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Topology of univoque sets in real base expansions

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ABSTRACT

Given a positive integer M and a real number $q \in (1, M+1]$, an expansion of a real number $x \in [0, M/(q-1)]$ over the alphabet $A = \{0, 1, \ldots, M\}$ is a sequence $(c_i) \in A^{\mathbb{N}}$ such that $x = \sum_{i=1}^{\infty} c_i q^{-i}$. Generalizing many earlier results, we investigate in this paper the topological properties of the set $\mathcal{U}_q$ consisting of numbers x having a unique expansion of this form, and the combinatorial properties of the set $\mathcal{U}'_q$ consisting of their corresponding expansions. We also provide shorter proofs of the main results of Baker in [3] by adapting the method given in [12] for the case $M = 1$.

1. Introduction and statement of the main results

Starting with a seminal paper of Rényi [22] many papers have been devoted to representations of real numbers x of the form

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$$x = \sum_{i=1}^{\infty} \frac{c_i}{q^i},$$

where the base $q > 1$ is a given real number, and (c_i) is a sequence of integers with $0 \le c_i \le M$ ($i \in \mathbb{N} := \mathbb{Z}_{\ge 1}$), where M is a given positive integer. The sequence (c_i) is often called an expansion of x. Such representations of real numbers have many intimate connections to combinatorics, number theory, probability and ergodic theory, topology and symbolic dynamics. See, e.g., the review papers [23,15,8].

In the 1990's, Erdős, Horváth and Joó [10] found bases $q \in (1, 2)$ such that $x = 1$ has exactly one representation of the above form with digits c_i belonging to $\{0, 1\}$. Following this discovery, the combinatorial and topological structure of (the set of) all numbers x having exactly one representation of the form $\sum_{i=1}^{\infty} c_i \cdot q^{-i}$ with digits $c_i \in \{0, 1, \dots, M\}$ – and the corresponding set of sequences (c_i) – was eventually clarified in [17,19,6,7] under the additional assumption that $M < q \le M + 1$. Later developments led to the necessity to relax this assumption. This was done in [9] for the expansions of $x = 1$. Building on the results of [9] here we clarify the structure of the set of real numbers x with a unique expansion for any choice of $M \ge 1$ and $q > 1$. Although the general research strategy is the same as in [6], some new arguments are needed and several new properties are uncovered.

In order to state our results we introduce some notation and terminology. In this paper we fix a positive integer M, and we consider the corresponding *alphabet* $A := \{0, 1, \dots, M\}$. The elements of A are often called *digits*. Since M is fixed, we usually do not indicate the dependence on M of the notions we are going to introduce.

A *sequence* always means an element of the set $A^{\mathbb{N}}$; it will often be written in the form (c_i) or $c_1 c_2 \cdots$. By a *block* or *word* we mean an element of $\cup_{k \in \mathbb{N}} A^k$. A block or word has *length* n if it belongs to A^n. We will also use the *conjugate* or *reflection* of any digit c, word $c_1 \cdots c_n$ or sequence (c_i), defined by

$$\overline{c} := M - c, \quad \overline{c_1 \cdots c_n} := \overline{c_1} \cdots \overline{c_n} \quad \text{and} \quad \overline{c_1 c_2 \cdots} := \overline{c_1}\,\overline{c_2} \cdots .$$

Finally, if $c \in A$, then we set $c^+ := c + 1$ if $c < M$, and $c^- := c - 1$ if $c > 0$, so that $c^+ \in A$ and $c^- \in A$. More generally, if $w = c_1 \cdots c_n$ is a word of length $n \ge 2$, then we write

$$w^+ = (c_1 \cdots c_n)^+ := c_1 \cdots c_{n-1} c_n^+$$

if $c_n < M$, and

$$w^- = (c_1 \cdots c_n)^- := c_1 \cdots c_{n-1} c_n^-$$

if $c_n > 0$.

We will use systematically the lexicographical order between sequences: we write $(a_i) < (b_i)$ or $(b_i) > (a_i)$ if there exists an index $n \in \mathbb{N}$ such that $a_i = b_i$ for $i < n$, and $a_n < b_n$. We also equip for each $n \in \mathbb{N}$ the set A^n of blocks of length n with the lexicographical order. We apply the usual notation from symbolic dynamics. For example, 0^∞ would indicate the sequence c with $c_i = 0$ for all $i \ge 1$, $(10)^\infty$ would indicate the sequence c with $c_{2i-1} = 1$ and $c_{2i} = 0$ for all $i \ge 1$, and so on. A sequence (c_i) is called

- *finite* if it has a last nonzero element, and *infinite* otherwise;
- *co-finite* if its conjugate is finite, and *co-infinite* otherwise;
- *doubly infinite* if it is both infinite and co-infinite.

This unusual terminology enables us to simplify the statements of many results. Note that 0^∞ does not have a last nonzero element and is thus infinite, hence doubly infinite. Similarly, M^∞ is a doubly infinite sequence. The other doubly infinite sequences are those sequences that have both infinitely many digits $c_i > 0$ and infinitely many digits $c_i < M$.

Given a real number $q > 1$, an *expansion in base q over the alphabet A* (or simply *expansion in base q* or *expansion* if there is no risk of confusion) of a real number x is a sequence $c = (c_i)$ satisfying the equality

$$\pi_q(c) := \sum_{i=1}^{\infty} \frac{c_i}{q^i} = x.$$

We sometimes write $\pi_q(c_1c_2\cdots)$ or $\pi_q((c_i))$ in place of $\pi_q(c)$.

If $q > M + 1$, then there exist numbers x satisfying the inequalities

$$\frac{M}{q^2} + \frac{M}{q^3} + \cdots < x < \frac{1}{q}, \tag{1.1}$$

and they have no expansions: for any sequence (c_i) we have $\pi_q(c_1c_2\cdots) < x$ if $c_1 = 0$, and $\pi_q(c_1c_2\cdots) > x$ if $c_1 > 0$. The inequalities (1.1) also imply that each $c \in A^{\mathbb{N}}$ is the unique expansion of $\pi_q(c)$. The topological structure of the set $\pi_q\left(A^{\mathbb{N}}\right)$ consisting of numbers with an expansion (which is always unique as we just observed) is in this case rather straightforward and resembles that of the classical triadic Cantor set $C := \left\{\sum_{i=1}^{\infty} a_i \cdot 3^{-i} : a_i \in \{0,2\}, i \geq 1\right\}$. The finite sequences in $A^{\mathbb{N}}$ should be compared with the right endpoints of the connected components of $[0,1] \setminus C$ and the co-finite sequences with its left endpoints. For this reason, we restrict ourselves in the sequel of this paper to bases $q \in (1, M+1]$ in which case $J_q := \pi_q\left(A^{\mathbb{N}}\right) = [0, M/(q-1)]$; see [22,21,12,2]. Moreover, every $x \in J_q$ has a lexicographically largest expansion $b(x,q)$ and a lexicographically largest infinite expansion $a(x,q)$, called the *greedy* and *quasi-greedy* expansions of x in base q, respectively. For example, in the case of the classical binary expansions (so $q = 2$ and $M = 1$), the fractions $x = \frac{k}{2^m} \in (0,1)$ with positive integers m and k have exactly two expansions: a finite and an infinite one; they are the greedy and quasi-greedy expansions of x, respectively. All other numbers $x \in J_2 = [0,1]$ have a unique expansion.

Of course, whether an expansion is greedy or quasi-greedy depends on q and M. However, when q and M are understood from the context, we simply speak of (quasi-) greedy expansions.

Let us give some examples, by describing the expansions of 1 in various bases over the alphabet $A = \{0,1\}$ (so $M = 1$); see [12,9] for details.

Examples 1.1.

(i) There exists a base $1 < q < 2$ such that $\pi_q(1(10)^\infty) = 1$. In this base $1(10)^\infty$ is the unique expansion of 1, and it is doubly infinite.

(ii) In the *Tribonacci base* $q \approx 1.839$, defined by the equation $q^3 = q^2 + q + 1$, 1 has $\aleph_0$ expansions: $(110)^\infty$, and the sequences

$$(110)^k(111)0^\infty, \quad k = 0, 1, \ldots.$$

Here $(110)^\infty$ is a doubly infinite expansion, and all other expansions are finite.

(iii) In the *Golden ratio base* $q = (1+\sqrt{5})/2$, defined by the equation $q^2 = q + 1$, 1 has $\aleph_0$ expansions again: $(10)^\infty$, and the sequences

$$(10)^k(11)0^\infty \quad \text{and} \quad (10)^k01^\infty, \quad k = 0, 1, \ldots.$$

Here $(10)^\infty$ is a doubly infinite expansion. There are many infinite expansions, but $(10)^\infty$ is the only doubly infinite expansion.

(iv) In every *small base* $1 < q < (1+\sqrt{5})/2$, 1 has $2^{\aleph_0}$ expansions; hence it has $2^{\aleph_0}$ doubly infinite expansions as well.

The choice of the alphabet A in the examples above and in general is pertinent. For instance, if $M = 2$, then 1 in the Golden ratio base has $2^{\aleph_0}$ expansions including all expansions (c_i) satisfying $c_{4k+1} \cdots c_{4k+4} \in \{1010, 0120\}$ for all $k \geq 0$.

In [9] we investigated the set of *univoque bases*

$$\mathcal{U} := \{q > 1 \ : \ 1 \text{ has a unique expansion in base } q\}$$

and the larger set

$$\mathcal{V} := \{q > 1 \ : \ 1 \text{ has a unique doubly infinite expansion in base } q\}.$$

It was shown that $\mathcal{V}$ is closed and that the closure $\overline{\mathcal{U}}$ of $\mathcal{U}$ is a *Cantor set*, i.e., a nonempty closed set having neither interior nor isolated points. Moreover, $\overline{\mathcal{U}}$ can be characterized as follows:

$$\overline{\mathcal{U}} = \{q > 1 \ : \ 1 \text{ has a unique infinite expansion in base } q\}.$$

In the following table we illustrate these notions by showing the number of doubly infinite expansions, infinite expansions and all expansions of 1 in the above four examples:

Examples 1.1	d.i. expansions	i. expansions	all expansions	q belongs to
(i)	1	1	1	$\mathcal{U}$
(ii)	1	1	∞	$\overline{\mathcal{U}} \setminus \mathcal{U}$
(iii)	1	∞	∞	$\mathcal{V} \setminus \overline{\mathcal{U}}$
(iv)	∞	∞	∞	$(1, (1+\sqrt{5})/2)$

The purpose of this paper is to carry out a similar study of the *univoque set*

$$\mathcal{U}_q := \{x \in J_q \ : \ x \text{ has a unique expansion in base } q\}$$

for each fixed base $q > 1$. We will prove for example that $\mathcal{U}_q$ is a closed set if and only if $q \notin \overline{\mathcal{U}}$. In order to state the main topological properties of the sets $\mathcal{U}_q$ we introduce the related sets $\mathcal{V}_q$ as follows: for $q \in (1, M+1)$, we set

$$\mathcal{V}_q := \{x \in J_q \ : \ x \text{ has a unique doubly infinite expansion in base } q\},$$

and for $q = M+1$ we set $\mathcal{V}_q := J_q = [0,1]$. If $q = M+1$, numbers $x \in J_q$ with a finite expansion have no doubly infinite expansion, while for $1 < q < M+1$, the quasi-greedy expansion $a(x,q)$ is always doubly infinite; see Proposition 2.1 (ii). Hence,

$$\mathcal{V}_q = \{x \in J_q \ : \ x \text{ has at most one doubly infinite expansion in base } q\},$$

for *each* $q \in (1, M+1]$.

The most important relations between the sets $\mathcal{U}_q$ and $\mathcal{V}_q$ are described in the following Theorems 1.2, 1.4 and 1.5:

Theorem 1.2.

(i) *If* $q \in \overline{\mathcal{U}}$, *then* $\overline{\mathcal{U}_q} = \mathcal{V}_q$.

(ii) *If $q \in \overline{\mathcal{U}}$, then $|\mathcal{V}_q \setminus \mathcal{U}_q| = \aleph_0$ and $\mathcal{V}_q \setminus \mathcal{U}_q$ is dense in $\mathcal{V}_q$.*
(iii) *If $q \in \mathcal{U}$, then each element $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ has exactly 2 expansions.*
(iv) *If $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$, then each element $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ has exactly $\aleph_0$ expansions.*

The proof of Theorem 1.2 will lead to some strengthened forms of (ii), (iii) and (iv). In order to state it we denote by A_q and B_q the elements x of $\mathcal{V}_q \setminus \mathcal{U}_q$ whose greedy expansions $b(x,q)$ are finite and infinite, respectively, so that

$$\mathcal{V}_q \setminus \mathcal{U}_q = A_q \cup B_q.$$

Given a base $q \in (1, M+1]$, we also introduce the *reflection map* $\ell : J_q \to J_q$, given by

$$\ell(x) = \frac{M}{q-1} - x, \quad x \in J_q.$$

In Part (iv) of the next proposition we refer to the expansions of 1 in a given base $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$. They are listed in Theorem 3.2 (vi).

Proposition 1.3. *Let $q \in (1, M+1]$ and write $(\alpha_i) := a(1,q)$.*

(i) *If $q = M+1$, then $\mathcal{V}_q = J_q = [0,1]$, $B_q = \varnothing$, and $A_q = \mathcal{V}_q \setminus \mathcal{U}_q$ is dense in $\mathcal{V}_q$.*
(ii) *If $q \in \overline{\mathcal{U}} \setminus \{M+1\}$, then both A_q and B_q are dense in $\mathcal{V}_q$. Moreover,*

$$B_q = \ell(A_q), \tag{1.2}$$

and the greedy expansion of each $x \in B_q$ ends with $\overline{(\alpha_i)}$.
(iii) *If $q \in \mathcal{U}$, then every $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ has exactly two expansions:*
   (a) *if $x \in A_q$ and $b(x,q) = b_1 \cdots b_n 0^\infty$ with $b_n > 0$, then $(b_1 \cdots b_n)^- \alpha_1 \alpha_2 \cdots$ is the other expansion of x;*
   (b) *if $x \in B_q$, then the expansions of x are the reflections of the expansions of $\ell(x) \in A_q$.*
(iv) *If $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$, then every $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ has exactly $\aleph_0$ expansions:*
   (a) *if $x \in A_q$ and $b(x,q) = b_1 \cdots b_n 0^\infty$ with $b_n > 0$, then the other expansions of x are of the form $(b_1 \cdots b_n)^- d_{n+1} d_{n+2} \cdots$, where $d_{n+1} d_{n+2} \cdots$ is one of the expansions of 1 in base q;*
   (b) *if $x \in B_q$, then the expansions of x are the reflections of the expansions of $\ell(x) \in A_q$.*

In Part (iv) of our next theorem we refer to the expansions of 1 in a given base $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$. They are listed in Theorem 3.3 (vi). We recall that if $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$, then there is a smallest integer $k \geq 1$ such that $\alpha_{k+1} \alpha_{k+2} \cdots = \overline{\alpha_1 \alpha_2 \cdots}$; see for instance Proposition 3.1. We also denote by $\tilde{q} = \tilde{q}(M)$ the smallest element of $\mathcal{V}$ (see Theorem 3.3 (ii)). We will use these notations in the statement of the following theorem.

Theorem 1.4. *Let $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$.*

(i) *The sets $\mathcal{U}_q$ and $\mathcal{V}_q$ are closed.*
(ii) *$|\mathcal{V}_q \setminus \mathcal{U}_q| = \aleph_0$ and $\mathcal{V}_q \setminus \mathcal{U}_q$ is a discrete set, dense in $\mathcal{V}_q$.*
(iii) *Each element $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ has exactly $\aleph_0$ expansions and a finite greedy expansion.*
(iv) *Let $x \in \mathcal{V}_q \setminus \mathcal{U}_q$, and let b_n be the last nonzero element of $(b_i) := b(x,q)$. Then x has exactly $\aleph_0$ other expansions of the form*

$$(b_1 \cdots b_n)^- d_{n+1} d_{n+2} \cdots ,$$

where $d_{n+1}d_{n+2}\cdots$ *is one of the expansions of* 1 *in base* q.
Furthermore, if $q = \tilde{q}$, M *is even and* $b_n \geq 2$, *then* $b_1 \cdots b_{n-1}(b_n - 2)M^\infty$ *is also an expansion of* x.
Finally, if

$$n > k, \quad b_{n-k} > 0 \quad \text{and} \quad b_{n-k+1} \cdots b_n = \overline{(\alpha_1 \cdots \alpha_k)^-},$$

then x *has one more expansion:*

$$(b_1 \cdots b_{n-k})^- M^\infty.$$

Theorem 1.5. *If* $q \in (1, M+1] \setminus \mathcal{V}$, *then* $\mathcal{U}_q = \overline{\mathcal{U}_q} = \mathcal{V}_q$.

Remarks 1.6.

(i) While $\mathcal{U}$, $\overline{\mathcal{U}}$ and $\mathcal{V}$ are three different sets, we infer from Theorems 1.2, 1.4 and 1.5 that at least two of the three sets $\mathcal{U}_q$, $\overline{\mathcal{U}_q}$ and $\mathcal{V}_q$ coincide for each $q \in (1, M+1]$.

(ii) The set $\mathcal{V}$ has Lebesgue measure zero and Hausdorff dimension one, because $\mathcal{V} \setminus \mathcal{U}$ is countable by [9] and $\mathcal{U}$ is a Lebesgue null set of Hausdorff dimension one by [11, Theorem 1] and [16, Theorem 1.6]. Theorem 1.5 implies that $\mathcal{U}_q = \overline{\mathcal{U}_q} = \mathcal{V}_q$ except for a set of bases $q \in (1, M+1]$ of Lebesgue measure zero and Hausdorff dimension equal to one.

(iii) If $q = M+1$, then $\overline{\mathcal{U}_q} = [0,1]$, and $\mathcal{U}_q$ has Lebesgue measure one. Suppose next that $q \in (1, M+1)$ is a non-integer. If $x \in \mathcal{U}_q$ has unique expansion $(c_i) \neq M^\infty$, then there exists an index N such that $(c_{N+i}) = c_{N+1}c_{N+2}\cdots$ is the unique expansion of a number belonging to $[0,1)$, whence $c_{N+i} < q$ for each $i \geq 1$. It follows from Proposition 4.1 in [7] (see also [14]) that $\mathcal{U}_q$ can be covered by countably many sets of the same Hausdorff dimension less than one. Hence the sets $\mathcal{U}_q$, $\overline{\mathcal{U}_q}$ and $\mathcal{V}_q$ have Hausdorff dimension less than one and are also nowhere dense because $\mathcal{V}_q \setminus \mathcal{U}_q$ is (at most) countable and $\mathcal{V}_q$ is closed. If $q \in (1, M+1)$ is an integer and $M < 2q-2$, then the unique expansion of a number $x \in \mathcal{U}_q$ has a tail belonging to $\{0^\infty, M^\infty\} \cup \{M-q+1, \ldots, q-1\}^\infty$ as follows, for instance, from Lemma 4.1. Conversely, a sequence belonging to $\{M-q+1, \ldots, q-1\}^\infty$ with no tail equal to $(M-q+1)^\infty$ or $(q-1)^\infty$ is the (unique) expansion of an element in $\mathcal{U}_q$. Straightforward alterations in the usual calculation of the Hausdorff dimension of the triadic Cantor set C show that $\dim_H(\mathcal{U}_q) = \log(2q-M-1)/\log(q)$, see for instance [13]. Hence, also in this case, $\mathcal{U}_q$, $\overline{\mathcal{U}_q}$ and $\mathcal{V}_q$ are nowhere dense and have Hausdorff dimension less than one. Finally, if $q \in (1, M+1)$ is an integer and $M \geq 2q-2$, then $\mathcal{U}_q = \{0, M/(q-1)\}$. More precise information about the Hausdorff dimension of $\mathcal{U}_q$ and the behavior of the function $q \mapsto \dim_H(\mathcal{U}_q)$ on $(1, M+1]$, albeit at the cost of a much more elaborate analysis, can be found in [16] and [1].

We mention two corollaries of Theorems 1.2, 1.4 and 1.5:

Corollary 1.7. *Let* $q \in (1, M+1]$. *Then* $\mathcal{U}_q$ *is closed if and only if* $q \notin \overline{\mathcal{U}}$.

Corollary 1.8. *Let* $q \in (1, M+1]$. *The following equivalences hold:*

$$q \in \mathcal{U} \Longleftrightarrow 1 \in \mathcal{U}_q,$$
$$q \in \overline{\mathcal{U}} \Longleftrightarrow 1 \in \overline{\mathcal{U}_q},$$
$$q \in \mathcal{V} \Longleftrightarrow 1 \in \mathcal{V}_q.$$

Let us denote by $\mathcal{U}_q'$ and $\mathcal{V}_q'$ the sets of quasi-greedy expansions in base q of all numbers $x \in \mathcal{U}_q$ and $x \in \mathcal{V}_q$, respectively. Note that $\mathcal{U}_q'$ is simply the set of unique expansions in base q. Elements of $\mathcal{U}_q'$ are frequently referred to as *univoque sequences (in base q)*.

If we endow the collection of subsets of $A^{\mathbb{N}}$ with the partial order $\subseteq$, then the maps $q \mapsto \mathcal{U}'_q$ and $q \mapsto \mathcal{V}'_q$ are non-decreasing by Lemma 4.3. However, on some intervals these maps are constant. We say that an interval $I \subseteq (1, M+1]$ of bases is a *stability interval* for $\mathcal{U}_q$ (resp. for $\mathcal{V}_q$) if $\mathcal{U}'_q = \mathcal{U}'_r$ (resp. $\mathcal{V}'_q = \mathcal{V}'_r$) for all $q, r \in I$. We call a stability interval I *maximal* if any interval $J \subseteq (1, M+1]$ that properly contains I is not a stability interval. The following theorem completes the investigation of stability intervals started by Daróczy and Kátai ([4,5]):

Theorem 1.9.

(i) *The maximal stability intervals for $\mathcal{U}_q$ are given by the singletons $\{q\}$ where $q \in \overline{\mathcal{U}}$, and the intervals $(q_1, q_2]$ where (q_1, q_2) is a connected component of $(1, M+1] \setminus \mathcal{V}$.*

(ii) *The maximal stability intervals for $\mathcal{V}_q$ are given by the singletons $\{q\}$ where $q \in \mathcal{U}$, the interval $(1, \tilde{q})$, and the intervals $[q_1, q_2)$ where (q_1, q_2) is a connected component of $(1, M+1] \setminus \mathcal{V}$ with $q_1 \neq 1$.*

In [19,6,9] we have clarified the topological structure of the complements $(1, M+1] \setminus \overline{\mathcal{U}}$ and $(1, M+1] \setminus \mathcal{V}$. The following theorem describes the topological structure of $J_q \setminus \overline{\mathcal{U}_q}$ and $J_q \setminus \mathcal{V}_q$ for all $q \in (1, M+1]$. We recall that $\mathcal{U}_q = \overline{\mathcal{U}_q} = \mathcal{V}_q$ if $q \notin \mathcal{V}$, and that $\mathcal{U}_q \subsetneq \mathcal{V}_q$ if $q \in \mathcal{V}$. Moreover, if we write the open set $(1, M+1] \setminus \mathcal{V}$ as a disjoint union of open intervals, then the set of left and right endpoint of these intervals are given by $\{1\} \cup (\mathcal{V} \setminus \mathcal{U})$ and $\mathcal{V} \setminus \overline{\mathcal{U}}$, respectively; see Theorem 3.3 (iii).

Theorem 1.10. *Let $q \in (1, M+1]$ and write $(\alpha_i) := a(1, q)$.*

(i) *If $q = M+1$, then $\overline{\mathcal{U}_q} = \mathcal{V}_q = J_q = [0, 1]$.*

(ii) *If $q \in \overline{\mathcal{U}} \setminus \{M+1\}$, then $J_q \setminus \overline{\mathcal{U}_q} = J_q \setminus \mathcal{V}_q$ is the union of infinitely many disjoint open sets (x_L, x_R). Furthermore, x_L and x_R run over A_q and B_q, respectively. More precisely, if $b(x_L, q) = b_1 \cdots b_n 0^\infty$ with $b_n > 0$, then $b(x_R, q) = b_1 \cdots b_n \overline{\alpha_1 \alpha_2 \cdots}$.*

(iii) *If $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$, then $J_q \setminus \overline{\mathcal{U}_q} = J_q \setminus \mathcal{U}_q$ is an open set. Furthermore, each connected component (x_L, x_R) of $J_q \setminus \mathcal{U}_q$ contains infinitely many elements of $\mathcal{V}_q$, forming an increasing sequence $(x_k)_{k=-\infty}^{\infty}$ satisfying*

$$x_k \to x_L \quad as \quad k \to -\infty, \qquad x_k \to x_R \quad as \quad k \to \infty. \tag{1.3}$$

Moreover, the relation between two subsequent numbers x_m and x_{m+1} in the sequence $(x_k)_{k=-\infty}^{\infty}$ is described as follows:

$$\text{if } b(x_m, q) = b_1 \cdots b_n 0^\infty \quad \text{with} \quad b_n > 0, \quad \text{then} \quad a(x_{m+1}, q) = b_1 \cdots b_n \overline{\alpha_1 \alpha_2 \cdots}. \tag{1.4}$$

(iv) *If $q \in (1, \tilde{q}]$, then $J_q \setminus \mathcal{U}_q = (0, M/(q-1))$.*

(v) *Let (q_1, q_2) be a connected component of $(1, M+1] \setminus \mathcal{V}$ with $q_1 \neq 1$ and $q \in (q_1, q_2]$. Then $\mathcal{U}'_q = \mathcal{V}'_{q_1}$ and the open sets $J_q \setminus \mathcal{U}_q$ and $J_{q_1} \setminus \mathcal{V}_{q_1}$ are homeomorphic.*

Remarks 1.11.

(i) If $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$ and $q \neq \tilde{q}$, then $J_q \setminus \mathcal{U}_q$ has infinitely many connected components. Indeed, there exists a connected component (q_1, q_2) of $(1, M+1) \setminus \mathcal{V}$ such that $q_1 \in \mathcal{V} \setminus \mathcal{U}$ and $q = q_2$. Then $J_q \setminus \mathcal{U}_q$ is homeomorphic to $J_{q_1} \setminus \mathcal{V}_{q_1}$ by Theorem 1.10 (v), and $J_{q_1} \setminus \mathcal{V}_{q_1}$ has infinitely many connected components by Theorem 1.10 (ii) and (iii).

(ii) If $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$ and (x_L, x_R) is a connected component of $J_q \setminus \mathcal{U}_q$, then all numbers in $(x_L, x_R) \cap \mathcal{V}_q$ have a finite greedy expansion, according to Theorem 1.4 (iii). The greedy expansion $b(x_{m+1}, q)$ of the number x_{m+1} occurring in Theorem 1.10 (iii) equals $(a_1(x_{m+1},q)\cdots a_k(x_{m+1},q))^+0^\infty$ where $k\ge 1$ is the smallest index such that $a_k(x_{m+1},q)<M$ and $(a_{k+i}(x_{m+1},q))=(\alpha_i)$; see Lemma 2.6 (ii).

(iii) For $q\in(1,\tilde q)$, we provide in Examples 4.4 (ii) a short proof of the following strengthening of Theorem 1.10 (iv): *each* $x\in(0,M/(q-1))$ has $2^{\aleph_0}$ expansions. This result was first established by Baker ([3]) using different ideas. We extract from our proof some other results of Baker ([3]) in Examples 4.4 (iii) and (iv).

We recall that a nonempty closed set is called *perfect* if it has no isolated points, and that x is a *condensation point* of a set $F\subseteq\mathbb{R}$ if every neighborhood of x contains uncountably many elements of F. By the Cantor–Bendixson theorem, the condensation points of an uncountable closed set F of real numbers form a perfect set P, and $F\setminus P$ is (at most) countable. In the following theorem we determine the isolated, accumulation and condensation points of the sets $\mathcal{U}_q$, $\overline{\mathcal{U}_q}$ and $\mathcal{V}_q$, and we list explicitly all cases where $\mathcal{U}_q$, $\overline{\mathcal{U}_q}$ and $\mathcal{V}_q$ is a Cantor set. We recall from Theorems 3.2 (iv) and 3.3 (iii) that if (q_0,q_0^*) is a connected component of $(1,M+1]\setminus\overline{\mathcal{U}}$, then $q_0\in\{1\}\cup\big(\overline{\mathcal{U}}\setminus\mathcal{U}\big)$, $q_0^*\in\mathcal{U}$, and $\mathcal{V}\cap(q_0,q_0^*)$ is formed by an increasing sequence $q_1<q_2<\cdots$, converging to q_0^*. We use this notation in the following theorem. We also recall that if $q_0=1$, then $q_1=\tilde q=\tilde q(M)$ and q_0^* is the *Komornik–Loreti constant* which we will denote by $q_{KL}=q_{KL}(M)$; see [17,18]. The precise value of $q_{KL}(M)$ is given in Theorem 3.2 (ii).

Theorem 1.12.

(i) *Let* $q\in\overline{\mathcal{U}}$.
 (a) *If* $q=M+1$, *then* $\mathcal{V}_q=\overline{\mathcal{U}_q}=[0,1]$ *is a perfect set.*
 (b) *If* $q\in\overline{\mathcal{U}}\setminus\{M+1\}$, *then* $\mathcal{V}_q=\overline{\mathcal{U}_q}$ *is a Cantor set.*

(ii) *Consider the connected component* $(1,q_{KL})$ *of* $(1,M+1]\setminus\overline{\mathcal{U}}$.
 (a) *If* $q\in(1,q_1]$, *then* $\mathcal{U}_q$ *is a two-point set.*
 (b) *If* $q\in(q_n,q_{n+1}]$ *for some* $n\ge 1$, *then* $\mathcal{U}_q$ *is countably infinite. Furthermore, its accumulation and isolated points form the sets*

$$\pi_q(\mathcal{U}'_{q_n})\quad \textit{and}\quad \pi_q(\mathcal{V}'_{q_n}\setminus\mathcal{U}'_{q_n}),$$

 respectively, and the isolated points of $\mathcal{U}_q$ *form a dense subset of* $\mathcal{U}_q$.

(iii) *Consider a connected component* (q_0,q_0^*) *of* $(1,M+1]\setminus\overline{\mathcal{U}}$ *with* $q_0\in\overline{\mathcal{U}}\setminus\mathcal{U}$.
 (a) *If* $q\in(q_0,q_1]$, *then* $\mathcal{U}_q=\overline{\mathcal{U}_q}$ *is a Cantor set, and* $\pi_q(\mathcal{V}'_{q_0}\setminus\mathcal{U}'_{q_0})$ *is dense in* $\mathcal{U}_q$.
 (b) *If* $q\in(q_n,q_{n+1}]$ *for some* $n\ge 1$, *then the condensation points, further accumulation points and isolated points of* $\mathcal{U}_q$ *form the sets*

$$\pi_q(\mathcal{V}'_{q_0}),\quad \pi_q(\mathcal{U}'_{q_n}\setminus\mathcal{V}'_{q_0})\quad \textit{and}\quad \pi_q(\mathcal{V}'_{q_n}\setminus\mathcal{U}'_{q_n}),$$

 respectively, and the isolated points of $\mathcal{U}_q$ *form a dense subset of* $\mathcal{U}_q$.

We recall (see, e.g., [20]) that a set $S\subseteq A^{\mathbb{N}}$ is called a *shiftspace* or *shift* if there exists a set $\mathcal{F}(S)\subseteq\cup_{k=1}^\infty A^k$ such that a sequence $(c_i)\in A^{\mathbb{N}}$ belongs to S if and only if none of the blocks $c_{i+1}\cdots c_{i+n}$ ($i\ge 0,n\ge 1$) belongs to $\mathcal{F}(S)$. A shift S is called a *shift of finite type* if one can choose $\mathcal{F}(S)$ to be finite. We endow the alphabet A with the discrete topology, and the set of expansions $A^{\mathbb{N}}$ with the Tychonov product topology, so that the corresponding convergence in $A^{\mathbb{N}}$ is the coordinate-wise convergence of sequences.

Theorem 1.13. *Let* $q>1$ *be a real number. The following statements are equivalent.*

(i) $q\in(1,M+1]\setminus\overline{\mathcal{U}}$.

(ii) $\mathcal{U}_q'$ *is a shift of finite type.*
(iii) $\mathcal{U}_q'$ *is a shift.*
(iv) $\mathcal{U}_q'$ *is a closed subset of* $A^{\mathbb{N}}$.

Finally we consider the *two-dimensional univoque set* $\mathbf{U}$, formed by the couples $(x,q) \in \mathbb{R}^2$ where $q > 1$ and x has a unique expansion in base q over the alphabet A:

$$\mathbf{U} := \left\{(x,q) \in \mathbb{R}^2 \ : \ q \in (1, M+1] \text{ and } x \in \mathcal{U}_q\right\}.$$

Setting

$$\mathbf{V} := \left\{(x,q) \in \mathbb{R}^2 \ : \ q \in (1, M+1] \text{ and } x \in \mathcal{V}_q\right\},$$

we have the following result:

Theorem 1.14. *The set* $\mathbf{U}$ *is not closed. Its closure* $\overline{\mathbf{U}}$ *equals* $\mathbf{V} \cup \{(0,1)\}$.

Hence, $\overline{\mathbf{U}} \cap \mathbf{J} = \mathbf{V}$, where the set

$$\mathbf{J} := \{(x,q) : q \in (1, M+1] \text{ and } x \in J_q\}$$

consists of all couples (x,q) such that x has an expansion in base q.

The rest of the paper is organized as follows. In Sections 2 and 3 we recall various properties of the greedy and quasi-greedy expansions and of the sets of bases $\mathcal{U}$, $\overline{\mathcal{U}}$ and $\mathcal{V}$. In Section 4 we deduce some elementary properties of the sets $\mathcal{U}_q$ and $\mathcal{V}_q$ and we reprove the main results in [3]. In Section 5 we prove two density results that will be important in the proof of our main theorems. In Section 6 we prove Theorem 1.2 and Proposition 1.3. Section 7 is devoted to the proof of Theorems 1.4, 1.5 and Corollaries 1.7, 1.8. We also give an intrinsic characterization of $\overline{\mathcal{U}_q}$ in Theorem 7.2. Our final Theorems 1.9, 1.10, 1.12, 1.13 and 1.14 are proved in Section 8. For the reader's convenience, a list of principal terminology and notations used in this paper is given in the final Section 9.

2. Greedy and quasi-greedy expansions

In this section we recall from [2,18,7,9] some results that we shall use very frequently in the sequel.

Given a base $q \in (1, M+1]$ and a real number $x \geq 0$, we define the *greedy sequence* $b(x,q) = (b_i(x,q))$ and the *quasi-greedy sequence* $a(x,q) = (a_i(x,q))$ in $A^{\mathbb{N}}$ as follows. If for some $n \in \mathbb{N}$, $b_i(x,q)$ is already defined for every $i < n$ (no condition if $n = 1$), then let $b_n(x,q)$ be the largest element of the digit set A such that

$$\sum_{i=1}^{n} \frac{b_i(x,q)}{q^i} \leq x.$$

Similarly, if $x > 0$ and if for some $n \in \mathbb{N}$, $a_i(x,q)$ is already defined for every $i < n$ (no condition if $n = 1$), then let $a_n(x,q)$ be the largest element of the digit set A such that

$$\sum_{i=1}^{n} \frac{a_i(x,q)}{q^i} < x.$$

Furthermore, we set $a(0,q) := 0^{\infty}$. It follows from the definitions that

$$\sum_{i=1}^{\infty} \frac{a_i(x,q)}{q^i} \le \sum_{i=1}^{\infty} \frac{b_i(x,q)}{q^i} \le x$$

for all $x \ge 0$. Moreover, $(b_i(x,q))$ is the lexicographically largest sequence in $A^{\mathbb{N}}$ satisfying

$$\sum_{i=1}^{\infty} \frac{b_i(x,q)}{q^i} \le x,$$

and $(a_i(x,q))$ is the lexicographically largest infinite sequence in $A^{\mathbb{N}}$ satisfying

$$\sum_{i=1}^{\infty} \frac{a_i(x,q)}{q^i} \le x.$$

If $x \in J_q$, then the sequences $a(x,q)$ and $b(x,q)$ are indeed expansions of x and are thus the quasi-greedy and greedy expansion of x respectively, as introduced in Section 1:

Proposition 2.1. *Let $(x,q) \in \mathbf{J}$. Then*

(i) *$a(x,q)$ and $b(x,q)$ are expansions of x, i.e.,*

$$\sum_{i=1}^{\infty} \frac{a_i(x,q)}{q^i} = \sum_{i=1}^{\infty} \frac{b_i(x,q)}{q^i} = x.$$

(ii) *If $q \ne M+1$, then $a(x,q)$ is doubly infinite.*

The expansions $b(x,q)$ and $a(x,q)$ have the following semi-continuity properties:

Lemma 2.2. *Let $(x,q) \in \mathbf{J}$ and $(y_n, r_n) \in \mathbf{J}$, $n \in \mathbb{N}$. Then*

(i) *for each positive integer m there exists a neighborhood $W \subseteq \mathbb{R}^2$ of (x,q) such that*

$$b_1(y,r)\cdots b_m(y,r) \le b_1(x,q)\cdots b_m(x,q) \quad \textit{for all} \quad (y,r) \in W \cap \mathbf{J}; \tag{2.1}$$

(ii) *if $y_n \downarrow x$ and $r_n \downarrow q$, then $b(y_n, r_n)$ converges (coordinate-wise) to $b(x,q)$.*

(iii) *for each positive integer m there exists a neighborhood $W \subseteq \mathbb{R}^2$ of (x,q) such that*

$$a_1(y,r)\cdots a_m(y,r) \ge a_1(x,q)\cdots a_m(x,q) \quad \textit{for all} \quad (y,r) \in W \cap \mathbf{J}; \tag{2.2}$$

(iv) *if $y_n \uparrow x$ and $r_n \uparrow q$, then $a(y_n, r_n)$ converges (coordinate-wise) to $a(x,q)$.*

Proof. (i) By definition of $b(x,q)$, we have the *strict* inequalities

$$\sum_{i=1}^{n} \frac{b_i(x,q)}{q^i} > x - \frac{1}{q^n} \quad \text{whenever} \quad b_n(x,q) < M.$$

Hence, if $(y,r) \in \mathbf{J}$ is sufficiently close to (x,q), then the *finitely* many inequalities

$$\sum_{i=1}^{n} \frac{b_i(x,q)}{r^i} > y - \frac{1}{r^n} : n \le m \text{ and } b_n(x,q) < M,$$

also hold. These inequalities imply (2.1). Note that the inequality (2.1) can be strict even if (y, r) is very close to (x, q) because it may be the case that

$$\sum_{i=1}^{n} \frac{b_i(x, q)}{r^i} > y$$

for some $n \leq m$.

(ii) If $y_n \geq x$ and $r_n \geq q$, we deduce from the definition of greedy sequences that

$$b(x, q) \leq b(y_n, r_n)$$

for every n. Equivalently, we have

$$b_1(x, q) \cdots b_m(x, q) \leq b_1(y_n, r_n) \cdots b_m(y_n, r_n)$$

for all positive integers m and n. It remains to notice that by the previous part the converse inequality also holds for each fixed m if n is large enough.

(iii) We may assume that $x \neq 0$. By definition we have

$$\sum_{i=1}^{n} \frac{a_i(x, q)}{q^i} < x \text{ for all } n = 1, 2, \ldots.$$

If $(y, r) \in \mathbf{J}$ is sufficiently close to (x, q), then

$$\sum_{i=1}^{n} \frac{a_i(x, q)}{r^i} < y, \quad n = 1, \ldots, m.$$

These inequalities imply (2.2).

(iv) If $y_n \leq x$ and $r_n \leq q$, we deduce from the definition of quasi-greedy sequences that

$$a(x, q) \geq a(y_n, r_n)$$

for every n. Equivalently, we have

$$a_1(x, q) \cdots a_m(x, q) \geq a_1(y_n, r_n) \cdots a_m(y_n, r_n)$$

for all positive integers m and n. It remains to notice that by the previous part the converse inequality also holds for each fixed m if n is large enough. □

Let $q \in (1, M + 1]$. Observe that for any fixed $x \in J_q$, the map $(c_i) \mapsto (M - c_i)$ is a strictly decreasing bijection between the expansions of x and $\ell(x)$. Therefore every $x \in J_q$ has a lexicographically smallest expansion, given by $(M - b_i)$, where $(b_i) = b(\ell(x), q)$. We often call this expansion the *lazy* expansion of x.

Lemma 2.3. *If $q \in (1, M + 1]$, then the lazy expansion of every $x \in J_q$ is infinite.*

Proof. Write $(\alpha_i) := a(1, q)$. If an expansion (c_i) of $x \in J_q$ has a last nonzero digit c_n, then $(c_1 \cdots c_n)^- \alpha_1 \alpha_2 \cdots$ is another expansion of x, smaller than (c_i). □

Since we will often use the greedy and quasi-greedy expansions of 1, we introduce for brevity the notation

$$\beta(q) = (\beta_i(q)) := b(1,q) \quad \text{and} \quad \alpha(q) = (\alpha_i(q)) := a(1,q).$$

We will often write (β_i) and (α_i) when the base q is known from the context. It will also be convenient to define

$$\beta(1) = (\beta_i(1)) := 10^\infty \quad \text{and} \quad \alpha(1) = (\alpha_i(1)) := 0^\infty.$$

Proposition 2.4.

(i) *The map $q \mapsto \beta(q)$ is a strictly increasing bijection from $[1, M+1]$ onto the set of all sequences (β_i), satisfying*

$$\beta_{n+1}\beta_{n+2}\cdots < \beta_1\beta_2\cdots \quad \textit{whenever} \quad \beta_n < M. \tag{2.3}$$

Furthermore, in case $q \in [1, M+1)$ the inequality (2.3) holds for all $n \geq 1$.

(ii) *The map $q \mapsto \alpha(q)$ is a strictly increasing bijection from $[1, M+1]$ onto the set of all* infinite *sequences (α_i), satisfying*

$$\alpha_{n+1}\alpha_{n+2}\cdots \leq \alpha_1\alpha_2\cdots \quad \textit{whenever} \quad \alpha_n < M. \tag{2.4}$$

Furthermore, the inequality (2.4) holds for all $n \geq 0$.

(iii) *$\alpha(q)$ is doubly infinite for every $q \in [1, M+1]$.*

Proposition 2.5. *Let $q \in (1, M+1]$, and write $(\alpha_i) = \alpha(q)$.*

(i) *The map $x \mapsto b(x,q)$ is a strictly increasing bijection from J_q onto the set of all sequences (b_i), satisfying*

$$b_{n+1}b_{n+2}\cdots < \alpha_1\alpha_2\cdots \quad \textit{whenever} \quad b_n < M. \tag{2.5}$$

Furthermore, the inequality (2.5) holds whenever $b_1 \cdots b_n \neq M^n$.

(ii) *The map $x \mapsto a(x,q)$ is a strictly increasing bijection from J_q onto the set of all* infinite *sequences (a_i), satisfying*

$$a_{n+1}a_{n+2}\cdots \leq \alpha_1\alpha_2\cdots \quad \textit{whenever} \quad a_n < M. \tag{2.6}$$

Furthermore, the inequality (2.6) holds whenever $a_1 \cdots a_n \neq M^n$.

We call a sequence (c_i) *periodic* if $(c_i) = (c_{k+i}) = c_{k+1}c_{k+2}\cdots$ for some $k \geq 1$. The smallest positive integer k for which a periodic sequence satisfies $(c_{k+i}) = (c_i)$ is called the *smallest period* of (c_i).

Lemma 2.6. *Let $(x,q) \in \mathbf{J}$, and set*

$$(b_i) = b(x,q), \quad (a_i) = a(x,q), \quad (\beta_i) = \beta(q), \quad (\alpha_i) = \alpha(q).$$

(i) *If $b(x,q)$ is infinite, then $a(x,q) = b(x,q)$.*

(ii) *If* (b_i) *has a last nonzero element* b_m, *then*

$$(a_i) = (b_1 \cdots b_m)^- \alpha_1 \alpha_2 \cdots .$$

(iii) *If* (β_i) *is infinite, then* $(\beta_i) = (\alpha_i)$ *is periodic only for* $q = M + 1$.

(iv) *If* (β_i) *has a last nonzero element* β_m, *then*

$$(\alpha_i) = ((\beta_1 \cdots \beta_m)^-)^\infty,$$

and m *is the smallest period of* (α_i).

Lemma 2.7. *Let* $q \in (1, M + 1]$, *and let* $(d_i) = d_1 d_2 \cdots$ *be a greedy or quasi-greedy sequence. Then for all* $N \geq 1$ *the truncated sequence* $d_1 \cdots d_N 0^\infty$ *is greedy.*

Proof. The statement follows at once from Proposition 2.5. □

Lemma 2.8. *Let* $q \in (1, M + 1]$, *and write* $(\alpha_i) = \alpha(q)$. *Let* $(b_i) \neq M^\infty$ *be a greedy sequence. Then*

(i) *There exists a sequence* $1 \leq n_1 < n_2 < \cdots$ *such that for each* $i \geq 1$,

$$b_{n_i} < M, \quad \text{and} \quad b_{m+1} \cdots b_{n_i} < \alpha_1 \cdots \alpha_{n_i - m} \quad \text{if} \quad 1 \leq m < n_i \quad \text{and} \quad b_m < M.$$

(ii) *For every positive integer* N, *there exists a greedy sequence* $(c_i) > (b_i)$ *such that*

$$c_1 \cdots c_N = b_1 \cdots b_N.$$

Proof. (i) See [9, Theorem 2.1].

(ii) This is a consequence of Lemma 2.2 (ii). □

3. A review of the sets of bases $\mathcal{U}, \overline{\mathcal{U}}$ and $\mathcal{V}$

In this section we recall several results from [9]. They generalized for $1 < q \leq M+1$ a number of theorems proved in [19] for $M < q \leq M + 1$.

The three sets $\mathcal{U}, \overline{\mathcal{U}}$ and $\mathcal{V}$ (which are clearly all contained in $(1, M+1]$) have the following lexicographic characterizations (the first one was proved in [12] for $M = 1$):

Proposition 3.1. *Let* $q \in (1, M + 1]$, *and write* $(\alpha_i) = \alpha(q)$, $(\beta_i) = \beta(q)$. *We have*

(i) $q \in \mathcal{U} \Longleftrightarrow \overline{(\beta_{n+i})} < (\beta_i)$ *whenever* $\beta_n > 0$;
(ii) $q \in \overline{\mathcal{U}} \Longleftrightarrow \overline{(\alpha_{n+i})} < (\alpha_i)$ *whenever* $\alpha_n > 0$;
(iii) $q \in \mathcal{V} \Longleftrightarrow \overline{(\alpha_{n+i})} \leq (\alpha_i)$ *whenever* $\alpha_n > 0$.

Moreover, in each case the inequalities are satisfied for all $n \geq 0$.

The main properties of $\mathcal{U}$ and $\overline{\mathcal{U}}$ are contained in the following theorem. In its statement we use the Thue–Morse sequence $(\tau_i)_{i=0}^\infty$, defined by the formulas $\tau_0 := 0$, and

$$\tau_{2^N+i} = 1 - \tau_i \quad \text{for} \quad i = 0, \ldots, 2^N - 1, \quad N = 0, 1, 2, \ldots .$$

We call a set $X \subseteq \mathbb{R}$ *closed from above (below)* if the limit of every bounded decreasing (increasing) sequence in X belongs to X. Alternatively, $X \subseteq \mathbb{R}$ is closed from above (below) if for each $x \in \mathbb{R} \setminus X$, there exists a $\delta = \delta(x) > 0$ such that

$$[x, x+\delta) \cap X = \varnothing \quad ((x-\delta, x] \cap X = \varnothing).$$

Theorem 3.2.

(i) *The set $\mathcal{U}$ is closed from above but not from below.*
(ii) *The smallest element $q_{KL} = q_{KL}(M)$ of $\mathcal{U}$ (called* Komornik–Loreti constant*) is determined by the formula*

$$\alpha_i(q') = \begin{cases} m-1+\tau_i & \text{if } M = 2m-1, \quad m = 1,2,\ldots; \\ m+\tau_i-\tau_{i-1} & \text{if } M = 2m, \quad m = 1,2,\ldots. \end{cases}$$

(iii) *The closure $\overline{\mathcal{U}}$ of $\mathcal{U}$ is a Cantor set. Moreover, $\overline{\mathcal{U}} \setminus \mathcal{U}$ is a countable dense set in $\overline{\mathcal{U}}$.*
(iv) *We have a disjoint union*

$$(1, M+1] \setminus \overline{\mathcal{U}} = \cup^*(q_0, q_0^*)$$

where q_0 runs over $\{1\} \cup (\overline{\mathcal{U}} \setminus \mathcal{U})$ and q_0^* runs over a proper subset $\mathcal{U}^*$ of $\mathcal{U}$.
(v) *If $q \in \overline{\mathcal{U}}$ and $(\alpha_i) = \alpha(q)$, then there exist arbitrarily large integers m such that for all k with $0 \leq k < m$,*

$$\overline{\alpha_{k+1} \cdots \alpha_m} < \alpha_1 \cdots \alpha_{m-k}.$$

(vi) *If $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$, then $(\alpha_i) = \alpha(q)$ is periodic. Furthermore, all expansions of 1 are given by (α_i) and the sequences*

$$(\alpha_1 \cdots \alpha_k)^N (\alpha_1 \cdots \alpha_k)^+ 0^\infty, \quad N = 0,1,\ldots,$$

where k is the smallest period of (α_i).

Next we list the main properties of the set $\mathcal{V}$:

Theorem 3.3.

(i) *$\mathcal{V}$ is compact, and $\mathcal{V} \setminus \overline{\mathcal{U}}$ is a countable dense subset of $\mathcal{V}$.*
(ii) *The smallest element $\tilde{q} = \tilde{q}(M)$ of $\mathcal{V}$ (called generalized Golden ratio) is given by the formulas*

$$\tilde{q} = \begin{cases} m+1 & \text{if } M = 2m, \\ \left(m + \sqrt{m^2+4m}\right)/2 & \text{if } M = 2m-1 \end{cases}$$

for $m = 1,2,\ldots$. Furthermore,

$$\begin{cases} \beta(\tilde{q}) = m^+0^\infty \quad \text{and} \quad \alpha(\tilde{q}) = m^\infty & \text{if } M = 2m, \\ \beta(\tilde{q}) = mm0^\infty \quad \text{and} \quad \alpha(\tilde{q}) = (mm^-)^\infty & \text{if } M = 2m-1. \end{cases}$$

(iii) *The set* $\mathcal{V} \setminus \overline{\mathcal{U}}$ *is discrete. Moreover, we have*

$$\mathcal{V} \cap (q_0, q_0^*) = \{q_n \ : \ n = 1, 2, \ldots\}$$

for each connected component (q_0, q_0^*) *of* $(1, M+1] \setminus \overline{\mathcal{U}}$*, where* (q_n) *is a (strictly) increasing sequence converging to* q_0^**. It follows from Theorem 3.2 (iv) that we have a disjoint union*

$$(1, M+1] \setminus \mathcal{V} = \cup^*(r_0, r_0^*)$$

where r_0 *runs over* $\{1\} \cup (\mathcal{V} \setminus \mathcal{U})$ *and* r_0^* *runs over* $\mathcal{V} \setminus \overline{\mathcal{U}}$*.*

(iv) *Given* $q_0 \in \{1\} \cup (\overline{\mathcal{U}} \setminus \mathcal{U})$*, the greedy expansions of the numbers* q_n *in (iii) have the form* $\beta(q_n) = s_n 0^\infty$ *with a sequence of words* s_n *defined recursively as follows. If* $q_0 \in \overline{\mathcal{U}} \setminus \mathcal{U}$*, then* $(\beta_i) := \beta(q_0)$ *has a last nonzero digit* β_m*, and we define*

$$s_0 := \beta_1 \cdots \beta_m, \quad \text{and} \quad s_{n+1} := s_n \overline{s_n^-}, \quad n = 0, 1, \ldots.$$

If $q_0 = 1$*, then* $q_1 = \tilde{q}$*, and we define* $s_0 := 1$*,*

$$s_1 := \begin{cases} m^+ & \text{if } M = 2m, \\ mm & \text{if } M = 2m-1, \end{cases}$$

and

$$s_{n+1} := s_n \overline{s_n^-}, \quad n = 1, 2, \ldots.$$

(v) *Let* $q \in \mathcal{V}$ *and* $(\alpha_i) = \alpha(q)$*. If for some* $k \geq 1$*,*

$$\overline{\alpha_{k+1} \cdots \alpha_{2k}} = \alpha_1 \cdots \alpha_k, \tag{3.1}$$

then

$$\alpha_k > 0 \quad \text{and} \quad (\alpha_i) = (\alpha_1 \cdots \alpha_k \overline{\alpha_1 \cdots \alpha_k})^\infty.$$

In particular, $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$*. Let, moreover,* n *be the smallest index* k *in (3.1). Then* $2n$ *is the smallest period of* (α_i)*, except if* $M = 2m$ *is even and* $q = m+1$ *(see (ii)).*

(vi) *If* $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$*, then* $(\alpha_i) = \alpha(q)$ *is periodic.*
If $M = 2m$ *is even and* $q = m+1$*, then all q-expansions are given by* $(\alpha_i) = m^\infty$ *and the sequences*

$$m^N m^+ 0^\infty \quad \text{and} \quad m^N m^- M^\infty, \quad N = 0, 1, \ldots.$$

Otherwise all q-expansions are given by (α_i) *and the sequences*

$$(\alpha_1 \cdots \alpha_{2n})^N (\alpha_1 \cdots \alpha_{2n})^+ 0^\infty, \quad N = 0, 1, \ldots$$

and

$$(\alpha_1 \cdots \alpha_{2n})^N (\alpha_1 \cdots \alpha_n)^- M^\infty, \quad N = 0, 1, \ldots,$$

where $2n$ *is the smallest period of* (α_i) *(see (v)).*

4. Preliminary results on $\mathcal{U}_q$ and $\mathcal{V}_q$

Given a base $q \in (1, M+1]$, there is a very useful lexicographic characterization of $\mathcal{U}_q'$, essentially obtained in [21,12,18]:

Lemma 4.1. *Let $q \in (1, M+1]$ and $(\alpha_i) = \alpha(q)$. A sequence $(c_i) \in A^{\mathbb{N}}$ belongs to $\mathcal{U}_q'$ if and only if the following two conditions are satisfied:*

$$c_{n+1}c_{n+2}\cdots < \alpha_1\alpha_2\cdots \quad \textit{whenever} \quad c_n < M, \tag{4.1}$$

$$\overline{c_{n+1}c_{n+2}\cdots} < \alpha_1\alpha_2\cdots \quad \textit{whenever} \quad c_n > 0. \tag{4.2}$$

Furthermore, these inequalities hold whenever $c_1\cdots c_n \neq M^n$ and $c_1\cdots c_n \neq 0^n$, respectively.

Proof. The sequence (c_i) is the unique expansion of a number $x \in J_q$ if and only if it is both the greedy and the lazy expansion of x. Moreover, (c_i) is the lazy expansion of x if and only if $\overline{(c_i)}$ is the greedy expansion of $\ell(x)$. Proposition 2.5 (i) yields at once the desired characterization of $\mathcal{U}_q'$. $\square$

Note that sequences satisfying (4.1) and (4.2) are always doubly infinite.

Motivated by Lemma 4.1, it will be convenient to change our definition of $\mathcal{V}_q'$ and $\mathcal{V}_q$:

Definition 4.2. Let $q \in (1, M+1]$ and $\alpha(q) = (\alpha_i)$. We denote by $\mathcal{V}_q'$ the set of *infinite* sequences $(c_i) \in A^{\mathbb{N}}$ satisfying the following two conditions:

$$c_{n+1}c_{n+2}\cdots \le \alpha_1\alpha_2\cdots \quad \text{whenever} \quad c_n < M, \tag{4.3}$$

$$\overline{c_{n+1}c_{n+2}\cdots} \le \alpha_1\alpha_2\cdots \quad \text{whenever} \quad c_n > 0, \tag{4.4}$$

and we define

$$\mathcal{V}_q := \left\{\pi_q((c_i)) \ : \ (c_i) \in \mathcal{V}_q'\right\}.$$

We will show later in Theorem 7.2 that Definition 4.2 is equivalent to the former one, given in the introduction. Note that sequences satisfying (4.3) and (4.4) are always doubly infinite, unless $q = M+1$. Moreover, if $(c_i) \in \mathcal{V}_q'$, then by Proposition 2.5 (ii), the inequalities (4.3) and (4.4) hold whenever $c_1\cdots c_n \neq M^n$ and $c_1\cdots c_n \neq 0^n$, respectively.

Lemma 4.3. *If $1 < p < q \le M+1$, then*

$$\mathcal{U}_p' \subseteq \mathcal{U}_q', \quad \mathcal{V}_p' \subseteq \mathcal{V}_q' \quad \textit{and} \quad \mathcal{V}_p' \subseteq \mathcal{U}_q'.$$

Proof. The inclusions $\mathcal{U}_p' \subseteq \mathcal{U}_q'$ and $\mathcal{V}_p' \subseteq \mathcal{V}_q'$ follow from Lemma 4.1, Definition 4.2 and Proposition 2.4 (ii). Moreover, since the map $q \mapsto \alpha(q)$ is *strictly* increasing, the inclusion $\mathcal{V}_p' \subseteq \mathcal{U}_q'$ holds as well. $\square$

The sets $\mathcal{U}_p'$ and $\mathcal{V}_p'$ only have a nontrivial structure if $p \in (\tilde{q}, M+1)$ and $p \in [\tilde{q}, M+1)$ respectively. Here $\tilde{q}$ is the generalized Golden ratio whose exact value (depending on M) is given in Theorem 3.3 (ii). We will illustrate this in the following examples. The results of Examples 4.4 (ii), (iii) and (iv) are already established by Baker ([3]). We provide short alternative proofs of these results by refining an idea contained in the proof of Theorem 3 in [12].

Examples 4.4.

(i) First, let $M = 2m$ and $q = m+1 = \tilde{q}$ for some $m \geq 1$, so that $\alpha(q) = m^\infty$. It follows from Definition 4.2 that

$$\mathcal{V}'_{\tilde{q}} = \{0^\infty, M^\infty\} \cup \{km^\infty \; : 0 < k < M\} \cup \left\{0^n l m^\infty, M^n \bar{l} m^\infty \; : \; n \geq 1, \; 1 \leq l \leq m\right\}.$$

If $M = 2m-1$ and $q = \tilde{q}$ for some $m \geq 1$, then $\alpha(q) = (mm^-)^\infty$. Definition 4.2 yields in this case that

$$\begin{aligned}\mathcal{V}'_{\tilde{q}} = \{0^\infty, M^\infty\} &\cup \left\{k(mm^-)^\infty, k(m^-m)^\infty \; : 0 < k < M\right\} \\ &\cup \left\{0^n l(mm^-)^\infty, 0^n r(m^-m)^\infty \; : n \geq 1, 1 \leq l < m, \; 1 \leq r \leq m\right\} \\ &\cup \left\{M^n l(mm^-)^\infty, M^n r(m^-m)^\infty \; : \; n \geq 1, m-1 \leq l < M, \; m \leq r < M\right\}.\end{aligned}$$

In both cases (i.e., for all $M \geq 1$), we infer from Lemma 4.1 that

$$\mathcal{U}'_{\tilde{q}} = \{0^\infty, M^\infty\}.$$

Hence, by Lemma 4.3, $\mathcal{U}_q = \mathcal{V}_q = \{0, M/(q-1)\}$ if $q \in (1, \tilde{q})$ and $\mathcal{V}_q \supseteq \mathcal{U}_q \supsetneq \{0, M/(q-1)\}$ if $q \in (\tilde{q}, M+1]$.

(ii) In the remaining Parts (ii), (iii) and (iv) of this example, $M \geq 1$ and $q \in (1, \tilde{q})$ are fixed but arbitrary, and m is defined as before: $m = (M+1)/2$ if M is odd, and $m = M/2$ if M is even. We deduced in (i) that $\mathcal{U}_q = \{0, M/(q-1)\}$. We now strengthen this result by proving that *each* x in the interior of J_q has $2^{\aleph_0}$ expansions. Let x be a number in the interior of J_q. From Theorem 3.3 (ii) and (vi) we infer that

$$1 = \frac{m-1}{\tilde{q}} + \sum_{j=2}^{\infty} \frac{M}{\tilde{q}^j}.$$

Hence there exists a positive integer $k \geq 2$ such that

$$1 < \frac{m-1}{q} + \frac{M}{q^2} + \cdots + \frac{M}{q^k}. \tag{4.5}$$

Let $n_1, n_2, n_3, \ldots$ be the strictly increasing sequence consisting of positive integers which are *not* multiples of k, i.e., $\{n_r : r \geq 1\} = \mathbb{N} \setminus k\mathbb{N}$. Furthermore, choose k large enough so that the inequalities

$$\sum_{j=1}^{\infty} \frac{m}{q^{kj}} \leq x \quad \text{and} \quad x < \sum_{j=1}^{\infty} \frac{M}{q^{n_j}} \tag{4.6}$$

hold as well. Let $(\delta_i) = \delta_1\delta_2\cdots$ be an arbitrary sequence in $\{0, m\}^{\mathbb{N}}$, and let

$$x' := x - \sum_{j=1}^{\infty} \frac{\delta_j}{q^{kj}}.$$

Note that $x' \geq 0$ by the first inequality of (4.6). We now define recursively an expansion $(\varepsilon_i) = \varepsilon_1\varepsilon_2\cdots$ of x' by applying a variant of the greedy algorithm as follows. If for some $n \geq 1$, the digits $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}$ are already defined (no condition if $n = 1$) and n is not a multiple of k, then let ε_n be the largest digit in the (whole) alphabet A such that

$$\left(\sum_{j=1}^{n-1} \frac{\varepsilon_j}{q^j}\right) + \frac{\varepsilon_n}{q^n} \le x'. \tag{4.7}$$

If n is a multiple of k, then let ε_n be the largest element in $\{0, \ldots, m-1\}$ for which (4.7) holds. By the second inequality of (4.6), there exists an index ℓ which is not a multiple of k such that $\varepsilon_\ell < M$. By (4.5), not a single block $\varepsilon_{r+1} \cdots \varepsilon_{r+k}$ of length k in the sequence $\varepsilon_{\ell+1}\varepsilon_{\ell+2} \cdots$ can be *maximal*, that is, it cannot be the case that both $\varepsilon_{r+i} = m-1$ for i such that $r+i$ is a multiple of k and $\varepsilon_{r+i} = M$ for the remaining indices $r+i$ $(1 \le i \le k)$. Hence there are infinitely many indices s such that

$$x' - \frac{1}{q^s} < \sum_{j=1}^{s} \frac{\varepsilon_j}{q^j} \le x'.$$

Letting $s \to \infty$ along these indices, we see that (ε_i) is indeed an expansion of x'. The expansion (c_i) with $c_i = \varepsilon_i$ if i is not a multiple of k and with $c_{kj} = \delta_j + \varepsilon_{kj}$ $(j \ge 1)$ is an expansion of x. It remains to observe that distinct sequences $(\delta_i) \in \{0, m\}^\infty$ give rise to distinct expansions (c_i) of x because $\delta_j = 0$ if and only if $c_{kj} < m$ $(j \ge 1)$.

(iii) Let $\mathcal{E}_q(x)$ be the set of all possible expansions of $x \in J_q$, and let $\mathcal{E}_q^n(x)$ be the set of all possible prefixes of length n $(n \ge 1)$ of sequences belonging to $\mathcal{E}_q(x)$:

$$\mathcal{E}_q^n(x) := \left\{(c_1, \ldots, c_n) \in A^n : \exists (c_{n+1}, c_{n+2}, \ldots) \in A^{\mathbb{N}} \text{ so that } (c_i) \in \mathcal{E}_q(x)\right\}.$$

Finally, let $\mathcal{N}_n(x,q) := |\mathcal{E}_q^n(x)|$. The analysis of (ii) can be used to show that $\mathcal{N}_n(x,q)$ grows exponentially fast as a function of n for x in the interior of J_q, in the rather strong sense that there exists a constant $c = c(q, M) > 0$ that does not depend on x so that for *each* x in the interior of J_q, the inequality

$$\liminf_{n\to\infty} \frac{\log_{M+1}(\mathcal{N}_n(x,q))}{n} \ge c$$

holds. Indeed, fix x in the interior of J_q, and let (c_i) and (d_i) be two distinct expansions of x. Thanks to (ii), we can assume that (c_i) and (d_i) do not end with M^∞. If $r = r(x)$ is the smallest index such that $c_r \ne d_r$, then $c_r c_{r+1} c_{r+2} \cdots$ and $d_r d_{r+1} d_{r+2} \cdots$ must be expansions of a number belonging to

$$[L, R) := \left[\frac{1}{q}, \frac{M-1}{q} + \frac{M}{q\,(q-1)}\right).$$

Hence, it is sufficient to show that the assertion holds for each x in $[L, R)$. It follows from the proof of (ii) that one may take $c = \log 2/(k \cdot \log(M+1))$, where k is the smallest positive integer such that (4.5) and the two inequalities

$$\sum_{j=1}^{\infty} \frac{m}{q^{kj}} \le \frac{1}{q} \quad \text{and} \quad \frac{M-1}{q} + \frac{M}{q\,(q-1)} \le \sum_{j=1}^{\infty} \frac{M}{q^{n_j}} \tag{4.8}$$

hold, where, as before, $n_1, n_2, n_3, \ldots$ is the strictly increasing sequence of all positive integers which are not multiples of k.

(iv) The Tychonov product topology on $A^{\mathbb{N}}$ is induced by the metric d defined as follows:

$$d((c_i),(d_i)) := \begin{cases} (M+1)^{-n} & \text{if } (c_i) \ne (d_i) \text{ and } n \text{ is the first index such that } c_n \ne d_n, \\ 0 & \text{if } (c_i) = (d_i). \end{cases}$$

We want to show that in the metric space $\left(A^{\mathbb{N}}, d\right)$, the Hausdorff dimension of the subset $\mathcal{E}_q(x)$ is, for *each* x in the interior of J_q, bounded below by the same constant $c(q, M)$ that we found in (iii). We may (and will) again assume that x belongs to the interval $[L, R)$ that we defined in (iii).

Let k be a positive integer satisfying (4.5) and both inequalities in (4.8), and define the Lipschitz map $f : A^{\mathbb{N}} \to A^{\mathbb{N}}$ by

$$(f(c_i))_n := \begin{cases} 0 & \text{if } n \text{ is not a multiple of } k, \\ 0 & \text{if } n \text{ is a multiple of } k \text{ and } c_n < m, \\ 1 & \text{if } n \text{ is a multiple of } k \text{ and } c_n \geq m. \end{cases}$$

We have seen in Part (ii) that $f\left(\mathcal{E}_q(x)\right) = f\left(A^{\mathbb{N}}\right)$. The bi-Lipschitz map $g : f\left(A^{\mathbb{N}}\right) \to [0, 1]$ given by $g(f(c_i)) = \sum_{n=1}^{\infty} (f(c_i))_n \cdot (M+1)^{-n}$ maps $f\left(A^{\mathbb{N}}\right)$ onto the attractor of the iterated function system that consists of the similarities $T : [0, 1] \to [0, 1]$ and $S : [0, 1] \to [0, 1]$, defined by $T(y) = y \cdot (M+1)^{-k}$ and $S(y) = (M+1)^{-k} + y \cdot (M+1)^{-k}$, $0 \leq y \leq 1$. By Propositions 9.6 and 9.7 in [13], the Hausdorff dimension of this attractor equals the solution of the equation $2 \cdot (M+1)^{-ks} = 1$. Hence

$$\dim_H \left(\mathcal{E}_q(x)\right) \geq \dim_H f\left(\mathcal{E}_q(x)\right) = \dim_H f\left(A^{\mathbb{N}}\right) = \dim_H g\left(f\left(A^{\mathbb{N}}\right)\right) = \frac{\log 2}{k \cdot \log(M+1)}.$$

We infer from Definition 4.2 the following useful characterizations of $\mathcal{V}_q$ which were already obtained in [7] in case $q \in (M, M+1]$:

Lemma 4.5. *Let $(x, q) \in \mathbf{J}$, and write $(\alpha_i) = \alpha(q)$, $(\beta_i) = \beta(q)$ and $(a_i) = a(x, q)$. The following conditions are equivalent:*

$$x \in \mathcal{V}_q; \tag{4.9}$$

$$\overline{a_{n+1}a_{n+2}\cdots} \leq \alpha(q) \quad \textit{whenever} \quad a_n > 0; \tag{4.10}$$

$$\overline{a_{n+1}a_{n+2}\cdots} \leq \beta(q) \quad \textit{whenever} \quad a_n > 0. \tag{4.11}$$

Proof. Assume (4.9). If $(c_i) \in \mathcal{V}'_q$ and $x = \pi_q((c_i))$, then $(c_i) = a(x, q)$ by (4.3) and Proposition 2.5 (ii), and then (4.10) follows from (4.4).

Conversely, if $a(x, q)$ satisfies (4.10), then $(c_i) := a(x, q)$ satisfies (4.4) by (4.10), and (4.3) by Proposition 2.5 (ii). Hence (4.9) is satisfied because $a(x, q)$ is always infinite.

Since $\alpha(q) \leq \beta(q)$, (4.10) implies (4.11). In order to show the converse implication, it suffices to show that if there exists a positive integer n such that

$$a_n > 0 \quad \text{and} \quad \overline{a_{n+1}a_{n+2}\cdots} > \alpha_1\alpha_2\cdots,$$

then there exists also a positive integer m such that

$$a_m > 0 \quad \text{and} \quad \overline{a_{m+1}a_{m+2}\cdots} > \beta_1\beta_2\cdots.$$

If the greedy expansion (β_i) is infinite, then $(\beta_i) = (\alpha_i)$ and we may choose $m = n$. If (β_i) has a last nonzero digit β_k, then $(\alpha_i) = (\alpha_1 \cdots \alpha_k)^{\infty}$ with $\alpha_1 \cdots \alpha_k = (\beta_1 \cdots \beta_k)^{-}$, and thus $\alpha_k < M$. Since we have

$$\overline{a_{n+1}a_{n+2}\cdots} > (\alpha_1 \cdots \alpha_k)^{\infty}$$

by assumption, there exists a nonnegative integer j satisfying

$$\overline{a_{n+1}\cdots a_{n+jk}} = (\alpha_1\cdots\alpha_k)^j \quad \text{and} \quad \overline{a_{n+jk+1}\cdots a_{n+(j+1)k}} > \alpha_1\cdots\alpha_k.$$

Putting $m := n + jk$ it follows that

$$a_m > 0 \quad \text{and} \quad \overline{a_{m+1}\cdots a_{m+k}} \geq \beta_1\cdots\beta_k.$$

It follows from our assumption $\overline{a_{n+1}a_{n+2}\cdots} > \alpha_1\alpha_2\cdots$ that $(\alpha_i) < M^\infty$ and $(a_i) < M^\infty$. By Proposition 2.1, (a_i) has no tail equal to M^∞, so that $\overline{a_{m+k+1}a_{m+k+2}\cdots} > 0^\infty$. We conclude that

$$\overline{a_{m+1}a_{m+2}\cdots} > \beta_1\beta_2\cdots. \quad \square$$

Lemma 4.6. *Let $q \in (1, M+1]$.*

(i) *We have $\mathcal{U}_q \subseteq \mathcal{V}_q$.*
(ii) *The sets $\mathcal{U}_q$ and $\mathcal{V}_q$ are* symmetric *in J_q, i.e., $\ell(\mathcal{U}_q) = \mathcal{U}_q$ and $\ell(\mathcal{V}_q) = \mathcal{V}_q$.*
(iii) *The set $\mathcal{V}_q$ is closed.*

Proof. (i) If $x \in \mathcal{U}_q$, then its unique expansion (c_i) is equal to $a(x, q)$, and (4.10) follows from (4.2).

(ii) The set $\mathcal{U}_q$ is symmetric because (c_i) is an expansion of x if and only if $(M - c_i)$ is an expansion of $\ell(x)$.

If $q \in (1, M+1)$ and $x \in J_q$, then $a(x, q)$ is doubly infinite by Proposition 2.1 (ii), and hence the sequence $\overline{a(x,q)}$ is also infinite. Using (4.10) and applying Proposition 2.5 (ii) hence we infer that $a(\ell(x), q) = \overline{a(x,q)}$, and then $\ell(x) \in \mathcal{V}_q$ by (2.6) and (4.10). If $q = M+1$, then $\mathcal{V}_q = J_q = [0, 1]$ is symmetric.

(iii) We prove that $\mathcal{V}_q$ is closed by showing that its complement in J_q is open. If (a_i) is the quasi-greedy expansion of some $x \in J_q \setminus \mathcal{V}_q$, then there exists an integer $n > 0$ such that

$$a_n > 0 \quad \text{and} \quad \overline{a_{n+1}a_{n+2}\cdots} > \alpha_1\alpha_2\cdots.$$

Let m be such that

$$\overline{a_{n+1}\cdots a_{n+m}} > \alpha_1\cdots\alpha_m, \tag{4.12}$$

and let

$$y = \sum_{i=1}^{n+m} \frac{a_i}{q^i}.$$

According to Lemma 2.7 the greedy expansion of y is given by $a_1\cdots a_{n+m}0^\infty$. Therefore the quasi-greedy expansion of each number $v \in (y, x]$ starts with the block $a_1\cdots a_{n+m}$. It follows from (4.12) that

$$(y, x] \cap \mathcal{V}_q = \varnothing.$$

Since $x \in J_q \setminus \mathcal{V}_q$ is arbitrary and $\mathcal{V}_q$ is symmetric, there also exists a number $z > x$ such that

$$[x, z) \cap \mathcal{V}_q = \varnothing. \quad \square$$

Lemma 4.7. *Let $q \in (1, M+1]$, $(\alpha_i) = \alpha(q)$, and let (b_i) be the greedy expansion of some $x \in J_q$. Suppose that for some $n \geq 1$,*

$$b_n > 0 \quad \textit{and} \quad \overline{b_{n+1}b_{n+2}\cdots} > \alpha_1\alpha_2\cdots .$$

Then:

(i) *there exists a number $z > x$ such that $[x, z] \cap \mathcal{U}_q = \varnothing$ and $(x, z] \cap \mathcal{V}_q = \varnothing$;*
(ii) *if $b_j > 0$ for some $j > n$, then there exists a number $y < x$ such that $[y, x] \cap \mathcal{U}_q = \varnothing$.*

Proof. (i) Choose a positive integer $M > n$ such that

$$\overline{b_{n+1}\cdots b_M} > \alpha_1\cdots\alpha_{M-n}.$$

Applying Lemma 2.8 choose a greedy sequence $(c_i) > (b_i)$ such that $c_1\cdots c_M = b_1\cdots b_M$. Then (c_i) is the greedy expansion of some $z > x$. If (d_i) is the greedy expansion of an element in $[x, z]$ or the quasi-greedy expansion of an element in $(x, z]$, then (d_i) also begins with $b_1\cdots b_M$ and hence

$$d_n > 0 \quad \text{and} \quad \overline{d_{n+1}\cdots d_M} > \alpha_1\cdots\alpha_{M-n}.$$

The assertion follows from Lemmas 4.1 and 4.5.

(ii) Suppose that $b_j > 0$ for some $j > n$. It follows from Lemma 2.7 that $(c_i) := b_1\cdots b_n 0^\infty$ is the greedy expansion of some $y < x$. If (d_i) is the greedy expansion of some element in $[y, x]$, then $(c_i) \le (d_i) \le (b_i)$ and $d_1\cdots d_n = b_1\cdots b_n$. Therefore

$$d_n > 0 \quad \text{and} \quad \overline{d_{n+1}d_{n+2}\cdots} \ge \overline{b_{n+1}b_{n+2}\cdots} > \alpha_1\alpha_2\cdots .$$

Invoking Lemma 4.1 again, we conclude that $[y, x] \cap \mathcal{U}_q = \varnothing$. $\square$

5. Two lemmas on density

The two results of this section are crucial for the proof of our main theorems. Their proofs are based on the construction of special convergent sequences, interesting in themselves.

Let $q \in (1, M + 1]$ and $(\alpha_i) = \alpha(q)$. We recall that

$$\alpha_{n+1}\alpha_{n+2}\cdots \le \alpha_1\alpha_2\cdots \quad \text{for all} \quad n \ge 0. \tag{5.1}$$

Furthermore, if $q \in \overline{\mathcal{U}}$, then

$$\overline{\alpha_{n+1}\alpha_{n+2}\cdots} < \alpha_1\alpha_2\cdots \quad \text{for all} \quad n \ge 0; \tag{5.2}$$

moreover, by Theorem 3.2 (v) there exist arbitrarily large integers m such that

$$\overline{\alpha_{k+1}\cdots\alpha_m} < \alpha_1\cdots\alpha_{m-k}, \quad k = 0, \ldots, m-1. \tag{5.3}$$

We recall from Section 1 that A_q denotes the set of elements $\mathcal{V}_q \setminus \mathcal{U}_q$ having a finite greedy expansion.

Lemma 5.1. *If $q \in \mathcal{V}$, then $|A_q| = \aleph_0$. Furthermore, for each $x \in \mathcal{U}_q$ there exists a sequence (x_i) of elements in A_q such that $x_i \to x$ and $a(x_i, q) \to a(x, q)$. Moreover, one may choose the sequence (x_i) to be increasing if $x \in \mathcal{U}_q \setminus \{0\}$.*

Proof. Let $x \in \mathcal{U}_q \setminus \{0\}$, and denote by (c_i) the unique expansion of x. Since $\overline{c_1 c_2 \cdots} \neq M^\infty$ is greedy, we infer from Lemma 2.8 (i) that there exists a sequence $1 \leq n_1 < n_2 < \cdots$, such that for each $i \geq 1$,

$$c_{n_i} > 0 \quad \text{and} \quad \overline{c_{m+1} \cdots c_{n_i}} < \alpha_1 \cdots \alpha_{n_i - m} \quad \text{if } m < n_i \text{ and } c_m > 0. \tag{5.4}$$

Now consider for each $i \geq 1$ the sequence (b_j^i), given by

$$(b_j^i) = c_1 \cdots c_{n_i} 0^\infty,$$

and define the number x_i by

$$x_i = \sum_{j=1}^{\infty} \frac{b_j^i}{q^j}.$$

Since (c_i) has infinitely many nonzero digits, we have $x_i \uparrow x$. According to Lemma 2.7 the sequence (b_j^i) is the finite greedy expansion of the number x_i, $i \geq 1$. Moreover, the increasing sequence $(x_i)_{i \geq 1}$ converges to x as i goes to infinity. We claim that $x_i \in A_q$ for each $i \geq 1$. Note that $x_i \notin \mathcal{U}_q$ because the quasi-greedy sequence (a_j^i), given by

$$(c_1 \cdots c_{n_i})^- \alpha_1 \alpha_2 \cdots ,$$

is another expansion of x_i. By Lemma 4.5 it remains to prove that

$$a_j^i > 0 \Longrightarrow \overline{a_{j+1}^i a_{j+2}^i \cdots} \leq \alpha_1 \alpha_2 \cdots . \tag{5.5}$$

If $j < n_i$ and $a_j^i > 0$, then

$$\overline{a_{j+1}^i \cdots a_{n_i}^i} = \overline{(c_{j+1} \cdots c_{n_i})^-} \leq \alpha_1 \cdots \alpha_{n_i - j}$$

by (5.4), and

$$\overline{a_{n_i+1}^i a_{n_i+2}^i \cdots} = \overline{\alpha_1 \alpha_2 \cdots} \leq \alpha_{n_i - j+1} \alpha_{n_i - j+2} \cdots$$

by Proposition 3.1. If $j \geq n_i$, then (5.5) follows directly from Proposition 3.1. Finally, note that $q^{-n} \to 0$ as $n \to \infty$, and that $q^{-n} \in A_q$ for each $n \in \mathbb{N}$ because $\alpha(q^{-n}) = 0^n \alpha(q)$. Since there are only countably many finite expansions, we conclude that $|A_q| = \aleph_0$. □

Lemma 5.2. *If $q \in \overline{\mathcal{U}}$, then for each $x \in A_q$ there exists a sequence (x_i) of elements in $\mathcal{U}_q$ such that $x_i \uparrow x$.*

Proof. Let $x \in A_q$. If b_n is the last nonzero element of $(b_i) = b(x, q)$, then

$$a(x, q) = (a_i) = (b_1 \cdots b_n)^- \alpha_1 \alpha_2 \cdots .$$

Since $q \in \overline{\mathcal{U}}$ by assumption, there exists a sequence $1 \leq m_1 < m_2 < \cdots$ such that (5.3) is satisfied with $m = m_i$ for all $i \geq 1$. We may assume that $m_1 \geq n$ and $m_{i+1} \geq 2m_i$ for each $i \geq 1$. Consider for each $i \geq 1$ the sequence (b_j^i), given by

$$(b_j^i) = (b_1 \cdots b_n)^- (\alpha_1 \cdots \alpha_{m_i} \overline{\alpha_1 \cdots \alpha_{m_i}})^\infty,$$

and define the number x_i by

$$x_i = \sum_{j=1}^{\infty} \frac{b_j^i}{q^j}.$$

Note that the sequence (x_i) converges to x as i goes to infinity. Next we show that $x_i \in \mathcal{U}_q$ for all $i \geq 1$. According to Lemma 4.1 it suffices to verify that

$$b_{m+1}^i b_{m+2}^i \cdots < \alpha_1 \alpha_2 \cdots \quad \text{whenever} \quad b_m^i < M \tag{5.6}$$

and

$$\overline{b_{m+1}^i b_{m+2}^i \cdots} < \alpha_1 \alpha_2 \cdots \quad \text{whenever} \quad b_m^i > 0. \tag{5.7}$$

According to (5.2) we have

$$\overline{\alpha_{m_i+1} \cdots \alpha_{2m_i}} \leq \alpha_1 \cdots \alpha_{m_i}.$$

Note that this inequality cannot be an equality, for otherwise it would follow from Theorem 3.3 (v) that

$$(\alpha_i) = (\alpha_1 \cdots \alpha_{m_i} \overline{\alpha_1 \cdots \alpha_{m_i}})^{\infty}.$$

However, this sequence does not satisfy (5.2) for $k = m_i$. Therefore

$$\overline{\alpha_{m_i+1} \cdots \alpha_{2m_i}} < \alpha_1 \cdots \alpha_{m_i}$$

or equivalently

$$\overline{\alpha_1 \cdots \alpha_{m_i}} < \alpha_{m_i+1} \cdots \alpha_{2m_i}. \tag{5.8}$$

If $m \geq n$, then (5.6) and (5.7) follow from (5.1), (5.3) and (5.8). Now we verify (5.6) and (5.7) for $m < n$. Fix $m < n$. If $b_m^i < M$, then

$$b_{m+1}^i \cdots b_n^i = (b_{m+1} \cdots b_n)^- < b_{m+1} \cdots b_n \leq \alpha_1 \cdots \alpha_{n-m},$$

where the last inequality follows from the fact that (b_i) is a greedy expansion and $b_m = b_m^i < M$. Hence

$$b_{m+1}^i b_{m+2}^i \cdots < \alpha_1 \alpha_2 \cdots .$$

Suppose that $b_m^i = a_m > 0$. Since

$$\overline{a_{m+1} a_{m+2} \cdots} \leq \alpha_1 \alpha_2 \cdots$$

by Lemma 4.5, and $b_{m+1}^i \cdots b_n^i = a_{m+1} \cdots a_n$, it suffices to verify that

$$\overline{b_{n+1}^i b_{n+2}^i \cdots} < \alpha_{n-m+1} \alpha_{n-m+2} \cdots .$$

This is equivalent to

$$\overline{\alpha_{n-m+1} \alpha_{n-m+2} \cdots} < (\alpha_1 \cdots \alpha_{m_i} \overline{\alpha_1 \cdots \alpha_{m_i}})^{\infty}. \tag{5.9}$$

Since $n \leq m_i$ for all $i \geq 1$, we infer from (5.3) that

$$\overline{\alpha_{n-m+1}\cdots\alpha_{m_i}} < \alpha_1\cdots\alpha_{m_i-(n-m)},$$

and (5.9) follows. It follows from (5.8) and the assumption $m_{i+1} \geq 2m_i$ $(i \geq 1)$ that the sequence (x_i) is strictly increasing. □

6. Proof of Theorem 1.2 and Proposition 1.3

First we prove the relation (1.2) between A_q and B_q as stated in Proposition 1.3 (ii):

Lemma 6.1. *If $q \in \overline{\mathcal{U}} \setminus \{M+1\}$, then*

$$B_q = \ell(A_q),$$

and the greedy expansion of each $x \in B_q$ ends with $\overline{\alpha(q)}$.

Proof. First suppose that $x \in A_q$ has a finite greedy expansion (b_i) with a last nonzero element b_n. Since $q \neq M+1$, the sequence

$$(c_i) := \overline{(b_1\cdots b_n)^-\alpha_1\alpha_2\cdots}$$

is an infinite expansion of $\ell(x)$. In order to show that (c_i) is the greedy expansion of $\ell(x)$, we verify that (c_i) satisfies the inequalities (2.5) of Proposition 2.5. By Proposition 3.1 it is sufficient to show that if $1 \leq k < n$ and $b_k > 0$, then

$$\overline{(b_{k+1}\cdots b_n)^-\alpha_1\alpha_2\cdots} < \alpha_1\alpha_2\cdots .$$

Since $x \in \mathcal{V}_q$ and $a(x,q) = (b_1\cdots b_n)^-\alpha_1\alpha_2\cdots$ by Lemma 2.6 (ii), we have (see (4.10))

$$\overline{(b_{k+1}\cdots b_n)^-\alpha_1\alpha_2\cdots} \leq \alpha_1\alpha_2\cdots .$$

We cannot have an equality here because it would imply $\overline{\alpha_{n-k+1}\alpha_{n-k+2}\cdots} = \alpha_1\alpha_2\cdots$, contradicting $q \in \overline{\mathcal{U}}$ (see Proposition 3.1 again). It follows from the symmetry of $\mathcal{U}_q$ and $\mathcal{V}_q$ (see Lemma 4.6) that $\ell(x) \in B_q$.

Conversely, suppose that $x \in B_q$ and let (b_i) be its infinite greedy expansion. Since $x \notin \mathcal{U}_q$, there exists a *smallest* positive integer n for which

$$b_n > 0 \quad \text{and} \quad \overline{b_{n+1}b_{n+2}\cdots} \geq \alpha_1\alpha_2\cdots . \tag{6.1}$$

By Lemma 4.5, we have necessarily $b_{n+1}b_{n+2}\cdots = \overline{\alpha_1\alpha_2\cdots}$. We claim that

$$(c_i) = \overline{(b_1\cdots b_n)^-}0^\infty$$

is the greedy expansion of $\ell(x) \in A_q$. We have to show that if $1 \leq k < n$ and $b_k > 0$, then

$$\overline{(b_{k+1}\cdots b_n)^-}0^\infty < \alpha_1\alpha_2\cdots .$$

Since (α_i) has infinitely many nonzero digits, it suffices to show that

$$\overline{b_{k+1}\cdots b_n} < \alpha_1\cdots\alpha_{n-k}.$$

By the minimality of n we have

$$\overline{b_{k+1}b_{k+2}\cdots} < \alpha_1\alpha_2\cdots .$$

Hence

$$\overline{b_{k+1}\cdots b_n} \le \alpha_1\cdots\alpha_{n-k},$$

and it remains to exclude the equality. However, in case of equality we would obtain the impossible relations

$$\alpha_1\alpha_2\cdots > \overline{b_{k+1}b_{k+2}\cdots} = \alpha_1\cdots\alpha_{n-k}\alpha_1\alpha_2\cdots \ge \alpha_1\cdots\alpha_{n-k}\alpha_{n-k+1}\alpha_{n-k+2}\cdots . \quad \square$$

Now we are ready for the proof of Theorem 1.2. The following proof will also establish the stronger properties stated in Proposition 1.3.

Proof of Theorem 1.2. (i) and (ii) Suppose that $q \in \overline{\mathcal{U}} \setminus \{M+1\}$. Since $\mathcal{U}_q \subseteq \mathcal{V}_q$ and $\mathcal{V}_q$ is closed by Lemma 4.6, we have $\overline{\mathcal{U}_q} \subseteq \mathcal{V}_q$. Conversely, it follows from Lemmas 5.2, 6.1 and the symmetry of $\mathcal{U}_q$ and $\mathcal{V}_q$ that $\mathcal{V}_q \setminus \mathcal{U}_q \subseteq \overline{\mathcal{U}_q}$, and this implies $\mathcal{V}_q \subseteq \overline{\mathcal{U}_q}$. We infer from Lemmas 5.1, 6.1 that both A_q and B_q are dense in $\overline{\mathcal{U}_q} = \mathcal{V}_q$, and $|A_q| = |B_q| = \aleph_0$. Therefore $|\mathcal{V}_q \setminus \mathcal{U}_q| = |A_q \cup B_q| = \aleph_0$. If $q = M+1$, the properties stated in Theorem 1.2 and Proposition 1.3 are well known.

(iii) and (iv) Fix $q \in \overline{\mathcal{U}}$ and let (b_i) be the greedy expansion of a number $x \in \mathcal{V}_q \setminus \mathcal{U}_q$. Let n be the *smallest* positive integer for which (6.1) holds and let (d_i) be another expansion of x. Then $(d_i) < (b_i)$, and hence there exists a *smallest* integer $j \ge 1$ for which $d_j < b_j$. First we show that $j \ge n$. Assume on the contrary that $j < n$. Then $b_j > 0$, and by minimality of n we have

$$b_{j+1}b_{j+2}\cdots > \overline{\alpha_1\alpha_2\cdots}.$$

From Proposition 3.1 (ii) we know that $\overline{\alpha_1\alpha_2\cdots}$ is the greedy expansion of $\ell(x)$, and thus

$$\sum_{i=1}^{\infty}\frac{d_{j+i}}{q^i} = b_j - d_j + \sum_{i=1}^{\infty}\frac{b_{j+i}}{q^i} > 1 + \sum_{i=1}^{\infty}\frac{\overline{\alpha_i}}{q^i} = 1 + \frac{M}{q-1} - 1 = \frac{M}{q-1},$$

which is impossible. If $j = n$, then $d_n = b_n^-$, for otherwise we have

$$2 \le \sum_{i=1}^{\infty}\frac{d_{n+i}}{q^i} \le \frac{M}{q-1},$$

which is also impossible. Indeed, the last condition would imply $q \le (M+2)/2$, while $q \in \overline{\mathcal{U}}$ implies $q > \tilde{q} \ge (M+2)/2$ by Theorem 3.3 (ii). Now we distinguish between two cases.

If $j = n$ and

$$\overline{b_{n+1}b_{n+2}\cdots} > \alpha_1\alpha_2\cdots , \tag{6.2}$$

then by Lemma 4.7 and (i) we have $b_r = 0$ for all $r > n$, from which it follows that (d_{n+i}) is an expansion of 1. Hence, if $q \in \mathcal{U}$ and (6.2) holds, then the only expansion of x starting with $(b_1\cdots b_n)^-$ is given by $(c_i) := (b_1\cdots b_n)^-\alpha_1\alpha_2\cdots$. If $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$ and (6.2) holds, then any expansion (c_i) starting with $(b_1\cdots b_n)^-$ is an expansion of x if and only if (c_{n+i}) is one of the expansions listed in Theorem 3.2 (vi).

If $j = n$ and

$$\overline{b_{n+1}b_{n+2}\cdots} = \alpha_1\alpha_2\cdots , \tag{6.3}$$

then

$$\sum_{i=1}^{\infty} \frac{d_{n+i}}{q^i} = 1 + \sum_{i=1}^{\infty} \frac{b_{n+i}}{q^i} = \sum_{i=1}^{\infty} \frac{M}{q^i}.$$

Hence, if (6.3) holds, then the only expansion of x starting with $(b_1 \cdots b_n)^-$ is given by $(b_1 \cdots b_n)^- M^\infty$.

Finally, if $j > n$, then (6.3) holds, for otherwise $b_r = 0$ for all $r > n$ again, and $d_j < b_j$ is impossible. Note that in this case $q \notin \mathcal{U}$, because otherwise (b_{n+i}) is the unique expansion of $\sum_{i=1}^{\infty} \overline{\alpha_i} q^{-i}$ and thus $(d_{n+i}) = (b_{n+i})$ which is impossible due to $j > n$. Hence, if $q \in \mathcal{U}$, then (b_i) is the only expansion of x starting with $b_1 \cdots b_n$. If $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$ and (6.3) holds, then any expansion (c_i) starting with $b_1 \cdots b_n$ is an expansion of x if and only if (c_{n+i}) is one of the conjugates of the expansions listed in Theorem 3.2 (vi). □

7. Proof of Theorems 1.4, 1.5 and Corollaries 1.7, 1.8

Fix $q \in (1, M+1]$ with $(\alpha_i) = \alpha(q)$. We recall from Proposition 2.5 that a sequence $(b_i) \in A^{\mathbb{N}}$ is greedy if and only if

$$b_{n+1} b_{n+2} \cdots < \alpha_1 \alpha_2 \cdots \quad \text{whenever} \quad n \geq 1 \quad \text{and} \quad b_1 \cdots b_n \neq M^n. \tag{7.1}$$

We also recall from Proposition 2.4 that

$$\alpha_{n+1} \alpha_{n+2} \cdots \leq \alpha_1 \alpha_2 \cdots \quad \text{for all} \quad n \geq 0. \tag{7.2}$$

Lemma 7.1. *Suppose that $q \in (1, M+1] \setminus \overline{\mathcal{U}}$. Then*

(i) *a greedy sequence (b_i) cannot end with $\overline{(\alpha_i)}$;*
(ii) *the set $\mathcal{U}_q$ is closed;*
(iii) *each element $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ has a finite greedy expansion.*

Proof. (i) Let $q \in (1, M+1]$, and assume that there exists a greedy sequence in base q ending with $\overline{\alpha_1 \alpha_2 \cdots}$. Then (7.1) implies that the sequence $\overline{\alpha_1 \alpha_2 \cdots}$ itself is also greedy in base q, and therefore

$$\overline{\alpha_{n+1} \alpha_{n+2} \cdots} < \alpha_1 \alpha_2 \cdots \quad \text{whenever} \quad \alpha_n > 0.$$

By Proposition 3.1 (ii) this implies that $q \in \overline{\mathcal{U}}$.

(ii) Let $x \in J_q \setminus \mathcal{U}_q$ and denote the greedy expansion of x in base q by (b_i). According to Lemma 4.1 there exists a positive integer n such that

$$b_n > 0 \quad \text{and} \quad \overline{b_{n+1} b_{n+2} \cdots} \geq \alpha_1 \alpha_2 \cdots.$$

Applying Lemma 4.7 and (i) we conclude that

$$[x, z] \cap \mathcal{U}_q = \varnothing$$

for some number $z > x$. It follows that $\mathcal{U}_q$ is closed from above. Since the set $\mathcal{U}_q$ is symmetric it is closed from below as well.

(iii) Assume on the contrary that $a(x, q) = b(x, q)$ for some $x \in \mathcal{V}_q \setminus \mathcal{U}_q$. Then it would follow from Lemmas 4.1 and 4.5 that for some positive integer n,

$$b_n(x,q) > 0 \quad \text{and} \quad \overline{b_{n+1}(x,q)b_{n+2}(x,q)\cdots} = \alpha_1\alpha_2\cdots,$$

contradicting (i). □

Recall from Proposition 3.1 that the set $\mathcal{V}$ consists of those numbers $q > 1$ for which the quasi-greedy expansion $(\alpha_i) = \alpha(q)$ satisfies

$$\overline{\alpha_{n+1}\alpha_{n+2}\cdots} \le \alpha_1\alpha_2\cdots \quad \text{for all} \quad n \ge 0. \tag{7.3}$$

If $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$, then (α_i) is of the form

$$(\alpha_i) = (\alpha_1 \cdots \alpha_k \overline{\alpha_1 \cdots \alpha_k})^\infty, \tag{7.4}$$

where k is the least positive integer satisfying

$$\overline{\alpha_{k+1}\alpha_{k+2}\cdots} = \alpha_1\alpha_2\cdots. \tag{7.5}$$

In particular, such a sequence is periodic. Note that $\alpha_k > 0$, for otherwise it would follow from (7.3) with $n = k-1$ and (7.4) that

$$M(\alpha_1\cdots\alpha_{k-1}) \le \alpha_1\cdots\alpha_{k-1}0,$$

which is impossible, because $M > 0$ and $\alpha_j \le \alpha_1$ for all $j \ge 1$ by Proposition 2.4 (ii). Any sequence of the form $(M^m 0^m)^\infty$, where m is a positive integer, is infinite and satisfies (7.2) and (7.3) but not (5.2). On the other hand, there are only countably many periodic sequences. Hence the set $\mathcal{V} \setminus \overline{\mathcal{U}}$ is countably infinite.

Now we are ready to prove Theorems 1.4 and 1.5.

Proof of Theorem 1.4. Throughout the proof $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$ is fixed but arbitrary, and k is the least positive integer satisfying (7.5) with $(\alpha_i) = \alpha(q)$.

(i) These are the statements of Lemmas 4.6 (iii) and 7.1 (ii).

(ii) $|\mathcal{V}_q \setminus \mathcal{U}_q| = |A_q| = \aleph_0$, by Lemmas 5.1 and 7.1 (iii). Lemma 5.1 also implies that A_q is dense in $\mathcal{V}_q$. It remains to show that all elements of $\mathcal{V}_q \setminus \mathcal{U}_q$ are isolated points of $\mathcal{V}_q$. Since the greedy expansion $b(x,q)$ of a number $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ is finite, by Lemma 4.7 (i) there exists a number $z > x$ such that $(x,z] \cap \mathcal{V}_q = \varnothing$. Since the sets $\mathcal{U}_q$ and $\mathcal{V}_q$ are symmetric, there also exists a number $y < x$ satisfying

$$[y,x) \cap \mathcal{V}_q = \varnothing.$$

(iii) and (iv) We already know from Lemma 7.1 that each $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ has a finite greedy expansion. It remains to show that each element $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ has exactly $\aleph_0$ expansions. Let $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ and let b_n be the last nonzero element of $(b_i) = b(x,q)$. If $1 \le j < n$ and $b_j = a_j(x,q) > 0$, then

$$\overline{a_{j+1}(x,q)\cdots a_n(x,q)} = \overline{(b_{j+1}\cdots b_n)^-} \le \alpha_1\cdots\alpha_{n-j}$$

by Lemma 4.5. Therefore

$$b_{j+1}\cdots b_n > \overline{\alpha_1\cdots\alpha_{n-j}}. \tag{7.6}$$

Let (d_i) be another expansion of x and let j be the *smallest* positive integer for which $d_j \ne b_j$. Since (b_i) is greedy, we have $d_j < b_j$ and $j \in \{1,\ldots,n\}$. First we show that $j \in \{n-k, n\}$. Assume on the contrary that $j \notin \{n-k, n\}$.

First assume that $n-k<j<n$. Then $b_j>0$ and by (7.6),

$$b_{j+1}\cdots b_n 0^\infty > \overline{\alpha_1\cdots\alpha_{n-j}(\alpha_{n-j+1}\cdots\alpha_k)^- 0^\infty}.$$

Since $(\alpha_1\cdots\alpha_k)^- M^\infty$ is the smallest expansion of 1 in base q (see Theorem 3.3 (vi)), $\overline{(\alpha_1\cdots\alpha_k)^-}0^\infty$ is the greedy expansion of $M/(q-1)-1$, and thus

$$\sum_{i=1}^\infty \frac{d_{j+i}}{q^i} = b_j - d_j + \sum_{i=1}^\infty \frac{b_{j+i}}{q^i} > \frac{M}{q-1},$$

which is impossible.

Next assume that $1\le j<n-k$. Rewriting (7.6) one gets

$$\overline{b_{j+1}\cdots b_n} < \alpha_1\cdots\alpha_{n-j}.$$

If we had

$$\overline{b_{j+1}\cdots b_{j+k}} = \alpha_1\cdots\alpha_k,$$

then

$$\overline{b_{j+k+1}\cdots b_n} < \alpha_{k+1}\cdots\alpha_{n-j}.$$

Hence

$$b_{j+k+1}b_{j+k+2}\cdots > \overline{\alpha_{k+1}\alpha_{k+2}\cdots} = \alpha_1\alpha_2\cdots .$$

Since in this case $b_{j+k}=\overline{\alpha_k}<M$, the last inequality contradicts the fact that (b_i) is a greedy sequence. Therefore

$$\overline{b_{j+1}\cdots b_{j+k}} < \alpha_1\cdots\alpha_k$$

or equivalently

$$b_{j+1}\cdots b_{j+k} \ge \overline{(\alpha_1\cdots\alpha_k)^-}.$$

Since $n>j+k$ and $b_n>0$, it follows that

$$b_{j+1}b_{j+2}\cdots > \overline{(\alpha_1\cdots\alpha_k)^-}0^\infty,$$

which leads to the same contradiction as we encountered in the case $n-k<j<n$. It remains to investigate what happens if $j\in\{n-k,n\}$.

If $j=n-k$, then it follows from (7.6) that

$$b_{n-k+1}\cdots b_n \ge \overline{(\alpha_1\cdots\alpha_k)^-}.$$

Equivalently,

$$b_{n-k+1}b_{n-k+2}\cdots = b_{n-k+1}\cdots b_n 0^\infty \ge \overline{(\alpha_1\cdots\alpha_k)^-}0^\infty,$$

and thus

$$\sum_{i=1}^{\infty} \frac{d_{n-k+i}}{q^i} \geq 1 + \sum_{i=1}^{\infty} \frac{b_{n-k+i}}{q^i} \geq \frac{M}{q-1}, \tag{7.7}$$

where both inequalities in (7.7) are equalities if and only if

$$d_{n-k} = b_{n-k}^-, \quad b_{n-k+1} \cdots b_n = \overline{(\alpha_1 \cdots \alpha_k)^-} \quad \text{and} \quad d_{n-k+1} d_{n-k+2} \cdots = M^\infty.$$

Hence $d_{n-k} < b_{n-k}$ is only possible in case $b_{n-k} > 0$ and $b_{n-k+1} \cdots b_n = \overline{(\alpha_1 \cdots \alpha_k)^-}$.

If $j = n$ and $d_n = b_n^-$, then (d_{n+i}) is one of the expansions of 1 in base q listed in Theorem 3.3 (vi). It follows from Theorem 3.3 (ii) that $\tilde{q} \geq (M+2)/2$ and that $\tilde{q} = (M+2)/2$ if and only if M is even. Since $q \geq \tilde{q}$, we have $M/(q-1) \leq 2$ and $M/(q-1) = 2$ if and only if $q = \tilde{q}$ and M is even. Hence, if M is even, $q = \tilde{q}$ and $b_n \geq 2$, the number x has one more expansion, namely $(b_1 \cdots b_{n-1})(b_n - 2)M^\infty$. □

Proof of Theorem 1.5. Fix $q \in (1, M+1] \setminus \mathcal{V}$. Since $\mathcal{U}_q \subseteq \overline{\mathcal{U}_q} \subseteq \mathcal{V}_q$ and every $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ has a finite greedy expansion by Lemma 7.1 (iii), the required relation $\mathcal{U}_q = \overline{\mathcal{U}_q} = \mathcal{V}_q$ will follow if we show that a number $x \in J_q$ with a finite greedy expansion does *not* belong to $\mathcal{V}_q$.

Let $x \in J_q$ be an element with a finite greedy expansion. Since $q \notin \mathcal{V}$, by Proposition 3.1 (iii) there exists a positive integer n such that

$$\alpha_n > 0 \quad \text{and} \quad \overline{\alpha_{n+1}\alpha_{n+2}\cdots} > \alpha_1 \alpha_2 \cdots .$$

Since $a(x,q)$ ends with $\alpha_1 \alpha_2 \cdots$, $x \notin \mathcal{V}_q$ by Lemma 4.5. □

We recall that in Sections 4–7 we have used Definition 4.2 of $\mathcal{V}_q$ and $\mathcal{V}_q'$ for $q \in (1, M+1]$. Part (iii) of our next theorem shows that this definition is equivalent to the earlier one given in the introduction:

Theorem 7.2. *For $(x,q) \in \mathbf{J}$ we have the following equivalences:*

(i) *$x \in \mathcal{U}_q$ if and only if x has a unique expansion.*
(ii) *$x \in \overline{\mathcal{U}_q}$ if and only if at least one of x and $M/(q-1) - x$ has a unique infinite expansion.*
(iii) *$x \in \mathcal{V}_q$ if and only if x has at most one doubly infinite expansion.*

Proof. (i) This is the definition of $\mathcal{U}_q$.

(iii) If $q = M+1$, then $\mathcal{V}_q = J_q = [0,1]$ and each number has a unique infinite expansion, hence at most one doubly infinite expansion. Henceforth we assume that $q \in (1, M+1)$. Suppose that $x \in \mathcal{V}_q$. If $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ and $q \in \mathcal{V}$, then by checking the list of expansions of x in Proposition 1.3 (iii) and (iv) and Theorem 1.4 (iv) we see that x has precisely one doubly infinite expansion. In all other cases we have $x \in \mathcal{U}_q$ by Theorem 1.5, so that x has a unique expansion.

Conversely, assume that $x \in J_q$ has at most one doubly infinite expansion. Since $q < M+1$, by Proposition 2.1 (ii) the quasi-greedy expansion $(a_i) := a(x,q)$ is doubly infinite, so that it is the unique doubly infinite expansion of x.

If (a_i) is also the lazy expansion of x, then its reflection is a greedy sequence, so that in particular

$$\overline{a_{n+1}a_{n+2}\cdots} \leq \alpha_1 \alpha_2 \cdots \quad \text{whenever} \quad a_n > 0 \tag{7.8}$$

by Proposition 2.5 (i), where $(\alpha_i) = \alpha(q)$. Hence $x \in \mathcal{V}_q$ by Definition 4.2.

Otherwise, the lazy expansion (c_i) of x is not doubly infinite, and, since it is infinite by Lemma 2.3, it has a last digit $c_k < M$. Then

$$(c_1 \cdots c_k)^+ \overline{\alpha_1 \alpha_2 \dots}$$

is also an expansion of x. Since this expansion ends with the reflection of (α_i), it is doubly infinite by Proposition 2.1 (ii). By our hypothesis it coincides with (a_i):

$$(a_i) = (c_1 \cdots c_k)^+ \overline{\alpha_1 \alpha_2 \dots}. \tag{7.9}$$

(Incidentally, (7.9) implies by Proposition 2.5 (ii) and Proposition 3.1 (iii) that $q \in \mathcal{V}$, but we do not need this in the sequel.) In view of Lemma 4.5 it remains to check the condition (7.8). Consider an index n such that $a_n > 0$.

If $n < k$, then $c_n = a_n > 0$, and therefore $(\overline{c_{n+i}}) < (\alpha_i)$ because (c_i) is lazy. Hence $\overline{c_{n+1} \cdots c_k} \le \alpha_1 \dots \alpha_{k-n}$, and using (7.9) the condition (7.8) follows:

$$\overline{a_{n+1} \dots a_k} = \overline{(c_{n+1} \cdots c_k)^+} < \overline{c_{n+1} \cdots c_k} \le \alpha_1 \cdots \alpha_{k-n}.$$

The case $n = k$ is obvious because then $(\overline{a_{n+i}}) = (\alpha_i)$. Finally, in case $n > k$ we argue as follows:

$$a_n > 0 \Longrightarrow \overline{\alpha_{n-k}} > 0 \Longrightarrow \alpha_{n-k} < M \Longrightarrow (\alpha_{n-k+i}) \le (\alpha_i) \Longrightarrow (\overline{a_{n+i}}) \le (\alpha_i).$$

(ii) For $q = M + 1$ the equivalence follows by observing that every $x \in \overline{\mathcal{U}_q} = J_q = [0, 1]$ has a unique infinite expansion. Henceforth we assume that $q \in (1, M + 1)$.

If $x \in \mathcal{U}_q$, then x has a unique expansion which must be infinite. If $x \in \overline{\mathcal{U}_q} \setminus \mathcal{U}_q$, then $q \in \overline{\mathcal{U}}$, and the lists in Proposition 1.3 (iii) and (iv) show again that x has a unique infinite expansion if and only if $x \in A_q$. By Lemma 6.1 exactly one of the numbers x and $M/(q-1) - x$ has a unique infinite expansion if $x \in \overline{\mathcal{U}_q} \setminus \mathcal{U}_q$.

Conversely, assume that either x or $M/(q-1) - x$ has a unique infinite expansion in base q (or both). Then x also has a unique doubly infinite expansion and therefore $x \in \mathcal{V}_q$ by the already proved Part (iii) of the present theorem. If $q \notin \mathcal{V} \setminus \overline{\mathcal{U}}$, then we conclude by noting that $\overline{\mathcal{U}_q} = \mathcal{V}_q$ in this case. If $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$, then $x \notin \mathcal{V}_q \setminus \mathcal{U}_q$. Indeed, if $q \in \mathcal{V} \setminus \overline{\mathcal{U}}$, then for every $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ both x and $M/(q-1) - x$ have infinitely many infinite expansions by Theorem 1.4 (iv), contradicting our assumption. Therefore $x \in \mathcal{U}_q = \overline{\mathcal{U}_q}$. □

Proof of Corollary 1.7. If $q \in \overline{\mathcal{U}}$, then $\mathcal{U}_q$ is not closed by Theorem 1.2 (i), (ii). If $q \in (1, M + 1] \setminus \overline{\mathcal{U}}$, then $\mathcal{U}_q$ is closed by Theorems 1.4 (i) and 1.5. □

Proof of Corollary 1.8. The relation $q \in \mathcal{U} \Longleftrightarrow 1 \in \mathcal{U}_q$ is evident and the relation $q \in \mathcal{V} \Longleftrightarrow 1 \in \mathcal{V}_q$ follows from Proposition 3.1 (iii) and Lemma 4.5. It remains to prove the relation $q \in \overline{\mathcal{U}} \Longleftrightarrow 1 \in \overline{\mathcal{U}_q}$. If $q \in \overline{\mathcal{U}}$, then

$$\alpha_n > 0 \Longrightarrow (\overline{\alpha_{n+i}}) < (\alpha_i)$$

by Proposition 3.1 (ii). In particular, we have

$$\alpha_n > 0 \Longrightarrow (\overline{\alpha_{n+i}}) \le (\alpha_i),$$

so that $1 \in \mathcal{V}_q$ by Lemma 4.5. We conclude by recalling from Theorem 1.2 (i) that $\mathcal{V}_q = \overline{\mathcal{U}_q}$. If $q \in (1, M + 1] \setminus \overline{\mathcal{U}}$, then $q \notin \mathcal{U}$, hence $1 \notin \mathcal{U}_q$ and thus $1 \notin \overline{\mathcal{U}_q}$ because $\mathcal{U}_q$ is closed by Lemma 7.1 (ii). □

8. Proof of Theorems 1.9, 1.10, 1.12 and 1.13

We recall from Lemma 4.3 the following inclusions:

$$\mathcal{U}'_p \subseteq \mathcal{U}'_q, \quad \mathcal{V}'_p \subseteq \mathcal{V}'_q \quad \text{and} \quad \mathcal{V}'_p \subseteq \mathcal{U}'_q \quad \text{for all} \quad 1 < p < q \leq M+1. \tag{8.1}$$

In the following lemma we exhibit some cases where these inclusions are not strict. For convenience, we define $\mathcal{V}'_1 := \{0^\infty, M^\infty\}$.

Lemma 8.1. *If (q_1, q_2) is a connected component of $(1, M+1] \setminus \mathcal{V}$, then*

$$\mathcal{U}'_q = \mathcal{V}'_{q_1} \quad \textit{for all} \quad q \in (q_1, q_2],$$

and

$$\mathcal{V}'_q = \mathcal{U}'_{q_2} \quad \textit{for all} \quad q \in [q_1, q_2).$$

Proof. The case $(q_1, q_2) = (1, \tilde{q})$ follows from Examples 4.4 (i). Henceforth we assume that $q_1 \in \mathcal{V} \setminus \mathcal{U}$; see Theorem 3.3 (iii). Let us write

$$\alpha(q_2) = (\alpha_1 \cdots \alpha_k \overline{\alpha_1 \cdots \alpha_k})^\infty$$

where k is chosen to be minimal; then $\alpha_k > 0$ by Theorem 3.3 (v). Due to (8.1), it is sufficient to show that $\mathcal{U}'_{q_2} \subseteq \mathcal{V}'_{q_1}$. Suppose that a sequence $(c_i) \in A^{\mathbb{N}}$ is univoque in base q_2, i.e.,

$$c_{n+1}c_{n+2} \cdots < (\alpha_1 \cdots \alpha_k \overline{\alpha_1 \cdots \alpha_k})^\infty \quad \text{whenever} \quad c_n < M \tag{8.2}$$

and

$$\overline{c_{n+1}c_{n+2} \cdots} < (\alpha_1 \cdots \alpha_k \overline{\alpha_1 \cdots \alpha_k})^\infty \quad \text{whenever} \quad c_n > 0. \tag{8.3}$$

If $c_n < M$, then by (8.2),

$$c_{n+1} \cdots c_{n+k} \leq \alpha_1 \cdots \alpha_k.$$

If we had

$$c_{n+1} \cdots c_{n+k} = \alpha_1 \cdots \alpha_k,$$

then

$$c_{n+k+1}c_{n+k+2} \cdots < (\overline{\alpha_1 \cdots \alpha_k}\alpha_1 \cdots \alpha_k)^\infty,$$

and by (8.3) (note that in this case $c_{n+k} = \alpha_k > 0$),

$$c_{n+k+1}c_{n+k+2} \cdots > (\overline{\alpha_1 \cdots \alpha_k}\alpha_1 \cdots \alpha_k)^\infty,$$

a contradiction. Hence

$$c_{n+1} \cdots c_{n+k} \leq (\alpha_1 \cdots \alpha_k)^-.$$

Note that $c_{n+k} < M$ in case of equality. It follows by induction that

$$c_{n+1}c_{n+2}\cdots \le ((\alpha_1\cdots\alpha_k)^-)^\infty.$$

Since a sequence (c_i) satisfying (8.2) and (8.3) is infinite, we conclude from Proposition 2.5 (ii) and Theorem 3.3 (iv) and (v) that (c_i) is the quasi-greedy expansion of some x in base q_1. Repeating the above argument for the sequence $\overline{c_1c_2\cdots}$, which is also univoque in base q_2, we conclude from Lemma 4.5 that $(c_i) \in \mathcal{V}'_{q_1}$. Hence $\mathcal{U}'_{q_2} \subseteq \mathcal{V}'_{q_1}$. $\square$

Proof of Theorem 1.9. It follows from Theorems 1.2 and 1.4 and the inclusions (8.1) that

$$\mathcal{V}'_s \subseteq \mathcal{U}'_q \subsetneq \mathcal{V}'_q \subseteq \mathcal{U}'_r \quad \text{if} \quad q \in \mathcal{V} \quad \text{and} \quad 1 < s < q < r \le M+1.$$

Hence the stability intervals $(q_1, q_2]$ of Lemma 8.1 for $\mathcal{U}_q$ and the stability intervals $(1, \tilde{q})$ and $[q_1, q_2)$ with $q_1 \in \mathcal{V}\setminus\mathcal{U}$ for $\mathcal{V}_q$, are maximal. By Theorem 3.3 (iii), these stability intervals for $\mathcal{U}_q$ and $\mathcal{V}_q$ cover $(1, M+1]\setminus\overline{\mathcal{U}}$ and $(1, M+1]\setminus\mathcal{U}$, respectively. We conclude the proof by recalling that $\overline{\mathcal{U}}$ has no interior points, and therefore $\mathcal{U}$ and $\overline{\mathcal{U}}$ do not contain any non-degenerate interval. $\square$

Proof of Theorem 1.10. (i) This is immediate from Theorem 1.2 (i) and Proposition 1.3 (i).

(ii) Since $\mathcal{V}_q$ is a closed set that contains the endpoints of J_q, the components of $J_q \setminus \mathcal{V}_q$ are open intervals (x_L, x_R) indeed, and their endpoints belong to $\mathcal{V}_q$. By Lemmas 4.6 (ii) and 5.1 the elements of $\mathcal{U}_q$ cannot be endpoints, hence the endpoints belong to $\mathcal{V}_q \setminus \mathcal{U}_q = A_q \cup B_q$. Note that $|A_q| = |B_q| = \aleph_0$ by Theorem 1.2 (ii), and Proposition 1.3 (ii).

If $x \in A_q$, then x is a right isolated point of $\mathcal{V}_q$ by Lemma 4.7 (i), and a left accumulation point of $\mathcal{V}_q$ by Lemma 5.2, so that x is a left endpoint x_L but not a right endpoint x_R. Applying Lemma 6.1, we infer that every $x \in B_q$ is a right endpoint x_R but not a left endpoint x_L.

It remains to show that if $b(x_L, q) = b_1\cdots b_n 0^\infty$ with $b_n > 0$, then $b(x_R, q) = b_1\cdots b_n\overline{\alpha(q)}$. First we show that $(d_i) := b_1\cdots b_n\overline{\alpha(q)}$ is a greedy sequence that belongs to $\mathcal{V}'_q \setminus \mathcal{U}'_q$. Since the sequence ends with $b_n\overline{\alpha(q)}$ and $b_n > 0$, it does not belong to $\mathcal{U}'_q$ by Lemma 4.1. Since (d_i) is infinite, it remains to verify that $d_k < M \Longrightarrow (d_{k+i}) < \alpha(q)$ and $d_k > 0 \Longrightarrow \overline{(d_{k+i})} \le \alpha(q)$. These implications hold true if $k \ge n$ because $q \in \overline{\mathcal{U}}$. Suppose that $1 \le k < n$. Assume first that $d_k = b_k < M$. Since $b(x_L, q) = b_1\cdots b_n 0^\infty$ is greedy, we have $b_{k+1}\cdots b_n 0^\infty < \alpha(q)$, and hence $b_{k+1}\cdots b_n \le \alpha_1\cdots\alpha_{n-k}$. It remains to observe that $\overline{\alpha(q)} < (\alpha_{n-k+i})$ by Proposition 3.1 (ii). Now assume that $d_k = b_k > 0$. Since $x_L \in \mathcal{V}_q$ and $a(x_L, q) = (b_1\cdots b_n)^-\alpha(q)$, we have $\overline{(b_{k+1}\cdots b_n)^-\alpha(q)} \le \alpha(q)$ by Lemma 4.5. This implies in particular the required relation $\overline{b_{k+1}\cdots b_n\alpha(q)} \le \alpha(q)$.

Since $b_1\cdots b_n\overline{\alpha(q)} \in \mathcal{V}'_q \setminus \mathcal{U}'_q$ and $a_1(x_R, q)\cdots a_n(x_R, q) \ge b_1\cdots b_n$, it remains to observe that if $(c_i) \in \mathcal{V}'_q$ starts with $b_1\cdots b_n$, then $(c_i) \ge b_1\cdots b_n\overline{\alpha(q)}$, which follows from (4.10) and the fact that $b_n > 0$.

(iii) Since $q \notin \overline{\mathcal{U}}, \mathcal{U}_q$ is closed and contains the endpoints of J_q, so that $J_q\setminus\mathcal{U}_q$ is the union of disjoint open intervals (x_L, x_R). Since $q \in \mathcal{V}\setminus\overline{\mathcal{U}}$, $\mathcal{V}_q \setminus \mathcal{U}_q$ is a discrete set by Theorem 1.4 (ii). The relation (1.3) follows from Lemmas 4.6 and 5.1. The relation (1.4) is the same as the relation between $b(x_L, q)$ and $b(x_R, q)$ in (ii) and can also be proved along the exact same lines, except that we now have to invoke Proposition 3.1 (iii) in place of Proposition 3.1 (ii).

(iv) This follows from Examples 4.4 (i).

(v) We already know from Lemma 8.1 that $\mathcal{U}'_q = \mathcal{V}'_{q_1}$. Write the set $J_q \setminus \mathcal{U}_q = \cup^*(x_L, x_R)$ as a disjoint union of open intervals (x_L, x_R), and define the map $h : J_q \to J_{q_1}$ as follows:

$$\begin{cases} h(x) = \sum_{i=1}^\infty c_i q_1^{-i} & \text{if } x = \sum_{i=1}^\infty c_i q^{-i} \in \mathcal{U}_q, \\ h(x) = \frac{x - x_L}{x_R - x_L}(h(x_R) - h(x_L)) + h(x_L) & \text{if } x \in (x_L, x_R). \end{cases}$$

The map h is strictly increasing by Proposition 2.5. It is clear that h restricted to each closed interval $[x_L, x_R]$ is continuous. It remains to observe that h cannot have a jump discontinuity at an accumulation point of $\mathcal{U}_q$ by Lemmas 2.7 and 2.8. Hence h is a strictly increasing bijection (and therefore a homeomorphism) that maps $J_q \setminus \mathcal{U}_q$ onto $J_{q_1} \setminus \mathcal{V}_{q_1}$. $\square$

Lemma 8.2. *Let $q \in (1, M+1]$. The set $\mathcal{V}'_q$ is a compact subset of $A^{\mathbb{N}}$ if and only if $q \neq M+1$.*

Proof. If $q \neq M+1$, then a sequence $(c_i) \in A^{\mathbb{N}}$ belongs to $\mathcal{V}'_q$ if and only if (4.3) and (4.4) hold. Hence $A^{\mathbb{N}} \setminus \mathcal{V}'_q$ is open, whence $\mathcal{V}'_q$ is closed and thus compact. For $n \geq 1$, the sequence 10^n1^∞ belongs to $\mathcal{V}'_{M+1}$. If $n \to \infty$, then 10^n1^∞ converges to 10^∞ which does not belong to $\mathcal{V}'_{M+1}$, i.e., $\mathcal{V}'_{M+1}$ is not closed. $\square$

Proof of Theorem 1.12. (ia) This is the classical integer base case.

(ib) If $\mathcal{V}_q$ had an interior point, then by Lemma 2.7, $\mathcal{V}_q$ would also have an interior point with a finite greedy expansion, contradicting Lemma 4.7. By Theorem 1.2 (ii), $\mathcal{V}_q \setminus \mathcal{U}_q$ is dense in $\mathcal{V}_q$. Hence every $x \in \mathcal{U}_q$ is an accumulation point of $\mathcal{V}_q$. Since the accumulation points of a set form a closed set, we infer that every $x \in \overline{\mathcal{U}_q} = \mathcal{V}_q$ is an accumulation point of $\mathcal{V}_q$.

(iia) See Examples 4.4 (i); here we have $q_1 = \tilde{q}$.

(iib) Theorem 1.4 (ii), Lemma 8.1 and induction on n show that $\mathcal{U}_q$ is countably infinite for each $q \in (\tilde{q}, q_{KL})$. Suppose that $q \in (q_n, q_{n+1}]$. According to Lemma 5.1, for each element $x \in \mathcal{U}_{q_n}$, there is a sequence (x_i) of elements in $\mathcal{V}_{q_n} \setminus \mathcal{U}_{q_n}$ such that $a(x_i, q_n) \to a(x, q_n)$ as $i \to \infty$. Since $\mathcal{U}'_q = \mathcal{V}'_{q_n}$, all elements of $\pi_q(\mathcal{U}'_{q_n})$ are accumulation points of $\mathcal{U}_q$ and can be approximated arbitrarily closely by elements of $\pi_q(\mathcal{V}'_{q_n} \setminus \mathcal{U}'_{q_n})$. A number $x \in \mathcal{V}_{q_n} \setminus \mathcal{U}_{q_n}$ is isolated in $\mathcal{V}_{q_n}$ by Theorem 1.4 (ii). Lemma 2.2 implies that $\pi_q(a(x, q_n))$ is isolated in $\mathcal{U}_q$ because univoque sequences are in particular greedy and quasi-greedy.

(iiia) The set $\mathcal{V}_{q_0}$ is perfect by Theorem 1.2 and hence consists entirely of condensation points. Suppose that a sequence (c_i) is univoque in base q, and let W be an arbitrary neighborhood of $\sum_{i=1}^{\infty} c_i q^{-i}$. If N is large enough, then each sequence starting with $c_1 \cdots c_N$ is the expansion in base q of a number in W. If (c_i) also belongs to $\mathcal{U}'_{q_0}$, then, since univoque sequences are in particular greedy and quasi-greedy, applying Lemma 2.2 and using the fact that $\mathcal{V}'_{q_0} \setminus \mathcal{U}'_{q_0}$ is countable (see Theorem 1.2), we conclude that there are uncountably many sequences in $\mathcal{U}'_{q_0}$ starting with $c_1 \cdots c_N$. Since, moreover, $\mathcal{U}'_{q_0} \subseteq \mathcal{U}'_q$, each number belonging to $\pi_q(\mathcal{U}'_{q_0})$ is a condensation point of $\mathcal{U}_q$. It follows easily from Lemmas 2.2, 5.2, 6.1 and 8.2 that $\overline{\mathcal{U}'_{q_0}} = \mathcal{V}'_{q_0}$ because $q_0 \in \overline{\mathcal{U}} \setminus \{M+1\}$. Since π_q is continuous and since the condensation points of $\mathcal{U}_q$ form a closed set, each element in $\mathcal{U}_q = \pi_q(\mathcal{V}'_{q_0})$ is a condensation point of $\mathcal{U}_q$. The set $\mathcal{U}_q$ has no interior points because $\mathcal{V}_q$ has none; see the proof of (ib). Hence $\mathcal{U}_q$ is a Cantor set. It follows from Lemma 5.1 that $\pi_q(\mathcal{V}'_{q_0} \setminus \mathcal{U}'_{q_0})$ is dense in $\mathcal{U}_q$.

(iiib) One shows exactly as in (iib) that elements of $\pi_q(\mathcal{V}'_{q_n} \setminus \mathcal{U}'_{q_n})$ are isolated points of $\mathcal{U}_q$ and form a dense subset of $\mathcal{U}_q$. Hence, elements of $\pi_q(\mathcal{U}'_{q_n})$ are accumulation points of $\mathcal{U}_q = \pi_q(\mathcal{V}'_{q_n})$. Since $\pi_q(\mathcal{V}'_{q_0})$ is compact by Lemma 8.2, and since $\mathcal{U}'_{q_n} \setminus \mathcal{V}'_{q_0}$ is countable, elements of $\pi_q(\mathcal{U}'_{q_n} \setminus \mathcal{V}'_{q_0})$ are not condensation points of $\mathcal{U}_q$. Numbers belonging to $\pi_q(\mathcal{V}'_{q_0})$ are condensation points of $\mathcal{U}_q$ as follows from the reasoning in (iiia). $\square$

Proof of Theorem 1.13. Since we have always (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv), it suffices to show that $\mathcal{U}'_q$ is a shift of finite type for every $q \in (1, M+1] \setminus \overline{\mathcal{U}}$, and that $\mathcal{U}'_q$ is not closed if $q \in \overline{\mathcal{U}}$. Fix $q \in (1, M+1] \setminus \overline{\mathcal{U}}$. Consider the connected component (q_1, q_2) of $(1, \infty) \setminus \mathcal{V}$, satisfying $q \in (q_1, q_2]$. Let us write

$$\alpha(q_2) = (\alpha_i) = (\alpha_1 \cdots \alpha_k \overline{\alpha_1 \cdots \alpha_k})^\infty,$$

where k is minimal, and set

$$\mathcal{F} = \left\{ja_1 \cdots a_k \in A^{k+1} \;:\; j < M \quad \text{and} \quad a_1 \cdots a_k \geq \alpha_1 \cdots \alpha_k\right\}.$$

It suffices to show that a sequence $(c_i) \in A^{\mathbb{N}}$ belongs to $\mathcal{U}'_q = \mathcal{U}'_{q_2}$ if and only if

$$c_j \cdots c_{j+k} \notin \mathcal{F} \quad \text{and} \quad \overline{c_j \cdots c_{j+k}} \notin \mathcal{F} \quad \text{for all} \quad j \geq 1. \tag{8.4}$$

It follows from the proof of Lemma 8.1 that $(c_i) \in \mathcal{U}'_{q_2}$ if and only if

$$c_j < M \Longrightarrow c_{j+1} \cdots c_{j+k} \leq (\alpha_1 \cdots \alpha_k)^- \text{ and } c_j > 0 \Longrightarrow \overline{c_{j+1} \cdots c_{j+k}} \leq (\alpha_1 \cdots \alpha_k)^-,$$

and this is equivalent to (8.4).

Finally, if $q \in \overline{\mathcal{U}}$, then $\mathcal{U}'_q$ is not closed as follows from Lemma 5.2. □

Proof of Theorem 1.14. First we show that $\mathbf{V} \subseteq \overline{\mathbf{U}} \cap \mathbf{J}$. Fix $(x, q) \in \mathbf{V}$. If $q = M + 1$, then $(x, q) \in \overline{\mathbf{U}}$ because $\overline{\mathcal{U}_q} = \mathcal{V}_q$. If $1 < q < M + 1$, then by Lemma 4.3, $a(x, q) \in \mathcal{U}'_r$ for every $r \in (q, M + 1]$, so that $\pi_r(a(x, q)) \in \mathcal{U}_r$. Since $\pi_r(a(x, q)) \to \pi_q(a(x, q)) = x$ as $r \downarrow q$, we conclude that $(x, q) \in \overline{\mathbf{U}} \cap \mathbf{J}$.

Since $\mathbf{U} \subseteq \mathbf{V}$, the converse inclusion $\overline{\mathbf{U}} \cap \mathbf{J} \subseteq \mathbf{V}$ will follow if we show that $\mathbf{V}$ is (relatively) closed in $\mathbf{J}$, i.e., if $(x, q) \in \mathbf{J} \setminus \mathbf{V}$, then $(x', q') \notin \mathbf{V}$ for all $(x', q') \in \mathbf{J}$ close enough to (x, q).

Henceforth we assume that $q \in (1, M + 1)$, and write $(\beta_i) = \beta(q)$, $(\beta'_i) = \beta(q')$, $(a_i) = a(x, q)$ and $(a'_i) = a(x', q')$. By Lemma 4.5 there exist two positive integers n and m such that

$$a_n > 0 \quad \text{and} \quad \overline{a_{n+1} \ldots a_{n+m}} > \beta_1 \ldots \beta_m. \tag{8.5}$$

It follows from the definition of quasi-greedy expansions that

$$\frac{a_1}{q} + \cdots + \frac{a_{j-1}}{q^{j-1}} + \frac{a_j^+}{q^j} + \frac{1}{q^{j+m}} > x \quad \text{whenever} \quad a_j < M.$$

Hence, if $(x', q') \in \mathbf{J}$ is sufficiently close to (x, q), then, applying also Lemma 2.2 (i), (iii),

$$\frac{a_1}{q'} + \cdots + \frac{a_{j-1}}{(q')^{j-1}} + \frac{a_j^+}{(q')^j} + \frac{1}{(q')^{j+m}} > x' \text{ whenever } j \leq n + m \text{ and } a_j < M, \tag{8.6}$$

$$\beta'_1 \ldots \beta'_m \leq \beta_1 \ldots \beta_m \quad \text{and} \quad a'_1 \ldots a'_{n+m} \geq a_1 \ldots a_{n+m}. \tag{8.7}$$

Now we distinguish between two cases.

If $a'_1 \ldots a'_{n+m} = a_1 \ldots a_{n+m}$, then we have

$$a'_n > 0 \quad \text{and} \quad \overline{a'_{n+1} \ldots a'_{n+m}} > \beta_1 \ldots \beta_m \geq \beta'_1 \ldots \beta'_m$$

by (8.5) and (8.7). This proves that $(x', q') \notin \mathbf{V}$.

If $a'_1 \ldots a'_{n+m} > a_1 \ldots a_{n+m}$, then let us consider the smallest j for which $a'_j > a_j$. It follows from (8.5), (8.6) and (8.7) that

$$a'_j = a_j^+ > 0 \quad \text{and} \quad \overline{a'_{j+1} \ldots a'_{j+m}} = M^m > \beta_1 \ldots \beta_m \geq \beta'_1 \ldots \beta'_m.$$

Hence $(x', q') \notin \mathbf{V}$ again. □

9. List of principal terminology and notations

- Page 2
  - $\mathbb{N} := \{1, 2, 3, \ldots\}$
  - *alphabet* $A := \{0, 1, \ldots, M\}$
  - *digit*: an element of the alphabet
  - *sequence*: an element of $A^{\mathbb{N}}$
  - *block* or *word*: a finite sequence of digits
  - *conjugate* or *reflection* of a digit, word, or a sequence:

$$\overline{c_i} := M - c_i, \quad \overline{c_1 \cdots c_n} := \overline{c_1} \ \cdots \ \overline{c_n}, \quad \overline{c_1 c_2 \cdots} := \overline{c_1} \ \overline{c_2} \ \cdots$$

  - $w^+ := c_1 \cdots c_{n-1}(c_n + 1)$ if $w = c_1 \cdots c_{n-1} c_n$ and $c_n < M$
  - $w^- := c_1 \cdots c_{n-1}(c_n - 1)$ if $w = c_1 \cdots c_{n-1} c_n$ and $c_n > 0$
  - *lexicographical order between words and sequences*
  - *finite, co-finite, infinite, co-infinite and doubly infinite sequences*
- Page 3
  - *expansion (of a number x in base q over the alphabet A)*:

$$(c_i) \in A^{\mathbb{N}} \text{ satisfying } x = \sum_{i=1}^{\infty} \frac{c_i}{q^i}$$

  - short notation:

$$\pi_q(c) = \pi_q(c_1 c_2 \cdots) := \sum_{i=1}^{\infty} \frac{c_i}{q^i}, \quad c = (c_i) \in A^{\mathbb{N}}$$

    We assume after Page 3 and in the remainder of this list that $1 < q \le M + 1$.
  - $J_q := [0, M/(q-1)]$: the set of numbers having an expansion in base q
  - $b(x, q)$: the *greedy* (or lexicographically largest) expansion of x in base q
  - $a(x, q)$: the *quasi-greedy* (or lexicographically largest infinite) expansion of x in base q
- Page 4
  - $\mathcal{U}$ is the set of *univoque bases*. A base q belongs to $\mathcal{U}$ if $x = 1$ has a unique expansion in base q.
  - $\mathcal{V}$ is the set of bases q in which $x = 1$ has a unique doubly infinite expansion.
  - A *Cantor set* is a nonempty closed set having neither interior nor isolated points.
  - $\overline{\mathcal{U}}$ is the topological closure of $\mathcal{U}$.
  - $\mathcal{U}_q$ is the set of numbers $x \in J_q$ having a unique expansion in base q.
  - $\mathcal{V}_q$ is the set of numbers $x \in J_q$ having at most one doubly infinite expansion in base q.
  - $\overline{\mathcal{U}_q}$ is the topological closure of $\mathcal{U}_q$.
- Page 5
  - A_q is the set of numbers $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ such that $b(x, q)$ is finite.
  - B_q is the set of numbers $x \in \mathcal{V}_q \setminus \mathcal{U}_q$ such that $b(x, q)$ is infinite.
  - $\ell : J_q \to J_q$ is the *reflection map* defined by $\ell(x) = M/(q-1) - x$.
  - The number $\tilde{q} := \min \mathcal{V}$ is the smallest element of $\mathcal{V}$; for $M = 1$, $\tilde{q}$ is the Golden ratio; see also Theorem 3.3.
- Page 6
  - $\mathcal{U}_q' \subseteq A^{\mathbb{N}}$: the set of expansions of the elements of $\mathcal{U}_q$
  - $\mathcal{V}_q' \subseteq A^{\mathbb{N}}$: the set of quasi-greedy expansions of the elements of $\mathcal{V}_q$

- Page 8
  - The *Komornik–Loreti constant* $q_{KL} := \min \mathcal{U}$ is the smallest element of $\mathcal{U}$; see also Theorem 3.2. For $M = 1$, $q_{KL} \approx 1.787$.
- Page 9
  - The two-dimensional analogues of $\mathcal{U}_q$, $\mathcal{V}_q$ and J_q are defined as follows.

$$\mathbf{U} := \left\{(x, q) \in \mathbb{R}^2 \ : \ q \in (1, M+1] \text{ and } x \in \mathcal{U}_q\right\},$$
$$\mathbf{V} := \left\{(x, q) \in \mathbb{R}^2 \ : \ q \in (1, M+1] \text{ and } x \in \mathcal{V}_q\right\},$$
$$\mathbf{J} := \left\{(x, q) \in \mathbb{R}^2 \ : \ q \in (1, M+1] \text{ and } x \in J_q\right\}.$$

- Page 12
  - $\beta(q) := b(1, q)$: greedy expansion of $x = 1$ in base q
  - $\alpha(q) := a(1, q)$: quasi-greedy expansion of $x = 1$ in base q
- Page 14
  - $X \subseteq \mathbb{R}$ is *closed from above (below)* if the limit of every bounded decreasing (increasing) sequence in X belongs to X.

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