Normality of closure of orthogonal nilpotent symmetric orbits

Marco Trevisiol

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Abstract

We study closures of conjugacy classes in the symmetric matrices of the orthogonal group and we determine which one are normal varieties. In contrast to the result for the symplectic group where all classes have normal closure, there is only a relatively small portion of classes with normal closure. We perform a combinatorial computation on top of the same methods used by Kraft-Procesi and Ohta.

0 Introduction

In a fundamental paper of Kostant [3], the adjoint action on a reductive Lie algebra $\mathfrak{g}$ defined over an algebrically closed field k of characteristic 0 is studied in detail. In the course of his analysis it arose the following problem. Let $A \in \mathfrak{g}$, let C_A be the conjugacy class of A and let $\overline{C_A}$ be the (Zarisky) closure of C_A. Is $\overline{C_A}$ always a normal variety?

In his paper he showed that if A is a regular nilpotent element of $\mathfrak{g}$, so that $\overline{C_A}$ is the nilpotent cone of $\mathfrak{g}$, the normality is always the case.

In a paper [2] by Hesselink, the problem of normality of $\overline{C_A}$ is reduced to the case in which A is nilpotent, possibly changing the Lie algebra $\mathfrak{g}$.

In a fundamental paper [5] of Kraft and Procesi, the normality of $\overline{C_A}$ is proved for any nilpotent A in the case $\mathfrak{g} = \mathfrak{gl}_n$. Their method consists in constructing an auxiliary variety Z which is a normal complete intersection such that $\overline{C_A}$ is a *quotient* of Z.

Kraft and Procesi extended their method to the case where $\mathfrak{g}$ is the orthogonal or symplectic Lie algebra. In that case not all nilpotent classes have normal closure. They obtained necessary and sufficient conditions on the partition of A in order to have the normality of $\overline{C_A}$, under suitable hypothesis on A.

In a subsequent paper by Sommers [12], the cases outside of this hypothesis (in [5]) are proved to always have normal closure. Sommers also solved the problem for the exceptional case E_6 in [11].

Kostant and Rallis generalized the study [3] of the adjoint action on $\mathfrak{g}$ to the analogue associate with a symmetric space in [4]. We briefly describe their setting. Let θ be a Lie algebra automorphism of $\mathfrak{g}$ of order 2. Then there exists a decomposition of $\mathfrak{g}$ in θ-eigenspaces given by

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

where $\mathfrak{k} = \{A \in \mathfrak{g} : \theta(A) = A\}$ and $\mathfrak{p} = \{A \in \mathfrak{g} : \theta(A) = -A\}$. Let G be the adjoint group of $\mathfrak{g}$ and $K \subseteq G$ be the subgroup of elements commuting with θ. We notice that $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ and $\operatorname{Lie} K = \mathfrak{k}$. Moreover the adjoint action of G on $\mathfrak{g}$ induces an action of K on $\mathfrak{g}$ and both $\mathfrak{k}$, $\mathfrak{p}$ are K-stable. The pair $(\mathfrak{g}, \mathfrak{k})$ is called a symmetric pair and $\mathfrak{p}$ is the symmetric space associate to it.

In their study, Kostant and Rallis focused on the action of K on $\mathfrak{p}$. Among the other things, they already observed that the nilpotent cone is not irreducible nor normal in general.

Vinberg further generalized the study of symmetric spaces to the notion of θ-groups in [13]. He also studied the orbits structure in the nilpotent cone of the symmetric spaces.

Their study brings the problem of the normality of $\overline{C_A}$ to the case of symmetric spaces.

In [10], Sekiguchi studied deeply the geometry of the nilpotent cone in symmetric spaces. In particular he proved several results for the principal nilpotent elements (analogue of the regular nilpotent elements for the symmetric spaces cases) when the symmetric space is the orthogonal symmetric space or the symplectic symmetric space, that is, the symmetric spaces of the symmetric pairs $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{gl}(n), \mathfrak{o}(n))$ and $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{gl}(2m), \mathfrak{sp}(m))$ respectively.

The method of the auxiliary variety Z developed by Kraft and Procesi in [5, 6, 7] was adapted by Ohta in [9] to the study of the singularities of the orbits in the orthogonal and symplectic symmetric spaces. In particular he proved that $\overline{C_A}$ is always normal in the case of symplectic symmetric space. Moreover he found several orbits C_A in the case of the orthogonal symmetric space which have a non-normal closure.

The main purpose of this paper is to give a necessary and sufficient condition on the partition of A in order to have the normality of $\overline{C_A}$ in the case A belong to the orthogonal symmetric space. Precisely, the main theorem is the following.

Theorem 1. *Let $\mathfrak{p}$ be the symmetric space of the symmetric pair $(\mathfrak{gl}(n), \mathfrak{o}(n))$ and let $A \in \mathfrak{p}$ be any nilpotent element. Let $\lambda = (\lambda_1, \ldots, \lambda_h)$ be the partition of A. Then $\overline{C_A}$ is normal if and only if*

$$\lambda_i - \lambda_{i+1} \leq 1 \quad \forall i = 1, \ldots, h \tag{1}$$

with the convention that $\lambda_{h+1} = 0$.

The proof of theorem 1 will be carried out in section 8. The *only if* part was already proved in [10, 9] (see section 8 for details).

We summarise the content of the rest of the paper.

In section 1 we recall some basic facts of symmetric nilpotent orbits for the orthogonal group. In section 2 we study the class of ab-diagrams occurring in our case. In section 3 we recall the construction of the variety Z and some of its properties which are needed for the proof. In section 4 we define a condition on partitions which plays an important role in our investigation. In section 5 we introduce some combinatorial description of pairs of partitions. In 7, resp. 8, we prove a complete intersection, resp. normality, condition for the variety Z using a combinatorial computation carried on in section 6. In fact, the core of this paper lies in section 6 as the previous sections describe the context used there to allow our computations, while the following sections contain simple corollaries to section 6.

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1 Symmetric nilpotent orbits

In this section we introduce the settings and the notation for the objects studied in the paper. We follow [7] and [4].

Let V be a vector space over $\mathbb{C}$ together with a symmetric non-degenerate bilinear form $(-,-)$. Let $G(V)$ be the isometry group with respect to $(-,-)$ and $\mathfrak{gl}(V)$ be the space of linear endomorphisms of V. Clearly $G(V)$ acts on $\mathfrak{gl}(V)$ by conjugation; moreover if D^* is the adjoint of an endomorphism D with respect to $(-,-)$, we have a decomposition in the eigenspaces of the map $\theta : D \mapsto -D^*$ as follow:

$$\mathfrak{gl}(V) = \mathfrak{k}(V) \oplus \mathfrak{p}(V)$$

therefore $\mathfrak{k}(V)$ is the Lie algebra of $G(V)$ and the action of $G(V)$ leaves both $\mathfrak{k}(V)$, $\mathfrak{p}(V)$ stable.

In this paper we are concerned with the study of nilpotent orbits of $G(V)$ in $\mathfrak{p}(V)$. It is well known ([10, Sez 3.1], [9, Sez 0]) that the nilpotent orbits in $\mathfrak{p}(V)$ are completely determined by their Jordan form and every partition of $n := \dim V$ gives rise to a nilpotent orbit. We denote by $P(n)$ the set of partitions of n and by $C_\lambda \subseteq \mathfrak{p}(V)$ the nilpotent orbit corresponding to λ.

For any partition $\lambda \in P(n)$, we denote by $|\lambda| := n$ and the dual partition by $\widehat{\lambda}$. We frequently identify a partition with its Young diagram. Therefore, if $\lambda = (\lambda_1, \ldots, \lambda_h)$, $|\lambda| = \lambda_1 + \cdots + \lambda_h$ and if $\widehat{\lambda} = (\widehat{\lambda}_1, \ldots, \widehat{\lambda}_t)$, λ_i are the rows of λ and $\widehat{\lambda}_j$ are the columns of λ.

We recall a dimension formula for the orbit C_λ from [9, Remark 8]:

$$\dim C_\lambda = \frac{1}{2}\left(n^2 - \sum_{i=1}^{t} \widehat{\lambda}_i^2\right). \tag{2}$$

The orbit closure $\overline{C_\lambda}$ is $G(V)$-stable, and the complement $\overline{C_\lambda} \setminus C_\lambda$ is a disjoint union of finitely many orbits. The relation $C_\mu \subseteq \overline{C_\lambda}$ produces a partial order on the partitions, called *dominance order* and denoted by $\mu \leq \lambda$, given by

$$\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i \quad \forall j$$

or, equivalently,

$$\sum_{k>j} \widehat{\lambda}_k \geq \sum_{k>j} \widehat{\mu}_k \quad \forall j.$$

We recall the notion of nilpotent pairs. We follow the same argument of [7]. Let V (resp. U) be a vector space over $\mathbb{C}$ equipped with a non-degenerate symmetric bilinear form $(-,-)_V$ (resp. $(-,-)_U$). We denote $L(V,U) := \mathrm{Hom}_{\mathbb{C}}(V,U)$, $L(V) := L(V,V)$ and we define $L_{V,U} := L(V,U) \times L(U,V)$. We can interpret $L_{V,U}$ as the representation variety of the quiver

$$Q = \begin{tikzcd} 1 \arrow[r, bend left] & 2 \arrow[l, bend left] \end{tikzcd}$$

with dimension vector $\underline{n} = (\dim V, \dim U)$. This means that

$$L_{V,U} = \left\{ \begin{tikzcd} V \arrow[r, bend left, "A"] & U \arrow[l, bend left, "B"] \end{tikzcd} : A \in L(V,U), B \in L(U,V) \right\}.$$

The group $G(V) \times G(U)$ acts on $L_{V,U}$ by change of basis.

For any $A \in L(V,U)$ we define the adjoint map $A^* \in L(U,V)$ as

$$(Av, u)_U = (v, A^* u)_V \tag{3}$$

for all $v \in V$, $u \in U$.

A pair $(A, B) \in L_{V,U}$ is nilpotent if the endomorphism AB (or equivalently BA) is nilpotent. A pair $(A, B) \in L_{V,U}$ is symmetric if $B = A^*$. We define $N_{V,U}$ as the cone in $L_{V,U}$ of symmetric nilpotent pairs.

As in [7], we have maps

$$\begin{tikzcd} L_{V,U} \arrow[r, "\pi"] \arrow[d, "\rho"'] & L(U) \\ L(V) & \end{tikzcd}$$

given by $\pi(A, B) = BA$, $\rho(A, B) = AB$. We can restrict π, ρ to the subspace of symmetric pairs, i. e. the image of $L(V, U) \to L_{V,U}$, $A \mapsto (A, A^*)$:

$$\begin{tikzcd} L(V,U) \arrow[r, "\pi"] \arrow[d, "\rho"'] & \mathfrak{p}(U) \\ \mathfrak{p}(V). & \end{tikzcd}$$

The subvariety $N_{V,U} \subseteq L_{V,U}$ is $G(V) \times G(U)$-stable. We call *nilpotent symmetric orbit* each of the $G(V) \times G(U)$-orbits in $N_{V,U}$. As shown in [5, Sec. 4.3], to each $(A, B) \in L_{V,U}$ it is associated its ab-diagram which determines its $GL(V) \times GL(U)$-orbit completely. An ab-diagram is a list of ab-strings i.e. strings on the letters a and b occurring on alternate positions [5, Sec. 4.2]. Each ab-diagram coming from elements in $L_{V,U}$ has $\dim V$ a's and $\dim U$ b's. If δ is the ab-diagram of an element $p \in N_{V,U}$, we can retrieve the Young diagram of $\pi(p)$ (resp. $\rho(p)$) by suppressing the b's (resp. the a's) from δ. We denote by $\pi(\delta)$ (resp. $\rho(\delta)$) the Young diagram obtained in this manner.

2 Ortho-symmetric ab-diagrams

In this section we give a combinatorial description of the ab-diagrams of the elements of $N_{V,U}$.

Let $X \in N_{V,U}$ be any nilpotent symmetric pair, let $G = G(V) \times G(U)$ and let $G.X$ be its orbit in $N_{V,U}$. We also consider $G' = \mathrm{GL}(V) \times \mathrm{GL}(U)$ acting on $L_{V,U}$, and the orbit $G'.X \subseteq L_{V,U}$.

The automorphism σ of G' defined by

$$(g_1, g_2)^\sigma = ((g_1^*)^{-1}, (g_2^*)^{-1})$$

has G as the set of fixed points. The automorphism (still denoted by σ) of $L_{V,U}$ defined by

$$\sigma(A, B) = (B^*, A^*)$$

has the symmetric pairs as the set of fixed points. It is straightforward to check that all hypotesis of [8, Prop. 2.1] hold. Thus we get that:

$$G'.X \cap N_{V,U} = G.X.$$

Let δ be an ab-diagram associated to an orbit $G'.Y$. We say that δ is *ortho-symmetric* if

$$G'.Y \cap N_{V,U} \neq \emptyset.$$

We are left with the following problem: which ab-diagrams are *ortho-symmetric*?

For any two pairs $(A, B) \in L_{V,U}$, $(A', B') \in L_{V',U'}$, it is defined the direct sum $(A, B) \oplus (A', B') = (A \oplus A', B \oplus B')$ as an element in $L_{V \oplus V', U \oplus U'}$.

If δ is the ab-diagram of (A, B) and δ' is the ab-diagram of (A', B'), the ab-diagram of the direct sum $(A, B) \oplus (A', B')$ is the disjoint union of δ and δ'.

We call a symmetric pair (A, A^*) indecomposable if it can not be written as a direct sum of two nontrivial symmetric pairs. We claim that the ab-diagrams of indecomposable nilpotent symmetric pairs are given in table 1. This table closely follows table II in [7, Sec. 6.3].

type	α_k	β_k	ε_k
	$abab \cdots a$	$baba \cdots b$	$abab \cdots b$ $baba \cdots a$
$\#a$	$k+1$	k	$2k$
$\#b$	k	$k+1$	$2k$
	-	-	$k \geq 1$

Table 1: Indecomposable ortho-symmetric ab-diagrams.

Proposition 2. *An ab-diagram δ is ortho-symmetric if and only if it is a disjoint union of finitely many ab-diagrams from table 1.*

Proof. For any linear map $X : V \to U$, we have that $\operatorname{rk} X = \operatorname{rk} X^*$. Let $(A, B) = (A, A^*)$ any symmetric pair; therefore we get that for any non-negative integer k the following must hold:

$$\operatorname{rk}(AB)^k A = \operatorname{rk}(BA)^k B. \tag{4}$$

Let δ be the ab-diagram of (A, B). By induction on the maximal length of a row inside δ, we can see that the condition (4) implies the following: for any even positive integer h, there must be an even number of rows in δ whose length is h and exactly half of them must start with a and half with b. This means that any ortho-symmetric ab-diagram can be obtained as a disjoint union of ab-diagrams in table 1.

On the other hand, we show that any ab-diagram in table 1 is ortho-symmetric. In order to show this, we just need to construct a symmetric pair (A, A^*) associated to each diagram α_k, β_k, ε_k.

α_k**:** Let $D : \mathbb{C}^{k+1} \to \mathbb{C}^{k+1}$ be any symmetric nilpotent endomorphism of nilpotent order exactly $k+1$. Let $D = I \circ X$ be the canonical decomposition of the map D through its image $D(\mathbb{C}^{k+1})$, so that $X : \mathbb{C}^{k+1} \to D(\mathbb{C}^{k+1})$ and $I : D(\mathbb{C}^{k+1}) \hookrightarrow \mathbb{C}^{k+1}$. Then D induces a non-degenerate bilinear symmetric form in $D(\mathbb{C}^{k+1})$, as shown in [7, Sec. 4], and $X = I^*$. As $\dim D(\mathbb{C}^{k+1}) = k$ and both X, I have maximal rank, we immediately get that the ab-diagram of the symmetric nilpotent pair (X, I) is α_k.

β_k**:** Let (A, A^*) be any pair with ab-diagram of type α_k. Then (A^*, A) is a pair with ab-diagram of type β_k.

ε_k**:** Let (A, B) be any nilpotent (not symmetric) pair between the spaces $(V, U) = (\mathbb{C}^k, \mathbb{C}^k)$ with ab-diagram $ab \cdots b$. Let (B^*, A^*) be the linear dual of (A, B), so that it is a nilpotent pair between the dual spaces (V^*, U^*) with ab-diagram $ba \cdots a$. Then we can equip $V \oplus V^*$ and $U \oplus U^*$ with the non-degenerate symmetric bilinear form given by the duality pairing. Therefore $(A, B) \oplus (B^*, A^*)$ is a symmetric nilpotent pair and its ab-diagram has type ε_k.

□

Example. Both δ_1, δ_2 are ab-diagrams, but only δ_1 is ortho-symmetric:

$$\delta_1 = \begin{matrix} abab \\ abab \\ baba \\ baba \\ a \end{matrix}, \qquad \delta_2 = \begin{matrix} abab \\ abab \\ baba \\ a \end{matrix}.$$

3 The variety Z

For any partition λ of n we are going to construct a variety $Z^{(\lambda)}$ (or just Z, if the partition λ is clear from the context) with the following property: there exists a quotient map $Z \to \overline{C_\lambda}$. In this way, we will be able to assert some properties of $\overline{C_\lambda}$ by proving them for the variety Z. This construction is analogous to the one carried out in [7] and it has already been described in [9]; here we just recall the definition.

Let $\lambda \in P(n)$ be any partition and let $t = \lambda_1$ be the number of columns of λ. As in [7], we define a variety Z using the representations of the quiver:

$$Q_t = \begin{tikzcd} 0 \arrow[r, bend left, "x"] & 1 \arrow[l, bend left, "y"] \arrow[r, bend left, "x"] & \cdots \arrow[l, bend left, "y"] \arrow[r, bend left, "x"] & t. \arrow[l, bend left, "y"] \end{tikzcd}$$

Let

$$n_i = \sum_{j>i}^{t} \widehat{\lambda}_j$$

so that $n_0 = n$ and $n_t = 0$. We fix the dimension vector $\underline{n} = (n_0, \ldots, n_t)$ and vector spaces V_i such that $\dim V_i = n_i$. We also fix a symmetric bilinear non-degenerate form on each V_i.

The variety Z consists of the representations of dimension vector $\underline{n}$ of the quiver Q_t with relations:

- $xy = yx$;
- $y = x^*$ (the adjoint map as in (3)).

Therefore, any point $z \in Z$ is a sequence of maps

$$(A_1, B_1; A_2, B_2; \ldots; A_t, B_t)$$

where $A_i \in L(V_{i-1}, V_i)$, $B_i \in L(V_i, V_{i-1})$, $A_i B_i = B_{i+1} A_{i+1}$ and $B_i = A_i^*$ for any i. As each B_i is adjoint to A_i, we get an inclusion

$$Z \hookrightarrow L(V_0, V_1) \times \cdots \times L(V_{t-1}, V_t)$$

mapping

$$(A_1, B_1; A_2, B_2; \ldots; A_t, B_t) \mapsto (A_1, \ldots, A_t).$$

As in [7], we can think of Z as a schematic fiber in the following way. Let

$$\begin{aligned} M &:= L(V_0, V_1) \times \cdots \times L(V_{t-1}, V_t), \\ N &:= \mathfrak{p}(V_1) \times \cdots \times \mathfrak{p}(V_{t-1}), \\ \Phi &: M \to N \\ &(A_1, \ldots, A_t) \mapsto (A_1 A_1^* - A_2^* A_2, \ldots, A_{t-1} A_{t-1}^* - A_t^* A_t); \end{aligned}$$

then we put $Z := \Phi^{-1}(0)$. As M is an affine space, this interpretation will be useful in section 7 to show that Z is a complete intersection variety (at least for some λ).

As in [7], let $G = G(V_0) \times \cdots \times G(V_t)$ be the group which acts on Z by change of basis and $H = G(V_1) \times \cdots \times G(V_t) \subseteq G$. We have a map

$$\begin{aligned} \Theta :& Z \to \overline{C_\lambda} \\ & (A_1, B_1; \ldots; A_t, B_t) \mapsto B_1 A_1. \end{aligned}$$

By the same argument as in [7], we have the following:

Proposition 3. *The map Θ splits through the quotient by the group H as*

$$\Theta : Z \longrightarrow Z/H \xrightarrow{\sim} \overline{C_\lambda};$$

moreover Θ is $G(V_0) = G/H$-equivariant.

(In particular, $\Theta^{-1}(C_\mu)$ is stable under the action of $G(V_0)$ for any partition $\mu \leq \lambda$.)

By restricting the quiver Q_t to two consecutive vertices $i-1, i$, we get a map $Z \to N_{V_{i-1},V_i}$. As all the orbits in N_{V_{i-1},V_i} are defined by their admissible ab-diagram, we can map any point $z \in Z$ and $i = 1, \ldots, t$ to its i-th ab-diagram τ_i. Therefore, for any string of ab-diagrams

$$\tau = (\tau_1, \ldots, \tau_t)$$

we define

$$Z_\tau = \{z \in Z : \forall i, \text{ the } i\text{-th } ab\text{-diagram of } z \text{ is } \tau_i\ \}.$$

The following are necessary conditions on τ in order to have $Z_\tau \neq \emptyset$:

1. τ_i must have exactly n_{i-1} a's and n_i b's,
2. $\rho(\tau_i) = \pi(\tau_{i+1})$ for $i = 1, \ldots, t-1$,
3. each τ_i must be ortho-symmetric.

We will denote by $\Lambda^{(\lambda)}$ (or, shortly, by Λ) the set of all strings τ which verify the three conditions above.

As in [5, Lemma 5.4], [7, Sez 8.2], $\Theta^{-1}(C_\lambda)$ is given by exactly one stratum Z_{τ^0}. The string of ab-diagrams τ^0 can be constructed in the following way: for all $i = 0, \ldots, t-1$, let λ_i be the partition of n_i given by the columns of λ with indices greater than i; then let τ_{i+1} be the ab-diagram built by setting one a on any box in the Young diagram of λ_i and by setting one b between any pair of consecutive a's.

Example. Let $\lambda = (3,1)$ be a partition. Then the string of ab-diagrams τ^0 is the following:

$$\tau^0 = (\tau_1^0, \tau_2^0, \tau_3^0) = \left(\begin{array}{c|c|c} ababa & aba & a \\ a & & \end{array} \right).$$

The stratum Z_{τ^0} is open in Z as it is the subvariety of the points

$$(A_1, B_1, \ldots, A_t, B_t)$$

in Z where each A_i, B_i has maximal rank. In a completely analogous manner of [7, p. 5.5] and [9, Prop. 8], we have

Proposition 4. *The subvariety Z_{τ^0} is contained in the smooth locus of Z.*

We recall a dimension formula for the strata Z_τ due to Otha [9, Prop. 6]. First, for each ab-diagram δ, we denote by a_i (resp. b_i) the number of rows of δ starting by a (resp. b) with length i. We define

$$\Delta(\delta) := \sum_{i \text{ odd}} a_i b_i$$

and

$$o(\delta) := \sum_{i \text{ odd}} a_i + b_i.$$

Proposition 5. *[9, Prop. 6] Let $\tau = (\tau_1, \dots, \tau_t)$. We have*

$$\dim Z_\tau = \frac{1}{2} \dim C_{\pi(\tau_1)} + \sum_{i=0}^{t-1} \left(\frac{1}{2} n_i n_{i+1} - \frac{1}{4}(n_i + n_{i+1}) \right) + \sum_{i=1}^{t} \left(\frac{1}{4} o(\tau_i) - \frac{1}{2} \Delta(\tau_i) \right). \quad (5)$$

4 s-step condition

In this section we define a condition which plays a pivotal role in the subsequent study of the nilpotent symmetric orbits.

Definition 6. Given a partition $\lambda = (\lambda_1, \dots, \lambda_h)$, we say that λ satisfies the *s-step condition* if for every $i = 1, \dots, h$ we have

$$\lambda_i \leq \lambda_{i+1} + s$$

with the convention that $\lambda_{h+1} = 0$.

We want to highlight some simple properties of this definition. First of all, s_1-step condition implies s_2-step condition for any $s_2 \geq s_1$. Secondly, if λ satisfies the s-step condition, then if we remove the first row or the first column from λ, the resulting partition still satisfies the s-step condition.

Example. The single line partition (n) satisfies the n-step condition, but not the $(n-1)$-step condition. The triangular partition $(n, n-1, \dots, 1)$ satisfies the 1-step condition. The *2-skew* triangular partition $(2n, 2(n-1), \dots, 2)$ satisfies the 2-step condition (but not the 1-step condition).

5 Differences between partitions

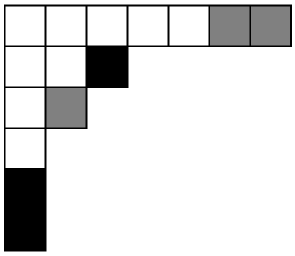
In this section we study some quantitative ways to address the difference between two partitions. Let $\lambda, \mu \in P(n)$ be two partitions of n such that $\lambda > \mu$.

Definition 7. If $\lambda = (\lambda_1, \dots, \lambda_k) \in P(n)$ and $\mu = (\mu_1, \dots, \mu_k) \in P(n)$, possibly allowing $\lambda_i = 0$ for some i, we define

$$q(\lambda, \mu) = \frac{1}{2} \sum_{i=1}^{k} |\lambda_i - \mu_i|.$$

Remark. By an easy computation modulo 2, it can be seen that $q(\lambda, \mu)$ is an integer.

Example. Let $\lambda = (7, 2, 2, 1)$ and $\mu = (5, 3, 1, 1, 1, 1)$. We can draw the Young diagram of both partitions and overlay one over the other, so that λ is given by white and gray boxes while μ is given by white and black boxes in the following diagram:



We can compute the q difference:

$$q(\lambda, \mu) = \frac{1}{2}\left((7-5) + (3-2) + (2-1) + (1-1) + 1 + 1\right) = 3.$$

We can check that in this case $q(\lambda, \mu)$ is the number of gray boxes or the number of black boxes. This is an easy fact (it is not even needed that $\lambda > \mu$) that can be derived from the following lemma.

Lemma 8. *Let $\lambda > \mu$ be two partitions of n. Then there exists partitions*

$$\lambda = \lambda^0 > \lambda^1 > \cdots > \lambda^{q(\lambda,\mu)} = \mu$$

such that $q(\lambda^i, \lambda^{i+1}) = 1$ for each $i = 0, \ldots, q(\lambda, \mu) - 1$.

Moreover, we can choose λ^i such that for each row r (and, resp., for each column c) the sequence of integers $\lambda_r^0, \ldots, \lambda_r^{q(\lambda,\mu)}$ (resp. $\widehat{\lambda}_c^0, \ldots, \widehat{\lambda}_c^{q(\lambda,\mu)}$) is monotone.

Proof. We will proceed by induction on $q = q(\lambda, \mu)$, being $q = 1$ a trivial base step as we already have $\lambda^1 = \mu$.

For the induction step, it is enough to build λ^1 such that $q(\lambda, \lambda^1) = 1$ and $q(\lambda^1, \mu) = q(\lambda, \mu) - 1$.

We build λ^1 starting from λ in this way. Let i be the first index such that $\lambda_i > \mu_i$ and let j be the first index such that $\lambda_j < \mu_j$. Clearly there must exists such indices because $\lambda \neq \mu$ and $|\lambda| = |\mu|$. Moreover, $i < j$ because $\lambda > \mu$. We define

$$\lambda^1 = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \ldots, \lambda_{j-1}, \lambda_j + 1, \lambda_{j+1}, \ldots, \lambda_k).$$

Therefore both the following hold: $q(\lambda, \lambda^1) = 1$ and $q(\lambda^1, \mu) = q - 1$. We apply induction on the pair (λ^1, μ) and we conclude the proof of the first part of the lemma.

In order to prove that the sequences of rows lengths are monotone, it is enough to notice that for any r if $\lambda_r > \mu_r$ (resp. $\lambda_r < \mu_r$) then the row r is never increased (resp. decreased) with the choice made. The same holds true for the columns lengths. $\square$

Example. Taking the previous example, the construction explained in the lemma will get:

$$(7, 2, 2, 1) > (6, 3, 2, 1) > (5, 3, 2, 1, 1) > (5, 3, 1, 1, 1).$$

Remark. In view of lemma 8 we can interpret $q(\lambda, \mu)$ as the minimum number of boxes needed to lower in order to obtain μ from λ.

We consider two other ways of comparing partitions. Given any partition λ, for each box b in the Young diagram of λ we define its column number $c(b)$ (resp. row number $r(b)$) by counting the columns (resp. rows) starting from the leftmost column (resp. uppermost row). For example, given $\lambda = (7,2,2,1)$ we have:

$$c(-)\colon \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 2 \\ \cline{1-2} 1 & 2 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array} \qquad r(-)\colon \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} 3 & 3 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$$

Definition 9. Given $\lambda \geq \mu$ as before, we define:

$$c(\lambda, \mu) = \sum_{b \in \lambda} c(b) - \sum_{b \in \mu} c(b),$$

where b is selected among the boxes in the Young diagrams of λ and μ and, similarly,

$$r(\lambda, \mu) = \sum_{b \in \mu} r(b) - \sum_{b \in \lambda} r(b).$$

Example. Let $\lambda = (7,2,2,1)$ and $\mu = (5,3,1,1,1,1)$. We easily get

$$\begin{aligned} c(\lambda, \mu) &= (28+3+3+1) - (15+6+1+1+1+1) = 10, \\ r(\lambda, \mu) &= (21+3+3+1+1) - (10+6+1+1+1+1+1) = 8. \end{aligned}$$

Remark. It is always guaranteed that $c(\lambda, \mu) \geq 0$ and $r(\lambda, \mu) \geq 0$ for any $\lambda \geq \mu$. We even have that

$$\lambda = \mu \iff c(\lambda, \mu) = 0 \iff r(\lambda, \mu) = 0. \tag{6}$$

In fact we have that $c(\lambda, \mu) \geq 0$ is equivalent to

$$\sum_i \binom{\lambda_i + 1}{2} \geq \sum_i \binom{\mu_i + 1}{2}. \tag{7}$$

As the function $n \mapsto \binom{n+1}{2}$ is convex, (7) is given by the Karamata's inequality ([1, Chapter 1, §28]).

Remark. Given partitions $\lambda \geq \mu \geq \nu$, by definition of c and r we have:

$$c(\lambda, \nu) = c(\lambda, \mu) + c(\mu, \nu), \tag{8}$$
$$r(\lambda, \nu) = r(\lambda, \mu) + r(\mu, \nu). \tag{9}$$

Remark. Using (6) we get that

$$c(\lambda, \mu) \geq q(\lambda, \mu) \tag{10}$$

when $q(\lambda, \mu) = 1$. In fact we can generalize this inequality for any $q(\lambda, \mu)$ because it is enough to apply lemma 8 and (8) as for each pair $(\lambda^i, \lambda^{i+1})$ we have $c(\lambda^i, \lambda^{i+1}) \geq 1$.

In the next result, we establish a useful inequality involving the functions r, c, q.

Lemma 10. *Let λ be a partition satisfying the s-step condition and $\mu < \lambda$ be any partition. Then*

$$s \cdot r(\lambda, \mu) \geq c(\lambda, \mu) + q(\lambda, \mu). \tag{11}$$

Moreover, (11) holds strictly if there exists a column index c such that $\widehat{\mu}_c > \widehat{\lambda}_c + 1$ or a row index r such that $\mu_r > \lambda_r + 1$.

Proof. Let $q = q(\lambda, \mu)$. We apply lemma 8 to obtain a sequence

$$\lambda = \lambda^0 > \cdots > \lambda^q = \mu.$$

We want to prove

$$s \cdot r(\lambda^i, \lambda^{i+1}) \geq c(\lambda^i, \lambda^{i+1}) + 1 \tag{12}$$

for all $i = 0, \ldots, q-1$. Once (12) is proved, the conclusion will follow by summing each of those inequalities. Indeed, by (9) and (8), we will get:

$$s \sum_{i=0}^{q-1} r(\lambda^i, \lambda^{i+1}) \geq \sum_{i=0}^{q-1} \big(c(\lambda^i, \lambda^{i+1}) + 1\big)$$
$$s \cdot r(\lambda^0, \lambda^q) \geq c(\lambda^0, \lambda^q) + q.$$

The resulting inequality will be strict as soon as there exists an i such that (12) holds strictly.

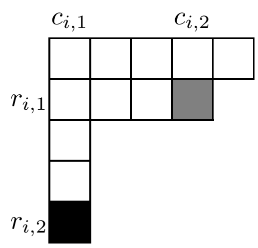
We start by noticing that the s-step condition implies that, for any pair of row indices $r_1 < r_2$,

$$\lambda_{r_1} - \lambda_{r_2} \leq s(r_2 - r_1).$$

Now, fix r_1 to be an index row such that $\lambda_{r_1} \geq \mu_{r_1}$ and fix r_2 to be an index row such that $\lambda_{r_2} \leq \mu_{r_2}$. Lemma 8 asserts that $\lambda^i_{r_j}$ is a monotone sequence for both $j = 1, 2$; therefore, for such r_1, r_2, we get $\lambda^i_{r_1} - \lambda^i_{r_2} \leq \lambda_{r_1} - \lambda_{r_2}$, so we get

$$\lambda^i_{r_1} - \lambda^i_{r_2} \leq s(r_2 - r_1). \tag{13}$$

As $q(\lambda^i, \lambda^{i+1}) = 1$, we already observed that the Young diagrams of λ^i, λ^{i+1} have only one box in different positions. This means that there exist exactly two columns $c_{i,1} < c_{i,2}$ which are not equal between λ^i, λ^{i+1}. Similarly, there exist exactly two rows indices $r_{i,1} < r_{i,2}$.



We have $c(\lambda^i, \lambda^{i+1}) = c_{i,2} - c_{i,1}$ and $r(\lambda^i, \lambda^{i+1}) = r_{i,2} - r_{i,1}$. Moreover, by definition of λ^i and λ^{i+1}, we have $\lambda^i_{r_{i,1}} = c_{i,2}$ and $\lambda^i_{r_{i,2}} = c_{i,1} - 1$. Therefore, by (13), we get

$$s \cdot r(\lambda^i, \lambda^{i+1}) = s(r_{i,2} - r_{i,1}) \geq \lambda_{r_{i,1}} - \lambda_{r_{i,2}} \geq \lambda^i_{r_{i,1}} - \lambda^i_{r_{i,2}} = c_{i,2} - c_{i,1} + 1 = c(\lambda^i, \lambda^{i+1}) + 1.$$

Finally, we consider the additional hypothesis of the existence of a column index c such that $\widehat{\mu}_c > \widehat{\lambda}_c + 1$ or a row index r such that $\mu_r > \lambda_r + 1$. We are going to prove that there exists i such that (13) holds strictly.

Let

$$I_{\text{cols}}(c) = \left\{ i \in \{0, \ldots, q-1\} : \widehat{\lambda^{i+1}}_c > \widehat{\lambda^i}_c \right\}.$$

As $q(\lambda^i, \lambda^{i+1}) = 1$, we have $|\widehat{\lambda^{i+1}}_c - \widehat{\lambda^i}_c| \leq 1$ for each i. Therefore, if there exists c such that $\widehat{\mu}_c > \widehat{\lambda}_c + 1$, we have $|I_{\text{cols}}(c)| \geq 2$. Let j be the minimum in $I_{\text{cols}}(c)$. We show that (13) holds strictly for the rows $r_{i,1}, r_{i,2}$ for any $i \in I_{\mathrm{cols}}(c)$ such that $i \neq j$. Indeed, as $\lambda^i_{r_{i,2}} = c = \lambda^j_{r_{j,2}}$, we have

$$\lambda^i_{r_{i,1}} - \lambda^i_{r_{i,2}} \leq \lambda^j_{r_{i,1}} - \lambda^j_{r_{j,2}} \leq \lambda_{r_{i,1}} - \lambda_{r_{j,2}} \leq s(r_{j,2} - r_{i,1}) \leq s(r_{i,2} - 1 - r_{i,1}) < s(r_{i,2} - r_{i,1}).$$

In a similar fashion, let

$$I_{\mathrm{rows}}(r) = \left\{ i \in \{0, \dots, q-1\} : \lambda^{i+1}{}_r > \lambda^i{}_r \right\}.$$

If there exists r such that $\mu_r > \lambda_r + 1$, then $|I_{\mathrm{rows}}(r)| \geq 2$ and we take $j \in I_{\mathrm{rows}}(r)$ to be the minimum and $i \in I_{\mathrm{rows}}(r)$ to be another index. We have $\lambda^i_{r_{i,2}} > \lambda^j_{r_{i,2}} = \lambda_{r_{i,2}}$, so we get

$$\lambda^i_{r_{i,1}} - \lambda^i_{r_{i,2}} < \lambda_{r_{i,1}} - \lambda_{r_{i,2}} \leq s(r_{i,2} - r_{i,1}).$$

In both cases (13) holds strictly for at least an i. Therefore (12) holds strictly for this i, and (11) must hold strictly as a consequence. Thus, the lemma is proven. $\square$

Example. Let

$$\lambda = (6, 4, 2), \quad \mu = (5, 3, 2, 1, 1), \quad \nu = (5, 3, 3, 1)$$

be three partitions of $n = 12$. We have that $\mu < \lambda$ and $\nu < \lambda$; moreover, in the pair (λ, μ) the first column differs by 2 boxes, while no column in (λ, ν) differs more than 1 box.

Once observed that λ satisfies the 2-step condition, we can compute each term involved in (11). For (λ, μ) we have: $2 \cdot 6 \geq 8 + 2$ (which holds strictly), while for (λ, ν) we have: $2 \cdot 4 \geq 6 + 2$.

6 Inequalities on the dimensions

In this section we will use the tools introduced in section 5 to effectively compute the dimensions of the strata in Z.

Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be any partition of $n := |\lambda|$ with $t := \lambda_1$ columns. We want to compare the dimension of Z_{τ^0} with the dimension of any other stratum Z_τ with $\tau \in \Lambda^{(\lambda)}$. Let $\mu = \pi(\tau)$, so that $\mu \leq \lambda$. The aim of this section is to give some sufficient combinatorial conditions on the pair of partitions (λ, μ) in order to secure a bound on the difference $\dim Z_{\tau^0} - \dim Z_\tau$.

The aim of the current section is to prove proposition 11 and proposition 16. In order to introduce the stronger inequality in proposition 16, we need an in-depth study of the combinatorics of Λ, so we prefer to introduce it later.

Proposition 11. *Let $\lambda \geq \mu$ be two partitions and let $Z = Z^{(\lambda)}$ be the variety built from λ. Let Z_{τ^0} be the only stratum of Z with $\pi(\tau^0_1) = \lambda$ and let Z_τ be any stratum with $\tau = (\tau_1, \dots, \tau_t)$ such that $\pi(\tau_1) = \mu$.*

For any real number x, we have that:

$$2r(\lambda, \mu) - c(\lambda, \mu) - q(\lambda, \mu) \geq 4x \quad \Longrightarrow \quad \dim Z_{\tau^0} - \dim Z_\tau \geq x.$$

The proof will proceed as follows: we take the dimension formula of any stratum of Z given by (5) then we will estimate each term on the difference. Each of the following lemmas deals with one of the terms.

Lemma 12. *Let $\lambda \geq \mu$ be two partitions of n. Let $t = \lambda_1$ be the number of columns of λ. Then*

$$\sum_{i=1}^{t} \widehat{\mu}_i^2 - \widehat{\lambda}_i^2 = 2r(\lambda, \mu).$$

(We allow $\widehat{\mu}_i = 0$ for i high enough if $\mu_1 < t$.)

Proof. For any integer m we have: $m^2 = 2\binom{m}{2} + m$. So we easily get:

$$\sum_{i=1}^{t} \widehat{\mu}_i^2 - \widehat{\lambda}_i^2 = \sum_{i=1}^{t} 2\binom{\widehat{\mu}_i}{2} + \widehat{\mu}_i - 2\binom{\widehat{\lambda}_i}{2} - \widehat{\lambda}_i = 2r(\lambda, \mu) + n - n.$$

$\square$

In order to carry on the computation, we examine carefully all the strata Z_τ with $\tau \neq \tau^0$. We do so by grouping the $\tau \in \Lambda$ which can appear in $\Theta^{-1}(C_\mu)$, for fixed μ. We already know that for every $\tau = (\tau_1, \ldots, \tau_t)$ such that $\Theta(Z_\tau) = C_\mu$, we have that $\pi(\tau_1) = \mu$.

The conditions $\tau \in \Lambda$ and $\pi(\tau_1) = \mu$ place important combinatorial restriction in the choice of the ab-diagrams $\tau_1, \ldots, \tau_t$. We will describe them now.

Let $Z^{(\mu)}$ be the variety Z built from the partition μ (rather than the partition λ). We will denote by σ^0 the unique stratum in $\Lambda^{(\mu)}$ such that $\Theta(Z^{(\mu)}_{\sigma^0}) = C_\mu$. As in the case of τ^0, every ab-diagram σ_i^0 has only rows starting and ending by a, in particular they all have odd length, so:

$$\sum_{i=1}^{t} o(\sigma_i^0) = n = \sum_{i=1}^{t} o(\tau_i^0). \tag{14}$$

The sequence of ab-diagrams σ^0 does not belong to $\Lambda^{(\lambda)}$ because the integers n_i computed from μ are different from those computed from λ.

Nonetheless, for any $\tau \in \Lambda^{(\lambda)}$ such that $\pi(\tau_1) = \mu$, the condition $\rho(\tau_{i-1}) = \pi(\tau_i)$ implies that each τ_i can be obtained from σ_i^0 by adding an adequate amount of a's and b's letters. In particular, for $i = 0$ we need no a's and possibly some b's, while for other i we could need more letters.

For each $i = 1, \ldots, t$, let d_i be the list of a's and b's we need to add to σ_i^0 in order to obtain τ_i. Let a_{d_i} (resp. b_{d_i}) be the number of a's (resp. b's) in d_i. We remark that a_{d_i} and b_{d_i} depend only on (λ, μ) and not on the particular τ. In fact, we can easily compute a_{d_i} by taking the differences

$$a_{d_i} = \sum_{j=i}^{t} \widehat{\lambda}_j - \widehat{\mu}_j; \tag{15}$$

and therefore we have $b_{d_i} = a_{d_{i+1}}$.

Example. Let $\tau \in \Lambda^{(\lambda)}$ and let $\sigma^0 \in \Lambda^{(\mu)}$. Suppose that the i-th diagrams are the following:

$$\sigma_i^0 = \begin{matrix} aba \\ aba \\ b \end{matrix}\ ; \qquad \tau_i = \begin{matrix} ababa \\ aba \\ ba \\ ab \end{matrix}\ .$$

Then, we have $d_i = (a, a, a, b, b)$, $a_{d_i} = 3$, $b_{d_i} = 2$.

Given an ab-diagram δ and a list d of a's and b's, we define $\mathrm{aug}_\delta(d)$ to be the set of orthosymmetric ab-diagrams obtainable from δ by adding the letters in d.

Example. Let $d = (a, b)$ and

$$\delta = \begin{matrix} aba \\ aba \\ b \end{matrix}\ .$$

Then $\operatorname{aug}_\delta(d)$ has three elements:

$$\operatorname{aug}_\delta(d) = \left\{ \begin{array}{l} ababa \\ aba \\ b \end{array}, \begin{array}{l} aba \\ aba \\ bab \end{array}, \begin{array}{l} aba \\ aba \\ a \\ b \\ b \end{array} \right\}.$$

Lemma 13. *Let δ^0 be any ab-diagram with rows only starting and ending by a. Let d be a list of a_d a's and b_d b's. Let $\delta \in \operatorname{aug}_{\delta^0}(d)$. Then the following holds:*

$$o(\delta) - 2\Delta(\delta) - o(\delta^0) \leq \max\{a_d, b_d\}.$$

Proof. The ab-diagram δ is obtained from δ^0 by adding a_d a's and b_d b's.

Let L (resp. S) be the set of rows of δ longer than 1 (resp. with length 1) built using only letters in d. Let S_a (resp. S_b) the rows of S starting with a (resp. b), so that $S = S_a \sqcup S_b$. Recall the notation a_i (resp. b_i) be the number of rows with length i starting with a (resp. b); in this particular ab-diagram we have $|S_b| = b_1$ and $|S_a| \geq a_1$.

$$\begin{aligned} o(\delta) - o(\delta^0) - 2\Delta(\delta) \leq& |L| + |S| - 2 \sum_{i \text{ odd}} a_i b_i \\ \leq& |L| + |S| - 2a_1 b_1 \\ \leq& |L| + |S_a| + |S_b| - 2|S_a||S_b| \\ \leq& |L| + \max\{|S_a|, |S_b|\}. \end{aligned}$$

Now, if $|S_a| \geq |S_b|$, then $|L| + |S_a| \leq a_{d_i}$ because each row in L or in S_a contains at least one a. If the opposite holds true, then $|L| + |S_b| \leq b_{d_i}$ for the same reason. Thus the lemma is proven. ◻

We can introduce a couple of small, but very important, improvements to lemma 13.

Lemma 14. *In the same setting of lemma 13 we assume furthermore that $d = (a, b)$, that is $a_d = b_d = 1$. We have the stronger inequality:*

$$o(\delta) - 2\Delta(\delta) - o(\delta^0) \leq \max\{a_d, b_d\} - 1 = 0.$$

Proof. We discuss all the possibilities in $\operatorname{aug}_{(a,b)}(\delta^0)$.

If both a and b extend rows in δ, then $o(\delta) \leq o(\delta^0)$ and we are done.

As all the rows in δ^0 have odd length, we cannot extend exactly one row with one letter, otherwise the resulting ab-diagram would not be ortho-symmetric.

In the remaining cases, both a, b has to be placed on new rows. Again, we cannot form a single row with even length, so the only case left is the following: δ has two additional single letter rows over δ^0, one a and one b. Therefore $o(\delta) - o(\delta^0) = 2$. However, $2\Delta(\delta) \geq 2(1 \cdot 1)$, so the conclusion follows. ◻

Lemma 15. *In the same setting of lemma 13 we assume furthermore that $d = (b)$, that is $a_d = 0$ and $b_d = 1$ and that δ^0 has l rows with length 1. We have the stronger equality:*

$$o(\delta) - 2\Delta(\delta) - o(\delta^0) = \max\{a_d, b_d\} - 2l = 1 - 2l.$$

Proof. As we already observed, we cannot extend exactly one row with one letter. So we must place b on a new row.

In this case, $o(\delta) - o(\delta^0) = 1$ and $\Delta(\delta) = a_1 b_1 = l \cdot 1$. $\square$

Using these lemmas we will be able to introduce the previously announced stronger version of proposition 11, namely:

Proposition 16. *Let $\lambda \geq \mu$ be two partitions and let $Z = Z^{(\lambda)}$ be the variety built from λ. Let Z_{τ^0} be the only stratum of Z with $\pi(\tau_1^0) = \lambda$ and let Z_τ be any stratum with $\tau = (\tau_1, \dots, \tau_t)$ such that $\pi(\tau_1) = \mu$. As in proposition 11, we suppose that there exists a real number x such that:*

$$2r(\lambda, \mu) - c(\lambda, \mu) - q(\lambda, \mu) \geq 4x.$$

Let u be the number of indices i for which $d_i = (a, b)$, then:

$$\dim Z_{\tau^0} \geq \dim Z_\tau + x + u/4.$$

If there exists an index i such that $d_i = (b)$ and σ_i^0 has l rows with length one, then:

$$\dim Z_{\tau^0} \geq \dim Z_\tau + x + l/2.$$

The proof of proposition 16 will be given together with the proof of proposition 11 at the end of this section.

We want an estimate of the sum of the terms $\max\{a_{d_i}, b_{d_i}\}$ depending only on λ, μ.

Lemma 17. *Let $\lambda \geq \mu$ be partitions, so that we can define a_{d_i}, b_{d_i} for all $i = 1, \dots, t$. The following holds:*

$$\sum_{i=1}^{t} \max\{a_{d_i}, b_{d_i}\} \leq c(\lambda, \mu) + q(\lambda, \mu). \tag{16}$$

Proof. First we prove the claim for $q(\lambda, \mu) = 1$.

In this case, μ is obtained from λ by moving down a single box x. Let us call c_1 (resp. c_2) the column where x lies in λ (resp. μ); so $c_1 < c_2$. Using (15), it is easy to check that $b_{d_i} = 1$ for each $c_1 \leq i < c_2$, and 0 otherwise, while $a_{d_i} = 1$ for each $c_1 < i \leq c_2$ and 0 otherwise. Therefore $\max\{a_{d_i}, a_{d_i}\} = 1$ for all $c_1 \leq i \leq c_2$ and 0 otherwise.

Therefore

$$\sum_{i=1}^{t} \max\{a_{d_i}, b_{d_i}\} = c_2 - c_1 + 1,$$

while

$$c(\lambda, \mu) = c_2 - c_1.$$

Therefore the conclusion holds in this case.

In the general case, by applying lemma 8, we get a sequence $\lambda^0 > \lambda^1 > \cdots > \lambda^{q(\lambda,\mu)}$ such that $q(\lambda^i, \lambda^{i+1}) = 1$.

Therefore, by the case $q(\lambda, \mu) = 1$, we have the inequalities (16) on $(\lambda^i, \lambda^{i+1})$ for each $i = 0, \dots, q(\lambda, \mu) - 1$ and we can sum them all. By the additivity of $c(\lambda, \mu)$, given by (8), we get the correct term on the right side. Also a_{d_i} and b_{d_i} are additive on (λ, μ), while taking the max is sub-additive, that is:

$$\max\{a_{d_i}, b_{d_i}\}(\lambda, \mu) \leq \sum_{j=0}^{q-1} \max\{a_{d_i}, b_{d_i}\}(\lambda^j, \lambda^{j+1}).$$

So we get the conclusion by summing over i. $\square$

We are finally able to prove both proposition 11 and proposition 16.

Proof of proposition 11 and proposition 16. We can start from the dimension formulas of the strata (5). Therefore

$$\begin{aligned}
4\big(\dim Z_{\tau^0} - \dim Z_\tau - x\big) &= 2(\dim C_{\pi(\tau_1^0)} - \dim C_{\pi(\tau_1)}) \\
&\quad + (o-2\Delta)(\tau^0) - (o-2\Delta)(\tau) - 4x \\
&= 2(C_\lambda - C_\mu) + o(\tau^0) - (o-2\Delta)(\tau) - 4x \\
&= \left(\sum_{i=1}^t \widehat{\mu}_i^2 - \widehat{\lambda}_i^2\right) + \left(\sum_{i=1}^t o(\tau_i^0) - (o-2\Delta)(\tau_i)\right) - 4x \\
\text{(by lemma 12)} &= 2r(\lambda,\mu) + \left(\sum_{i=1}^t o(\tau_i^0) - (o-2\Delta)(\tau_i)\right) - 4x \\
\text{(by (14))} &= 2r(\lambda,\mu) + \left(\sum_{i=1}^t o(\sigma_i^0) - (o-2\Delta)(\tau_i)\right) - 4x \\
&\geq 2r(\lambda,\mu) - 4x + \\
&\quad - \sum_{i=1}^t \max_{\tau_i \in \operatorname{aug}_{\sigma_i^0}(d_i)} \big(o(\tau_i) - 2\Delta(\tau_i) - o(\sigma_i^0)\big) \\
\text{(by lemma 13)} &\geq 2r(\lambda,\mu) - \sum_{i=1}^t \max\{a_{d_i}, b_{d_i}\} - 4x \\
\text{(by lemma 17)} &\geq 2r(\lambda,\mu) - c(\lambda,\mu) - q(\lambda,\mu) - 4x.
\end{aligned}$$

Therefore $\dim Z_{\tau^0} - \dim Z_\tau \geq x$ is guaranteed as soon as $2r(\lambda,\mu) - c(\lambda,\mu) - q(\lambda,\mu) \geq 4x$.

In order to obtain the sharper result of proposition 16, it is enough to use lemma 14 or lemma 15 in place of lemma 13 in the second to last step. □

7 Complete intersection conditions

In this section we want to give condition under which we can make sure that Z is a complete intersection variety.

Proposition 18. *Let λ be any partition and recall that $Z = \Phi^{-1}(0) \subseteq M$. We have*

$$\operatorname{codim}_M(Z_{\tau^0}) = \dim N.$$

Proof. In order to prove the equality of the required dimensions, we will work with the dimension formula given by (5). We are going to prove that

$$\dim Z_{\tau^0} = \dim M - \dim N. \tag{17}$$

We immediately have:

$$\dim M = \sum_{i=1}^t \dim L(V_{i-1}, V_i) = \sum_{i=1}^t n_{i-1} n_i$$

and

$$\dim N = \sum_{i=1}^{t-1} \dim \mathfrak{p}(V_i) = \sum_{i=1}^{t-1} \frac{1}{2} n_i^2 + \sum_{i=1}^{t-1} \frac{1}{2} n_i,$$

so

$$\dim M - \dim N = \sum_{i=1}^{t} n_{i-1} n_i - \frac{1}{2} \sum_{i=1}^{t-1} n_i^2 - \frac{1}{2} \sum_{i=1}^{t-1} n_i.$$

On the other hand, we have:

$$\dim Z_{\tau^0} = \frac{1}{2} \dim C_{\pi(\tau_1^0)} + \sum_{i=0}^{t-1} \left(\frac{1}{2} n_i n_{i+1} - \frac{1}{4}(n_i + n_{i+1}) \right) + \sum_{i=1}^{t} \frac{1}{4} o(\tau_i^0) - \frac{1}{2} \Delta(\tau_i^0).$$

We have that $\pi(\tau_1^0) = \lambda$. We then use formula (2).

We also have that $o(\tau_i^0)$ is precisely the number of rows of τ_i^0 (as they all have odd length), the number of rows of τ_i^0 is equal to the number of rows of $\lambda^i := \pi(\tau_i^0)$, the partition λ^i has clearly $\widehat{\lambda^i}_1$ rows and $\widehat{\lambda^i}_1 = \widehat{\lambda}_i$, that is the i-th column of λ.

Finally, we have that $\Delta(\tau_i^0) = 0$, because there are no rows in τ_i^0 starting with b.

Therefore:

$$\dim Z_{\tau^0} = \frac{1}{4} \left(n_0^2 - \sum_{i=1}^{t} \widehat{\lambda}_i^2 \right) + \sum_{i=0}^{t-1} \left(\frac{1}{2} n_i n_{i+1} - \frac{1}{4}(n_i + n_{i+1}) \right) + \frac{1}{4} \sum_{i=1}^{t} \widehat{\lambda}_i;$$

we recall that $\widehat{\lambda}_i = n_{i-1} - n_i$ and that $n_0 = n = |\lambda|$, $n_t = 0$; so

$$\begin{aligned} \dim Z_{\tau^0} =& \frac{1}{4} \left(n_0^2 - \sum_{i=1}^{t} (n_{i-1} - n_i)^2 \right) + \\ &+ \frac{1}{2} \sum_{i=0}^{t-1} n_i n_{i+1} - \frac{1}{4} n_0 - \frac{1}{2} \sum_{i=1}^{t-1} n_i + \frac{1}{4} \sum_{i=1}^{t} n_{i-1} - n_i \\ =& -\frac{1}{2} \sum_{i=1}^{t-1} n_i^2 + \frac{1}{2} \sum_{i=1}^{t-1} n_{i-1} n_i + \frac{1}{2} \sum_{i=0}^{t-1} n_i n_{i+1} - \frac{1}{2} \sum_{i=1}^{t-1} n_i \\ =& \dim M - \dim N. \end{aligned}$$

□

We can proceed to the main result of this section.

Proposition 19. *If λ satisfies the 2-step condition, then Z is a complete intersection variety.*

The argument is basically the same used in [5, Theorem in 3.3] and [7, Theorem in 5.3]. Here we merely put the pieces together. One of these pieces is the following technical lemma.

Lemma 20. *If the partition λ satisfies the 2-step condition and $\tau \in \Lambda$ is any string of ab-diagrams different from τ^0, then*

$$\dim Z_{\tau^0} > \dim Z_\tau.$$

Proof of proposition 19. We start recalling proposition 4 and proposition 18.

As $Z \setminus Z_{\tau^0}$ consists of finitely many strata Z_τ and, because of lemma 20, $\operatorname{codim}_Z(Z_\tau) \geq 1$, we deduce that $Z = \overline{Z_{\tau^0}}$. That implies that Z is a complete intersection variety smooth in codimension 0 and Z is reduced as a scheme. □

Proof of lemma 20. Let $\mu = \pi(\tau)$, so in particular $\mu < \lambda$, as $\tau \neq \tau^0$. Therefore lemma 10 implies

$$2r(\lambda, \mu) \geq c(\lambda, \mu) + q(\lambda, \mu) \tag{18}$$

and, by combining it with proposition 11 we immediately get

$$\dim Z_{\tau^0} \geq \dim Z_\tau.$$

So our concern is reduced to obtain a strict inequality. We will gain the strict inequality either by proving that (18) holds strictly for the pair (λ, μ) or by using proposition 16 with $l > 0$.

Let c be the first column such that $\widehat{\mu}_c > \widehat{\lambda}_c$ and let $r = \widehat{\mu}_c$. This means that the first $c - 1$ columns of λ and μ are the same and $\mu_r > \lambda_r$. On the basis of c and r, we distinguish three cases.

$\widehat{\mu}_c - \widehat{\lambda}_c \geq 2$: in this case we can use lemma 10 with the column c to obtain that

$$2r(\lambda, \mu) > c(\lambda, \mu) + q(\lambda, \mu).$$

$\widehat{\mu}_c - \widehat{\lambda}_c = 1$**,** $\mu_r - \lambda_r \geq 2$: we can still use lemma 10, this time with the row r, to obtain that

$$2r(\lambda, \mu) > c(\lambda, \mu) + q(\lambda, \mu).$$

$\widehat{\mu}_c - \widehat{\lambda}_c = 1$**,** $\mu_r - \lambda_r = 1$: in this case we can use proposition 16 with $i = c$ and $l \geq 1$. Indeed, the following two facts are immediately seen: $d_c = (b)$; the ab-diagram σ_c^0 has at least one row equal to a single a, namely the $\widehat{\mu}_c$-th row.

□

8 Normality conditions

In this section we prove theorem 1. We start by proving the necessary condition as a consequence of the works by Ohta and Sekiguchi.

Theorem 21. *Let λ be a partition of n. Suppose that λ does not satisfy the 1-step condition. Then $\overline{C_\lambda} \subseteq \mathfrak{p}$ is not normal.*

Proof. If λ does not satisfy the 1-step condition, then there is an i such that $\lambda_i \geq \lambda_{i+1} + 2$. Let

$$\mu = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1} + 1, \ldots, \lambda_h)$$

be another partition of n, so that $\mu < \lambda$ and μ is a minimal degeneration of λ (in the same sense of [6, 9, etc.]). Let $m = \lambda_i - \lambda_{i+1} \geq 2$ and let us define the following two partitions of m:

$$\lambda' = (m),$$
$$\mu' = (m - 1, 1).$$

The degeneration $\mu' < \lambda'$ is obtained from $\mu < \lambda$ by removing the common rows and columns (of μ and λ). By theorem [9, Theorem 2], the singularity C_μ of $\overline{C_\lambda}$ is normal if and only if the singularity $C_{\mu'}$ of $\overline{C_{\lambda'}}$ is normal.

Moreover, by construction, $C_{\lambda'}$ is the variety of the principal (or regular) nilpotent elements of $\mathfrak{p}$ (in dimension m); therefore we also have that $\overline{C_{\lambda'}} = \mathcal{N}(\mathfrak{p})$, that is the cone of the nilpotent elements in $\mathfrak{p}$. As μ' is the minimal degeneration of λ', $C_{\mu'}$ is the variety of subregular nilpotent elements of $\mathfrak{p}$. The theorem [10, §3.1, Theorem 7.] proves that the singularity $C_{\mu'}$ is not normal in $\mathcal{N}(\mathfrak{p})$. So this concludes the *only if* part. □

We turn to the sufficient condition. We will make use of the construction of the variety Z.

Theorem 22. *If the partition λ satisfies the 1-step condition, then Z is a normal variety. In particular $\overline{C_\lambda}$ is a normal variety.*

As λ satisfies the 1-step condition, also λ satisfies the 2-step condition, so, by proposition 19, Z is a complete intersection variety. At this point the proof of theorem 22 is a direct consequence of the following lemma, as in [5, Section 3.7].

Lemma 23. *If the partition λ satisfies the 1-step condition, then* $\dim(Z \setminus Z_{\tau^0}) \leq \dim Z - 2$.

Proof. Let $\tau \in \Lambda$ such that $\tau \neq \tau^0$. We need to prove that

$$\dim Z_\tau + 2 \leq \dim Z_{\tau^0}.$$

Let $\mu = \pi(\tau)$, so $\mu < \lambda$, therefore lemma 10 gives us $r(\lambda, \mu) \geq c(\lambda, \mu) + q(\lambda, \mu)$. Therefore

$$2r(\lambda, \mu) \geq c(\lambda, \mu) + q(\lambda, \mu) + x$$

where $x = c(\lambda, \mu) + q(\lambda, \mu)$. By applying proposition 11 we immediately get

$$\dim Z_{\tau^0} \geq \dim Z_\tau + \frac{x}{4},$$

so we are done if $x > 4$.

Since now, we assume that $x \leq 4$. We recall (6), that is $\lambda > \mu$ if and only if both $c(\lambda, \mu), q(\lambda, \mu)$ are strictly positive, so $c(\lambda, \mu) \geq 1$ and $q(\lambda, \mu) \geq 1$. We also recall (10), so we even get $c(\lambda, \mu) \geq q(\lambda, \mu)$.

Let us consider the case $q(\lambda, \mu) \geq 3$. In this case we have that $x = c(\lambda, \mu) + q(\lambda, \mu) \geq 6$ and we are already done in this case.

Similarly, if $q(\lambda, \mu) = 2$, we get $c(\lambda, \mu) + q(\lambda, \mu) \geq 4$. So it remains only the case $q(\lambda, \mu) = 2$ and $c(\lambda, \mu) = 2$.

If $q(\lambda, \mu) = 1$, we have $c(\lambda, \mu) \geq 1$. Therefore, the left cases are $1 \leq c(\lambda, \mu) \leq 3$.

Similarly to the proof of lemma 20, we want to use proposition 16 in each of these four left cases to gain stronger inequalities and show the thesis. In particular we try to get stronger inequalities by looking for adequate $l > 0$ or $u > 0$.

$q = 2$, $c = 2$: here λ is different from μ in 4 columns, let $c, c+1, c', c'+1$ be these columns. This means that the lists d_i are not empty only for $i = c, c+1, c', c'+1$, in particular they are: $d_c = d_{c'} = (b)$ and $d_{c+1} = d_{c'+1} = (a)$. We can apply proposition 16 with $l \geq 3$ and $i = c$.

$q = 1$, $c = 3$: here the 4 columns are consecutive: let $c, c+1, c+2, c+3$ be these columns. Then, the lists d_i are not empty only for $i = c, \ldots, c+3$, in particular they are: $d_i = (b)$, $d_{c+1} = d_{c+2} = (a, b)$, $d_{c+3} = (a)$. We can apply proposition 16 with $u = 2$ and $i = c+1, c+2$.

$q = 1$, $c = 2$: here the 3 columns are consecutive: let $c, c+1, c+2$ be these columns. Similar to the previous case we can apply proposition 16 with $u = 1$ on $i = c+1$ and $l \geq 1$ on $i = c$.

$q = 1$, $c = 1$: here there are only 2 columns: let $c, c+1$ be these columns. Similar to the case $q = 2$, $c = 2$ we can apply proposition 16 with $l \geq 3$ on $i = c$.

□

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