Precision mathematics and approximation mathematics: the conceptual and educational role of their comparison.

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Abstract. The relationship between applied and pure mathematics is of utmost concern for Klein. Examples from Volume III of his "Elementarmathematik" illustrate how, starting from an intuitive and sometimes practical approach, Klein develops abstract concepts working in rich "mathematical environments". The examples concern the concept of empirical function, point sets obtained through circular inversion, and the "continuous" transformation of curves.

The lecture course of Felix Klein

Felix Klein's "Präzisions- und Approximationsmathematik", appeared in 1928 as volume III of the seminal series of lecture notes on elementary mathematics from a higher standpoint (*Elementarmathematik von einem höheren Standpunkte aus;* Klein, 1928). The 1928 edition was in its turn a re-edition of a lecture course delivered by Klein in 1901 with the title "Anwendung der Differential- und Integralrechnung auf die Geometrie: eine Revision der Prinzipien", published in 1902 in lithographic form (Klein, 1902; a reprint of 1908, edited by Conrad Heinrich Müller, left the text essentially unchanged).

In this third volume Klein explores the relationship between *precision mathematics and approximation mathematics.* He crosses between various fields of mathematics – from functions in one and two variables to practical geometry to space curves and surfaces – always underlining the relationship between the exactness of the idealised concepts and the approximations to be considered in applications.

The point of view is not that of mathematics as a service subject, rather that of

"... the heuristic value of the applied sciences as an aid to discovering new truths in mathematics" (The Evanston colloquium, lecture VI; Klein, 1894, p. 46).

Of course there is also "*a universal pedagogical principle to be observed in all mathematical instruction",* namely that

"It is not only admissible, but absolutely necessary, to be less abstract at the start, to have constant regard to the applications, and to refer to the refinements only gradually as the student becomes able to understand them" (ibid., p. 50).

Therefore, the logical procedures that lead to theorems are confronted with the way in which concepts are formed starting from observations.

The final part of the book concerns gestalt relations of curves and surfaces, and shows how *Klein* masters the art of describing geometrical forms; Klein appeals to intuition leading the reader to think at continuous transformations of the geometric objects considered.

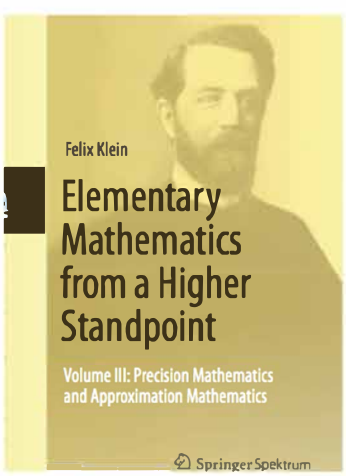


Figure 1. The 2016 edition of Precision Mathematics and Approximation Mathematics, translated from the third German edition (1928) by Marta Menghini with Anna Baccaglini-Frank as collaborator and Gert Schubring as advisor.

Volume III was translated for the first time in English in 2016 (Klein, 2016; see Fig. 1). It is not clear why it was not translated jointly with the first two volumes in the 1930s. Maybe its value for the training of teachers, clearly recognized as concerns the two first volumes, had not yet been understood (see Kilpatrick, this volume, Menghini & Schubring, 2016). Or, maybe, the decision was made because of the different role played by Klein in the 1928 edition of the third volume: Klein participated, together with Fritz Seyfarth, in the whole project of re-editing the three volumes on *Elementary mathematics from a higher standpoint*, but he died in 1925, after the first two volumes had appeared. The third volume was, therefore, edited only by Seyfarth; changes and insertions had been nevertheless discussed with Klein, as Seyfarth writes in his preface to the third edition (see Klein 2016, xiii). Probably for this same reason the third volume contains no particular indications of its importance for the training of the future mathematics teachers.

A translation after nearly a century, joined with a new edition of the two first volumes, must not be considered strange: this translation has an historical value, since Klein is one of the greatest mathematicians of history (books by Felix Klein are still in use today: Klein's *Nicht-euklidische Geometrie* has been re-edited in 2006, the English version of the *Lectures on the Ikosahedron* re-edited in 2007, and his *Development of mathematics in the nineteenth century* was translated in 1979); it also has a mathematical value – because of the interesting approaches and the links to applications. But, above all, it has a didactical value, concerning the training of mathematics teachers, for the reasons given at the beginning of this chapter, which we will try to explain more in depth through some examples.

The third volume focuses on precision and approximation mathematics, that is on the link between mathematics and its applications:

Precision mathematics includes all the propositions that can be logically deduced from the axioms of geometry or of analysis – obtained by abstraction from experience;

Approximation mathematics includes the results that can be obtained from experience with a certain degree of approximation.

Klein starts by considering those properties that applied mathematicians take for granted when studying certain phenomena from a mathematical point of view. These properties must be seen as supplementary conditions for the ideal objects of pure mathematics. In the meantime, these very properties prove to be the more intuitive ones. Therefore the comparison moves towards another field: it is a comparison between properties that can be considered only in the theoretical field of abstract mathematics and properties that can be grasped by intuition. This distinction still has repercussions in mathematics education today.

First example: empirical and idealised curve.

Klein makes an important distinction between functions arising out of applications of mathematics and functions as abstractions in their own right. This topic had been introduced by Klein in his *American conferences* of 1984:

"In imagining a line, we do not picture to ourselves "length without breadth", but a strip of a certain width. Now such a strip has of course always a tangent (Fig. 2); i.e. we can always imagine a straight strip having a small portion (element) in common with the curved strip; similarly with respect to the osculating circle. The definitions in this case are regarded as holding only approximately, or as far as may be necessary (Klein, 1984, p. 98)".

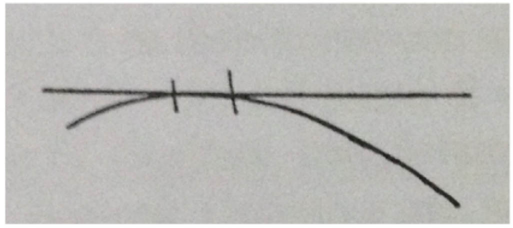


Figure 2. Taken from: Klein, 1984, pag. 98.

So we need to examine which restrictions we have to put to an idealized curve $y = f(x)$ so as to obtain that it corresponds to the concept of an empirical curve.

We have the perception that the empirical curve

a) is connected in its smallest parts, that is, it *takes on all values between the ordinates of two of its points*

b) encloses a specific *area* between the x-axis and the ordinates of two of its points

c) has everywhere a *slope* (and a *curvature*, ...)

d) has a finite number of maxima and minima

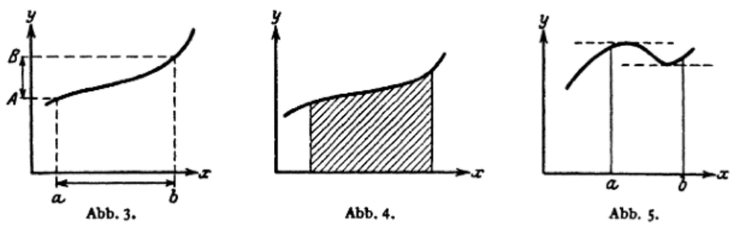


Figure 3. Taken from Klein, 1928, p. 22

How can we translate these intuitive properties into mathematical properties of $f(x)$ (Fig. 3)? The connectedness of point a) can be expressed by saying that $f(x)$ must be "continuous" (even if there is not a complete equivalence of the two meanings); point b) is simply translated into the fact that $f(x)$ is integrable; point c) means that $f(x)$ has a first (and second, …) derivative at any point; finally point d) is expressed by the property that *$f(x)$ must split – in the given interval – into a finite number of monotonous parts.*

So, for instance, a function as $y = \sin 1/x$ (Fig. 4) exists only in the ideal field: it belongs only to precision mathematics.

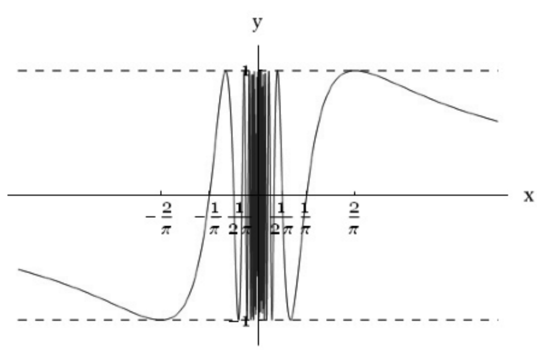


Figure 4. The function $y = \sin 1/x$

But the properties (theorems) that we can study mathematically on our idealized curve $f(x)$ – once we have considered the restrictions suggested by our experience – can then apply also to our empirical curve. So we go back from abstraction to applications.

In 1913/14 the Italian geometer Guido Castelnuovo gave a series of lectures at the University of Roma entitled "Matematica di precisione e matematica delle approssimazioni". The course was explicitly inspired by the course delivered by Felix Klein in 1902.

Castelnuovo states clearly at the beginning of his lecture course that the teaching and learning of mathematics would be more successful if it included, besides the logical procedures that lead to the theorems, also the way in which concepts are formed starting from observations, and how they can be verified in practice.

In 1911 the Liceo Moderno was established (Legge, 21 July 1911, n. 860), which effectively started in 1913. In this school, the preparation towards university studies (not necessarily of a scientific nature) was achieved through the study of Latin, modern languages, and the sciences (Marchi & Menghini, 2013). Mathematics is presented as an apt language for describing natural phenomena and a part of the programmes is concerned with approximation mathematics and its heuristic nature. Its mathematics programs were ascribed to Castelnuovo (see Castelnuovo, 1912):

"The renovation of the mathematics of the 17th century is linked to the blooming of the natural sciences. Within this context, the teacher will have to explain how the fundamental concepts of modern mathematics, particularly the concept of function, are implied by the observational sciences, and – being then rendered precise by mathematics – have in turn had a positive influence on the development of the latter (Castelnuovo, 1912, p. 124)".

Castelnuovo's course on precision and approximation mathematics is therefore very apt to train the teachers of his modern lyceé, which he hopes to become widespread. In particular, the comparison between empirical and idealized function describes very well the reciprocal aid of observational sciences and pure mathematics mentioned in the above quotation.

Second example: Iterated inversion with respect to three touching circles.

One of the reasons for putting applications at the beginning of a teaching sequence is, as already said, *the heuristic value of applied sciences.* Even when starting from a practical approach Klein develops more abstract concepts working in rich "mathematical environments", which form the core of a pertinent program for mathematics teacher education. In this case we are *not* necessarily interested in going back to applications: we take the idea from applications, and we work as mathematicians in the ideal field.

Let us start from an example taken from physics:

The method of image charges (also known as the method of images or method of mirror charges) is a technique to solve problems in electrostatics. The name comes from the fact that charged objects in the original problem are replaced with equivalent imaginary discrete point-charges, still satisfying the boundary conditions associated with the problem.

A simple case of the method of image charges is that of a point charge q, which we can consider located at the point $(0, 0, a)$ above an infinite grounded (i.e.: $V = 0$) conducting plate in the xy-plane. The problem can be simplified by replacing the plate of equipotential with a charge $-q$, located at $(0, 0, -a)$.

The method of images may also be applied to a sphere or to a cylinder. In fact, the case of image charges above a conducting plate in a plane can be considered as a particular case of images for a sphere. In this case a point-charge q lying inside the sphere at a distance l_1 from the origin has as its image another point-charge lying outside the sphere at a distance of R^2/ l_1 from the origin. So, the relation between the two charges is given by *circular inversion*. The potential produced by the two charges is zero on the surface of the sphere (Fig. 5).

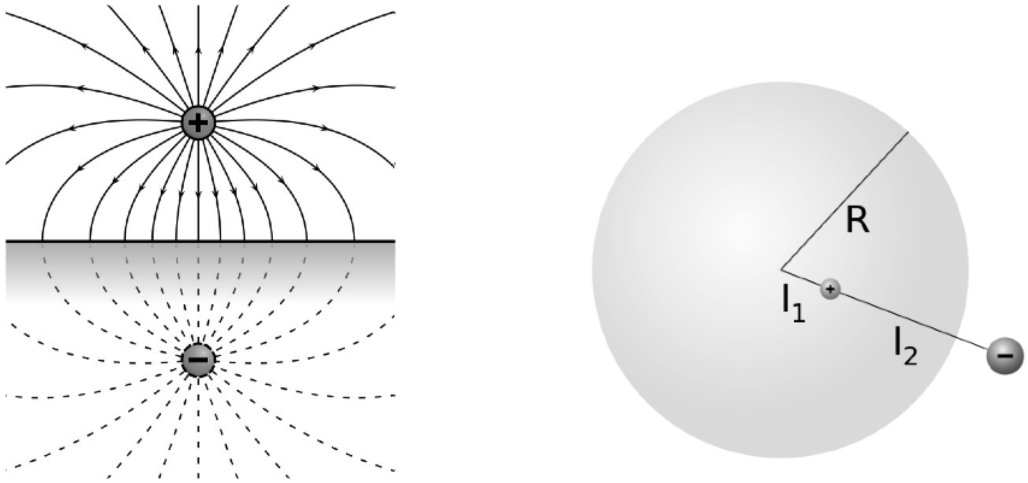


Figure 5. Image charges in the case of a plate (left) and of a sphere (right)

Klein states that already in 1850 *William Thomson* and *Bernhard Riemann*, when studying the equilibrium of charges on three *rotating cylinders* with parallel axes, observed that the "method of image charges" leads to the generation of a certain point set (see Thomson, 1853; we could not find anything explicit on this topic in Riemann's legacy).

We leave now the field of applications and turn completely to the *ideal field* considering three disjoint circles (which correspond to the normal section of the three rotating cylinders).

We consider the point set obtained from the points of a given region (the one outside the 3 circles) by applying any combination of the inversions in the three given circles, that is applying to the region the whole group of "transformations" that arises from the three "generators".

The text of Klein contains wonderful and clear drawings on this subject, but the use of a software like *Geogebra*, which has circular inversion in its menu, helps in following the probable development of the explanation that was delivered to Klein's students during the lecture course. In the following the drawings are either taken from the text of Klein or made with Geogebra. In this second case the "historic overview" of the various constructions is didactically important. Here we can only show some *snapshots*.

So, let us start from two circles (Fig. 6). In the first step we apply circular inversion with respect to the left circle, that is we reflect in the left circle the whole region outside, including the right circle. All the points of the infinite region outside the left circle are transformed in the points internal to the circle, and the right circle is transformed in a smaller circle inside the left circle.

To simplify our language we will from now on *only speak of the reflection of the circles*, without mentioning all the points of the regions contained in or outside them. So, the next step is the reflection of the left circle into the right one

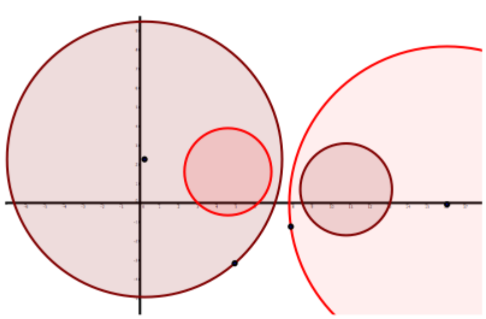


Figure 6

Now we can proceed in any order, for instance reflecting again the right circle with its internal circle into the smaller circle at the the left (Fig. 7)

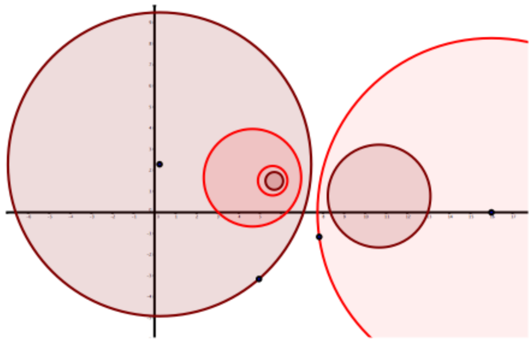


Figure 7

It is easy to understand that we obtain a configuration made by chains of circles each inside the other (Fig. 8).

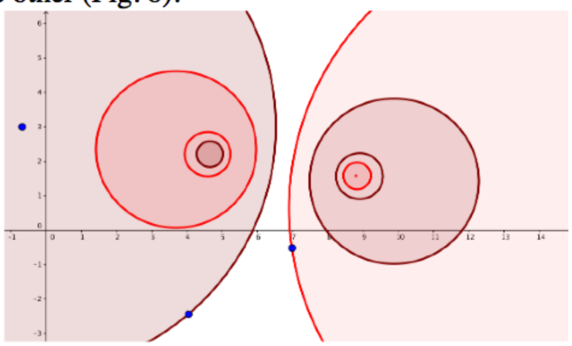


Figure 8

We then add the third circle, and continue to apply the circular inversion reflecting the whole configuration in the new circle, and also reflecting the new circle in the two former ones. We see that many chains of circles are appearing. In fact, an infinite number of chains (Fig. 9).

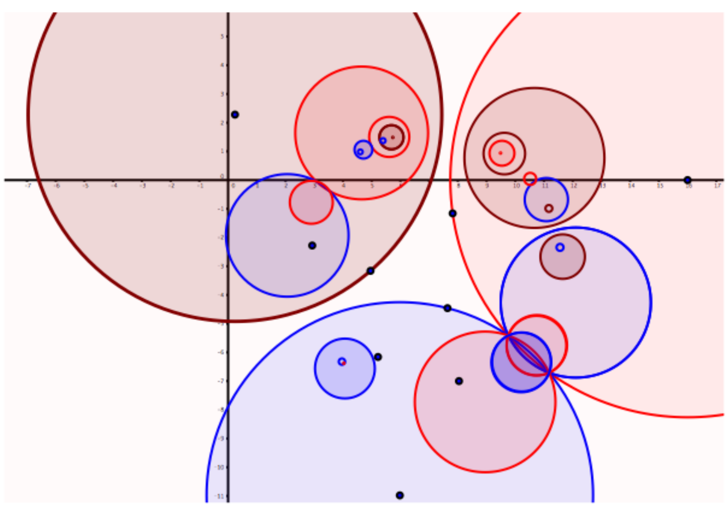


Figure 9

And now we have the fundamental transition from an empirical construction to the idealised field. We use the *axiom of the nested intervals*: *For every indefinitely decreasing sequence of closed intervals (segments, parts of curves, plane regions, parts of spaces), each of which contains all the following ones, there exists one and only one point common to all the intervals. This point is therefore univocally defined by the interval sequence.*

The axiom of the nested intervals allows us to say that each chain (sequence) has a *limit point.*

Speaking again of the regions transformed, we can say that we obtain a net of regions that fill the plane except for an infinite set of limit points: What can we say about this limit point set? We find out that it is *nowhere dense, but nevertheless perfect (it contains all of its accumulation points) and has the cardinality of the continuum* (like the Cantor set).

Now we continue the *joke* and consider three *touching* circles. And again we repeat the inversions. In the first step (Fig. 10) each circle contains the image of the two others. Each circle, as well as the external regions, contains curvilinear triangles.

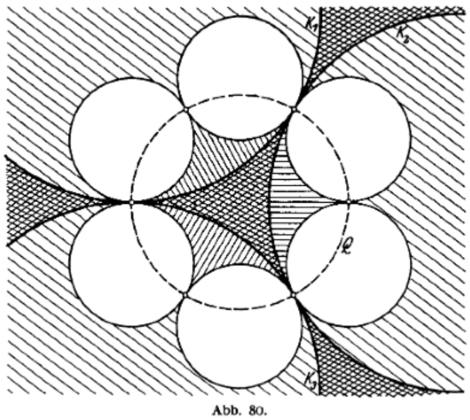


Figure 10.
Taken from Klein, 1928, p. 140
Taken from Klein, 1928, p. 141

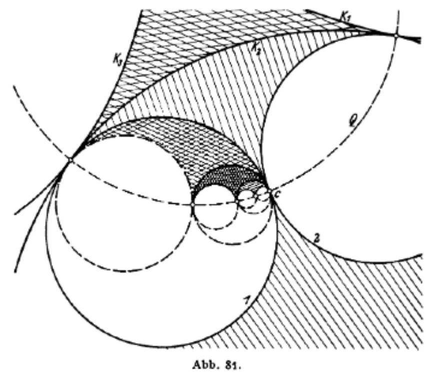


Figure 11.

The points of contact of the arising curvilinear triangles (and of the arising circles) accumulate on the orthogonal circle, that is the circle that is orthogonal to the three given ones (Fig. 10 and Fig. 11), as was also suggested by Fig. 9. *Proceeding with the construction, the orthogonal circle is filled more and more densely with the points of contact of two circles* (Fig. 12)

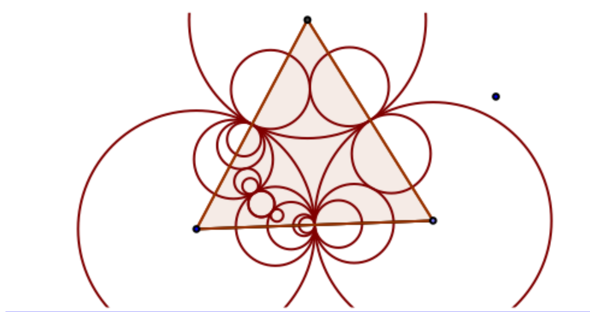


Figure 12

The next question is then: What can be said about the set of points of contact? *It is easy to understand that in the set of the points of contact each point is an accumulation point for the others, and that the set is everywhere dense on the periphery of the orthogonal circle.*

"*Until now I have spoken of these things in a somewhat indeterminate manner, because I did not refer to any quantitative relations but only to the figure as such. However, it is easy to give to the figure a form that allows an arithmetic interpretation of everything*" (Klein 2016, p. 157).

Following the idea of Klein, let us consider a circle with its centre on the contact point of two of the three touching circles. If we reflect the two circles in it, these are transformed into lines, due to the rules of circular inversion. The third circle is instead transformed into a smaller circle (Fig. 13). Now we add the orthogonal circle, which is in its turn reflected into a line, perpendicular to the former two (fig. 14), which divides the smaller circle in two parts.

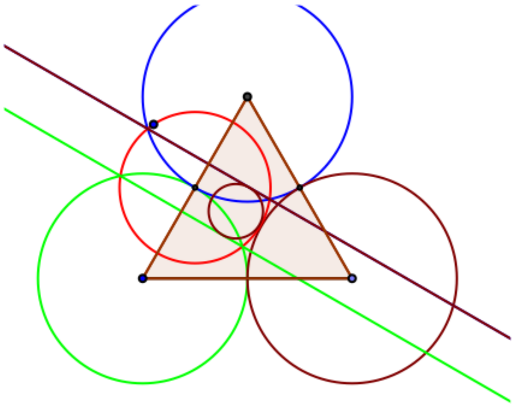


Figure 13

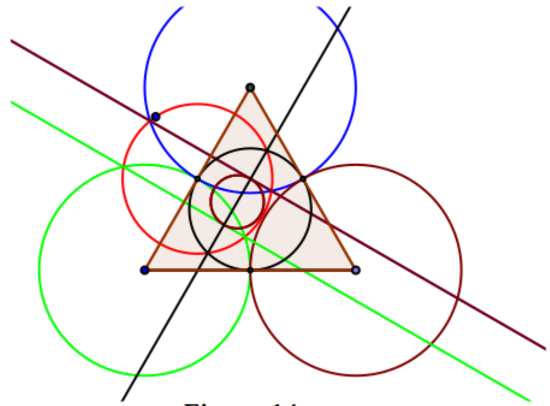


Figure 14

We thus obtain a curvilinear triangle delimited by two parallel straight lines and by a semi-circle touching these lines (Fig. 15).

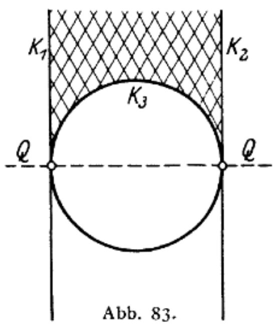


Figure 15. Taken from Klein, 1928, p. 143

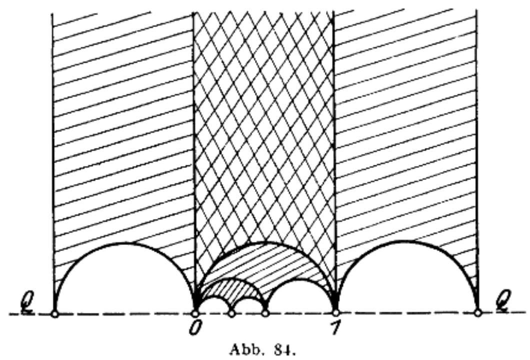


Figure 16. Taken from Klein, 1928, p. 144

Now we choose the line *QQ* (as called by Klein in Fig. 16) as *x*-axis, and the origin and the scale so that the points $x = 0$, $x = 1$ fall at the two finite vertices. Then the whole figure is easy to construct, mirroring the figure 15 unlimitedly on the right and on the left, and reflecting the sequence of infinitely many triangles such obtained in each of the occurring semi-circles. In this way all the points of contact can be found on the *x*-axis, and it is easy to calculate exactly arithmetically what had been explained in our first figure only with the help of our immediate geometric sense. Indeed, the formulas of the circular inversion indicate that a circle with rational centre and rational radius is transformed in another circle with rational centre and rational radius:

More precisely, given the equation of circular inversion in its simple vector form

$$OP \bullet OP' = r$$

Where *O* is the centre of the circumference, *r* its radius, *P* and *P'* two corresponding points collinear with *O*. It is clear that a point on the circumference corresponds to itself. Moreover, the image of the centre is a point at infinity, and – since the correspondence is an involution – all points at infinity have the centre as their image point.

Let us consider the circle C_1 whose diameter is the segment [0;1] on the number line. Its centre has abscissa ½. The straight line $x = 0$ (circle with infinite radius), perpendicular to the line *QQ*', has as its correspondent with respect to C_1 the circle C_2, which has as end points of its diameter 0 and ½. In its turn, the image of C_1 with respect to C_2 is a circle C_3, whose diameter has as endpoints 0 and 1/3, and so on (Fig. 17 and 18).

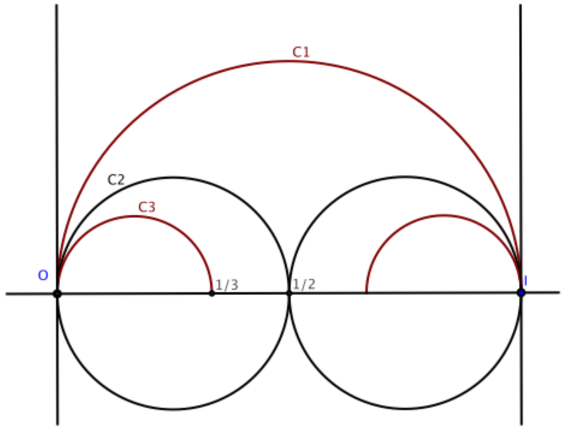


Figure 17

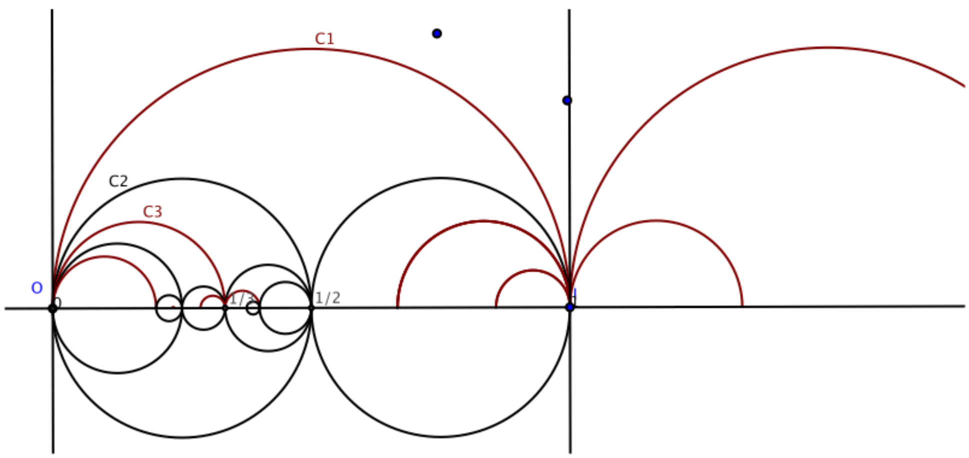


Figure 18

Therefore all the points of contact have rational abscissas *x* and every point with a rational *x* becomes a contact point. *So, there is not only an analogy between the set of the points of contact and the set of the rational points on the x-axis, but there exists an identity. In particular it turns out that the set of the points of contact is denumerable.* We have found a method of construction of the rational numbers on the real line, which provides a wonderful mental image of the relationship between rational numbers and real numbers on the number line.

Third example: *Gestalt relations* of curves

The final part of Volume III concerns *gestalt relations* of curves in space and surfaces and shows Klein to be the master of the art of description of geometric forms. In part II, concerning – as above - the *free geometry of plane curves* (that is a geometry independent from a coordinate system), Klein discusses "the possibility to deduce properties of the idealised curve from the empirical shape" (Klein, 2016, p. 189) Klein considers again the relation between an empirical and an idealised curve and poses the question:

Can I now deduce from the gestalt relations of the empirical curve, which I see before my eyes, the corresponding properties of the idealised curve?

To answer this question, Klein's conception is necessarily decisive, that the idealised curve is something that goes beyond sensorial intuition and exists only on the base of definitions. So we cannot appeal only to intuition. "Rather, we always need to reflect on whether, respectively why, things that we roughly see – so to speak – before our eyes in an empirical construction can be rigorously transferred to the idealised object *thanks to the given definitions*."

As an example Klein considers the following figure (Fig. 19), namely a closed convex curve cut by a straight line:

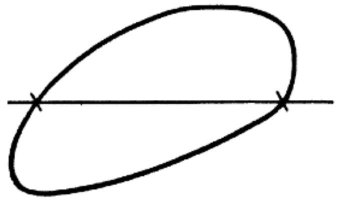


Figure 19. Taken from Abb. 103. Klein, 1928, p. 176

In our mind, we substitute the straight line of the figure by an idealised straight line, and the drawn curve – at first – by a regular curve (a Jordan-curve). Intuition teaches us that the idealised straight line passes inside and outside the Jordan curve, giving rise to two intersection points. But, even admitting that the drawn curve is a continuous curve, we cannot deduce from the figure that any closed curve has two intersections with a straight line. It is, for instance, possible that in the neighbourhood of the point in which the empirical figure shows only one intersection, the idealised figure presents three or five of them. This is in fact not excluded, if we do not further limit with definitions the type of curve.

Now we pass to considering an algebraic curve and let us suggest general properties of the algebraic curves by looking at some of them, starting from simple representations, and passing from one curve to the other by *continuous* transformations. As Klein says, we perform *a proof by continuity*. What do we mean by continuous transformation of a curve? Let us follow Klein's interesting idea:

The general equation of a curve of the *n*-th degree C_n has $\frac{n(n+3)}{2}$ constants. For instance, the general equation of a conic section has 2(2+3)/2 = 5 constants (in fact, six parameters which can be multiplied or divided by one of them):

$$a x^2 + b y^2 + cxy + dx + ey + f = 0$$

We consider these as coordinates of a point in a higher dimensional space and call this point the "*representational point*" belonging to the curves C_n. If the C_n assumes all possible shapes, that is if the coefficients vary arbitrarily, the representational point varies in the *whole* $\frac{n(n+3)}{2}$ -*dimensional space*.

It is very useful to choose for support such a space and to consider it alongside all the C_n. Indeed, Klein states, we do not know exactly what it means that a curve varies with continuity, but we can easily imagine a point moving continuously in the space.

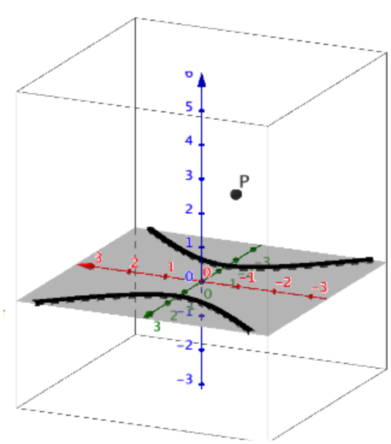
To help furthermore imagination, let us take as a first example (which is not present in Klein) a conic section centred in the origin (with only three constants):

$$a x^2 + b y^2 = c.$$

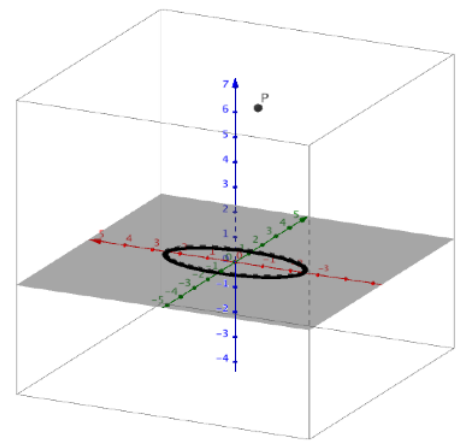
In this case the representational point *P* has coordinates *a, b, c* and varies in the 3D-space. We realise, with the help of Geogebra[1], a simulation in which the point *P* moves in space while the conic section takes on the coefficients of the point P (Fig. 20 a, b)

According to the values of *a*, *b*, *c* we get all kind of conic sections, but not the parabola as we do not have a term of the first degree.

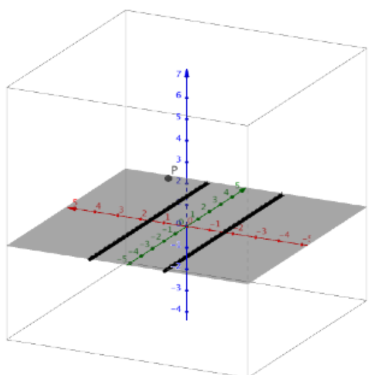
We can also observe that we have the degenerate conics (a point or two lines) when the point *P* crosses the *xy* plane (that is, the constant *c* is = 0, as in Fig. 20 c). During this transition a hyperbola becomes an ellipse or vice-versa. This can be used in schools to show the way in which conic sections are transformed one into the other by continuity.



[1] I thank Anna Baccaglini-Frank for her collaboration in creating



a b



c Figure 20 a, b, c

A second example is considered by Klein himself. It is a quartic. Klein is not worried about the number of constants: even if the representational space is of higher dimension Klein wants to stimulate our *intuition*: we have to imagine point P moving in the space, we have to understand how the quartic varies accordingly, understand that we find double points when P crosses certain surfaces, and that we have higher singularities (which can be avoided in our "walk in the space") crossing certain curves on the surfaces.

Nevertheless we tried to simulate the situation with the help of Geogebra using, once again, only three coordinates. This does not change Klein's example.

$$(a\,x^2 + b\,y^2 - a\,b)\,(b\,x^2 + a\,y^2 - a\,b) = c$$

The starting point, corresponding to $c = 0$, is constituted by two ellipses (Fig. 21).

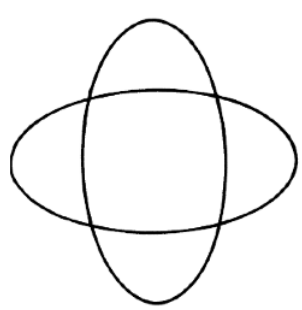


Abb. 116.

Figure 21. Taken from Klein, 1928, p. 191

Moving *P* in space, the quartic takes on various forms (Fig. 22). For instance, when *c* becomes negative, the quartic becomes like the one of Fig. 23.

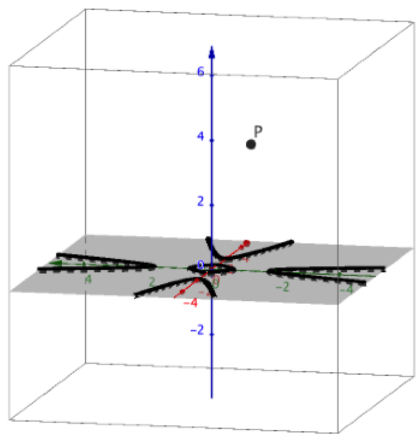


Figure 22

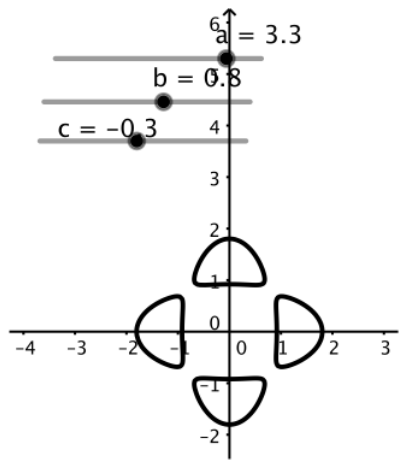
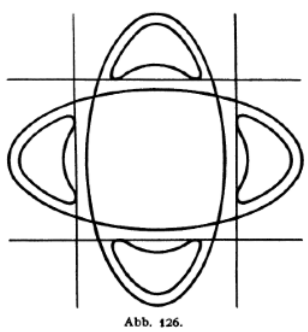


Figure 23



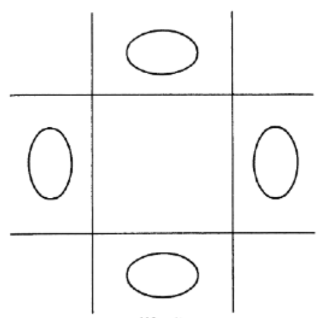


Figure **24** Taken from Klein, 1928, p. 202 **Figure 25** Taken from Klein, 1928, p. 202

Klein observes how inflection points, double points or bitangents (namely tangents to two points of the curve) change when transforming the quartic, in particular Klein proves by continuity certain regularities concerning their number. For instance, the number of real inflection points (8 in figure 24) added to twice the number of isolated bitangents (4 in figure 25) is constant.

But from a didactic point of view, to look at the continuous transformation of such a quartic is sufficient.

Conclusion

The history of mathematics education presents at different times and in different countries "utilitarian" periods in which applications of mathematics are considered an *end* of the curriculum, and even mathematical subject matter which cannot be linked to external use comes under attack (Niss, 2008). On the other hand, applications can be regarded as a *means* to support learning, by providing interpretation and meaning. When considering the two aspects of applied mathematics, a means or an aim, we are not faced with a contradiction but, as Niss states, with a duality. Klein tries to support an even stronger conception of a continuous exchange between empirical observation and formalised objects.

The third part of Klein's work has the interesting title "*About the perception of idealised structures by means of drawings and models"*. In this last section Klein presents the *collection of models* in Göttingen with the related explanations. Klein continues to reason by continuity, appealing to intuition. He writes, at the beginning of this section, that a main theme within the topics treated in his lecture course has been the distinction between empirical space intuition, with its limited precision, and the idealised conceptions of precision geometry. As soon as one becomes aware of this difference, one can choose his way unilaterally in one or in the other direction. But both directions seem to be equally *unfruitful*.

Klein strongly advocates *the need to maintain a connection between the two directions, once their differences are clear in one's mind*:

"A wonderful stimulus seems to lay in such a connection. This is why I have always fought in favour of clarifying abstract relations also by reference to empirical models".

The examples shown above surely support this statement.

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