplus A_1$ (Diagram I)




We take as form of $D(2,1;2)$ the restriction of $(\cdot,\cdot)_{G(3)}$. (It turns out that every choice for the bilinear form brings at the same result, due to $h^\vee = 0$).The bilinear form of $F(4)$ restrict to the normalized form of $A_1$. The orthogonal complement is generated by

$$\{\mathfrak{g}_\alpha \mid \alpha = 1/2(\pm\delta \pm \varepsilon_3 \pm (\varepsilon_2 - \varepsilon_1)), \pm\varepsilon_1, \pm\varepsilon_2 \pm \varepsilon_1 \pm \varepsilon_3, \pm\varepsilon_2 \pm \varepsilon_3\}$$

As $\mathfrak{g}^0$-module, $\mathfrak{g}^1$ has only one maximal root: $-\varepsilon_2 + \varepsilon_3$:

$$\mathfrak{g}^1 = V_{\mathfrak{g}^0}(\varepsilon_2 - \varepsilon_3) = V_{D(2,1;2)}(\varepsilon_3 - 1/2(\varepsilon_2 + \varepsilon_3)) \otimes V_{A_1}(1/2(\varepsilon_1 - \varepsilon_2))$$

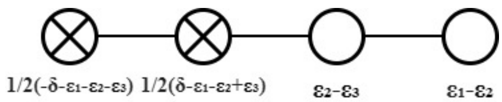
| | $u(k)$ | $2\rho$ | $\omega$ | $\theta$ | $(2\rho+\omega,\omega)$ | $h^\vee$ |
|---|---|---|---|---|---|---|
| $D(2,1;2)$ | $k$ | $-\delta+\varepsilon_1+\varepsilon_2+\varepsilon_3$ | $\varepsilon_3 - 1/2(\varepsilon_2+\varepsilon_3)$ | $-\varepsilon_1-\varepsilon_2$ | $3/2$ | $0$ |
| $A_1$ | $k$ | $\varepsilon_1-\varepsilon_2$ | $(\varepsilon_1-\varepsilon_2)/2$ | $\varepsilon_1-\varepsilon_2$ | $3/2$ | $2$ |

The AP criterion reads

$$\frac{3}{4k} + \frac{3/2}{2(k+2)} = 1,$$

solved by $k = 1$ and $k = -3/2$.

### 8.3.4 $\mathfrak{g}^0 = A(3,0)$ (Diagram II)




The bilinear form of $G(3)$ restricts to the normalized form of $A(2,0)$. The orthogonal complement $\mathfrak{g}^1$ is generated by

$$\{\mathfrak{g}_\alpha \mid \alpha = \pm\delta, \pm\varepsilon_i, \pm 1/2(-\delta+\varepsilon_1+\varepsilon_2+\varepsilon_3), \pm 1/2(\delta-\varepsilon_1+\varepsilon_2+\varepsilon_3),$$
$$\pm 1/2(\delta+\varepsilon_1-\varepsilon_2+\varepsilon_3), \pm 1/2(\delta+\varepsilon_1+\varepsilon_2-\varepsilon_3)\}$$

The maximal weight of $\mathfrak{g}^1$ wrt $\mathfrak{g}^0$ is $1/2(\delta-\varepsilon_1-\varepsilon_2-\varepsilon_3)$

$$\mathfrak{g}^1 = V_{A(3,0)}(1/2(\delta-\varepsilon_1-\varepsilon_2-\varepsilon_3)).$$

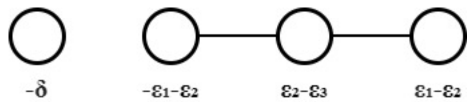
| | $u(k)$ | $2\rho$ | $\omega$ | $(2\rho+\omega,\omega)$ | $h^\vee$ |
|---|---|---|---|---|---|
| $A(3,0)$ | $k$ | $-\delta-\varepsilon_1+\varepsilon_2-3\varepsilon_3$ | $1/2(\delta-\varepsilon_1-\varepsilon_2-\varepsilon_3)$ | $3$ | $3$ |

Then the conformal level $k$ must satisfies

$$\frac{3}{2(k+3)}=1$$

Then $k=-3/2$

### 8.3.5 $\mathfrak{g}^0=A_1\oplus A_3$ (Diagram III)




The bilinear form of $G(3)$ restricts to the normalized form of $A_3$ and

$$\frac{(\cdot,\cdot)_{A_1}}{(\cdot,\cdot)_{F(4)}}=-\frac{2}{3}$$

The orthogonal complement is generated by

$$\{\mathfrak{g}_\alpha\mid\alpha=1/2(\pm\delta\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3),\pm\varepsilon_i\}$$

The maximal weights of $\mathfrak{g}^1$ wrt $\mathfrak{g}^0$ are $-\varepsilon_3, 1/2(-\delta+\varepsilon_1-\varepsilon_2-\varepsilon_3)$ and $1/2(-\delta-\varepsilon_1-\varepsilon_2-\varepsilon_3)$. Then the action splits as

$$\mathfrak{g}^1=V_{A_3}(-\varepsilon_3)\oplus V_{A_1}(-\delta/2)\otimes V_{A_3}(1/2(\varepsilon_1-\varepsilon_2-\varepsilon_3))\oplus V_{A_1}(-\delta/2)\otimes V_{A_2}(1/2(-\varepsilon_1-\varepsilon_2-\varepsilon_3))$$

| | $u(k)$ | $2\rho$ | $\omega$ | $(2\rho+\omega,\omega)$ | $h^\vee$ |
|---|---|---|---|---|---|
| $A_1$ | $-2/3k$ | $-\delta$ | $-\delta/2$ | $3/2$ | $2$ |
| $A_3$ | $k$ | $-2\varepsilon_2-4\varepsilon_3$ | $-\varepsilon_3$ | $5$ | $4$ |
| $A_3$ | $k$ | $-2\varepsilon_2-4\varepsilon_3$ | $1/2(\varepsilon_1-\varepsilon_2-\varepsilon_3)$ | $15/4$ | $4$ |
| $A_3$ | $k$ | $-2\varepsilon_2-4\varepsilon_3$ | $-1/2(\varepsilon_1+\varepsilon_2+\varepsilon_3)$ | $15/4$ | $4$ |

Then the conformal level $k$ must satisfies

$$\begin{cases}\dfrac{5}{2(k+4)}=1\\ \dfrac{15/4}{2(k+4)}+\dfrac{3/2}{2(k+2)}=1\end{cases}$$

The solution is $k=-3/2$.

### 8.3.6 Conclusions

We summarize the results in the following table:

| Subalgebra | Conformal levels | Maximality |
|---|---|---|
| $A_1 \oplus B_3$ | $1, -3/2$ | Yes |
| $A(1,0) \oplus A_2$ | $1, -3/2$ | Yes |
| $D(2,1;3) \oplus A_1$ | $1, -3/2$ | Yes |
| $A(3,0)$ | $-3/2$ | Yes |
| $A_1 \oplus A_3$ | $-3/2$ | No |

# Chapter 9

# Embeddings of Regular Subalgebras in $D(2,1;\alpha)$

## 9.1 Affine Dynkin Diagrams of $D(2,1;\alpha)$

Let $\mathfrak{g} = D(2,1;\alpha)$, using the notation of Chapter 1 the roots are

$$\Delta_0 = \{\pm 2\varepsilon_i (i = 1, \cdots, 3\}$$
$$\Delta_1 = \{\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3\}$$

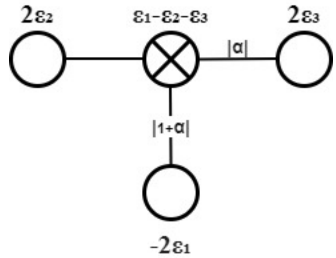
The invariant bilinear form on $D(2,1;\alpha)$ is

$$(\varepsilon_1, \varepsilon_1)_{D(2,1;\alpha)} = -(1+\alpha)/2$$
$$(\varepsilon_2, \varepsilon_2)_{D(2,1;\alpha)} = 1/2$$
$$(\varepsilon_3, \varepsilon_3)_{D(2,1;\alpha)} = \alpha/2$$
$$(\varepsilon_i, \varepsilon_j)_{D(2,1;\alpha)} = 0 \qquad i \neq j$$

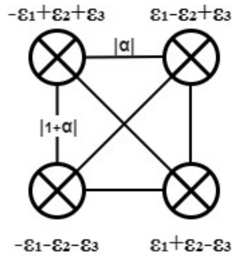
Taking as simple system of roots

$$\Pi = \{2\varepsilon_2, \varepsilon_1 - \varepsilon_2 - \varepsilon_3, 2\varepsilon_3\},$$

the minimum root is $-2\varepsilon_1$. The corresponding affine Dynkin Diagram is



There is, up to isomorphism, only one other affine diagram, obtained by applying the odd reflection w.r.t. the simple odd root $\varepsilon_1-\varepsilon_2-\varepsilon_3$.




## 9.2 Regular Maximal subalgebras

Deleting the central root from the first diagram we obtain the even algebra $\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$. Deleting one of the other three root give us the whole algebra $D(2,1,\alpha)$.

In the second diagram, if we delete one root it remains the diagram of $D(2,1;\alpha)$.

The conclusion is that there is only one equal rank regular maximal subalgebra, that is the even algebra $\mathfrak{g}_0$.

## 9.3 Conformal levels of $\mathfrak{g}_0\subset\mathfrak{g}_1$

The orthogonal complement of $\mathfrak{g}_0$ is $\mathfrak{g}_1$. Taking the order such that $\varepsilon_i>0$ for all $i$, the maximal $\alpha\in\mathfrak{g}_1$ is $\varepsilon_1+\varepsilon_2+\varepsilon_3$, then

$$\mathfrak{g}_1=V_{\mathfrak{g}_0}(\varepsilon_1+\varepsilon_2+\varepsilon_3)=V_1(\omega)\oplus V_2(\omega)\oplus V_3(\omega),$$

where $V_i(\omega)=V_{\mathfrak{sl}(2)}(\varepsilon_i)$.

We compute the ratios between the invariant form of $D(2,1;\alpha)$ and the three $\mathfrak{sl}(2)$:

$$\begin{aligned}
u_1(k)&=\frac{(\varepsilon_1,\varepsilon_1)_{\mathfrak{sl}(2)}}{(\varepsilon_1,\varepsilon_1)_{D(2,1;\alpha)}}k=-\frac{k}{1+\alpha}\\
u_2(k)&=\frac{(\varepsilon_2,\varepsilon_2)_{\mathfrak{sl}(2)}}{(\varepsilon_2,\varepsilon_2)_{D(2,1;\alpha)}}k=k\\
u_1(k)&=\frac{(\varepsilon_1,\varepsilon_1)_{\mathfrak{sl}(2)}}{(\varepsilon_3,\varepsilon_3)_{D(2,1;\alpha)}}k=\frac{k}{\alpha}
\end{aligned}$$

Furthermore,

$$(2\rho_{\mathfrak{sl}(2)} + \omega, \omega)_{\mathfrak{sl}(2)} = 3/2$$
$$h^{\vee} = 2$$

The the equation for the conformal levels is

$$\frac{3/2}{2(-k(1+\alpha)+2)} + \frac{3/2}{2(k+2)} + \frac{3/2}{2(k/\alpha+2)} = 1$$

The solutions are $k = 1$,$k = -1-\alpha$ and $k = \alpha$.

**Exceptional cases**

We have to exclude those levels that satisfy $k = -\check{h}$. When $\alpha = 1$ (which is the same as $\alpha = -2$) the only conformal level is $k = 1$.

# Chapter 10

# Decomposition in the Vertex Algebras

In this Chapter, we will apply the fusion rule argument (1.8.1) to compute the decomposition of $V_k(\mathfrak{g})$ as $(V)_k(\mathfrak{g}^0)$ when $\mathfrak{g}$ is of type $B(m,n)$ and $k$ is a conformal level and when $\mathfrak{g}$ is of type $D(m,n)$ and $k = 1$ (for the other conformal level $k = 2 - m + 2$ the criterion is not satisfied).

To apply such criterion, we need to define an automorphism of finite order $\sigma$ such that its space of fixed points is $\mathfrak{g}^0$. A suitable $\sigma$ is the involution that acts as $(-1)^i\mathrm{id}$ on $\mathfrak{g}^i$, $i = 0,1$.

## 10.1 Decomposition for $\mathfrak{g} = B(m,n)$

In order to apply Theorem 1.8.1 we need to find a decomposition of $\mathfrak{g}^1 \otimes \mathfrak{g}^1$ as $\mathfrak{g}^0$-module.

For both Lie algebras of type $B$ and $D$, it is known that the following equality holds:

$$V(\omega_1) \otimes V(\omega_1) = V(2\omega_1) \oplus V(\omega_1 + \omega_2) \oplus V(0)$$

For the algebras $\mathfrak{osp}(m,2n)$, we have a completely analogous statement from Moseneder that we report in the following.

Let $\mathfrak{g} = \mathfrak{osp}(m,2n)$ and, using the usual notation, we take as simple root system the ordered sets:

$$(\delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, \delta_n - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m) \qquad \text{if } m \text{ is odd}$$
$$(\delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, \delta_n - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_{m_1} + \varepsilon_{m+1}) \quad \text{if } m \text{ is even}$$

**Proposition 10.1.1** *With the setup described above, let $V$ the irreducible representation of $\mathfrak{g} = \mathfrak{osp}(m,2n)$ of maximal weight $\delta_1$. Then $V \otimes V$ decomposes as*

*i)* $V(2\delta_1) \oplus V(\delta_1 + \delta_2) \oplus V(0)$ *if $n \geq 2$ and $m \neq 2$. If $m = 2n$, $V(0)$ occurs with multiplicity 2.*

*ii)* $V(2\delta) \oplus V(\delta) \oplus V(0)$ *if $n = m = 1$*

*iii)* $V(2\delta) \oplus V(\delta + \varepsilon_1) \oplus V(0)$ *if $n = 1$ and $m > 2$*

*iv)* $V(2\delta_1) \oplus V(\delta_1 + \delta_2) \oplus V(0)$ *if $n \geq 2$ and $m = 2$.*

*v)* $V(2\delta) \oplus V(\delta + \varepsilon_1) \oplus V(\delta - \varepsilon_1) \oplus 2V(0)$ *if $n = 1$ and $m = 2$*

**Proof:**

i) If $v$ is a primitive vector in $V \otimes V$, then, clearly, it is a singular vector for $\mathfrak{g}_{\bar{0}}$. Since $V = \mathbb{C}^m \oplus \mathbb{C}^{2n}$ as $\mathfrak{g}_{\bar{0}}$-module, one can compute easily the decomposition of $V \otimes V$ as $\mathfrak{g}_{\bar{0}}$-module. The outcome is that $V_{\mathfrak{g}_{\bar{0}}}(0)$ occurs with multiplicity 2 while, letting $HW$ be the set of nonzero highest weights occurring in the decomposition,

$$HW = \begin{cases} \{2\delta_1, \delta_1 + \delta_2, \delta_1\} & \text{if } m = 1 \\ \{2\varepsilon_1, \varepsilon_1, 2\delta_1, \delta_1 + \delta_2, \varepsilon_1 + \delta_1\} & \text{if } m = 3 \\ \{2\varepsilon_1, \varepsilon_1 + \varepsilon_2, \varepsilon_1 - \varepsilon_2, 2\delta_1, \delta_1 + \delta_2, \varepsilon_1 + \delta_1\} & \text{if } m = 4 \\ \{2\varepsilon_1, \varepsilon_1 + \varepsilon_2, 2\delta_1, \delta_1 + \delta_2, \varepsilon_1 + \delta_1\} & \text{if } m \geq 5 \end{cases}$$

By Proposition 2.3 of [18], only $2\delta_1, \delta_1 + \delta_2$ and $\delta_1$ are dominant weights for $\mathfrak{g}$. Since $V_{\mathfrak{g}_{\bar{0}}}(2\delta_1)$ and $V_{\mathfrak{g}_{\bar{0}}}(\delta_1 + \delta_2)$ both occur with multiplicity 1 in $V \otimes V$, $V(2\delta_1)$ and $V(\delta_1 + \delta_2)$ occur with multiplicity at most 1 in $V \otimes V$. Clearly $V(2\delta_1)$ occurs in $V \otimes V$. Since $2\delta_1$ is the highest root, $V(2\delta_1)$ is the adjoint representation, so, as $\mathfrak{g}_{\bar{0}}$-module

$$V(2\delta_1) = \begin{cases} V_{\mathfrak{g}_{\bar{0}}}(2\delta_1) \oplus V_{\mathfrak{g}_{\bar{0}}}(\delta_1) & \text{if } m = 1 \\ V_{\mathfrak{g}_{\bar{0}}}(2\delta_1) \oplus V_{\mathfrak{g}_{\bar{0}}}(\varepsilon_1) \oplus V_{\mathfrak{g}_{\bar{0}}}(\varepsilon_1 + \delta_1) & \text{if } m = 3 \\ V_{\mathfrak{g}_{\bar{0}}}(2\delta_1) \oplus V_{\mathfrak{g}_{\bar{0}}}(\varepsilon_1 + \varepsilon_2) \oplus V_{\mathfrak{g}_{\bar{0}}}(\varepsilon_1 - \varepsilon_2) \oplus V_{\mathfrak{g}_{\bar{0}}}(\varepsilon_1 + \delta_1) & \text{if } m = 4 \\ V_{\mathfrak{g}_{\bar{0}}}(2\delta_1) \oplus V_{\mathfrak{g}_{\bar{0}}}(\varepsilon_1 + \varepsilon_2) \oplus V_{\mathfrak{g}_{\bar{0}}}(\varepsilon_1 + \delta_1) & \text{if } m \geq 5 \end{cases}$$

Since $\delta_1 + \delta_2 > \delta_1$, it follows that $V(\delta_1 + \delta_2)$ does occur with multiplicity 1. If $m = 1$, we can compute the dimension of $V(\delta_1 + \delta_2)$ using formula (2.4) of [18]. It turns out that $\dim V(\delta_1 + \delta_2) = 2n^2 + n$ so $dimV(2\delta_1) + dimV(\delta_1 + \delta_2) = 4n^2 + 4n = (2n+1)^2 - 1$. This gives the statement in this case. If $m > 1$, it remains only to check the multiplicity of $V(0)$. Since the multiplicity of $V_{\mathfrak{g}_{\bar{0}}}(0)$ is 2, the multiplicity of $V(0)$ is at most 2. Note that $V \cong V^*$ as $\mathfrak{g}$-module, so $V \otimes V \cong \mathfrak{gl}(m,2n)$ and $\mathfrak{g}$ acts trivially on $\mathbb{C}Id$. It follows that $V(0)$ occurs with multiplicity at least 1.

Let us assume $m \neq 2n$ first. The supertrace defines a nondegenerate invariant form on $\mathfrak{gl}(m,2n)$. Since the form is nondegenerate on $\mathbb{C}Id \oplus \mathfrak{g}$ we have $\mathfrak{gl}(m,2n) = \mathbb{C}Id \oplus \mathfrak{g} \oplus U$. Since V is irreducible, by Schur's lemma, $\dim Hom_{\mathfrak{g}}(V,V) = 1$, so $V(0)$ cannot be in the socle of $U$. It follows that $V(\delta_1+\delta_2) \subset U$ and $\dim U/V(\delta_1+\delta_2) \leq 1$. If the form is degenerate on $V(\delta_1+\delta_2)$, then $V(\delta_1+\delta_2)$ is isotropic and, since $\dim V(\delta_1+\delta_2) > 1$, this contradicts the fact the form is nondegenerate on $U$ and $\dim U/V(\delta_1+\delta_2) \leq 1$. It follows that the form is nondegenerate on $V(\delta_1+\delta_2)$. Then, by Schur's lemma again, $V(\delta_1+\delta_2)^{\perp} \cup U$ cannot be one-dimensional so $V(\delta_1+\delta_2) = U$. The case $m = 2n$ is easier, since, as $sl(m.2n)$-module, the trivial representation occurs in $\mathfrak{gl}(m,2n)/(\mathbb{C}Id)$ so $V(0)$ has multiplicity 2.

ii),iii) We argue as in the proof of $i)$ to deduce that $V_{\mathfrak{g}_{\bar{0}}}(0)$ occurs with multiplicity 2 while

$$HW = \begin{cases} \{2\delta_1, \delta_1\} & \text{if } m = 1 \\ \{2\varepsilon_1, \varepsilon_1, 2\delta_1, \varepsilon_1+\delta_1\} & \text{if } m = 3 \\ \{2\varepsilon_1, \varepsilon_1+\varepsilon_2, \varepsilon_1-\varepsilon_2, 2\delta_1, \varepsilon_1+\delta_1\} & \text{if } m = 4 \\ \{2\varepsilon_1, \varepsilon_1+\varepsilon_2, 2\delta_1, \varepsilon_1+\delta_1\} & \text{if } m \geq 5 \end{cases}$$

By Proposition 2.3 of [18], only $2\delta_1$, $\delta_1+\varepsilon_1$, and $\delta_1$ are dominant weights for $\mathfrak{g}$. Clearly $V(2\delta_1)$ occurs in $V \otimes V$ with multiplicity 1 since $V_{\mathfrak{g}_{\bar{0}}}(2\delta_1)$ occurs with multiplicity 1. Since $2\delta_1$ is the highest root, $V(2\delta_1)$ is the adjoint representation, so, as $\mathfrak{g}_{\bar{0}}$-module,

$$V(2\delta_1) = \begin{cases} V_{\mathfrak{g}_{\bar{0}}}(2\delta_1) \oplus V_{\mathfrak{g}_{\bar{0}}}(\delta_1) & \text{if } m = 1 \\ V_{\mathfrak{g}_{\bar{0}}}(2\delta_1) \oplus V_{\mathfrak{g}_{\bar{0}}}(\varepsilon_1) \oplus V_{\mathfrak{g}_{\bar{0}}}(\varepsilon_1+\delta_1) & \text{if } m = 3 \\ V_{\mathfrak{g}_{\bar{0}}}(2\delta_1) \oplus V_{\mathfrak{g}_{\bar{0}}}(\varepsilon_1+\varepsilon_2) \oplus V_{\mathfrak{g}_{\bar{0}}}(\varepsilon_1-\varepsilon_2) \oplus V_{\mathfrak{g}_{\bar{0}}}(\varepsilon_1+\delta_1) & \text{if } m = 4 \\ V_{\mathfrak{g}_{\bar{0}}}(2\delta_1) \oplus V_{\mathfrak{g}_{\bar{0}}}(\varepsilon_1+\varepsilon_2) \oplus V_{\mathfrak{g}_{\bar{0}}}(\varepsilon_1+\delta_1) & \text{if } m \geq 5 \end{cases}$$

Since $V_{\mathfrak{g}_{\bar{0}}}(\delta_1+\varepsilon_1)$ occurs with multiplicity 2, $V(\delta_1+\varepsilon_1)$ occurs with multiplicity 1. Likewise, if $m = 1$, $V(\delta_1)$ occurs with multiplicity 1. To check the multiplicity of $V(0)$ we argue as in the proof of $i)$.

iv) We argue as in the proof of $i)$ to deduce that $V_{\mathfrak{g}_{\bar{0}}}(0)$ occurs with multiplicity 2 while

$$HW = \{2\varepsilon_1, -2\varepsilon_1, 2\delta_1, \delta_1+\delta_2, \varepsilon_1+\delta_1, -\varepsilon_1+\delta_1\}.$$

Clearly $V(2\delta_1)$ occurs in $V\otimes V$ with multiplicity 1. Also, by Schur's Lemma, $V(0)$ occurs with multiplicity at least 1. Since $2\delta_1$ is the highest root, $V(2\delta_1)$ decomposes as $V_{\mathfrak{g}_{\bar 0}}(2\delta_1)\oplus V_{\mathfrak{g}_{\bar 0}}(0)\oplus V_{\mathfrak{g}_{\bar 0}}(\delta_1+\varepsilon_1)\oplus V_{\mathfrak{g}_{\bar 0}}(\delta_1-e_1)$. It follows that $V(\delta_1+\delta_2)$ must occur in the composition series of $V\otimes V$ . If $n=2$, we compute $\dim V(\delta_1+\delta_2)$ using (2.4) of [18], which gives $\dim V(\delta_1+\delta_2)=16$. Since $\dim V(2\delta_1)=19$ the claim follows in this case. For $n>2$, then $V(\delta_1+\delta_2)$ is tame of atypicality 1, that is the following character formula holds:

$$\mathrm{ch}\, V(\delta_1+\delta_2)=e^{-\rho}R^{-1}\sum_{w\in W}\varepsilon(w)w\frac{e^{\delta_1+\delta_2+\rho}}{1+e^{-\delta_n+\varepsilon_1}},$$

(see Proposition 3.2 of [27]; notation as in loc. cit.). From this formula it follows that $\dim V(\delta_1+\delta_2)=2n^2+3n+2$, hence $\dim V(2\delta_1)+\dim V(\delta_1+\delta_2)=4n^2+8n+3=(2n+2)^2-1$; the result follows.

v) We argue as in the proof of $i)$ to deduce that $V_{\mathfrak{g}_{\bar 0}}(0)$ occurs with multiplicity 2 while

$$HW=\{2\varepsilon_1,-2\varepsilon_1,2\delta_1,\varepsilon_1+\delta_1,-\varepsilon_1+\delta_1\}.$$

Clearly $V(2\delta_1)$ occurs in $V\otimes V$ with multiplicity 1. Since $2\delta_1$ is the highest root, $V(2\delta_1)$ decomposes as $V_{\mathfrak{g}_{\bar 0}}(2\delta_1)\oplus V_{\mathfrak{g}_{\bar 0}}(0)\oplus V_{\mathfrak{g}_{\bar 0}}(\delta_1+\varepsilon_1)\oplus V_{\mathfrak{g}_{\bar 0}}(\delta_1-\varepsilon_1)$. Since $V_{\mathfrak{g}_{\bar 0}}(\delta_1\pm\varepsilon_1)$ occur with multiplicity 2, $V(\delta_1\pm\varepsilon_1)$ must occur in the composition series of $V\otimes V$. The modules $V(\delta_1\pm\varepsilon_1)$ are tame of atypicality 1, that is the following character formula holds:

$$\mathrm{ch}\, V(\delta_1\pm\varepsilon_1)=e^{-\rho}R^{-1}\sum_{w\in W}\varepsilon(w)w\frac{e^{\delta_1\pm\varepsilon_1+\rho}}{1+e^{-\delta_1\mp\varepsilon_1}},$$

(see Proposition 3.2 of [27]; notation as in loc. cit.). In particular $\dim V(\delta_1\pm\varepsilon_1)=3$, hence $\dim V(2\delta_1)+\dim V(\delta_1+\varepsilon_1)+\dim V(\delta_1-\varepsilon_1)=8+3+3=4^2-2$. Since the trivial representation occurs with multiplicity 2 in the composition series of $\mathfrak{gl}(2,2)$ seen as a $sl(2,2)$-module, $V(0)$ occurs with multiplicity 2 in $V\otimes V$. The result follows.

Thanks to this Proposition we can test the condition of Theorem 1.8.1. We will see that for the algebra $B(m,n)$ that condition holds for both the conformal levels, so we can write a decomposition of $V_k(B(m,n))$.

In the following, we analyze the condition for each subalgebra of $B(m,n)$ found in Chapter 4 and we will use the same notation and the same choices of simple root systems.

### 10.1.1 $\mathfrak{g}^0 = D(p,q) \oplus B(m-p,n-q)$

In Chapter 4, we obtained

$$\mathfrak{g}^1 = V_{D(p,q)}(\delta_1) \otimes V_{B(m-p,n-q)}(\delta_{q+1}).$$

Thanks to the Proposition 10.1.1 we have an explicit decomposition of $\mathfrak{g}^1 \otimes \mathfrak{g}^1$. In what follows, for this subalgebra and the others, we will show the computations when the two components of $\mathfrak{g}^0$ are either even or satisfy the condition $i$) of Proposition 10.1.1. The other cases give exactly the same results.

The decomposition is

$$\begin{aligned}\mathfrak{g}^1 \otimes \mathfrak{g}^1 = \big(&V_{D(p,q)}(2\delta_1) \oplus V_{D(p,q)}(\delta_1+\delta_2) \oplus V_{D(p,q)}(0)\big) \otimes \\ &\otimes \big(V_{B(m-p,n-q)}(2\delta_{q+1}) \oplus V_{B(m-p,n-q)}(\delta_{q+1}+\delta_{q+2}) \oplus V_{B(m-p,n-q)}(0)\big)\end{aligned}$$

In the following table we summarize the data needed to test the conditions of Theorem 1.8.1.

| subalgebra | weight | $k$ | $\dfrac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}$ |
|---|---|---|---|
| $D(p,q)$ | $2\delta_1$ | 1 | $\dfrac{2q-2p+2}{2q-2p+1}$ |
| $D(p,q)$ | $\delta_1+\delta_2$ | 1 | $\dfrac{2q-2p}{2q-2p+1}$ |
| $B(m-p,n-q)$ | $2\delta_{q+1}$ | 1 | $\dfrac{2m-2n+2q-2p-1}{2m-2n+2q-2p}$ |
| $B(m-p,n-q)$ | $\delta_{q+1}+\delta_{q+2}$ | 1 | $\dfrac{2m-2n+2q-2p+1}{2m-2n-2p+2q}$ |
| $D(p,q)$ | $2\delta_1$ | $(3-2m+2n)/2$ | $\dfrac{4q-4p+4}{4q-4p+1+2m-2n}$ |
| $D(p,q)$ | $\delta_1+\delta_2$ | $(3-2m+2n)/2$ | $\dfrac{4q-4p}{4q-4p+1+2m-2n}$ |
| $B(m-p,n-q)$ | $2\delta_{q+1}$ | $(3-2m+2n)/2$ | $\dfrac{4m-4n+4q-4p-2}{2m-2n+4q-4p+1}$ |
| $B(m-p,n-q)$ | $\delta_{q+1}+\delta_{q+2}$ | $(3-2m+2n)/2$ | $\dfrac{4m-4n+4q-4p+2}{2m-2n-4p+4q+1}$ |

For each choice of a weight of $D(p,q)$ and $B(m-p,n-q)$ we test the criterion:

$\boxed{k=1}$

| $D(p,q)$ | $B(m-p,n-q)$ | $\frac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}+\frac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}$ | $\in\mathbb{Z}_{>0}$? |
|---|---|---|---|
| $0$ | $0$ | $0$ | no |
| $0$ | $2\delta_{q+1}$ | $\frac{2m-2n+2q-2p-1}{2m-2n+2q-2p}$ | no |
| $0$ | $\delta_{q+1}+\delta_{q+2}$ | $\frac{2m-2n+2q-2p+1}{2m-2n-2p+2q}$ | no |
| $2\delta_1$ | $0$ | $\frac{2q-2p+2}{2q-2p+1}$ | no |
| $2\delta_1$ | $2\delta_{q+1}$ | $2+\frac{1}{2q-2p+1}-\frac{1}{2m-2n+2q-2p}$ | no |
| $2\delta_1$ | $\delta_{q+1}+\delta_{q+2}$ | $2+\frac{1}{2q-2p+1}+\frac{1}{2m-2n+2q}$ | no |
| $\delta_1+\delta_2$ | $0$ | $\frac{2q-2p}{2q-2p+1}$ | no |
| $\delta_1+\delta_2$ | $2\delta_{q+1}$ | $2-\frac{1}{2q-2p+1}-\frac{1}{2m-2n+2q-2p}$ | no |
| $\delta_1+\delta_2$ | $\delta_{q+1}+\delta_{q+2}$ | $2-\frac{1}{2q+1}+\frac{1}{2m-2n+2q}$ | no |

$\boxed{k=(3-2m+2n)/2}$

| $D(p,q)$ | $B(m-p,n-q)$ | $\frac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}+\frac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}$ | $\in\mathbb{Z}_{>0}$? |
|---|---|---|---|
| $0$ | $0$ | $0$ | no |
| $0$ | $2\delta_{q+1}$ | $\frac{4m-4n+4q-4p-2}{2m-2n+4q-4p+1}$ | no (even/odd) |

| | | | |
|---|---|---|---|
| $0$ | $\delta_{q+1}+\delta_{q+2}$ | $\dfrac{4m-4n+4q-4p+2}{2m-2n-4p+4q+1}$ | no |
| $2\delta_1$ | $0$ | $\dfrac{4q-4p+4}{2m-2n+4q-4p+1}$ | no |
| $2\delta_1$ | $2\delta_{q+1}$ | $\dfrac{4m-4n+8q-8p+2}{2m-2n-4p+4q+1}$ | no |
| $2\delta_1$ | $\delta_{q+1}+\delta_{q+2}$ | $\dfrac{4m-4n++8q-8p+6}{2m-2n-4p+4q+1}$ | no |
| $\delta_1+\delta_2$ | $0$ | $\dfrac{4q-4p}{2m-2n-4p+4q+1}$ | no |
| $\delta_1+\delta_2$ | $2\delta_{q+1}$ | $\dfrac{4m-4n+8q-8p-2}{2m-2n-4p+4q+1}$ | no |
| $\delta_1+\delta_2$ | $\delta_{q+1}+\delta_{q+2}$ | $\dfrac{4m-4n+8q-8p+2}{2m-2n-4p+4q+1}$ | no |

### 10.1.2 $\mathfrak{g}^0 = C_q \oplus B(m,n-q)$

In Chapter 4, we obtained

$$\mathfrak{g}^1 = V_{C_q}(\delta_1)\otimes V_{B(m,n-q)}(\delta_{q+1}).$$

The decomposition of $\mathfrak{g}^1\otimes\mathfrak{g}^1$ is

$$\begin{aligned}\mathfrak{g}^1\otimes\mathfrak{g}^1 = &\big(V_{C_q}(2\delta_1)\oplus V_{C_q}(\delta_1+\delta_2)\oplus V_{C_q}(0)\big)\otimes\\ &\otimes\big(V_{B(m,n-q)}(2\delta_{q+1})\oplus V_{B(m,n-q)}(\delta_{q+1}+\delta_{q+2})+V_{B(m,n-q)}(0)\big).\end{aligned}$$

In the following table we summarize the data needed to test the conditions of Theorem 1.8.1

| subalgebra | weight | $k$ | $\frac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}$ |
|---|---|---|---|
| $C_q$ | $2\delta_1$ | $1$ | $\frac{2q+2}{2q+1}$ |
| $C_q$ | $\delta_1+\delta_2$ | $1$ | $\frac{2q}{2q+1}$ |
| $B(m,n-q)$ | $2\delta_{q+1}$ | $1$ | $\frac{2m-2n+2q-1}{2m-2n+2q}$ |
| $B(m,n-q)$ | $\delta_{q+1}+\delta_{q+2}$ | $1$ | $\frac{2m-2n+2q+1}{2m-2n+2q}$ |
| $C_q$ | $2\delta_1$ | $(3-2m+2n)/2$ | $\frac{4q+4}{4q+1+2m-2n}$ |
| $C_q$ | $\delta_1+\delta_2$ | $(3-2m+2n)/2$ | $\frac{4q}{4q+1+2m-2n}$ |
| $B(m,n-q)$ | $2\delta_{q+1}$ | $(3-2m+2n)/2$ | $\frac{4m-4n+4q-2}{4q+1+2m-2n}$ |
| $B(m,n-q)$ | $\delta_{q+1}+\delta_{q+2}$ | $(3-2m+2n)/2$ | $\frac{4m-4n+4q+2}{4q+1+2m-2n}$ |

For each choice of a weight of $C_q$ and $B(m,n-q)$ we test the criterion:

$\boxed{k=1}$

| $C_q$ | $B(m,n-q)$ | $\frac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}+\frac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}$ | $\in\mathbb{Z}_{>0}$? |
|---|---|---|---|
| $0$ | $0$ | $0$ | no |
| $0$ | $2\delta_{q+1}$ | $\frac{2m-2n+2q-1}{2m-2n+2q}$ | no |
| $0$ | $\delta_{q+1}+\delta_{q+2}$ | $\frac{2m-2n+2q+1}{2m-2n+2q}$ | no |
| $2\delta_1$ | $0$ | $\frac{2q+2}{2q+1}$ | no |
| $2\delta_1$ | $2\delta_{q+1}$ | $2+\frac{1}{2q+1}-\frac{1}{2m-2n+2q}$ | no |
| $2\delta_1$ | $\delta_{q+1}+\delta_{q+2}$ | $2+\frac{1}{2q+1}+\frac{1}{2m-2n+2q}$ | no |

| | | | |
|---|---|---|---|
| $\delta_1+\delta_2$ | $0$ | $\frac{2q}{2q+1}$ | no |
| $\delta_1+\delta_2$ | $2\delta_{q+1}$ | $2-\frac{1}{2q+1}-\frac{1}{2m-2n+2q}$ | no |
| $\delta_1+\delta_2$ | $\delta_{q+1}+\delta_{q+2}$ | $2-\frac{1}{2q+1}+\frac{1}{2m-2n+2q}$ | no |

$\boxed{k=(3-2m+2n)/2}$

| $C_q$ | $B(m,n-q)$ | $\frac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}+\frac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}$ | $\in\mathbb{Z}_{>0}$? |
|---|---|---|---|
| $0$ | $0$ | $0$ | no |
| $0$ | $2\delta_{q+1}$ | $\frac{4n-4m-4q+2}{4q+1+2m-2n}$ | no |
| $0$ | $\delta_{q+1}+\delta_{q+2}$ | $\frac{4n-4m-4q-2}{4q+1+2m-2n}$ | no |
| $2\delta_1$ | $0$ | $\frac{4q+4}{4q+1+2m-2n}$ | no |
| $2\delta_1$ | $2\delta_{q+1}$ | $\frac{4n-4m+6}{4q+1+2m-2n}$ | no |
| $2\delta_1$ | $\delta_{q+1}+\delta_{q+2}$ | $\frac{4n-4m+2}{4q+1+2m-2n}$ | no |
| $\delta_1+\delta_2$ | $0$ | $\frac{4q}{4q+1+2m-2n}$ | no |
| $\delta_1+\delta_2$ | $2\delta_{q+1}$ | $\frac{4n-4m+2}{4q+1+2m-2n}$ | no |
| $\delta_1+\delta_2$ | $\delta_{q+1}+\delta_{q+2}$ | $\frac{4n-4m-2}{4q+1+2m-2n}$ | no |

### 10.1.3 $\mathfrak{g}^0=D_p\oplus B(m-p,n)$

In Chapter 4, we obtained

$$\mathfrak{g}^1=V_{D_p}(\varepsilon_1)\otimes V_{B(m-p,n)}(\delta_1).$$

The decomposition of $\mathfrak{g}^1 \otimes \mathfrak{g}^1$ is

$$\begin{aligned}\mathfrak{g}^1 \otimes \mathfrak{g}^1 = &\big(V_{D_p}(2\varepsilon_1) \oplus V_{D_p}(\varepsilon_1+\varepsilon_2) \oplus V_{D_p}(0)\big) \otimes \\ &\otimes \big(V_{B(m-p,n)}(2\delta_1) \oplus V_{B(m-p,n)}(\delta_1+\delta_2) + V_{B(m-p,n)}(0)\big)\,.\end{aligned}$$

In the following table we summarize the data needed to test the conditions of Theorem 1.8.1

| subalgebra | weight | $k$ | $\dfrac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}$ |
|---|---|---|---|
| $D_p$ | $2\varepsilon_1$ | 1 | $\dfrac{2p}{2p-1}$ |
| $D_p$ | $\varepsilon_1+\varepsilon_2$ | 1 | $\dfrac{2p-2}{2p-1}$ |
| $B(m-p,n)$ | $2\delta_1$ | 1 | $\dfrac{2n-2m+2p+1}{2m-2p-2n}$ |
| $B(m-p,n)$ | $\delta_1+\delta_2$ | 1 | $\dfrac{2n-2m+2p-1}{2m-2p-2n}$ |
| $D_p$ | $2\varepsilon_1$ | $(3-2m+2n)/2$ | $\dfrac{4p}{4p-1-2m+2n}$ |
| $D_p$ | $\varepsilon_1+\varepsilon_2$ | $(3-2m+2n)/2$ | $\dfrac{4p-4}{4p-1-2m+2n}$ |
| $B(m-p,n)$ | $2\delta_1$ | $(3-2m+2n)/2$ | $\dfrac{4n-4m+4p+2}{4p-1-2m+2n}$ |
| $B(m-p,n)$ | $\delta_1+\delta_2$ | $(3-2m+2n)/2$ | $\dfrac{4n-4m+4p-2}{4p-1-2m+2n}$ |

For each choice of a weight of $D_p$ and $B(m-p,n)$ we test the criterion:

$\boxed{k=1}$

| $D_p$ | $B(m-p,n)$ | $\frac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}+\frac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}$ | $\in\mathbb{Z}_{>0}$? |
|---|---|---|---|
| $0$ | $0$ | $0$ | no |
| $0$ | $2\delta_1$ | $\frac{2n-2m+2p+1}{2m-2p-2n}$ | no |
| $0$ | $\delta_1+\delta_2$ | $\frac{2n-2m+2p-1}{2m-2p-2n}$ | no |
| $2\varepsilon_1$ | $0$ | $\frac{2p}{2p-1}$ | no |
| $2\varepsilon_1$ | $2\delta_1$ | $\frac{1}{2p-1}+\frac{1}{2m-2n-2p}$ | no |
| $2\varepsilon_1$ | $\delta_1+\delta_2$ | $\frac{1}{2p-1}-\frac{1}{2m-2n-2p}$ | no |
| $\varepsilon_1+\varepsilon_2$ | $0$ | $\frac{2p-2}{2p-1}$ | no |
| $\varepsilon_1+\varepsilon_2$ | $2\delta_1$ | $-\frac{1}{2p-1}+\frac{1}{2m-2n-2p}$ | no |
| $\varepsilon_1+\varepsilon_2$ | $\delta_1+\delta_2$ | $-\frac{1}{2p-1}-\frac{1}{2m-2n-2p}$ | no |

$\boxed{k=(3-2m+2n)/2}$

| $D_p$ | $B(m-p,n)$ | $\frac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}+\frac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}$ | $\in\mathbb{Z}_{>0}$? |
|---|---|---|---|
| $0$ | $0$ | $0$ | no |
| $0$ | $2\delta_1$ | $\frac{4n-4m+4p+2}{4p-1-2m+2n}$ | no |
| $0$ | $\delta_1+\delta_2$ | $\frac{4n-4m+4p-2}{4p-1-2m+2n}$ | no |
| $2\varepsilon_1$ | $0$ | $\frac{4p}{4p-1-2m+2n}$ | no |
| $2\varepsilon_1$ | $2\delta_1$ | $\frac{4n-4m+8p+2}{4p-1-2m+2n}$ | no |

| | | | |
|---|---|---|---|
| $2\varepsilon_1$ | $\delta_1+\delta_2$ | $\dfrac{4n-4m+8p-2}{4p-1-2m+2n}$ | no |
| $\varepsilon_1+\varepsilon_2$ | $0$ | $\dfrac{4p-4}{4p-1-2m+2n}$ | no |
| $\varepsilon_1+\varepsilon_2$ | $2\delta_1$ | $\dfrac{4n-4m+8p-2}{4p-1-2m+2n}$ | no |
| $\varepsilon_1+\varepsilon_2$ | $\delta_1+\delta_2$ | $\dfrac{4n-4m+8p-6}{4p-1-2m+2n}$ | no |

### 10.1.4 $\mathfrak{g}^0 = C_n \oplus B_m$

In Chapter 4, we obtained

$$\mathfrak{g}^1 = V_{C_n}(\delta_1) \otimes V_{B_n}(\varepsilon_1).$$

The decomposition of $\mathfrak{g}^1 \otimes \mathfrak{g}^1$ is

$$\begin{aligned}\mathfrak{g}^1 \otimes \mathfrak{g}^1 = &(V_{C_n}(2\delta_1) \oplus V_{C_n}(\delta_1+\delta_2) \oplus V_{C_n}(0)) \otimes \\ &\otimes (V_{B_m}(2\varepsilon_1) \oplus V_{B_m}(\varepsilon_1+\varepsilon_2) + V_{B_m}(0)).\end{aligned}$$

In the following table we summarize the data needed to test the conditions of Theorem 1.8.1

| subalgebra | weight | $k$ | $\dfrac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}$ |
|---|---|---|---|
| $C_n$ | $2\delta_1$ | $1$ | $\dfrac{2n+2}{2n+1}$ |
| $C_n$ | $\delta_1+\delta_2$ | $1$ | $\dfrac{2n}{2n+1}$ |
| $B_m$ | $2\varepsilon_1$ | $1$ | $\dfrac{2m+1}{2m}$ |
| $B_m$ | $\varepsilon_1+\varepsilon_2$ | $1$ | $\dfrac{2m-1}{2m}$ |
| $C_n$ | $2\delta_1$ | $(3-2m+2n)/2$ | $\dfrac{4n+4}{1+2n+2m}$ |
| $C_n$ | $\delta_1+\delta_2$ | $(3-2m+2n)/2$ | $\dfrac{4n}{1+2n+2m}$ |

| | | | |
|---|---|---|---|
| $B_m$ | $2\varepsilon_1$ | $(3-2m+2n)/2$ | $\dfrac{4m+2}{1+2m+2n}$ |
| $B_m$ | $\varepsilon_1+\varepsilon_2$ | $(3-2m+2n)/2$ | $\dfrac{4m-2}{1+2m+2n}$ |

For each choice of a weight of $C_n$ and $B_m$ we test the criterion:

$\boxed{k=1}$

| $C_n$ | $B_m$ | $\dfrac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}+\dfrac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}$ | $\in \mathbb{Z}_{>0}$? |
|---|---|---|---|
| $0$ | $0$ | $0$ | no |
| $0$ | $2\varepsilon_1$ | $\dfrac{2m+1}{2m}$ | no |
| $0$ | $\varepsilon_1+\varepsilon_2$ | $\dfrac{2m-1}{2m}$ | no |
| $2\delta_1$ | $0$ | $\dfrac{2n+2}{2n+1}$ | no |
| $2\delta_1$ | $2\varepsilon_1$ | $2+\dfrac{1}{2n+1}+\dfrac{1}{2m}$ | no |
| $2\delta_1$ | $\varepsilon_1+\varepsilon_2$ | $2+\dfrac{1}{2n+1}-\dfrac{1}{2m}$ | no |
| $\delta_1+\delta_2$ | $0$ | $\dfrac{2n}{2n+1}$ | no |
| $\delta_1+\delta_2$ | $2\varepsilon_1$ | $2-\dfrac{1}{2n+1}+\dfrac{1}{2m}$ | no |
| $\delta_1+\delta_2$ | $\varepsilon_1+\varepsilon_2$ | $2-\dfrac{1}{2n+1}-\dfrac{1}{2m}$ | no |

$\boxed{k=(3-2m+2n)/2}$

| $C_n$ | $B_m$ | $\dfrac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}+\dfrac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}$ | $\in \mathbb{Z}_{>0}$? |
|---|---|---|---|
| $0$ | $0$ | $0$ | no |
| $0$ | $2\varepsilon_1$ | $\dfrac{4m+2}{1+2m+2n}$ | no |
| $0$ | $\varepsilon_1+\varepsilon_2$ | $\dfrac{4m-2}{1+2m+2n}$ | no |
| $2\delta_1$ | $0$ | $\dfrac{4n+4}{1+2m+2n}$ | no |
| $2\delta_1$ | $2\varepsilon_1$ | $\dfrac{4n+4m+6}{1+2m+2n}$ | no |
| $2\delta_1$ | $\varepsilon_1+\varepsilon_2$ | $\dfrac{4n+4m+2}{1+2m+2n}$ | no |
| $\delta_1+\delta_2$ | $0$ | $\dfrac{4n}{1+2m+2n}$ | no |
| $\delta_1+\delta_2$ | $2\varepsilon_1$ | $\dfrac{4n+4m+2}{1+2m+2n}$ | no |
| $\delta_1+\delta_2$ | $\varepsilon_1+\varepsilon_2$ | $\dfrac{4n+4m-2}{1+2m+2n}$ | no |

### 10.1.5 $\mathfrak{g}^0 = D(m,n)$

In Chapter 4, we obtained

$$\mathfrak{g}^1 = V_{D(m,n)}(\delta_1).$$

The decomposition of $\mathfrak{g}^1 \otimes \mathfrak{g}^1$ is

$$\mathfrak{g}^1 \otimes \mathfrak{g}^1 = V_{D(m,n)}(2\delta_1) \oplus V_{D(m,n)}(\delta_1+\delta_2) \oplus V_{D(m,n)}(0)$$

In the following table we test the conditions of Theorem 1.8.1.

| subalgebra | weight | $k$ | $\dfrac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}$ | $\in \mathbb{Z}_{>0}$? |
|---|---|---|---|---|
| $D(m,n)$ | $2\delta_1$ | $(3-2m+2n)/2$ | $\dfrac{4m-4n-4}{2m-2n-1}$ | no |
| $D(m,n)$ | $\delta_1+\delta_2$ | $(3-2m+2n)/2$ | $\dfrac{4m-4n}{2m-2n-1}$ | no |

### 10.1.6 Conclusions

When $\mathfrak{g} = B(m,n)$, every maximal regular subalgebra satisfy the hypothesis of Theorem 1.8.1, for both conformal levels. Then the following decompositions hold:

$$\begin{aligned}
V_1(B(m,n)) =& V_1(D(p,q)) \oplus V_1(B(m-p,n-q)) \oplus L_{D(p,q)}(\delta_1) \oplus L_{B(m-p,n-q)}(\delta_{q+1})\\
=& V_{-1/2}(C_q) \oplus V_1 B(m,n-q) \oplus L_{C_q}(\delta_1) \oplus L_{B(m,n-q)}(\delta_{q+1})\\
=& V_1(D_p) \oplus V_1(B(m-p,n)) \oplus L_{D_p}(\varepsilon_1) \oplus L_{B(m-p,n)}(\delta_1)\\
=& V_{-1/2}(C_n) \oplus V_1(B_m) \oplus L_{C_n}(\delta_1) \oplus L_{B_m}(\varepsilon_1)\\
=& V_1(D(m,n)) \oplus L_{D(m,n)}(\delta_1)\\
V_{(3-2m+2n)/2}(B(m,n)) =& V_{(3-2m+2n)/2}(D(p,q)) \oplus V_{(3-2m+2n)/2}(B(m-p,n-q)) \oplus\\
& \oplus L_{D(p,q)}(\delta_1) \oplus L_{B(m-p,n-q)}(\delta_{q+1})\\
=& V_{-(3-2m+2n)/4}(C_q) \oplus V_{(3-2m+2n)/2} B(m,n-q) \oplus L_{C_q}(\delta_1) \oplus L_{B(m,n-q)}(\delta_{q+1})\\
=& V_{(3-2m+2n)/2}(D_p) \oplus V_{(3-2m+2n)/2}(B(m-p,n)) \oplus L_{D_p}(\varepsilon_1) \oplus L_{B(m-p,n)}(\delta_1)\\
=& V_{-1(3-2m+2n)/4}(C_n) \oplus V_{(3-2m+2n)/2}(B_m) \oplus L_{C_n}(\delta_1) \oplus L_{B_m}(\varepsilon_1)\\
=& V_{(3-2m+2n)/2}(D(m,n)) \oplus L_{D(m,n)}(\delta_1)
\end{aligned}$$

## 10.2 Decomposition for $\mathfrak{g} = D(m,n)$

When $\mathfrak{g}^0 \subset \mathfrak{g} = D(m,n)$ is a regular equal rank subalgebra, we can decompose $\mathfrak{g}^{\otimes}\mathfrak{g}^1$ using Proposition 10.1.1. Unfortunately, the hypotheses of Theorem 1.8.1 hold only whit the conformal level $k = 1$.

### 10.2.1 $\mathfrak{g}^0 = D(p,q) \oplus D(m-p,n-q)$

In Chapter 5, we obtained

$$\mathfrak{g}^1 = V_{D(p,q)}(\delta_1) \oplus V_{D(m-p,n-q)}(\delta_{q+1})$$

The decomposition of $\mathfrak{g}^1 \otimes \mathfrak{g}^1$ is

$$\begin{aligned}
\mathfrak{g}^1 \otimes \mathfrak{g}^1 =& (V_{D(p,q)}(2\delta_1) \oplus V_{D(p,q)}(\delta_1+\delta_2) \oplus V_{D(p,q)}(0))\\
& \otimes (V_{D(m-p,n-q)}(2\delta_{q+1}) \oplus V_{D(m-p,n-q)}(\delta_{q+1}+\delta_{q+2}) + V_{D(m-p,n-q)}(0)).
\end{aligned}$$

| subalgebra | weight | $k$ | $\frac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}$ |
|---|---|---|---|
| $D(p,q)$ | $2\delta_1$ | 1 | $\frac{2(q-p+1)}{2q-2p+1}$ |
| $D(p,q)$ | $\delta_1+\delta_2$ | 1 | $\frac{2(q-p)}{2q-2p+1}$ |
| $D(m-p,n-q)$ | $2\delta_{q+1}$ | 1 | $\frac{2(n-m-q+p+1)}{2n-2m+2p-2q+1}$ |
| $D(m-p,n-q)$ | $\delta_{q+1}+\delta_{q+2}$ | 1 | $\frac{2(n-m-q+p)}{2n-2m+2p-2q+1}$ |

For each choice of a weight of $D(p,q)$ and $D(m-p,n-q)$ we test the criterion:

| $D(p,q)$ | $D(m-p,n-q)$ | $\frac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}+\frac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}$ | $\in\mathbb{Z}_{>0}$? |
|---|---|---|---|
| 0 | 0 | 0 | no |
| 0 | $2\delta_{q+1}$ | $\frac{2(n-m-q+p+1)}{2n-2m+2p-2q+1}$ | no |
| 0 | $\delta_{q+1}+\delta_{q+2}$ | $\frac{2(n-m-q+p)}{2n-2m+2p-2q+1}$ | no |
| $2\delta_1$ | 0 | $\frac{2(q-p+1)}{2q-2p+1}$ | no |
| $2\delta_1$ | $2\delta_{q+1}$ | $2+\frac{1}{2q-2p+1}+\frac{1}{2n-2m+2p-2q+1}$ | no |
| $2\delta_1$ | $\delta_{q+1}+\delta_{q+2}$ | $2+\frac{1}{2q-2p+1}-\frac{1}{2n-2m+2p-2q+1}$ | no |
| $\delta_1+\delta_2$ | 0 | $\frac{2(q-p)}{2q-2p+1}$ | no |
| $\delta_1+\delta_2$ | $2\delta_{q+1}$ | $2-\frac{1}{2q-2p+1}+\frac{1}{2n-2m+2p-2q+1}$ | no |
| $\delta_1+\delta_2$ | $\delta_{q+1}+\delta_{q+2}$ | $2-\frac{1}{2q-2p+1}-\frac{1}{2n-2m+2p-2q+1}$ | no |

### 10.2.2 $\mathfrak{g}^0 = D(p,n) \oplus D_{m-p}$

In Chapter 5, we obtained

$$\mathfrak{g}^1 = V_{D(p,n)}(\delta_1) \otimes V_{D_{m-p}}(\varepsilon_{p+1}).$$

The decomposition of $\mathfrak{g}^1 \otimes \mathfrak{g}^1$ is

$$\begin{aligned}\mathfrak{g}^1 \otimes \mathfrak{g}^1 = &(V_{D(p,n)}(2\delta_1) \oplus V_{D(p,n)}(\delta_1+\delta_2) \oplus V_{D(p,n)}(0))\\ &\otimes (V_{D_{m-p}}(2\varepsilon_{p+1}) \oplus V_{D_{m-p}}(\varepsilon_{p+1}+\varepsilon_{p+2}) + V_{D_{m-p}}(0))\end{aligned}$$

| subalgebra | weight | $k$ | $\dfrac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}$ |
|---|---|---|---|
| $D(p,n)$ | $2\delta_1$ | 1 | $\dfrac{2(n-p+1)}{2n-2p+1}$ |
| $D(p,n)$ | $\delta_1+\delta_2$ | 1 | $\dfrac{2(n-p)}{2n-2p+1}$ |
| $D_{m-p}$ | $2\varepsilon_{p+1}$ | 1 | $\dfrac{2(m-p)}{2m-2p-1}$ |
| $D_{m-p}$ | $\varepsilon_{p+1}+\varepsilon_{p+2}$ | 1 | $\dfrac{2(m-p)}{2m-2p-1}$ |

For each choice of a weight of $D(p,n)$ and $D_{m-p}$ we test the criterion:

| $D(p,n)$ | $D_{m-p}$ | $\dfrac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)} + \dfrac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}$ | $\in \mathbb{Z}_{>0}$? |
|---|---|---|---|
| 0 | 0 | 0 | no |
| 0 | $2\varepsilon_{p+1}$ | $\dfrac{2(m-p)}{2m-2p-1}$ | no |
| 0 | $\varepsilon_{p+1}+\varepsilon_{p+2}$ | $\dfrac{2(m-p+1)}{2m-2p-1}$ | no |
| $2\delta_1$ | 0 | $\dfrac{2(n-p+1)}{2n-2p+1}$ | no |
| $2\delta_1$ | $2\varepsilon_{p+1}$ | $2+\dfrac{1}{2p-2p+1}+\dfrac{1}{2m-2p-1}$ | no |

| | | | |
|---|---|---|---|
| $2\delta_1$ | $\varepsilon_{p+1}+\varepsilon_{p+2}$ | $2+\dfrac{1}{2n-2p+1}-\dfrac{1}{2m-2p-1}$ | no |
| $\delta_1+\delta_2$ | $0$ | $\dfrac{2(n-p)}{2n-2p+1}$ | no |
| $\delta_1+\delta_2$ | $2\varepsilon_{p+1}$ | $2-\dfrac{1}{2n-2p+1}+\dfrac{1}{2m-2p-1}$ | no |
| $\delta_1+\delta_2$ | $\varepsilon_{p+1}+\varepsilon_{p+2}$ | $2-\dfrac{1}{2n-2p+1}-\dfrac{1}{2m-2p-1}$ | no |

### 10.2.3 $\mathfrak{g}^0 = D(m,q) \oplus C_{n-q}$

In Chapter 5, we obtained

$$\mathfrak{g}^1 = V_{D(m,q)}(\delta_1) \oplus V_{C_{n-q}}(\delta_{q+1})$$

The decomposition of $\mathfrak{g}^1 \otimes \mathfrak{g}^1$ is

$$\begin{aligned}\mathfrak{g}^1 \otimes \mathfrak{g}^1 = &\left(V_{D(m,q)}(2\delta_1) \oplus V_{D(m,q)}(\delta_1+\delta_2) \oplus V_{D(m,q)}(0)\right) \otimes \\ &\otimes \left(V_{C_{n-q}}(2\delta_{q+1}) \oplus V_{C_{n-q}}(\delta_{q+1}+\delta_{q+2}) + V_{C_{n-q}}(0)\right)\end{aligned}$$

| subalgebra | weight | $k$ | $\dfrac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}$ |
|---|---|---|---|
| $D(m,q)$ | $2\delta_1$ | 1 | $\dfrac{2(m-q+1)}{2m-2q-1}$ |
| $D(m,q)$ | $\delta_1+\delta_2$ | 1 | $\dfrac{2(m-q)}{2m-2q-1}$ |
| $C_{n-q}$ | $2\delta_{q+1}$ | 1 | $\dfrac{2(n-q+1)}{2n-2q+1}$ |
| $C_{n-q}$ | $\delta_{q+1}+\delta_{q+2}$ | 1 | $\dfrac{2(n-q)}{2n-2q+1}$ |

For each choice of a weight of $D(m,q)$ and $C_{n-q}$ we test the criterion:

| $D(m,q)$ | $C_{n-q}$ | $\frac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}+\frac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}$ | $\in\mathbb{Z}_{>0}$? |
|---|---|---|---|
| $0$ | $0$ | $0$ | no |
| $0$ | $2\delta_{q+1}$ | $\frac{2(n-q+1)}{2n-2q+1}$ | no |
| $0$ | $\delta_{q+1}+\delta_{q+2}$ | $\frac{2(n-q)}{2n-2q+1}$ | no |
| $2\delta_1$ | $0$ | $\frac{2(m-q-1)}{2m-2q-1}$ | no |
| $2\delta_1$ | $2\delta_{q+1}$ | $2-\frac{1}{2m-2q-1}+\frac{1}{2n-2q+1}$ | no |
| $2\delta_1$ | $\delta_{q+1}+\delta_{q+2}$ | $2-\frac{1}{2m-2p-1}-\frac{1}{2n-2q+1}$ | no |
| $\delta_1+\delta_2$ | $0$ | $\frac{2(m-q)}{2m-2q-1}$ | no |
| $\delta_1+\delta_2$ | $2\delta_{q+1}$ | $2+\frac{1}{2m-2q-1}+\frac{1}{2n-2q+1}$ | no |
| $\delta_1+\delta_2$ | $\delta_{q+1}+\delta_{q+2}$ | $2+\frac{1}{2m-2q-1}-\frac{1}{2n-2q+1}$ | no |

### 10.2.4 $\mathfrak{g}^0 = D_m \oplus C_n$

In Chapter 5, we obtained

$$\mathfrak{g}^1 = V_{D_m}(\varepsilon_1)\oplus V_{C_n}(\delta_1)$$

The decomposition of $\mathfrak{g}^1\otimes\mathfrak{g}^1$ is

$$\mathfrak{g}^1\otimes\mathfrak{g}^1 = (V_{D_m}(2\varepsilon_1)\oplus V_{D_m}(\varepsilon_1+\varepsilon_2)\oplus V_{D_m}(0))\otimes(V_{C_n}(2\delta_1)\oplus V_{C_n}(\delta_1+\delta_2)+V_{C_n}(0))$$

| subalgebra | weight | $k$ | $\frac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}$ |
|---|---|---|---|
| $D_m$ | $2\varepsilon_1$ | 1 | $\frac{2(m-q+1)}{2m-2q-1}$ |
| $D_m$ | $\varepsilon_1+\varepsilon_2$ | 1 | $\frac{2(m-q)}{2m-2q-1}$ |

| | | | |
|---|---|---|---|
| $C_n$ | $2\delta_1$ | 1 | $\dfrac{2(n-q+1)}{2n-2q+1}$ |
| $C_n$ | $\delta_1+\delta_2$ | 1 | $\dfrac{2(n-q)}{2n-2q+1}$ |

For each choice of a weight of $D(m,q)$ and $C_{n-q}$ we test the criterion:

| $D(m,q)$ | $C_{n-q}$ | $\dfrac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}+\dfrac{(2\rho+\omega,\omega)}{2(u(k)+h^\vee)}$ | $\in\mathbb{Z}_{>0}$? |
|---|---|---|---|
| 0 | 0 | 0 | no |
| 0 | $2\delta_{q+1}$ | $\dfrac{2(n-q+1)}{2n-2q+1}$ | no |
| 0 | $\delta_{q+1}+\delta_{q+2}$ | $\dfrac{2(n-q)}{2n-2q+1}$ | no |
| $2\delta_1$ | 0 | $\dfrac{2(m-q-1)}{2m-2q-1}$ | no |
| $2\delta_1$ | $2\delta_{q+1}$ | $2-\dfrac{1}{2m-2q-1}+\dfrac{1}{2n-2q+1}$ | no |
| $2\delta_1$ | $\delta_{q+1}+\delta_{q+2}$ | $2-\dfrac{1}{2m-2p-1}-\dfrac{1}{2n-2q+1}$ | no |
| $\delta_1+\delta_2$ | 0 | $\dfrac{2(m-q)}{2m-2q-1}$ | no |
| $\delta_1+\delta_2$ | $2\delta_{q+1}$ | $2+\dfrac{1}{2m-2q-1}+\dfrac{1}{2n-2q+1}$ | no |
| $\delta_1+\delta_2$ | $\delta_{q+1}+\delta_{q+2}$ | $2+\dfrac{1}{2m-2q-1}-\dfrac{1}{2n-2q+1}$ | no |

### 10.2.5 Conclusions

When $\mathfrak{g}=D(m,n)$, every maximal regular subalgebra satisfy the hypothesis of Theorem 1.8.1 at the conformal level $k=1$. Then the following decompositions hold:

$$\begin{aligned}
V_1(D(m,n)) =&V_1(D(p,q)) \oplus V_1(D(m-p,n-q)) \oplus L_{D(p,q)}(\delta_1) \oplus L_{D(m-p,n-q)}(\delta_{q+1}) \\
=&V_1(D(p,n)) \oplus V_1(D_{m-p}) \oplus L_{D(p,n)}(\delta_1) \oplus L_{D_{m-p}}(\varepsilon_{p+1}) \\
=&V_1(D(m,q)) \oplus V_{-1/2}(C_{n-q}) \oplus L_{D(m,q)}(\delta_1) \oplus L_{C_{n-q}}(\delta_{q+1}) \\
=&V_1(D_m) \oplus V_{-1/2}(C_n) \oplus L_{D_m}(\varepsilon_1) \oplus L_{C_n}(\delta_1)
\end{aligned}$$

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