

# The loop expansion around the Bethe solution at zero temperature predicts an upper critical dimension equal to 8 for spin glass models in a field

Maria Chiara Angelini,<sup>1</sup> Carlo Lucibello,<sup>2</sup> Giorgio Parisi,<sup>1,3,4</sup> Gianmarco Perrupato,<sup>1</sup> Federico Ricci-Tersenghi,<sup>1,3,4</sup> and Tommaso Rizzo<sup>5,1</sup>

<sup>1</sup>*Dipartimento di Fisica, Sapienza Università di Roma, P.le Aldo Moro 5, 00185 Rome, Italy*

<sup>2</sup>*Bocconi Institute for Data Science and Analytics (BIDSA), Bocconi University, Via Sarfatti 25, 20100 Milan, Italy*

<sup>3</sup>*Istituto Nazionale di Fisica Nucleare, Sezione di Roma I, P.le A. Moro 5, 00185 Rome, Italy*

<sup>4</sup>*Institute of Nanotechnology (NANOTEC) - CNR, Rome unit, P.le A. Moro 5, 00185 Rome, Italy*

<sup>5</sup>*Institute of Complex Systems (ISC) - CNR, Rome unit, P.le A. Moro 5, 00185 Rome, Italy*

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The spin-glass transition in a field in finite dimension is analyzed directly at zero temperature using a perturbative loop expansion around the Bethe lattice solution. The loop expansion is generated by the  $M$ -layer construction whose first diagrams are evaluated numerically and analytically. The Ginzburg criterion, from both the paramagnetic and spin-glass phase, reveals that the upper critical dimension below which mean-field theory fails is  $D_U = 8$ , at variance with the classic results  $D_U = 6$  yielded by finite-temperature replica field theory. The different outcome follows from two crucial properties: finite-connectivity of the lattice and zero temperature. They both also lead to specific features of the replica-symmetry-breaking phase.

Spin glass (SG) models are the prototype of disordered systems: the Hamiltonian is very simple (two spin interactions plus an external field), but the presence of quenched disorder in the couplings generates a very complex behavior. The fully connected (FC) mean-field (MF) version of the model, introduced by Sherrington and Kirkpatrick (SK) in ref. [1], was solved forty years ago [2]: the SK model undergoes a phase transition from a paramagnetic phase at high temperatures to an SG phase below a critical line in the temperature-field ( $T$ - $h$ ) plane, called de Almeida-Thouless (dAT) line [3], that diverges for  $T \rightarrow 0$ : i.e. at  $T = 0$  the SK model is in the SG phase, no matter how strong the external field is.

The solution to the SK model requires the introduction of  $n$  copies of the system, called *replicas*, with  $n$  that goes to zero at the end of the computation. This mathematical trick, useful to carry on the computation, was shown to have a deeper, physical implication: the SG phase is precisely characterized by the spontaneous breaking of replica symmetry. In the SG phase, an exponential number of pure states are organized in an ultrametric structure and the order parameter is the so-called replica-overlap, a quantity that describes the similarities between the different pure states. At the dAT line, the paramagnetic solution becomes unstable. To identify the dAT line one can compute the fluctuations around the paramagnetic solution, via the study of the spectrum of the Hessian of the replicated free energy. One can identify three sub-spaces or sectors of Hessian eigenvectors, that are called replicon, longitudinal and anomalous [4, 5]. On the dAT line, the replicon eigenvalue becomes critical and stays critical in the whole SG phase, which is thus a marginally stable phase. This highly non-trivial solution has been proved to be rigorously exact [6].

Beyond MF, things are much less clear. In particular,

it is not exactly known whether the finite-dimensional SG model has a transition to an SG phase when the external field is present. Numerical simulations suggest a positive answer for  $D = 4$  [7], but for  $D = 3$  the results are inconclusive due to the huge finite-size effects and the very large equilibration times [8, 9]: at the state of the art, it is impossible to decide if a transition exists just based on numerical results.

Usually, in statistical mechanics, the finite-dimensional behavior of models can be deduced using the powerful method of Renormalization Group (RG) [10]. One can set up a field theory for the order parameter associated with the desired transition, constructing a Lagrangian that is the most general one compatible with the symmetries of the problem. The basic approximation is the so-called tree approximation or the Landau-Ginzburg (LG) theory. It corresponds to the assumption that there are no fluctuations in the field and it results to be exact for the MF-FC model. The next step is to see how the fluctuations, associated with the finite-range interactions, will modify the MF picture. Choosing to perform this task perturbatively leads to a *loop-expansion* around the LG solution. Looking at when the one-loop correction becomes important, one identifies the upper critical dimension  $D_U$  at which the MF theory does not predict the correct critical behavior anymore: this is the so-called *Ginzburg criterium*. At this point, a perturbative expansion around the MF solution can be constructed, with a small parameter  $\epsilon = D_U - D$ , to see how the MF transitions are modified at dimension  $D$  below  $D_U$ .

Unfortunately, this program cannot be carried out so simply for SG models in a field. The MF theory in the high-temperature phase and the first-order perturbative expansion around it were analyzed in different papers [11–16]. Let us stress that the Lagrangian is very com-

plicated in this case, having three bare masses, associated with the three sectors, and eight cubic vertices involving the replica fields, that are all the possible invariants under the replica symmetry and particular care should be taken to handle the limit of the number of replicas  $n$  going to 0. The results of the various works are however not sufficient to fully understand which is the fate of the transition in finite dimension. For  $D > D_U^{\text{FC}} = 6$ , the MF-FC Fixed Point (FP) is stable, however, its basin of attraction shrinks to zero approaching  $D_U^{\text{FC}}$  from above. The main problem is the absence of a perturbative stable FP below six dimensions [11, 13]. However, this lack is not a proof for the non-existence of an SG phase in lower dimensions and many scenarios on the fate of the SG transition have been put forward. Some authors have tried to extract information from the perturbative analysis nonetheless [15, 16] possibly including quartic interactions [17] that are known to have a non-trivial role [18]. It could also be possible that a non-perturbative FP exists, that governs the SG phase for low enough dimensions [19]. Recently, the perturbative expansion was computed up to the second-order [20, 21]. The authors find a strong-coupling FP that could in principle be stable at any dimension, even above  $D_U^{\text{FC}}$ . This new fixed-point is in a way a “non-perturbative” one because it cannot be reached continuously from the MF-FC one just lowering the dimension. However, the perturbative analysis in the strong-coupling regime is uncontrolled: for this reason, the existence and relevance of this new FP cannot be stated just with the methods of Refs. [20, 21].

Another approach is the use of non-perturbative real-space RG methods: their use is the natural choice if we are looking for a non-perturbative FP in finite dimensions. The Ensemble RG (ERG)[22] and the Migdal-Kadanoff (MK) RG [23] were applied to the SG in a field, finding a critical FP for high enough dimensions ( $D \gtrsim 8$ ), while resulting in no SG phase at lower dimensions. The new FP found is a  $T = 0$  FP, thus different from the MF-FC one. We remind that in the FC model there is no transition at  $T = 0$  as a consequence of the diverging connectivity, an unrealistic feature that is not present in finite-dimensional models. However, the MK and the ERG RG flows are obtained after some crude approximations, as usually done when using non-perturbative RG, that are not exact. Thus they can provide useful indications, but cannot offer a definite answer to the problem.

Recently, a new loop expansion around the MF Bethe solution has been proposed in Ref. [24]. The Bethe lattice (BL) is a different type of MF lattice on which the SG in a field can be solved. At variance to the FC lattice, the finite connectivity in the BL allows for local fluctuations of the order parameter. This is an important feature shared with finite-dimensional systems.

The loop expansion around the Bethe solution is obtained via the  $M$ -layer construction [24]. One introduces  $M$  copies of the original finite-dimensional lattice and

generates a new lattice through a local random rewiring of the links. For large  $M$  the resulting  $M$ -layer lattice looks locally like a BL (and thus all observables tend to their MF BL values with small  $1/M$  corrections), while at large distances the lattice retains its finite-dimensional character. This has important consequences for critical behavior: close to the MF critical point the system displays MF critical behavior until the correlation length reaches a size where the finite-dimensional nature of the model is dominant and the correct non-MF exponents are observed due to universality. The  $1/M$  expansion (for  $M = 1$  one recovers the original model) takes the form of a diagrammatic loops expansion with appropriate rules [24] and it is very useful to study critical phenomena. Similarly to field-theoretical loops expansion, one can apply the Ginzburg criterion and identify the upper critical dimension  $D_U$  where the corrections alter the MF behavior. For  $D < D_U$  the expansion can then be used to obtain the critical exponents through standard RG treatments.

The expansion around the BL solution has the same advantages as standard field-theoretical loop expansions, but has a larger range of applicability, as it can be used for any problem that displays a continuous phase transition on the BL. Recent applications include the Random Field Ising model (RFIM) at zero temperature [25], the bootstrap percolation [26] and the glass crossover [27]. It has also been applied to the SG in a field in the limit of high connectivity for  $T > 0$  [28], showing that in such a limit the expansion is completely equivalent to the standard expansion around the MF-FC solution [11, 13]. This is in agreement with the fact discussed in Ref. [24] that the  $1/M$  expansion and the standard field theoretical expansion are completely equivalent if the physics of the model on the BL is like the one on the FC lattice.

In this paper, we study the  $M$ -layer BL expansion of the SG in a field at  $T = 0$  from both the paramagnetic and the SG phase. We show that finite connectivity and zero temperature lead to a critical behavior different from the one of the replicated field theory expansion at finite temperature. In particular, the Ginzburg criterion leads to an upper critical dimension  $D_U = 8$  from both sides of the critical point.

To be concrete we consider the model Hamiltonian

$$H = - \sum_{(ij) \in E} J_{ij} \sigma_i \sigma_j - h \sum_i \sigma_i, \quad (1)$$

where the spins take the values  $\sigma_i = \pm 1$ ,  $h$  is a constant external field [29] and the quenched couplings  $J_{ij}$  have a Gaussian distribution with  $\bar{J} = 0$ ,  $\overline{J^2} = \frac{1}{z-1}$ ,  $z$  being the (fixed) connectivity of the model. The first sum is over the set of edges  $E$  of a  $D$ -dimensional lattice.

Approaching the transition from the paramagnetic side the order parameter is zero and we analyze, as usual, the behavior of spin correlations. Working at  $T = 0$  it is worth considering the response function  $R_{ij}$  defined via

the following procedure: being  $\sigma^*$  the ground state (GS) configuration; compute the new GS under the constraint  $\sigma_i = -\sigma_i^*$ ; if also  $\sigma_j$  flips, then  $R_{ij} = 1$ , otherwise  $R_{ij} = 0$ . Following Ref. [24], a generic correlation or response function  $G(x)$  between two points at distance  $x$  on the original lattice is given at leading order in  $1/M$  by

$$G(x) = \frac{1}{M} \sum_{L=1}^{\infty} \mathcal{N}(x, L) G^{\text{BL}}(L), \quad (2)$$

where  $\mathcal{N}(x, L)$  is the number of non-backtracking paths of length  $L$  connecting the two points at distance  $x$  on the original lattice ( $M = 1$ ) and  $G^{\text{BL}}(L)$  is the correlation function between two spins at distance  $L$  on a BL with connectivity  $z = 2D$ . While  $\mathcal{N}(x, L)$  is known [24]

$$\mathcal{N}(x, L) \propto (2D - 1)^L \exp(-x^2/(4L)) L^{-D/2}, \quad (3)$$

the crucial model-dependent quantity to be computed is  $G^{\text{BL}}(L)$ . One can show (see SM) that the average response function on the BL can be computed exactly by applying  $L$  times an integral operator. Consequently, its behavior at large  $L$  is given by

$$R^{\text{BL}}(L) \propto \lambda^L, \quad (4)$$

where  $\lambda$  is the largest eigenvalue of the integral operator and goes to  $\lambda_c = \frac{1}{2D-1}$  at the critical point of the BL, such that the total response diverges and the paramagnetic solution is no longer stable [30]. Inserting Eqs. (4) and (3) into Eq. (2), we obtain for the Fourier transform of the response function in the small momentum region as

$$\begin{aligned} R(p) &\propto \frac{1}{M} \sum_{L=1, \infty} [\lambda \cdot (2D - 1)]^L \exp(-Lp^2) \\ &\simeq \frac{1}{M} \int_0^{\infty} dL \exp(-L(p^2 + \tau)) = \frac{1}{M} \frac{1}{p^2 + \tau} \end{aligned} \quad (5)$$

with  $\tau \equiv -\log(\lambda(2D - 1))$ . Note that  $\tau \rightarrow 0$  when  $\lambda \rightarrow \lambda_c$ : at leading order the response has the form of the bare propagator in a field theory and becomes critical at the BL critical point.

Let us now look at the  $1/M^2$  corrections to the bare propagator. According to Ref. [24], this is given by the sum of the contributions coming from all the paths that connect the two points at distance  $x$  on the original lattice containing just one topological loop. The contribution of a specific topological diagram in Fourier space is

$$\tilde{G}_{\text{loop}}(p) = \frac{1}{M^2} \sum_{\vec{L}} \mathcal{N}(p, \vec{L}) G_{\text{loop}}^{\text{BL}}(\vec{L}), \quad (6)$$

where  $\vec{L}$  is a vector containing the lengths of each line in the topological diagram and the factor  $\mathcal{N}(p, \vec{L})$  accounts for the number of such topological diagrams on the original regular lattice with  $M = 1$ . The term  $G_{\text{loop}}^{\text{BL}}(\vec{L})$  is

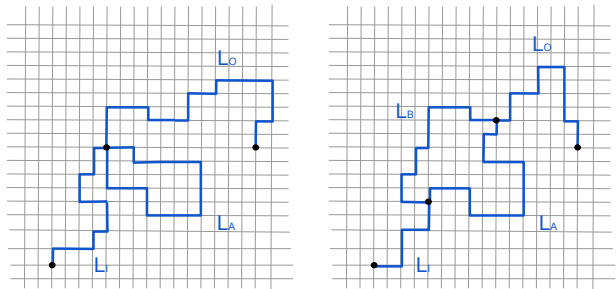


Figure 1: One loop topological diagrams relevant for the first order correction around the BL: the “quartic loop” on the left has a vertex with four lines, while the “cubic loop” on the right has only vertices with three lines.

again the only term depending on the model: it is the so-called *line-connected* value [24] that the observable takes on a BL in which the analyzed topological loop has been manually inserted. The term “line connected” means that one should add the value of the observable evaluated on each of the subgraphs that are obtained from the original structure by sequentially removing its lines times a factor  $-1$  for each line removed.

Let us point out two crucial differences between this expansion and the standard expansion around LG theory:

- the latter has just cubic vertices, while in the BL expansion vertices of all degrees can be present;
- the diagrams of the BL expansion have a clear physical meaning while the Feynmann diagrams of the standard expansion are just a smart way to compute the desired corrections.

At one loop we consider the two diagrams shown in Fig. 1. The left one has a quartic vertex, for this reason, it is not included in the standard cubic theory. We compute  $G_{\text{loop}}^{\text{BL}}(\vec{L})$  on this diagram with the same tools as for the 0-loop term (all the details in the SM). The resulting contribution to the response function coming from this quartic loop is  $R_{4\text{-loop}}^{\text{BL}}(\vec{L}) \propto L_A \lambda^{\Sigma(\vec{L})}$ , where  $\Sigma(\vec{L})$  is the sum of all  $L$ 's, i.e.  $\Sigma(\vec{L}) = L_A + L_I + L_O$  in this diagram, and  $\lambda$  is the same eigenvalue on the BL as in the previous discussion. The cubic loop (on the right in Fig. 1) has cubic vertices and is already present in the LG theory. Its behavior should be analyzed when  $L_A$  and  $L_B$  are large, because we checked that when one of the two internal legs is short, the diagram reduces to the quartic loop. For large  $L_A$  and  $L_B$ , we obtain  $R_{3\text{-loop}}^{\text{BL}}(\vec{L}) \propto \frac{L_A L_B}{L_A + L_B} \lambda^{\Sigma(\vec{L})}$ , with  $\Sigma(\vec{L}) = L_A + L_B + L_I + L_O$ .

The term  $\mathcal{N}(p, \vec{L})$  has already been computed [25] and

it reads respectively for the quartic and cubic loops

$$\mathcal{N}(p, \vec{L}) \propto \frac{(2D-1)^{\Sigma(\vec{L})}}{L_A^{D/2}} e^{-(L_I+L_O)p^2}, \quad (7)$$

$$\mathcal{N}(p, \vec{L}) \propto \frac{(2D-1)^{\Sigma(\vec{L})}}{(L_A+L_B)^{D/2}} e^{-(L_I+L_O+\frac{L_AL_B}{L_A+L_B})p^2}. \quad (8)$$

Inserting the above expression and  $R_{3\text{-loop}}^{\text{BL}}(\vec{L})$  in Eq. (6), we obtain the correction to the response given by the cubic loop. In order to apply the Ginzburg criterion, it is more convenient to consider the inverse susceptibility

$$(MR(p))^{-1} = \tau + p^2 + \frac{c}{M} \sum_{L_A, L_B} \frac{L_AL_B}{(L_A+L_B)^{D/2+1}} e^{-L_A\tau - L_B\tau - \frac{L_AL_B}{L_A+L_B}p^2},$$

that can be rewritten as

$$(MR(p))^{-1} = A(\tau - \tau_c) + Bp^2 + O(p^4), \quad \text{with} \quad \tau_c = \frac{c}{M} \sum_{L_A, L_B} \frac{L_AL_B}{(L_A+L_B)^{D/2+1}}, \quad (9)$$

$$A = 1 - \frac{c}{M} \sum_{L_A, L_B} \frac{L_AL_B}{(L_A+L_B)^{D/2}}, \quad (10)$$

$$B = 1 - \frac{c}{M} \sum_{L_A, L_B} \frac{L_A^2 L_B^2}{(L_A+L_B)^{D/2+2}}. \quad (11)$$

We see that for large but finite  $M$ , the  $M$ -layer lattice has the same critical behavior of the BL ( $M = \infty$ ), with small  $O(1/M)$  shifts of the critical temperature and of the constants  $A$  and  $B$ . However, the above sums over  $L_A$  and  $L_B$  are divergent respectively for  $D \leq 6$ ,  $D \leq 8$  and  $D \leq 8$  and thus the Ginzburg criterion tells us that the critical exponents cannot be those of the Gaussian theory below  $D_U = 8$ . The same argument applied to the quartic loop would give a critical dimension equal to 6 (the diagram indeed appears in the computation of the connected correlation of the RFIM [25]) and allows to neglect the quartic loop with respect to the cubic one.

To go below the upper critical dimension we rescale lengths as  $L = x/\tau$  and momenta as  $p^2 = k^2\tau$ , obtaining

$$(MR(p))^{-1}/\tau = 1 + k^2 + \frac{c\tau^{D/2-4}}{M} \times \int_{\tau/\Lambda}^{\infty} dx_A \int_{\tau/\Lambda}^{\infty} dx_B \frac{x_A x_B e^{-x_A - x_B - \frac{x_A x_B}{x_A + x_B} k^2}}{(x_A + x_B)^{D/2+1}}. \quad (12)$$

The above expression shows that loop corrections are not negligible for  $D < D_U = 8$  when  $\tau \rightarrow 0$ . Indeed for  $D < D_U$  the integral would be divergent at short distances if not for the lattice cutoff  $\Lambda$ . One should check if, by standard mass, field, and coupling constant renormalization the above 1-loop diagrams and higher-order diagrams as well can be made finite in the limit  $\Lambda \rightarrow \infty$ . Then the critical exponents can be computed by standard

methods [31–33] provided an  $O(\epsilon)$  non-trivial FP of the  $\beta$  function can be identified (at variance with the  $T > 0$  case [11]): this program is currently underway. An interesting question is if this putative zero-temperature FP describes also the  $T > 0$  physics, i.e. if the temperature is an irrelevant operator in the Wilson RG sense.

We now consider the Ginzburg criterion coming from the Replica Symmetry Breaking (RSB) phase. At zero temperature RSB implies the presence of many local ground states (LGS), i.e. configurations whose energy cannot be decreased by flipping any finite number of spins [34]. An essential property is that the lowest LGSs differ from the GS by an extensive number of spins, but have energy differences of order one. As a consequence, the effective local field acting on a site for a given realization of the disorder depends on the LGS considered. However, at variance with the  $T > 0$  case, at  $T = 0$  and finite field we found that only a finite fraction of the sites (that we call the RSB cluster) displays RSB, while sites not on the RSB cluster have the same effective local field on all relevant LGSs. The probability  $p_{\text{RSB}}$  that a given site is in the RSB cluster goes to zero continuously when the dAT line is approached from within the RSB phase [35]. This picture is compatible with what is found in ref. [23], where, analyzing the exponents connected to the  $T = 0$  critical FP identified with the Migdal-Kadanoff RG, one finds that the system is ordered but only on a fractal system-size set.

The Ginzburg criterion was originally formulated in the phase where the order parameter is non-zero and prescribes to compare its fluctuations (on the correlation length scale) with the square of its average. In the RSB phase, a convenient local order parameter is the RSB cluster indicator function equal to one if a site is on the RSB cluster and zero otherwise. The average order parameter is thus  $p_{\text{RSB}}$  and its fluctuations are given by the probability  $p_{\text{RSB}}(x, y)$  that sites at positions  $x$  and  $y$  on the lattice are both on the RSB cluster. On the  $M$ -layer lattice, the fluctuations  $p_{\text{RSB}}(x, y)$  are expressed by Eq. (2) where  $G^{\text{BL}}(L)$  must be replaced by the probability  $p_{\text{RSB}}^{\text{BL}}(L)$  that two sites at distance  $L$  on the BL are both on the RSB cluster. One finds [35]

$$p_{\text{RSB}}^{\text{BL}}(L) \propto p_{\text{RSB}}^2 L^3 \lambda^L, \quad (13)$$

where  $\lambda(z-1) = 1 - ap_{\text{RSB}}$  ( $a$  is a constant) close to the dAT line. This implies that performing the summation over  $L$  in momentum space as in eq. (5) we obtain an expression proportional to  $(ap_{\text{RSB}} + p^2)^{-4}$  due to the  $L^3$  factor. Going back to real space we can now compare the fluctuations on the scale of the correlation length  $\xi \propto p_{\text{RSB}}^{-1/2}$  with  $p_{\text{RSB}}^2$ :

$$\frac{1}{M} \frac{G(b\xi)}{p_{\text{RSB}}^2} \propto \frac{1}{M} p_{\text{RSB}}^{D/2-4} \int_0^\infty \frac{d\alpha}{\alpha^{D/2-3}} e^{-\alpha - \frac{b^2}{4\alpha}}. \quad (14)$$

This noise-to-signal ratio is small due to the  $1/M$  prefactor, but it diverges in the critical region ( $p_{\text{RSB}} \ll 1$ ) for

$D \leq 8$ , confirming from the RSB side of the transition that the upper critical dimension is  $D_U = 8$ .

We already mentioned that the expansion around the BL was applied to the SG in a field for  $T > 0$  and in the limit of large  $z$  in Ref. [28]. Even if we take the limit  $T \rightarrow 0$  of that expansion, the 1-loop correction results to be of the standard form (the detailed computation is in the SM). Finite connectivity is thus a crucial ingredient in the computation, and the limits  $z \rightarrow \infty$  and  $T \rightarrow 0$  cannot be exchanged. This is a clear indication that for SG models the expansion around the FC model cannot describe the behavior of finite-dimensional systems.

We emphasize that the FP we have found in this work by expanding around the BL is different from the finite temperature MF-FC one even for  $D > D_U$ . Indeed when  $T > 0$  one can demonstrate that the critical behavior of all the possible correlation functions is the same (mainly because they all receive a critical contribution by the only critical eigenvalue that is the replicon [28, 36]). However, if the relevant FP is a  $T = 0$  one, different correlation functions could decay differently (this effect is linked to the degeneracy of the three eigenvalues that become all critical at  $T = 0$ ), so one should look at them all. This is what happens in the RFIM, whose physics is governed by a  $T = 0$  FP and whose correlation function associated with disorder fluctuations decays more slowly than the one associated with thermal fluctuations [36]. The same behavior is predicted by the MK RG of Ref. [23] for the SG in a field. We leave the analysis of the disorder correlation function to future work.

A final remark on the value  $D_U = 8$ , that was already special in the standard replica field theory: the dangerously irrelevant quartic coupling become singular under the RG for  $D < 8$ , and the standard MF-FC theory has to be corrected to take this into account [18]. Moreover, in Ref. [23] the upper critical dimension was found to be  $D \simeq 8$  with the MK RG method, while for  $D < 8$  no stable SG phase was found. The role of dimension  $D = 8$  needs to be better investigated looking at the finite-size scaling of observables at  $T = 0$ . Long-range SG models [37] are natural candidates to perform this analysis.

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## Supporting Information for

# “The loop expansion around the Bethe solution at zero temperature predicts an upper critical dimension equal to 8 for spin glass models in a field”

### SPIN GLASS MODELS ON THE BETHE LATTICE

We consider a spin glass model of  $N$  Ising spins,  $\sigma_i = \pm 1$ , with Hamiltonian

$$\mathcal{H} = - \sum_{(ij) \in E} J_{ij} \sigma_i \sigma_j - H \sum_i \sigma_i .$$

The edge set  $E$  defines the interaction graph, which is a random regular graph of fixed degree  $z$ , also known as Bethe Lattice (BL),  $H$  is a constant external field and the quenched couplings  $J_{ij}$  are random variables extracted from a Gaussian distribution with  $\mathbb{E}_J[J] = 0$  and  $\mathbb{E}_J[J^2] = \frac{1}{z-1}$  (this scaling ensures a well-defined Hamiltonian in the  $z \rightarrow \infty$  limit).

A complete description of this model, even at  $T = 0$ , can be found in Ref. [38], however, we report here some of the main ingredients useful for the subsequent computations. To solve the model in the high-temperature region, we consider cavity fields  $h_{i \rightarrow j}$  and  $u_{i \rightarrow j}$  defined on each edge of the graph. They parametrize, respectively, the marginal probability distribution on  $\sigma_i$  in the cavity graph where edge  $(ij)$  has been removed, and the marginal probability distribution on  $\sigma_j$  just considering the information coming from the edge  $(ij)$ , in other words the marginal probability in the cavity graph where all edges involving vertex  $j$ , but  $(ij)$ , have been removed.

The BL has the special property that in the large  $N$  limit the loops of finite length have a vanishing density. In other words, the BL is locally tree-like. For this reason, the different cavity fields  $u_{i \rightarrow j}$  arriving in  $j$  from its neighbors can be considered as independent in the large  $N$  limit: this property makes the BL a mean-field solvable model. In fact, one can write self-consistent equations involving the cavity fields that at  $T = 0$  read

$$h_{i \rightarrow j} = H + \sum_{k \in \partial i \setminus j} u_{k \rightarrow i} \quad (15a)$$

$$u_{i \rightarrow j} = \text{sign}(h_{i \rightarrow j} J_{ij}) \min(|h_{i \rightarrow j}|, |J_{ij}|) \quad (15b)$$

where  $\partial i$  is the set of neighbors of  $i$ . These equations allow us to solve the model on a given (locally tree-like) graph. However, in the large  $N$  limit, if we are interested in computing a self-averaging observable, like a free-energy or a correlation function, it is enough to know the probability distribution of the cavity field that satisfies the following self-consistency equation

$$P_B(u) = \mathbb{E}_J \int \prod_{i=1}^{z-1} P_B(u_i) du_i \delta \left( u - \text{sign} \left( J \left( H + \sum_i u_i \right) \right) \min \left( |J|, \left| H + \sum_i u_i \right| \right) \right) \quad (16)$$

This distribution gives the correct statistical description for the cavity messages that in turn provide the correct marginal probabilities in the paramagnetic phase of a spin glass model defined on a very large  $z$ -regular random graph (a BL).

In the main text, an expansion around the BL for spin glass models on finite-dimensional lattices was used to compute the critical behavior of the two-point connected correlation function. Due to the symmetry of the coupling distribution, the first non-trivial two spins correlations for the SG in a field are the squared correlations. In particular, the connected squared correlation between two points at distance  $x$  is defined as

$$G_c(x) = \mathbb{E}_J \left[ \langle \sigma_0 \sigma_x \rangle^2 - \langle \sigma_0 \rangle^2 \langle \sigma_x \rangle^2 \right] . \quad (17)$$

where  $\langle \cdot \rangle$  denotes the thermal average while  $\mathbb{E}_J$  is the average over the quenched disordered couplings, as before.  $G_c(x)$  is the correlation function associated to the thermal fluctuations and it goes to 0 when  $T \rightarrow 0$  as  $G_c(x) = O(T^2)$ . For this reason, in the following we will define a rescaled connected correlation function that stays finite at  $T = 0$ . The associated susceptibility is the so-called spin-glass susceptibility and diverges at the dAT line in the MF solution. While in Ref. [28] the computation of  $G^{\text{BL}}(L)$ , and of its 1-loop correcting term  $G_{\text{loop}}^{\text{BL}}(\vec{L})$ , was done analytically for

all the possible two-points squared correlations in the limit  $z \rightarrow \infty$  and  $T > 0$  for the SG in a field, the analytical computation is unfeasible when  $z$  is finite.

To obtain the zero-order expansion, one needs to compute the correlation between two points at distance  $L$  on a BL. Since on a BL there exists only one path of finite length between two given spins  $\sigma_1$  and  $\sigma_2$ , we can obtain an effective two-spins Hamiltonian by integrating out all the internal spins along the path

$$\mathcal{H}[\sigma_1, \sigma_2] = -h_1\sigma_1 - J_{12}\sigma_1\sigma_2 - h_2\sigma_2, \quad (18)$$

The effective Hamiltonian is fully determined by a triplet  $(h_1, h_2, J_{12})$  of effective fields and effective coupling. At zero temperature, the Gibbs measure is concentrated on the ground state  $(\sigma_1^*, \sigma_2^*)$  of the effective Hamiltonian that can be easily computed from Eq. (18). Since we are at  $T = 0$ , the connected correlation function in Eq. (17) is ill-defined; therefore, we work with its rescaled version that we called response function [25]

$$R_{ij} = \mathbb{P}[\langle \sigma_i \rangle_j = -\sigma_i^*], \quad (19)$$

where  $\langle \cdot \rangle_j$  denotes the expectation over the ground state of the system conditioned to the flipping of the spin  $\sigma_j$ , i.e.  $\langle \sigma_j \rangle_j = -\sigma_j^*$ . In practice,  $R_{ij} = 1$  if  $\sigma_i$  flips due to the flipping of  $\sigma_j$ , and  $R_{ij} = 0$  otherwise.

Table I: Rules for computing the ground state configuration  $(\sigma_1^*, \sigma_2^*)$  of the Hamiltonian in Eq. (18) given the triplet of cavity messages  $(h_1, h_2, J_{12})$ .

	$\sigma_1^*$	$\sigma_2^*$
$ h_1  < \min( J_{12} ,  h_2 )$	$\text{sign}(J_{12}h_2)$	$\text{sign}(h_2)$
$ h_2  < \min( J_{12} ,  h_1 )$	$\text{sign}(h_1)$	$\text{sign}(J_{12}h_1)$
$ J_{12}  < \min( h_1 ,  h_2 )$	$\text{sign}(h_1)$	$\text{sign}(h_2)$

Making use of the rules in Tab. I to compute the ground state of the two-spins effective Hamiltonian, it is quite easy to show that, in terms of the effective triplet  $(h_1, h_2, J_{12})$ , the response function can be written as

$$R_{12} = \mathbb{P}[|J_{12}| > |h_1|], \quad (20)$$

The crucial quantities for the computation of the response  $R^{\text{BL}}(L)$  between two spins at distance  $L$  on a BL are thus the triplets of effective coupling and fields at distance  $L$ . We keep track of the distribution of triplets in two different ways: in Sec. we explain how to perform unbiased Monte Carlo sampling to propagate an empirical distribution of triplets along a line; in Sec. instead, we will approximate the distribution introducing an analytical Ansatz that we argue to be exact in the large  $L$  limit. In both cases, we build on previous approaches proposed for the Random Field Ising Model (RFIM) in Ref. [25].

## NUMERICAL COMPUTATION OF THE DISTRIBUTION OF EFFECTIVE TRIPLETS ON A LINE

Triplets can be computed in a recursive fashion. Let us join two chains, the first one between  $\sigma_1$  and  $\tau$ , characterized by the triplet  $(u_1, u_{\tau,1}, J_1)$ , and the second one between  $\tau$  and  $\sigma_2$  identified by  $(u_{\tau,2}, u_2, J_2)$ . In order to compute the triplet describing the effective Hamiltonian between  $\sigma_1$  and  $\sigma_2$  we need to sum over  $\tau$  and keep only the lowest energy term because we are working at  $T = 0$ :

$$\begin{aligned} \mathcal{H}(\sigma_1, \sigma_2) &= -\sigma_1 u_1 - \sigma_2 u_2 + \min_{\tau} [-J_1 \sigma_1 \tau - h\tau - J_2 \tau \sigma_2] \\ &\equiv E - (u_1 + u'_1)\sigma_1 - J_{12}\sigma_1\sigma_2 - (u_2 + u'_2)\sigma_2 \end{aligned} \quad (21)$$

where  $h = H + u_{\tau,1} + u_{\tau,2} + \sum_{k \in \partial\tau \setminus \{1,2\}} u_{k \rightarrow \tau}$  is the total field acting on spin  $\tau$ , while  $u_{k \rightarrow \tau}$  are  $z - 2$  independent random variables extracted from  $P_B(u)$  and

$$\begin{aligned} u'_1 &= (A - B + C - D)/4, \\ u'_2 &= (A - B - C + D)/4, \\ J_{12} &= (A + B - C - D)/4, \end{aligned} \quad (22)$$

with  $A = |J_1 + J_2 + h|$ ,  $B = |J_1 + J_2 - h|$ ,  $C = |J_1 - J_2 + h|$ ,  $D = |-J_1 + J_2 + h|$ .



In practice, we start from a population  $P_{L=1}(u_0, u_1, J_1)$  of random triplets  $(0, 0, J)$ , with  $J$  extracted from a Gaussian distribution. We evolve the population  $P_{L-1}$  into the population  $P_L$  following the rules summarized in Eq. (22), where each triplet of the population  $P_{L-1}$  is joined to a random triplet  $(0, 0, J)$  and  $z-2$  cavity fields  $u_{k \rightarrow \tau}$  extracted from  $P_B(u)$  are added on the central spin.

Unfortunately, this procedure is very ineffective, because at each step a constant fraction of the population (the one satisfying the condition  $|h| > |J_{L-1}| + |J|$ ) produces a new triplet with  $J_L = 0$ : in this way the part of the population keeping information about branches with non-zero effective couplings shrinks very fast during the iterations.

To amplify this signal, one could evolve two populations of the same size: one population stores the triplets corresponding to the branches with  $J_L \neq 0$ , while a second population keeps the pairs  $(u_0, u_L)$  along branches with  $J_L = 0$ . This is what has been done in the RFIM case. However, for the SG, looking carefully at the evolution rules in eq.(22), one can notice that among the triplets  $(u_0, u_L, 0)$ , the pair of fields could become independent in some situations. To further amplify the signal then, we evolve three populations of the same size: the population  $\mathcal{A}_L$  stores the triplets along branches with  $J_L \neq 0$ , the population  $\mathcal{B}_L$  keeps correlated pairs  $(u_0, u_L)$  along branches with  $J_L = 0$ , while the population  $\mathcal{C}_L$  keeps pairs  $(u_0, u_L)$  of independent fields along branches with  $J_L = 0$ . When an effective triplet  $T_{L-1}$  at length  $L-1$  is joined with a triplet  $(0, 0, J)$  to form a new triplet  $T_L$ , we encounter the following different cases:

- $T_{L-1} \in \mathcal{A}_{L-1}$  and  $|h| < |J_{L-1}| + |J| \rightarrow T_L \in \mathcal{A}_L$
- $T_{L-1} \in \mathcal{A}_{L-1}$  and  $|h| > |J_{L-1}| + |J| \rightarrow T_L \in \mathcal{B}_L$
- $T_{L-1} \in \mathcal{B}_{L-1}$  and  $|h| < |J| \rightarrow T_L \in \mathcal{B}_L$ .

In all the other cases  $T_L \in \mathcal{C}_L$ . At the same time, we measure the probabilities  $p_L$  and  $c_L$  that are the weights of the  $\mathcal{A}_L$  and  $\mathcal{B}_L$  populations respectively.

As shown in Fig. 2, both  $p_L$  and  $c_L$  decay exponentially fast in  $L$ , as

$$p_L = aL\lambda^L + b\lambda^L + o(\lambda^L), \quad c_L = cL^2\lambda^L + dL\lambda^L + e\lambda^L + o(\lambda^L),$$

where  $\lambda$  is the largest eigenvalue of the linear operator associated to the linearization of the BP equations (15) around the fixed point

$$\lambda g(u) = \mathbb{E}_J \int \prod_{i=1}^{z-2} P_B(u_i) du_i g(u') du' \mathbb{I} \left[ |H + \sum_i u_i + u'| < |J| \right] \delta \left( u - \text{sign}(J) |H + \sum_i u_i + u'| \right) \quad (23)$$

with  $\mathbb{I}[\cdot]$  being the indicator function. At the critical point,  $H = H_c$ ,  $\lambda(H_c) = 1/(z-1)$  holds. One can calculate  $\lambda$  numerically in the Bethe solution as the growing factor associated with the evolution of a perturbation.

Once we have a population of triplets at each length  $L$ , it is quite simple to compute the response function making use of Eq. (20).

Notice that only events with a non-zero effective coupling contribute to the response function: this is the reason why amplifying the population of cavity messages with  $J_L \neq 0$  is mandatory to have a precise measurement of correlations in the  $T = 0$  limit. In the same way, amplifying the population of cavity messages with  $J_L = 0$  and correlated fields will improve the signal for the response function at one loop (see Sec. ).

In Fig. 3, we show the response function  $R^{\text{BL}}(L)$  at distance  $L$  averaged over the population of the triplets generated as explained above, in a BL with fixed connectivity  $z = 3$ , at zero temperature and critical field  $H_c$ : it decays as  $R^{\text{BL}}(L) \propto \lambda^L$ , with  $\lambda = \frac{1}{z-1}$ , as already found analytically in the  $z \rightarrow \infty$  limit [28].

## SEMI-ANALYTICAL COMPUTATION OF THE DISTRIBUTION OF EFFECTIVE TRIPLETS ON A LINE

In this section, we reproduce the numerical results of the previous section introducing an Ansatz  $P_L(u_0, u_L, J)$  for the leading behavior at large  $L$  of the joint distribution of the effective coupling and fields between two spins at distance  $L$  in a BL at  $T = 0$ . The Ansatz has the same form of one for the RFIM [25], except for the fact that the effective coupling  $J$  can take both positive and negative values. The reason why it cannot be different is that in the paramagnetic phase on a tree the RFIM solution at  $T = 0$  can be mapped into the SG solution in a field through a

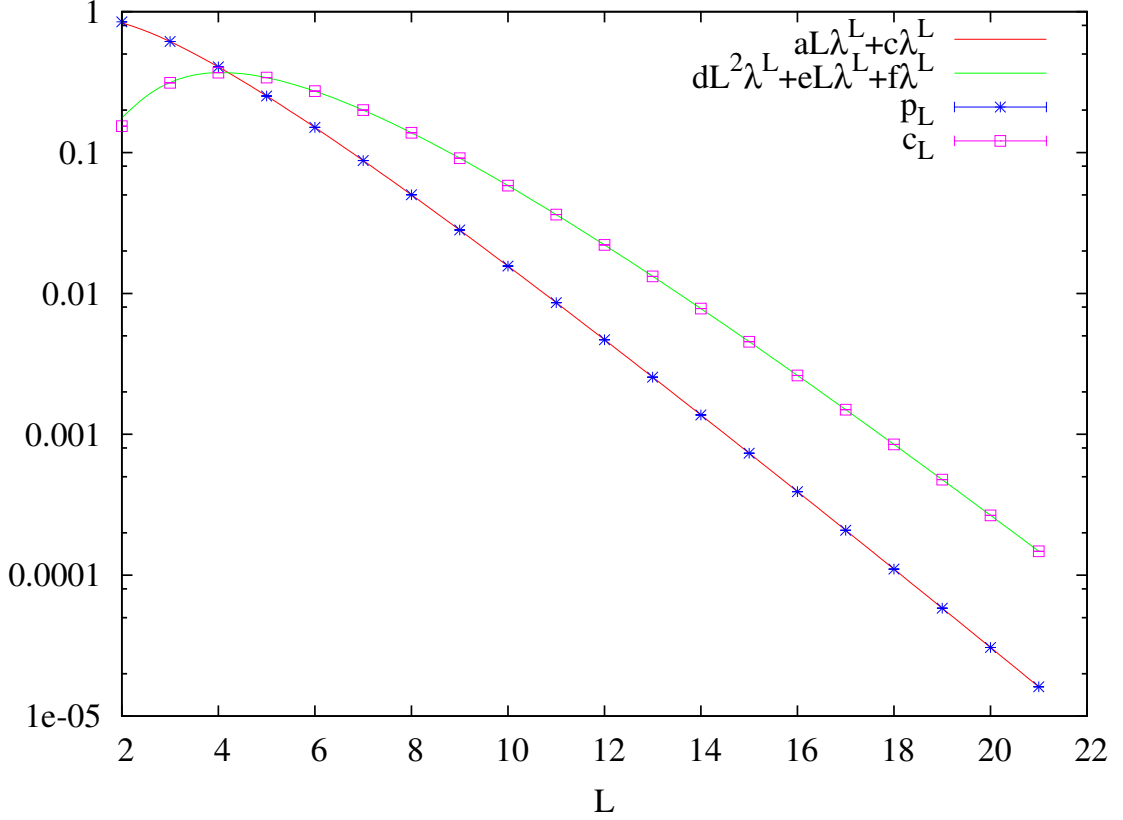


Figure 2: Probability  $p_L$  to have  $J_L \neq 0$  and probability  $c_L$  to have null coupling and correlated fields  $(u_0, u_L)$  on a chain of length  $L$  in a Bethe lattice: both decay exponentially in  $L$ . Measurements are taken at the critical field  $H_c = 0.358$  for  $z = 3$ . Errors are smaller than symbols.

simple transformation. The Ansatz, at leading order  $L\lambda^L$ , is the following:

$$\begin{aligned}
 P_L(u_0, u_L, J) = & \delta(J) \left[ P_B(u_0)P_B(u_L) - bL\lambda^L g(u_0)g(u_L) + \right. \\
 & \left. - c_1 L\lambda^L g'(u_0)g'(u_L) - c_2 L\lambda^L g''(u_0)g''(u_L) \right] + \\
 & + aL^2 \lambda^L \rho e^{-\rho|J|L} g(u_0)g(u_L)
 \end{aligned} \tag{24}$$

Imposing the normalization of the Ansatz,  $\int du_0 du_L dJ P_L(u_0, u_L, J) \equiv 1$ , we obtain the condition  $b = 2a$ . The normalized Ansatz thus becomes:

$$\begin{aligned}
 P_L(u_0, u_L, J) = & \delta(J) \left[ P_B(u_0)P_B(u_L) - 2aL\lambda^L g(u_0)g(u_L) + \right. \\
 & \left. - c_1 L\lambda^L g'(u_0)g'(u_L) - c_2 L\lambda^L g''(u_0)g''(u_L) \right] + \\
 & + aL^2 \lambda^L \rho e^{-\rho|J|L} g(u_0)g(u_L)
 \end{aligned} \tag{25}$$

At this point, we impose the self-consistency of the part with  $J \neq 0$ : joining two chains of length  $L_1$  and  $L_2$ , the  $J \neq 0$  part has to keep the same form with the only substitution of  $L$  with  $L_1 + L_2$ . This is true if the condition

$$a = \frac{\rho}{4\hat{P}(0)} \tag{26}$$

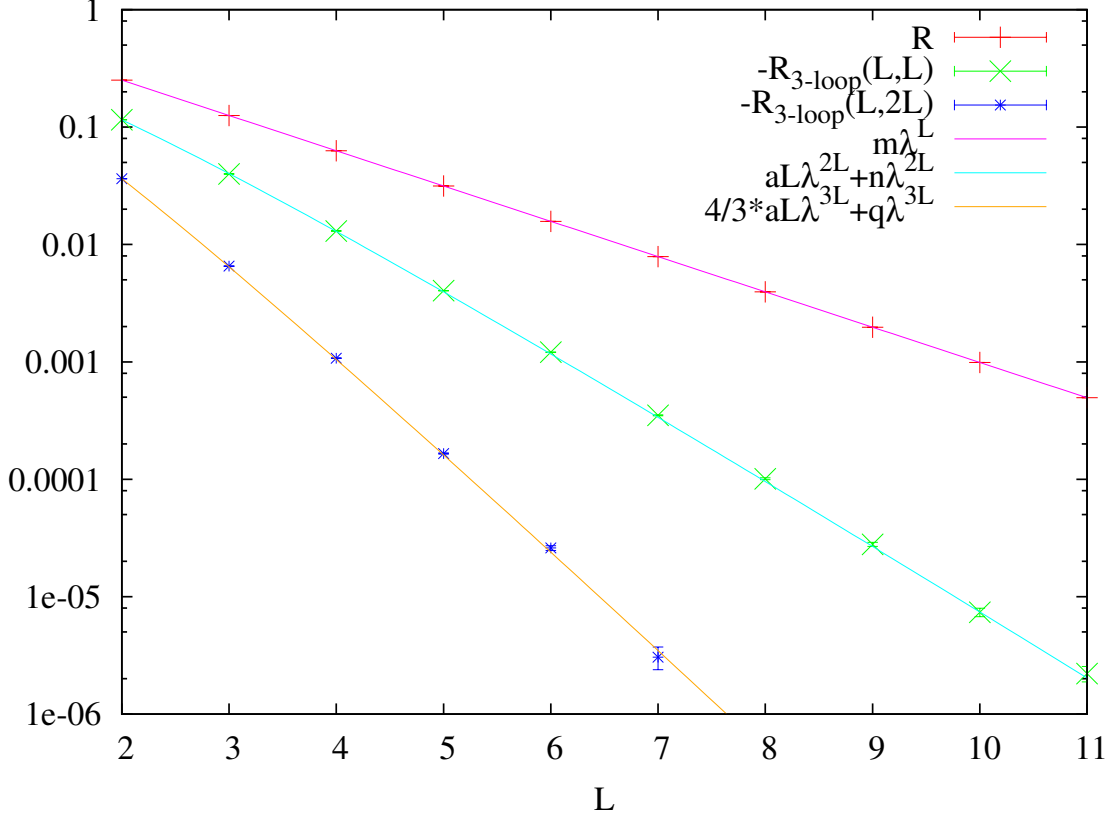


Figure 3: Response as a function of the distance  $L$  on the Bethe lattice, and amputated response on the loop of Fig. 1 with  $L_1 = L_2 = L$ . The leading behaviour on a line is  $R \propto \lambda^L$ , while that on the cubic loop is  $R_{3\text{-loop}} \propto L\lambda^{2L}$ . Measurements are taken at the critical field  $H_c = 0.358$  for  $z = 3$ .

holds, where

$$\widehat{P}(h) = \int g(u) du g(v) dv \prod_{i=1}^{z-2} P_B(u_i) du_i \delta \left( h - (H + u + v + \sum_i u_i) \right). \quad (27)$$

We then impose the self-consistency of the whole Ansatz: joining on a central spin  $\tau$  two chains of length  $L_1$  and  $L_2$ , for which the distribution of fields and coupling is given in eq. (25), following the rules in eq. (22) for the computation of the resulting new triplet of effective fields and coupling once one sums over  $\tau$ , one should obtain a distribution that has the same form as the one in eq. (25) with  $L = L_1 + L_2$ . This is true only if  $c_1 = c_2 = 0$ , neglecting contribution  $O(\lambda^L)$ , that are already ignored in eq. (25). The final Ansatz thus takes the following form

$$P_L(u_0, u_L, J) = \delta(J) \left[ P_B(u_0)P_B(u_L) - 2aL\lambda^L g(u_0)g(u_L) \right] + aL^2\lambda^L \rho e^{-\rho|J|L} g(u_0)g(u_L). \quad (28)$$

We stress that all the quantities entering this Ansatz can be computed analytically [38]:  $P_B(u)$  from Eq. (16),  $g(u)$  from Eq. (23) and  $\rho$  from the decay of the mean coupling on branches with  $J \neq 0$  or from the properties of the linear operator discussed in detail in the SI of Ref. [25].

As for the RFIM, the Ansatz is encoding the fact that at  $T = 0$  the quantity  $J_L$  is either exactly 0 or of order  $1/L$  with a probability of order  $L\lambda^L$ . This continuous distribution plus a peak at  $J = 0$  for the renormalized coupling was already displayed in the finite-dimensional lattice analyzed with the MK RG near the  $T = 0$  critical point in Ref. [39].

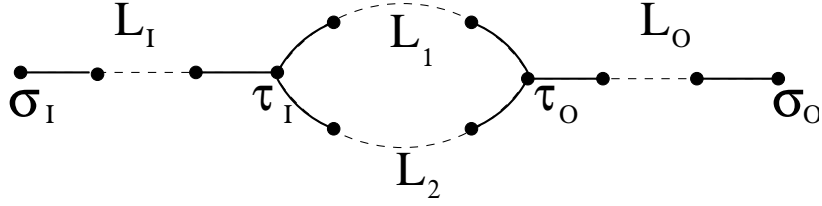


Figure 4: Topological loop responsible for the leading correction to the critical value of the response function

### COMPUTATION OF THE ONE LOOP CONTRIBUTION AT FINITE $z$

Now we aim at extending the above Ansatz to describe the joint probability distribution of the cavity fields relevant in the computation of the correlation in a loop of the type shown in Fig. 4. Some considerations can help us. First of all, the path between the two spins should be connected (i.e. made on non-zero effective couplings) if we want a non-zero response function, thus on the external legs, the only possible term that we can put is of the form  $\lambda^{L_I+L_O}$ . For this reason, we will just compute what in jargon is called the *amputated* correlation function on the internal branches of the loop. In the internal loop we have to compute the convolution of two triplets  $(J_1, u_i^1, u_o^1)$  and  $(J_2, u_i^2, u_o^2)$  of effective fields and coupling at the end of chains of lengths  $L_1$  and  $L_2$

$$P_{L_1, L_2}(u_i, u_o, J) = \int du_i^1 du_i^2 du_o^1 du_o^2 dJ_1 dJ_2 P(u_i^1, u_o^1, J_1) P(u_i^2, u_o^2, J_2) \delta(u_i - u_i^1 - u_i^2) \delta(u_o - u_o^1 - u_o^2) \delta(J - J_1 - J_2). \quad (29)$$

We have already said that the loop contribution to the observable is the *connected* one, given by the value of the observable computed on the loop minus the observable computed on the two paths  $L_I + L_1 + L_O$  and  $L_I + L_2 + L_O$  considered as independent. We can easily obtain this loop correction disregarding the asymptotic term  $P_B(u_I)P_B(u_O)$  in the Ansatz for the two internal branches of length  $L_1$  and  $L_2$ .

There are two relevant contributions to the connected loop: the first one, which we call  $P^A$ , is the one obtained by a loop in which both  $J_1 \neq 0$  and  $J_2 \neq 0$ . The second one, called  $P^B$ , have  $J_1 \neq 0$  and  $J_2 = 0$  or  $J_1 = 0$  and  $J_2 \neq 0$ . The third contribution with  $J_1 = J_2 = 0$  reduces to a disconnected loop, giving no contribution to the connected correlation. Performing the integral in eq. (29), we obtain for the two interesting contributions:

$$P_{L_1, L_2}^A(u_i, u_o, J) = 2\lambda^{L_1+L_2} \widehat{P}(u_i) \widehat{P}(u_o) a^2 \rho L_1^2 L_2^2 \frac{e^{-|J|L_2\rho} L_1 - e^{-|J|L_1\rho} L_2}{L_1^2 - L_2^2} \quad (30)$$

$$P_{L_1, L_2}^B(u_i, u_o, J) = -2\lambda^{L_1+L_2} \widehat{P}(u_i) \widehat{P}(u_o) a^2 \rho \left( L_1 e^{-|J|L_1\rho} + L_2 e^{-|J|L_2\rho} \right)$$

with  $\widehat{P}(h)$  defined in eq. (27). At this point we average the response over the loop probability distribution obtaining:

$$R^A \equiv \mathbb{P}[|J| > |u_o|; (u_i, u_o, J) \sim P_{L_1, L_2}^A] = 2a \lambda^{L_1+L_2} \frac{L_1^3 - L_2^3}{L_1^2 - L_2^2} \quad (31)$$

$$R^B \equiv \mathbb{P}[|J| > |u_o|; (u_i, u_o, J) \sim P_{L_1, L_2}^B] = -2a \lambda^{L_1+L_2} (L_1 + L_2)$$

For the RFIM,  $R^A = -R^B$  and the leading contribution to the response coming from the loop is null, while for the SG it is

$$R_{3\text{-loop}}^{\text{BL}}(\vec{L}) \equiv \lambda^{L_I+L_O} (R^A + R^B) = -2a \lambda^{\Sigma(\vec{L})} \frac{L_1 L_2}{L_1 + L_2} \quad (32)$$

with  $\Sigma(\vec{L}) = L_I + L_1 + L_2 + L_O$ . The two branches act as resistors in parallel. We can check this result numerically by computing the amputated one-loop response using random triplets for the two branches obtained from the enriched populations discussed above. Putting  $L_1 = L_2 = L$  in Eq. (32), we have a leading behaviour of the type  $R_{3\text{-loop}}(L, L) = aL\lambda^{2L}$ , that perfectly describes the data in Fig. 3. To check the dependence on  $L_1$  and  $L_2$ , we numerically verified that putting  $L_1 = 2L_2 = 2L$  one obtains  $R_{3\text{-loop}}(L, 2L) = -\frac{4}{3}aL\lambda^{3L}$ , as shown in Fig. 3.

**THE  $T = 0$  LIMIT OF THE SOLUTION OBTAINED WITH  $T > 0$  AND  $z \rightarrow \infty$**

In Ref. [28] we computed at positive temperatures ( $T > 0$ ) the analytical expression for the connected correlation function on a BL with a manually inserted cubic loop in the  $z \rightarrow \infty$  limit (to recover previous results obtained in the fully-connected model). The result can be summarized as follows:

$$R_{3\text{-loop}}^{z=\infty} = \lambda_R^{L_I+L_O} \left[ \left( L_2 \lambda_{L/A}^{L_2-1} \lambda_R^{L_1} + L_1 \lambda_{L/A}^{L_1-1} \lambda_R^{L_2} \right) b_1 \beta^2 + \lambda_{L/A}^{L_1+L_2} b_2 \right. \\ \left. + (\lambda_{L/A}^{L_2} \lambda_R^{L_1} + \lambda_{L/A}^{L_1} \lambda_R^{L_2}) b_3 + \lambda_R^{L_1+L_2} b_4 \right] \quad (33)$$

with  $\lambda_R$  the eigenvalue associated to the replicon sector,  $\lambda_{L/A}$  the degenerate longitudinal-anomalous eigenvalues and  $\beta = 1/T$  the inverse temperature. The coefficients read

$$\begin{aligned} b_1 &= -32(2m_2 - 3m_4)(1 - 7m_2 + 11m_4 - 5m_6)^2 \\ b_2 &= 64(1 - 7m_2 + 11m_4 - m_6)^2 \\ b_3 &= -80(1 + 35m_2^2 + 77m_4^2 + m_4(18 - 68m_6) - 2m_2(6 + 52m_4 - 23m_6) - 8m_6 + 15m_6^2) \\ b_4 &= 32(1 + 44m_2^2 + 101m_4^2 + m_4(22 - 90m_6) - 2m_2(7 + 67m_4 - 30m_6) - 10m_6 + 20m_6^2) \end{aligned} \quad (34)$$

with  $m_2, m_4$  and  $m_6$  the moments of order 2, 4, 6 of the magnetization. Their definition is the following

$$m_a = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} \tanh^a \left( \beta(\sqrt{Q}z + H) \right) dz, \quad (35)$$

with  $Q$  solution of the self-consistency equation  $Q = m_2$ . One thus could look at the  $T = 0$  limit of these expressions and compare them to what we have presented above.

Let us emphasize that for any positive temperature on the dAT line  $\lambda_R$  is the only critical eigenvalue and thus the leading term in the response function is of the type  $\lambda_R^{\Sigma(\bar{L})}$ . At  $T = 0$ , however,  $\lambda_{L/A} \rightarrow \lambda_R$  and the leading terms in the one-loop response could change.

To compute the  $T = 0$  limit, we should substitute in the equations the values for the moments of the magnetization in the zero temperature limit. Expanding Eq. (35) around  $T = 0$  we find

$$\begin{aligned} m_a &= 1 - C_1(a) \left[ \frac{T e^{-H^2/2}}{\sqrt{2\pi}} - \left( \frac{T e^{-H^2/2}}{\sqrt{2\pi}} \right)^2 (H^2 - 1) + \left( \frac{T e^{-H^2/2}}{\sqrt{2\pi}} \right)^3 \frac{3H^4 - 10H^2 + 5}{2} \right] + \\ &\quad - C_2(a) \frac{T^3 e^{-H^2/2}}{2\sqrt{2\pi}} (H^2 - 1), \end{aligned} \quad (36)$$

with  $C_1(a) \equiv \int_{-\infty}^{\infty} (1 - \tanh^a(x)) dx$  and  $C_2(a) \equiv \int_{-\infty}^{\infty} x^2 (1 - \tanh^a(x)) dx$ , whose numerical values are  $C_1(2) = 2$ ,  $C_1(4) = 8/3$ ,  $C_1(6) = 46/15$ ,  $C_2(2) = \pi^2/6$ ,  $C_2(4) = 2(3 + \pi^2)/9$ ,  $C_2(6) = 4/3 + 23\pi^2/90$ .

Substituting expression (36) in the coefficients defined in Eq. (34) we find  $b_1 = O(T^6)$ ,  $b_2 = O(T^6)$ ,  $b_3 = O(T^4)$  and  $b_4 = O(T^2)$ , implying that in the  $T = 0$  limit the dominating term for the response function is the one coming from the replicon, as for  $T \neq 0$ , that has a behaviour  $\lambda_R^{\Sigma(\bar{L})}$ . This result is different from what we computed directly at  $T = 0$  and finite  $z$ .

One may attempt to improve this result, obtained in the  $z \rightarrow \infty$  limit, by using the actual values of the magnetization moments computed at finite  $z$  in the Bethe solution. This would represent the simplest idea to go beyond the  $z = \infty$  result. However, we find that the coefficients entering the expansions up to the second order in  $T$  of the magnetization moments,

$$m_2 = 1 + m_2^{(1)} T + m_2^{(2)} T^2 + O(T^3), \quad (37)$$

$$m_4 = 1 + m_4^{(1)} T + m_4^{(2)} T^2 + O(T^3), \quad (38)$$

$$m_6 = 1 + m_6^{(1)} T + m_6^{(2)} T^2 + O(T^3), \quad (39)$$

do satisfy the following relations

$$\frac{m_4^{(1)}}{m_2^{(1)}} = \frac{m_4^{(2)}}{m_2^{(2)}} = \frac{4}{3} \quad \frac{m_6^{(1)}}{m_2^{(1)}} = \frac{m_6^{(2)}}{m_2^{(2)}} = \frac{23}{15}, \quad (40)$$

independently on the distribution of the cavity fields. This implies that using the non-Gaussian cavity fields obtained at finite  $z$  would not change the result obtained in the  $z \rightarrow \infty$  limit, that is using Gaussian fields. This is true up to second order in  $T$ , and this is enough as the leading term  $b_4$  is of that order of magnitude. We checked this result numerically for a BL with a finite and small value for  $z$ .

The results of this Section prove that computing the leading term of the correlation at  $T > 0$ , where the only critical eigenvalue is the replicon, inevitably produces a wrong result in the  $T \rightarrow 0$  limit, where the degeneracy among different eigenvalue plays an important role. This is true both in the fully connected model ( $z \rightarrow \infty$ ) and also for  $z$  finite within the Bethe solution. The only way to perform the right computation is by working directly at  $T = 0$  as we have done in the present work.