International Journal of Statistics and Probability; Vol. 5, No. 4; July 2016
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

On the Dichotomy between the Psychological or Empirical Aspect of Subjective Probability and the Logical or Geometrical One

Pierpaolo Angelini[1] & Antonio Maturo[2]

[1] ITA “Emilio Sereni”, Roma.

[2] Università “Gabriele D’Annunzio”, Chieti-Pescara.

Correspondence: Prof. Pierpaolo Angelini, ITA “E. Sereni”, via Prenestina n. 1395, Roma. Tel: 39-340-9099-992.

E-mail: pp.angelini@virgilio.it

Received: April 12, 2016 Accepted: May 11, 2016 Online Published: xxx, 2016

doi:10.5539/ijsp.v5n4pxx URL: http://dx.doi.org/10.5539/ijsp.v5n4pxx

Abstract

In the domain of the logic of certainty we examine the objective notions of the subjective probability with the clear aim of identifying their fundamental characteristics before the assignment, by the individual, of the probabilistic evaluation. Probability is an additional and subjective notion that one applies within the range of possibility, thus giving rise to those gradations, more or less probable, that are meaningless in the logic of certainty. Each criterion for evaluations under conditions of uncertainty is a device or instrument for obtaining a measurement; it furnishes an operational definition of probability or prevision **P** and together with the corresponding conditions of coherence can be taken as a foundation for the entire theory of probability. When we examine these criteria and their corresponding conditions of coherence we show the inevitable dichotomy between the subjective or psychological or empirical aspect of probability and the objective or logical or geometrical one.

Keywords: random entity, coherence, convex hull, barycenter, vector space, metric.

1. Introduction

According to the subjectivistic conception of probability, the concept of probability and the foundations of probability theory have a psychological value: such a theory is rigidly deduced and reconstructed on the sole basis of a psychological interpretation and formulation because the mathematical principles are always the same from whatever point of view one starts (de Finetti, 1931b). In analyzing the objective meaning of the notion of coherence, it is necessary to point out in which way some probability evaluations may be incoherent or intrinsically contradictory and the rules of probabilistic logic, as those of formal logic in the field of propositions, are essential in order to teach us how to reason in the field of probability evaluations. Since all probability evaluations have and can have only an essentially and exclusively psychological value, it is necessary to separate what in a problem is logical from what is essentially of a merely empirical value and nature. So, one will be able to say whether every other problem is logically determined or undetermined (de Finetti, 1930a, 1930b). Evidently, this separation is fundamental in order to be able to deepen the criticism of principles of any mathematical theory and, in particular, of probability theory (de Finetti, 1931a). Probability, as an individual’s psychological perception, is subject to certain laws. If an event has an objective probability, all the individuals who will conform their psychological position to it, can be said to be judging correctly, while the others to be wrong. Apart from this, the laws are the same for everybody and, in particular, hold for objective probabilities, so it is not true that if everything is subjective, everything must be arbitrary and no law can be valid. Essentially, it needs to characterize the whole of the formally admissible opinions, without bothering if reasons exist of any other type which might cause someone to consider any one of them more or less right. In fact, such reasons are beyond the merely logical or objective aspect of the problem which only mathematics can and must deal with: thus a clear separation of the two phases, the formal phase and the practical or empirical phase, appears appropriate and inevitable. The formal phase, that is to say, the characterization of the not incoherent opinions, is to be dealt with mathematically; the practical or empirical phase, that is to say, the choice of one among such possible opinions, has to be left to good sense and judgement of every single individual. The only difference between those who follow the subjectivistic conception and those who follow the objectivistic one is that while such a choice is free and arbitrary for the former, it can be right in only one way for the latter. Therefore, the subjectivistic approach takes into consideration, along with the objectively right evaluation of probability, all those evaluations which are not contradictory by themselves, although wrong according to the objectivistic viewpoint. A person who does not share the subjectivistic viewpoint believes in the existence of an objective value of probability which cannot be maintained except for a certain more or less limited field; it would only be to events of a certain type, more or less schematic and artificial, that he will assign an objective probability, while in practical life he would be incessantly guided to think or say that a certain event appears more or less easily, is more or less probable or verisimilar and on such judgements he will found his decisions, also in areas which, according to his way of thinking, would be precluded from probability theory. Clearly, in order to justify conceptually such judgements, it needs to conform to the subjectivistic viewpoint whose validity field is not subject to any restriction (de Finetti & Minisola, 1961; de Finetti & Emanuelli, 1967a; de Finetti, 1955, 1963, 1969, 1970).

2. Logic of Certainty

When a given individual, according to his state of information, defines a set more or less large of possible alternatives, of which one and only one is necessarily true, he finds himself into the domain of the logic of certainty. We denote by $\mathcal{S}$ the abstract space of alternatives and by $\mathcal{Q}$, subset of $\mathcal{S}$, the space of the only alternatives possible for a certain individual; in fact, it may be convenient to think of $\mathcal{Q}$ as embedded in a larger and more manageable space $\mathcal{S}$. However, his information as well as his knowledge could also allow him to eliminate a part of the alternatives that can be imagined because he believes that they are impossible; vice versa, all the others will be possible. After all, a rather crude analysis can be made if all the possible alternatives are collected in order to obtain an unique and certain alternative. The possibility, unlike probability, has no gradations, thus the domain of the logic of certainty is objective; it is equally possible, for a given individual at a certain time, that the next FIFA world cup is won by a very weak national football team, that the next President of the Italian Republic is a woman, that the unemployment rate falls by three percentage points at the end of next year in Italy. Into the domain of the logic of certainty, only true and false exist as final and certain answers and certain and impossible and possible as options with regard to the temporary knowledge of any individual; into this domain we study the objective notions of subjective probability with the clear aim of identifying their fundamental characteristics before the assignment, by the individual, of the probabilistic evaluation. Probability is an additional and subjective notion that one applies within the range of possibility, thus giving rise to those gradations, more or less probable, that are meaningless in the logic of certainty. The field of the logic of certainty is objective because the elements of $\mathcal{Q}$ do not depend on the individual's opinions but only on his degree of ignorance (de Finetti, 1967b, 1970).

3. Events and Random Numbers

An event E is a statement which we do not know yet to be true or false; the event which is certain and the one which is impossible can be taken as a limit case. The statements of which we can say if they are true or false on the basis of an ascertainment well determined and always possible, at least conceptually, have objective meaning. Such objective statements are said propositions if one is thinking more in terms of the expressions in which they are formulated or, equally, events if one is thinking more in terms of the situations and circumstances to which their being true or false corresponds (de Finetti 1954). For any individual who does not know with certainty the value of a number X, which is random in a non-redundant usage for him, there are two or more than two, a finite or infinite number, possible values for X, where the set of these values is I(X): in any case, only one is the true value of each random number (de Finetti, 1970).

Remark 1 Events are also questions whose wordings, unambiguous and exhaustive, have the aim of removing any opportunity to complaining in case that a bet is based upon them: they admit two answers, yes = 1 or no = 0, true = 1 or false = 0, where such answers are always alternative. Also the random numbers can be identified by questions whose wordings are indisputably clear and complete; unlike events, they contain two or more than two answers which consist only of numbers, only one of which is the one that actually occurs.

Remark 2 For the representation of random numbers it is useful to think of a set $\mathcal{S}$, whose subset $\mathcal{Q}$ is constituted by the only possible alternatives for a certain individual at a given time. Sometimes, $\mathcal{S}$ can coincide with a manifold less extensive of the linear ambit or linear space $\mathcal{A}$ in which $\mathcal{S}$ is contained: in the case of two random numbers, $\mathcal{S}$ can coincide with a curve of the Cartesian plane $\mathcal{A}$, otherwise, if the numbers are three, $\mathcal{S}$ can coincide with a surface of the three-dimensional space $\mathcal{A}$. Then, the possible points of $\mathcal{Q}$ would be positioned on the curve of the Cartesian plane or on the surface of the three-dimensional space and such points may be all the points or a part or a few points of $\mathcal{S}$ according to the individual's knowledge at a given time and the existence of other restrictions and conditions. We could have $\mathcal{A} = \mathbb{R}^2$ or $\mathcal{A} = \mathbb{R}^3$ under one-to-one correspondence between the points of the two-dimensional or three-dimensional space and the ordered lists of two or three real numbers. If $\mathbb{R}^2$ and $\mathbb{R}^3$ are equipped with a scalar product positive-definite, they would be Euclidean spaces or metric spaces. However, since every vector space may be considered as an affine space over itself, $\mathcal{A}$ could also be an affine space and this, theoretically, would be the best thing by virtue of the fact that the affine properties are more general than the metric ones. The affine properties are the basis of essential concepts of probability theory and only they make sense, being independent of the choice of a coordinate system; however, the importance of the metric properties appears in order to represent analytical conditions of coherence (de Finetti, 1931b, 1954, 1970).

Remark 3 The space of alternatives $\mathcal{S}$ of a random number X coincides with the real line x on which it is possible to consider $\mathcal{Q}$, subset of $\mathcal{S}$, which consists of the only possible values or points for a certain individual. Every point of the real line is assumed to correspond to a real number and every real number to a point of it, so the real line is a vector space of dimension 1 over the field $\mathbb{R}$ of real numbers, that is to say, over itself: there is an one-to-one correspondence between points on the real line and real numbers. The set $\mathbb{R}$ of real numbers is a Euclidean space because it has a standard scalar product which is simply ordinary multiplication of real numbers and the standard norm on it is simply the absolute value function. Every real number of the x-axis is a point of $\mathcal{S}$. Since every possible value of X is a random event, all the possible values of X are events, all together and implicitly considered into $\mathcal{Q}$. In conformity with the possible values of X which constitute the set $\mathcal{Q}$, X can belong to a half-line, $X \geq x$, or to an interval, $x_1 \leq X \leq x_2$, or to an arbitrary set, $X \in \mathcal{I}$.

Remark 4 If we consider two random numbers, X and Y, $\mathcal{S}$ coincides with the Cartesian plane whose element, in general, is (x, y). For (X, Y), $\mathcal{Q}$ consists of pairs of possible values for X and Y. If we consider three random numbers, X, Y and Z, $\mathcal{S}$ coincides with the ordinary space whose element, in general, is (x, y, z) and if we consider more than three random numbers, the only restriction for $\mathcal{S}$ is that it is not visually intuitive to go beyond the third dimension.

4. Random Entities

Random points, random vectors, random matrices, random sets and random functions are random entities. An objective scheme of representation for random entities is given by the set $\mathcal{S}$ of "points" whose elements can be a finite or infinite number. Such points are not geometric points, but they are simply elements of $\mathcal{S}$, that is to say, they may be points in two-dimensional Euclidean space or in three-dimensional Euclidean space, vectors, matrices, sets of points and functions if $\mathcal{S}$ is, respectively, a set of points or vectors or matrices or sets of points or functions. Clearly, we need to consider each "point" of $\mathcal{Q}$ or $\mathcal{S}$ like a random event which is, a posteriori, true or false: among such "points", there is a very important "point" representing the alternative which, a posteriori, will really occur. It is, a priori, uncertain and for this reason it constitutes the essence of every problem concerning the alternatives $\mathcal{Q}$ which are contained in $\mathcal{S}$ in which $\mathcal{Q}$ is embedded.

Remark 5 On a plane the point which would be hit in firing at a target is a random point, with the geometric representation of this problem which is independent of any coordinate system. Similarly, in ordinary space the point where, at a precise moment, a stolen car is, such a car being equipped with a satellite alarm, is random. When the theft occurs, this alarm sends to a control center a radio signal through which it is possible to determine the exact position of the vehicle. The space of alternatives $\mathcal{S}$, corresponding to the usual physical space extended in length, width and height and in which bodies move or place themselves, provides an immediate geometric image which does not depend on coordinates.

Remark 6 A vector is an ordered list of n real numbers, $(x_1, \ldots, x_n) \in \mathbb{R}^n$, where n is a non-negative integer: real numbers $x_1, \ldots, x_n$ are called scalar components in the n-dimensional Euclidean space, with the number x_i which is the i-th scalar component of $(x_1, \ldots, x_n)$. Thus, the list of known unit prices of ten articles which are for sale in a given department store is the decuple $(p_1, \ldots, p_{10})$. Given n, for a certain individual, a vector is random when he does not know all scalar components of the finite ordered list of n real numbers, such a list being the true vector. For the same individual, different n-tuples of $\mathbb{R}^n$, which constitute $\mathcal{Q}$, are possible. The space of alternatives $\mathcal{S}$ is a vector space over the field $\mathbb{R}$ of real numbers because it coincides with all the n-tuples of $\mathbb{R}^n$. Evidently, each n-tuple of $\mathbb{R}^n$, belonging to $\mathcal{S}$, is a point of $\mathcal{S}$.

Remark 7 A matrix (a_{ij}) $m \times n$, with $m, n \geq 1$, is a rectangular array of mn numbers, $(a_{ij}) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$, whose elements are arranged in m rows and n columns. The numbers of every row could represent known unit prices of n given articles which are for sale in m different department stores. The whole of all rectangular arrays of mn real numbers is a vector space over the field $\mathbb{R}$ and an isomorphism exists between it and $\mathbb{R}^{mn}$ because every array of mn numbers can be arranged into a row vector or column vector of $\mathbb{R}^{mn}$. For a certain individual, a matrix which has predetermined rows and columns is random when he does not know the real numbers of every row or column of the true matrix. For the same individual, possible matrices which constitute $\mathcal{Q}$ and all those of the vector space $\mathcal{S}$ over $\mathbb{R}$ have the same predetermined number of rows and columns. Clearly, each matrix of $\mathcal{S}$ is a "point" of $\mathcal{S}$.

Remark 8 Random curves and random sets on surfaces are random sets which give a non-linear structure to $\mathcal{S}$. The unknown path of an airplane, from takeoff to landing, is a random curve: every trajectory can be thought of as a set which consists of infinite points and it is a "point" of $\mathcal{S}$. On the other hand, if a given individual does not know the part of the Italian territory, viewable via satellite map, on which rain fell in the last twelve hours starting from a certain instant, we have a random set on surface: each part of the Italian territory is a set of infinite points and it is a "point" of $\mathcal{S}$. Moreover, among the different parts of the Italian territory which constitute the abstract space of alternatives $\mathcal{S}$, there are both the empty part, that corresponds to the alternative according to which rain did not fall on the Italian territory in the last twelve hours, and the total part, that corresponds to the alternative according to which rain fell on the whole Italian territory in the last twelve hours.

Remark 9 For a certain individual, a function Y(t), with the variable t which is time, is random when its behavior is unknown to him, for which it can be said that chance exists instant by instant. If one knows the values of Y(t) because they have been calculated at any number of instants $t = t_1, \ldots, t_n$, however large the finite n, the value of Y(t) at a different instant t will still be uncertain. Every measurable function, where its values are $Y(t_1), \ldots, Y(t_n)$, is a "point" of $\mathcal{S}$. When we ask whether or not the numerical values of a function Y(t) of the set $\mathcal{S}$ at given instants fall inside fixed sets $a_h \leq Y(t_h) \leq b_h$ (h = 1, …, n) defined by two freely determined coordinates, random events given by $a_1 \leq Y(t_1) \leq b_1, \ldots, a_n \leq Y(t_n) \leq b_n$ can be true or false according to whether they occur or not inside intervals $[a_1, b_1], \ldots, [a_n, b_n]$.

Evidently, each problem concerning the possible alternatives of $\mathcal{Q}$ is usefully visualized by means of $\mathcal{S}$ whose nature is always and unequivocally objective (de Finetti, 1970).

5. Arithmetic and Boolean Operations

Putting the logical values true and false equal to the idempotent numbers 1 and 0 for which we have $1^n = 1,\ 0^n = 0$, an event E is always a random number which can admit these two numbers, called indicators of E. Arithmetic and Boolean operations must be unified by applying arithmetic operations even to events and Boolean operations even to random numbers. For events, the arithmetic product is the same as the logical product $\wedge$, the arithmetic sum is the number of successes $Y = E_1 + \ldots + E_n$ and complementation is negation, that is to say, $\overline{E} = 1 - E$. Obviously, Y can yield a result outside the {0, 1} set. The logical sum $\vee$ can be expressed by $A \vee B = 1 - (1 - A)(1 - B)$, where we must consider $A \vee B = \overline{(\overline{A} \wedge \overline{B})}$, with A and B which are random events. In $\mathbb{R}$ we can make the following definitions: $a \vee b = \max(a, b)$, $a \wedge b = \min(a, b)$ and $\bar{a} = 1 - a$, where a and b are real numbers; then, in case a and b have as values 1 or 0, the logical product, the logical sum and the negation are recovered. Moreover, it needs to unify the notation for the probability of an event E and for the mathematical expectation or prevision of a random number X; in fact, it is adopted **P**(E) for probability of E and **P**(X) for prevision of X, where **P** is linear, that is, additive and homogeneous.

6. Logic of Uncertainty

The subjectivistic conception of probability, through psychological analysis, vivifies notions that are mathematically correct but that is not sufficient to consider from the formal point of view. In fact, the instrument really propulsive of scientific thinking is not classical logic or, in the specific instance, logic of certainty that, as such, involves no affective demonstration, no judgement by anyone, but is probability and probability calculus. Therefore, when we consider any problem concerning the assignment of probability among possible cases and how to define it and to express it quantitatively, we find ourselves into the field, personal and subjective, of logic of uncertainty, clearly distinct and separate from that one of logic of certainty (de Finetti, 1931a). Indeed, when we say that we are not satisfied of logic of certainty, we mean that we are not satisfied of agnostic and undifferentiated attitude towards uncertainty. For all those things which, not being known to us with certainty, are uncertain or possible, any individual feels a more or less strong propensity to expect that some cases possible are true rather than others, to believe that the answer to a given question is no rather than yes, to estimate that the unknown value of a certain quantity is small rather than large. Evidently, these attitudes express, in the domain of uncertainty, different degrees of subjective probability, each of which is assigned to one of the possible alternatives identified by a given individual on the basis of his knowledge. So, to find oneself into the field of logic of prevision means to examine carefully desires or hopes that certain alternatives occur, anxieties and fears regarding the occurrence of unfavourable alternatives and to weigh up the pros and cons of each choice trying to reason about it in order to distribute, among all the possible alternatives and in the way which will appear most appropriate, one's own sensations of probability (de Finetti & Minisola 1961; de Finetti, 1955, 1963, 1969).

Remark 10 When a particular individual chooses to be guided only by the logic of certainty, after having distinguished a set more or less large of possible alternatives in the way which seems to him most effective, he has to stop because the question is closed. Remaining within the logic of certainty, the only thing that he could make is a prophecy, that is to say, among the cases that he believes possible, he might venture to guess the alternative that, according to him, will occur, transforming in this way, but unreasonably, the uncertainty in illusory certainty (de Finetti, 1967b, 1970).

Remark 11 The space of n random numbers coincides with the n-dimensional vector space $\mathcal{A}$ after the introduction of a coordinate system $x_1, \ldots, x_n$ in $\mathcal{A}$; by virtue of the fact that each event is a random number, a set of n possible events $E_1, \ldots, E_n$ is embedded in $\mathcal{A}$. From such a set other events, called constituents, are originated: they are identified by particular ordered lists of n numbers $(x_1, \ldots, x_n)$, with $x_i = 0$ or $x_i = 1$, i = 1, …, n, each of which is a possible point of $\boldsymbol{Q}$ contained in the vector space $\boldsymbol{\mathcal{A}}$. Such considerations make clear, from the point of view of the logic of certainty, why the probability of an event is automatically incorporated in the prevision of a random number. In fact, going beyond the domain of the logic of certainty, we enter into the field of the logic of uncertainty and in the event that X is a random number, **P**(X) is the prevision of X: if $I(X) = \{x_1, \ldots, x_n\}$, when we assign to each value x_i of X the probability p_i (i = 1, …, n), with $0 \leq p_i \leq 1$ and $\sum p_i = 1$, it turns out to be $\mathbf{P}(X) = x_1 p_1 + \ldots + x_n p_n$. The prevision of X coincides with the probability of an event E when and only when X, admitting only two possible values, 1 and 0, is an event, thus prevision and probability are two different words that express the same concept extra-logical, subjective and personal (de Finetti & Emanuelli, 1967a).

Remark 12 The die symmetry and the knowledge of an observed frequency are elements which any individual carefully examines to express his opinion from which the subjective probability is originated. According to the subjectivistic conception, the only probability that exists in any case is the subjective probability. It must be understood as the degree of belief of a certain individual in the occurrence of a specific event; anyway, probability of an event E is not an intrinsic characteristic of E because it depends on the information that the individual making the probabilistic evaluation has, so it is always subordinate to his present state of knowledge which can change for the possible attainment of new essential information and for the passage of time (de Finetti, 1963).

Remark 13 A probabilistic evaluation, known over a set of whatever events, always expresses the opinion of a given individual, real or hypothetical; the only admissible restriction is that this opinion is coherent, consequently, if it is not coherent it should be corrected by the individual in order to make it coherent (de Finetti, 1930a, 1930b).

7. Objective Statements and Subjective Evaluations

We reason in accordance with probability theory, although without awareness and in a rather approximate way, when we incessantly make our forecasts and assumptions which constitute the usual object of our thinking in all the practical circumstances of life, more than the much rarer judgements which are logically certain. In fact, we reason in all the circumstances of life, where we base ourselves on probabilities, by applying without awareness the two fundamental theorems of probability calculus, the theorem of total probability and the theorem of composite probability: our way of thinking is not forced by logical requirements but is only suggested by psychological motives when we judge on the probability that it will rain or not in order to decide to take or not the umbrella, on the probability for an individual to arrive in time at the postal office on foot in order to decide to go by bus or taxi or not, on the probability that different performances which have been announced for tonight are more or less interesting to decide whether to go and where and so on. Regarding these examples, nobody can certainly think that they are cases of objective probability because it is not be able to solve such problems. Instead, according to the subjectivistic viewpoint, every question has an exact and satisfactory answer, because it is always based on the psychological degree of confidence of a certain individual in relation to a certain assumption. In all cases, including the gambling games or statistics or molecular physics cases or any other case whose objective probability coincides with the subjective one, it is evidently only a matter of a pure psychological feeling. Anyway, the theorems of probability theory are always valid, thus justifying one of our most important empirical ways of reasoning.

Hence, any statement of probability calculus has an objective or logical meaning unlike probability evaluations whose meaning can only be empirical. For example, we consider a deck of Italian playing cards which consists of 40 cards divided into 4 suits; in particular, Neapolitan playing cards are divided into swords, cups, coins and clubs, whose 3 face cards per suit are knave or fante in Italian, knight or cavallo in Italian, king or re in Italian. Thus, if we suppose that the probability of drawing a fante or cavallo or re is $\mathbf{P}(E_f) = \mathbf{P}(E_c) = \mathbf{P}(E_r) = 1/10$, then we conclude that the probability of E, where E consists in drawing a face card, is given by $\mathbf{P}(E) = \mathbf{P}(E_f) + \mathbf{P}(E_c) + \mathbf{P}(E_r) = 3/10$: we make a purely logical reasoning because it is logically true that the three considered events are mutually exclusive and under such a condition it is logically certain that the theorem of total probability is valid. However, probability evaluations have an empirical or subjective meaning: if the probability of drawing a fante is $1/10$ for us, we always express a subjective opinion. In accordance with the subjectivistic viewpoint, we do not believe that the probability of any event E, **P**(E), with $0 \leq \mathbf{P}(E) \leq 1$, is objectively determined because we consider, on the contrary, all the functions **P** as formally admissible laws when they are not in conflict with theorems of probability calculus. Evidently, the choice of one of these functions is left to each individual who chooses according to his subjective opinion. Regarding the previous example, we consider admissible all the ∞^3 functions **P** for which it turns out to be $\mathbf{P}(E_f) = x$, $\mathbf{P}(E_c) = y$, $\mathbf{P}(E_r) = z$, $\mathbf{P}(E) = x + y + z$, with x, y, $z \geq 0$ and $x + y + z \leq 1$. The choice of functions for which we have $\mathbf{P}(E_f) = \mathbf{P}(E_c) = \mathbf{P}(E_r) = 1/10$, although suggested by spontaneous and universally approved remarks, is a very particular case and it is not forced by logical requirements of which mathematics can or must be interested. Obviously, recognizing if certain premises are sufficient or not in order to involve a certain conclusion becomes very difficult when the problem under consideration is not as simple as in the previous example. However, such a problem is never solved when there is not a clear separation between all that is logical or objective and all that is empirical or subjective (de Finetti, 1930a, 1930b).

8. Criteria for the Probabilistic Evaluations

It is representative of one of the primary necessities of science the fact that it must not run the risk of taking as notions illusory combinations of terms of a metaphysical nature, but it must work with concepts of verified validity in a practical meaning. Therefore, its definitions must be operational, that is to say, must reduce a scientific concept not simply to sentences having only an apparent meaning, but to real experiences which are at least theoretically possible. Thus, the criteria which may be used to reveal concretely **P**(X) or, in particular, **P**(E) according to the opinion of a certain individual are two and entirely equivalent: they are based upon the identification of the practical consequences that a given individual knows to accept and accepts when he expresses his evaluation of **P**(X) or **P**(E) and, if coherently applied, lead to the same **P**(X) = $\bar{x}$ in the event that X is estimated or to the same **P**(E) = p in the case that E is evaluated. If X is evaluated, both criteria contain the random magnitude $X - \bar{x}$, expressed by the difference between the real value X and the one chosen by a certain individual at his own will, **P**(X) = $\bar{x}$. The first criterion provides that, after the subjective choice of $\bar{x}$, the individual is obliged to accept any bet unilaterally determined by an opponent, whose gain is $c(X - \bar{x})$, with c any betting amount, positive or negative, equally determined by the opponent; in particular, if c = 1, the gain of the bet is $(X - \bar{x})$, while if c = − 1, it is $(\bar{x} - X)$. On the contrary, the second criterion provides that, after choosing $\bar{x}$, the individual must suffer the penalty $(X - \bar{x})^2$, positively proportional to the square of the difference between X and $\bar{x}$ (de Finetti, 1970). In particular, if an event E is evaluated, both criteria contain the magnitude E − p given by the difference between the real value E, 1 or 0 according to whether E occurs or does not occur, and the one chosen by a certain individual according to his subjective opinion, **P**(E) = p. The first criterion provides that, after the choice of p by a determined individual, he is obliged to accept any bet unilaterally determined by an opponent, whose gain is c(E − p), where c is any betting amount, positive or negative, established by the opponent; in particular, if c = 1 , the gain is (E − p), while if c = − 1, it is given by (p − E). On the contrary, the second criterion provides that, after the subjective choice of p, the individual must suffer the penalty $(E - p)^2$. Evidently, in order to measure subjective probabilities, that is to say, to translate our degree of uncertainty, regarding judgements, into numerical determinations, the degree of confidence that we have in the occurrence of events is expressed by the conditions at which one would bet. There is a difference between judging if a bet is fair and judging how convenient it is for a certain individual, at a certain time, under certain circumstances, to accept it; moreover, the convenience will be judged differently, depending on the character of the individual and his love of risk. In other words, there is an essential difference between the case of one occasional and well defined betting and the abnormal case of an individual who would consistently and interminably be driven to betting.

9. Necessary and Sufficient Conditions of Coherence

The choice of **P**(X) or **P**(E), even if it is subjective, should not be contradictory and takes place within the set of coherent previsions of X or in that one of coherent probabilities of E; both the sets contain values objectively admissible which are independent of the personal views of any individual and also of the judgements about others' opinions. The necessary and sufficient conditions for coherence are two and completely equivalent, one for each evaluation criterion (de Finetti, 1970).

Regarding the first definition of coherence, it is assumed that the individual who subjectively evaluates **P**(X_i) or **P**(E_i), with i = 1, …, n, does not want to make bets on X_i or E_i that give him an inevitable loss, therefore a set of his previsions or probabilities is not intrinsically contradictory when and only when, among the linear combinations of bets that he is obliged to accept, there are not combinations with gains all uniformly negative. Analytically, this means that for the numerical values of the random magnitude $Y = c_1(X_1 - \bar{x}_1) + \ldots + c_n(X_n - \bar{x}_n)$ or the random magnitude $Y = c_1(E_1 - p_1) + \ldots + c_n(E_n - p_n)$ must not be, objectively, that sup I(Y) is negative; conversely, we have that inf I(Y) cannot be positive. Even if the bets are an infinite number, Y is always linear combination of a finite number of them.

Regarding the second definition of coherence, it is assumed that the individual who subjectively evaluates **P**(X_i) or **P**(E_i), with i = 1, …, n, does not prefer a given penalty if he can choose another penalty certainly smaller, therefore a set of his previsions or probabilities is coherent when and only when he could not choose them in order to make his penalty certainly and uniformly smaller. Analytically, this means that there are not any evaluations **P***(X_i) or **P***(E_i) that replaced with the evaluations **P**(X_i) or **P**(E_i) are such that for all the possible points, $(X_1, \ldots, X_n)$ or $(E_1, \ldots, E_n)$, the penalty $L^* = \sum_i (X_i - \mathbf{P}^*(X_i))^2 \cdot (1/k_i)^2$ is uniformly smaller than the penalty $L = \sum_i (X_i - \mathbf{P}(X_i))^2 \cdot (1/k_i)^2$ or the penalty $L^* = \sum_i (E_i - \mathbf{P}^*(E_i))^2 \cdot (1/k_i)^2$ is uniformly smaller than $L = \sum_i (E_i - \mathbf{P}(E_i))^2 \cdot (1/k_i)^2$, with $k_1, \ldots, k_n$ which are arbitrarily predetermined and homogeneous towards X_i or E_i.

A prevision **P** is coherent if its use cannot lead to an inadmissible decision such that a different possible decision would have certainly led to better results, whatever happened. If the sets of possible values for X and Y are, respectively, I(X)

$= \{x_1, \ldots, x_n\}$ and $I(Y) = \{y_1, \ldots, y_n\}$, when we assign the same weights p_i $(i = 1, \ldots, n)$, where we have $0 \leq p_i \leq 1$ and $\sum p_i = 1$, to each x_i and y_i we will have $\mathbf{P}(X + Y) = \mathbf{P}(X) + \mathbf{P}(Y)$, that is to say, **P** is additive; a prevision **P** of the random number X must satisfy the inequality $\inf I(X) \leq \mathbf{P}(X) \leq \sup I(X)$, that is, $\mathbf{P}(X)$ must not be less than the lower bound of the set of possible values for X, which is $\inf I(X)$, nor greater than the upper bound, which is $\sup I(X)$. A prevision **P** of X must also be linear, that is, we have $\mathbf{P}(aX) = a\mathbf{P}(X)$, for every real number a. In general, we have $\mathbf{P}(aX + bY + cZ + \ldots) = a\mathbf{P}(X) + b\mathbf{P}(Y) + c\mathbf{P}(Z) + \ldots$, with a, b, c, … whatever real numbers, for any finite number of summands. So, coherence reduces to linearity, which contains additivity property, and convexity. Similarly, if E is an event, when we have $0 \leq \mathbf{P}(E) \leq 1$, its evaluation is coherent; if $E_1, \ldots, E_n$ are mutually exclusive events, their evaluations are coherent when we have $\mathbf{P}(E_1 + \ldots + E_n) = \mathbf{P}(E_1) + \ldots + \mathbf{P}(E_n)$ (de Finetti, 1970).

10. Geometric Interpretation of Conditions of Coherence

Given in $\mathcal{A}$ n random numbers $X_1, \ldots, X_n$, with $\mathcal{A}$ n-dimensional vector space having coordinate system $x_1, \ldots, x_n$, every prevision, coherent or not, of each random number X_i is always a point $(\mathbf{P}(X_1), \ldots, \mathbf{P}(X_n))$ of $\mathcal{A}$. In this space, moreover, the coordinates of the points Q of the set $\mathcal{Q}$ of possible points are identified by ordered lists $(x_1, \ldots, x_n)$ of n real numbers, with x_1 that is a possible value of $X_1, \ldots, x_n$ that is a possible value of X_n. Thus, on the basis of the geometric interpretation of the necessary and sufficient conditions for coherence, the set $\mathcal{P}$ of coherent previsions **P** is the closed convex hull of the set $\mathcal{Q}$ of the possible points Q of $\mathcal{A}$ (de Finetti, 1970).

Remark 14 The first condition of coherence involves that a point P of $\mathcal{A}$, with coordinates $(\mathbf{P}(X_1), \ldots, \mathbf{P}(X_n))$, is an admissible prevision if and only if no hyperplane separates it from the set $\mathcal{Q}$ of the possible points Q of $\mathcal{A}$: this characterizes the points of the convex hull, for which it is said that every linear equation between the numbers $X_i, c_1X_1 + \ldots + c_nX_n = c$, must also apply to the previsions $\mathbf{P}(X_i)$, $c_1\mathbf{P}(X_1) + \ldots + c_n\mathbf{P}(X_n) = c$, as well as any inequation between them, $c_1X_1 + \ldots + c_nX_n \geq c$, must also be satisfied by the previsions, $c_1\mathbf{P}(X_1) + \ldots + c_n\mathbf{P}(X_n) \geq c$.

Remark 15 The vector space $\mathcal{A}$ is Euclidean when it is provided with a scalar product positive-definite: by virtue of the metric $\rho^2 = \sum_i (x_i/k_i)^2$, it results $L = (P - Q)^2$, that is to say, the penalty L coincides with the square of the distance between the prevision-point P and the outcome-point Q. Thus, regarding the second condition of coherence, the points of the convex hull also enjoy the property according to which P cannot be moved in such a way as to reduce its distance from all points Q.

The points which are admissible in terms of coherence can be obtained as barycentres of, at most, n + 1 points Q_i of $\mathcal{Q}$ in the n-dimensional space or they are adherent points of $\mathcal{Q}$, but not belonging to $\mathcal{Q}$. More explicitly, every prevision-point P of $\mathcal{P}$ is admissible in terms of coherence when it is a barycentre of possible points Q_i of $\mathcal{Q}$, with non-negative weights, summing to 1: however, if all the weights are concentrated at a unique point Q_i, also the possible points turn out to be coherent previsions (de Finetti, 1970).

11. Conclusions

Probability exists only in our own judgement because it is always the degree of belief of a given individual for the occurrence of a given event. Nevertheless, when it needs, any individual can assess the probability of an event on the basis of an observed frequency or dividing the number of favourable outcomes to it by the total number of possible outcomes which are equally possible. In fact, the subjectivistic theory is not in contrast with any other provided that such different interpretations accept the role of particular criteria for the evaluation of the probability and give up the pretence of leading to a definition of probability. Each criterion for subjective evaluations furnishes an operational definition of probability or prevision **P** and together with the corresponding conditions of coherence can be taken as a foundation for the entire theory of probability. When we study this we show the dichotomy between the subjective or psychological aspect of probability and the objective or logical or geometrical one. Analytically and objectively, the first definition of coherence is similar to the property of stable equilibrium of the barycentre, while the second definition is similar to the property of minimum of the moment of inertia which characterizes the barycentre once again. When the properties of the barycentre are not satisfied, the set of previsions of a given individual cannot be coherent. Given the probabilities of the possible values, finite in number, of X, its barycentre, which is $\mathbf{P}(X)$, can be expressed as a function of them; the prevision of X does not presuppose the introduction of the concept of continuous probability distribution that, extending to the general case the concept of mathematical expectation or mean value of X, requires the use of mathematical tools more advanced than necessary.

References

de Finetti, B. (1930). Fondamenti logici del ragionamento probabilistico. *Bollettino dell'Unione Matematica Italiana*, 9, 258-261.

de Finetti, B. (1930). Problemi determinati e indeterminati nel calcolo delle probabilità. *Rendiconti della R. Accademia Nazionale dei Lincei*, 12, 367-373.

de Finetti, B. (1931). *Probabilismo. Saggio critico sulla teoria delle probabilità e sul valore della scienza (Biblioteca di Filosofia, diretta da A. Aliotta).* Napoli-Città di Castello, F. Perrella.

de Finetti, B. (1931). Sul significato soggettivo della probabilità. *Fundamenta Mathematicae*, 17, 298-329.

de Finetti, B. (1954). La nozione di evento. *Pubblicazioni delle Facoltà di Scienze e di Ingegneria dell'Università di Trieste,* Serie B 130, 170-174.

de Finetti, B. (1955). La probabilità e le scienze sociali. *L'industria,* 4, 467-495.

de Finetti, B., & Minisola, F. (1961). *La matematica per le applicazioni economiche.* Roma, Cremonese.

de Finetti, B. (1963). La decisione nell'incertezza. *Scientia*, 88, 61-68.

de Finetti, B., & Emanuelli, F. (1967). *Economia delle assicurazioni.* Trattato Italiano di Economia, vol. XVI, Torino, UTET.

de Finetti, B. (1967). L'adozione della concezione soggettivistica come condizione necessaria e sufficiente per dissipare secolari pseudoproblemi. In D. Fürst, & G. Parenti, *I fondamenti del calcolo delle probabilità. Atti della tavola rotonda tenuta a Poppi nei giorni 11-12 giugno 1966* (pp. 57-94). Scuola di Statistica dell'Università, Firenze.

de Finetti, B. (1969). *Un matematico e l'economia.* Milano, F. Angeli.

de Finetti, B. (1970). *Teoria delle probabilità, voll. I e II.* Torino, Einaudi.