Lipschitz-Killing Curvatures for Arithmetic Random Waves

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Abstract

In this paper, we show that the Lipschitz-Killing Curvatures for the excursion sets of Arithmetic Random Waves (toral Gaussian eigenfunctions) are dominated, in the high-frequency regime, by a single chaotic component. The latter can be written as a simple explicit function of the threshold parameter times the centered norm of these random fields; as a consequence, these geometric functionals are fully correlated in the high-energy limit. The derived formulae show a clear analogy with related results on the round unit sphere and suggest the existence of a general formula for geometric functionals of random eigenfunctions on Riemannian manifolds.

Keywords and Phrases: Lipschitz-Killing Curvatures, Arithmetic Random Waves, Wiener chaos, Gaussian Kinematic Formula, Limit Theorems.

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1 Introduction and general framework

1.1 Toral eigenfunctions and Arithmetic Random Waves

Arithmetic Random Waves (i.e., toral Gaussian eigenfunctions) were introduced nearly a decade ago in [ORW08, RW08] and have been investigated very widely ever since, see for instance [GW18, KKW13, MPRW16] and more recently [BMW17, BW16, Cam19, DNPR19, RW16, RWY16]; see also [ADP19, APP18, BCP19] for related results on random trigonometric polynomials. Interest in their investigation is motivated both by mathematical physics applications, and by the rich interplay of probability, geometry and even number theory that characterizes the behaviour of geometric functionals of their excursion sets.

Let us start recalling their definition; for an integer $d \geq 2$, let $f : \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}$ be the real-valued functions satisfying the eigenvalue equation

$$\Delta f + Ef = 0, \tag{1.1}$$

where $E > 0$ and Δ is the Laplace-Beltrami operator on $\mathbb{T}^d$; the spectrum of Δ is totally discrete. (For $d = 2$ we will often write $\mathbb{T}$ in place of $\mathbb{T}^2$.) Indeed, the eigenspaces of the Laplacian on the torus are related to the theory of lattice points on $(d-1)$-dimensional spheres: let

$$S := \{n \in \mathbb{Z} : n = n_1^2 + \cdots + n_d^2, \text{ for some } n_1, \ldots, n_d \in \mathbb{Z}\}$$

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be the collection of all numbers expressible as a sum of d squares. The sequence of eigenvalues, or *energy levels*, are all numbers of the form $E_n = 4\pi^2 n$, $n \in S$. In order to describe the Laplace eigenspace corresponding to E_n, we introduce the set of frequencies Λ_n; for $n \in S_n$ let

$$\Lambda_n = \{\lambda \in \mathbb{Z}^d : \ ||\lambda||^2 = n\}.$$

Λ_n is the frequency set corresponding to E_n. Using the notation $e(t) := \exp(2\pi i t)$ for $t \in \mathbb{R}$, the $\mathbb{C}$-eigenspace $\mathcal{E}_n$ corresponding to E_n is spanned by the L^2-orthonormal set of functions $\{e(\langle \lambda, \cdot \rangle)\}_{\lambda \in \Lambda_n}$. We denote the dimension of $\mathcal{E}_n$

$$\mathcal{N}_n = \dim \mathcal{E}_n = |\Lambda_n|,$$

that is equal to the number of different ways n may be expressed as a sum of d squares. In particular, for $d = 2$, $\mathcal{N}_n$ is subject to large and erratic fluctuations; it grows *on average* [La08] as $\sqrt{\log n}$, but could be as small as 8 for an infinite sequence of prime numbers $p \equiv 1 \pmod 4$, or as large as a power of $\log n$.

The frequency set Λ_n can be identified with the set of lattice points lying on a $(d-1)$-dimensional sphere with radius $\sqrt{n}$, the sequence of spectral multiplicities $\{\mathcal{N}_n\}_{n \in S}$ is unbounded. It is natural to consider properties of *generic* or *random* eigenfunctions $f_n \in \mathcal{E}_n$, in the high-energy asymptotics regime. More precisely, let $f_n : \mathbb{T}^d \to \mathbb{R}$ be the Gaussian random field of (real valued) $\mathcal{E}_n$-functions with eigenvalue E_n, i.e. the random linear combination

$$f_n(x) := \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n} a_\lambda e(\langle \lambda, x \rangle), \quad x \in \mathbb{T}^d, \tag{1.2}$$

where the coefficients $\{a_\lambda\}_{\lambda \in \Lambda_n, n \in S}$ are complex-Gaussian random variables[2] verifying the following properties:

1. every a_λ has the form $a_\lambda = \mathrm{Re}(a_\lambda) + i\,\mathrm{Im}(a_\lambda)$ where $\mathrm{Re}(a_\lambda)$ and $\mathrm{Im}(a_\lambda)$ are two independent real-valued, centred, Gaussian random variables with variance $1/2$,

2. the a_λ's are stochastically independent, save for the relations $a_{-\lambda} = \overline{a}_\lambda$ in particular making f_n real-valued.

By definition, f_n is stationary, i.e. the law of f_n is invariant under all translations

$$f(\cdot) \to f(y + \cdot), \qquad y \in \mathbb{T}^d;$$

in fact f_n is a centred Gaussian random field with covariance function

$$\mathbb{E}[f_n(x) f_n(y)] = \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} e(\langle \lambda, x - y \rangle), \qquad x, y \in \mathbb{T}.$$

Note that the normalizing factor in (1.2) is chosen so that f_n has unit-variance.

[2]defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$

1.2 Notation

We will use $\lambda, \lambda_1, \lambda_2 \dots$ and in general λ_i, $i = 1, 2, \dots$ to denote elements of Λ_n, while $\lambda_{(\ell)}$ and $\lambda_{i,(\ell)}$ with $\ell = 1, \dots, d$, will denote the ℓ-th component of the vectors λ and $\lambda_i \in \Lambda_n$ respectively. The indices j, ℓ always run from 1 to d.

For $\ell = 1, \dots d$, we denote with $\partial_\ell f_n(x)$ the derivative of $f_n(x)$ with respect to x_ℓ. A straightforward differentiation of (1.2) gives

$$\partial_\ell f_n(x) = \frac{2\pi i}{\sqrt{\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n} a_\lambda \lambda_{(\ell)} e(\langle \lambda, x \rangle), \tag{1.3}$$

in view of [RW16, Lemma 2.3], see formula (C.105) below, the random field $\partial_\ell f_n$ has variance

$$\begin{aligned}
\mathrm{Var}(\partial_\ell f_n(x)) &= \frac{2^2\pi^2(-1)}{\mathcal{N}_n} \sum_{\lambda_1, \lambda_2 \in \Lambda_n} \mathbb{E}[a_{\lambda_1} a_{\lambda_2}]\, \lambda_{1,(\ell)}\, \lambda_{2,(\ell)}\, e(\langle \lambda_1, x \rangle)\, e(\langle \lambda_2, x \rangle) \\
&= \frac{2^2\pi^2}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \lambda_{(\ell)}^2 = \frac{2^2\pi^2 n}{d} = \frac{E_n}{d},
\end{aligned}$$

we introduce then the normalized derivative $f_{n,\ell}(x)$ defined by

$$f_{n,\ell}(x) := \frac{\partial_\ell f_n(x)}{2\pi\sqrt{\frac{n}{d}}} = i\sqrt{\frac{d}{n\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n} \lambda_{(\ell)} a_\lambda e(\langle \lambda, x \rangle).$$

Note that $f_{n,\ell}(x)$ is real-valued since $f_{n,\ell}^2(x) = f_{n,\ell}(x)\overline{f_{n,\ell}}(x)$. Analogously, we denote with $\partial_{j\ell}^2 f_n$ the second derivative of f_n with respect to x_ℓ and x_j

$$\partial_{j\ell}^2 f_n(x) = -\frac{4\pi^2}{\sqrt{\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n} a_\lambda \lambda_{(\ell)} \lambda_{(j)} e(\langle \lambda, x \rangle). \tag{1.4}$$

We note that conditions 1) and 2) in (1.2) immediately imply that

$$\mathbb{E}[a_\lambda^2] = \mathbb{E}[(\mathrm{Re}(a_\lambda))^2] - \mathbb{E}[(\mathrm{Im}(a_\lambda))^2] = 0,$$

and that $2|a_\lambda|^2$ has a chi-squared distribution with 2 degrees of freedom:

$$\mathbb{E}[|a_\lambda|^2] = 1, \qquad \mathbb{E}[(|a_\lambda|^2 - 1)^2] = \mathrm{Var}(|a_\lambda|^2) = 1, \qquad \mathbb{E}[|a_\lambda|^4] = 2.$$

For some of the arguments to follow, we shall make a heavy use of results and notation recently introduced in the number theory literature by [KKW13]. In particular, let μ_n be the probability measure on the circle $\mathcal{S}^1 := \{z \in \mathbb{C} : \ ||z|| = 1\}$ defined by

$$\mu_n := \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \delta_{\lambda/\sqrt{n}} \ ,$$

that is to say, the empirical measure for the distribution of integers corresponding to the eigenvalue $4\pi^2 n$; an important role is going to be played by the fourth Fourier coefficient of this distribution, i.e.

$$\hat{\mu}_n(4) := \int_{\mathcal{S}^1} z^4 d\mu_n(z) = \frac{1}{n^2\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} (\lambda_1 + i\lambda_2)^4. \tag{1.5}$$

This coefficient will not appear in our statements, but inspection of the proof reveals its important role.

2 Main results

The purpose of this paper is to provide a full characterization for the asymptotic behaviour in the high-energy limit of Lipschitz-Killing Curvatures computed on the excursion sets of two-dimensional Arithmetic Random Waves defined in (1.2) for $d = 2$, and to compare the results with those recently derived in the case of random spherical eigenfunctions in [CM18]; see also [AT07, AW09, CX16, EL16, FA17, MRW20, NPR19, PV20] and the references therein for background material and a number of recent results on Lipschitz-Killing Curvatures in Euclidean settings or on the unit round sphere.

It is well-known (see e.g., [KKW13]) that there exists a density-1 subsequence $\{n_j\}_j \subset S$ such that for every $x, y \in \mathbb{T}^2$

$$\mathbb{E}[f_n(x/\sqrt{n_j})f_n(y/\sqrt{n_j})] \to J_0(2\pi\|x-y\|), \qquad n_j \to +\infty, \tag{2.6}$$

where J_0 denotes the Bessel function of the first kind of order zero. To fix notation, let us recall first that the excursion sets of f_n are defined by

$$A_u(f_n; \mathbb{T}) := \{x \in \mathbb{T} : f_n(x) \geq u\}, \qquad u \in \mathbb{R}.$$

In the two-dimensional case, it is well-known that the three Lipschitz-Killing Curvatures $\mathcal{L}_k$, $k = 0, 1, 2$ correspond to the area functional $k = 2$, half the boundary length $k = 1$ and the Euler-Poincaré Characteristic $k = 0$, i.e., the number of connected components minus the number of "holes". We are interested in these geometric functionals evaluated at excursion sets of Arithmetic Random Waves: for $u \in \mathbb{R}$,

$$\mathcal{L}_k(n; u) := \mathcal{L}_k(A_u(f_n; \mathbb{T})), \qquad k = 0, 1, 2; \tag{2.7}$$

in particular in their asymptotic behavior as $n \to +\infty$ such that $\mathcal{N}_n \to +\infty$. To the best of our knowledge, all results concerning $\mathcal{L}_0(n; u)$, i.e. the Euler-Poincaré characteristic, are new.

The expected values of (2.7) are given in the following lemma (see Appendix C). As usual, we use ϕ, Φ to denote the standard Gaussian density and distribution function, respectively.

Lemma 2.1. *The expected values for the Lipschitz-Killing Curvatures on excursion sets of Arithmetic Random Waves are given by*

$$\mathbb{E}[\mathcal{L}_k(n; u)] = m_k(u) \left(\sqrt{\frac{E_n}{2}}\right)^{2-k}, \qquad k = 0, 1, 2 \tag{2.8}$$

for $n \in S$ and $u \in \mathbb{R}$, where

$$m_2(u) := 1 - \Phi(u), \quad m_1(u) := \sqrt{\frac{\pi}{8}}\phi(u), \quad m_0(u) := \frac{1}{2\pi}u\phi(u). \tag{2.9}$$

The next Theorem, which is the main result of this paper, provides a full characterization of asymptotic fluctuations (in the high-energy limit) around these expected values. Let us introduce the following notation for centred functionals: for $n \in S$ and $u \in \mathbb{R}$

$$\overline{\mathcal{L}}_k(n; u) := \mathcal{L}_k(n; u) - \mathbb{E}[\mathcal{L}_k(n; u)], \qquad k = 0, 1, 2;$$

we will need also the following subset of the set of frequencies: if n is not a square we set

$$\Lambda_n^+ := \{\lambda \in \Lambda_n : \lambda_{(2)} > 0\},$$

otherwise $\Lambda_n^+ := \{\lambda \in \Lambda_n : \lambda_{(2)} > 0\} \cup \{(\sqrt{n}, 0)\}$. Note that, for every $n \in S$, $\{a_\lambda\}_{\lambda\in\Lambda_n^+}$ are i.i.d. random variables, and $|\Lambda_n^+| = \mathcal{N}_n/2$.

In order to state our main results, we will need some more notation. Let $Q_{0,n} := [0, 1/\sqrt{E_n})^2$, and denote by $\mathcal{L}_0(u; Q_{0,n})$ the Euler-Poincaré characteristic of the intersection between the excursion set $\{f_n \geq u\}$ and the square $Q_{0,n}$.

Condition 2.2. *For $n \in S$*

$$\mathbb{E}[\mathcal{L}_0(u; Q_{0,n})(\mathcal{L}_0(u; Q_{0,n}) - 1)] = O(1), \tag{2.10}$$

where the constant involved in the O-notation is absolute.

Note that Condition 2.2 only concerns the zero-th Lipschitz-Killing curvature.

Remark 2.3. The estimate (2.10) holds for a density-1 subsequence of eigenvalues in the high-energy limit (thanks to (2.6) and [CMW16]), and we do believe (2.10) to be true for every $n \in S$.

Theorem 2.4. *For $k = 0, 1, 2$, $n \in S$ it holds that*

$$\overline{\mathcal{L}}_k(n; u) = \frac{c_k(u)}{\sqrt{\mathcal{N}_n/2}} \left(\sqrt{\frac{E_n}{2}}\right)^{2-k} \frac{1}{\sqrt{\mathcal{N}_n/2}} \sum_{\lambda\in\Lambda_n^+} (|a_\lambda|^2 - 1) + \mathcal{R}_k(n; u), \tag{2.11}$$

where

$$c_2(u) := \frac{1}{2} u\phi(u), \quad c_1(u) := \frac{1}{2}\sqrt{\frac{\pi}{8}} u^2\phi(u), \quad c_0(u) := \frac{1}{2}(u^2 - 1)u\phi(u)\frac{1}{2\pi}, \tag{2.12}$$

and under Condition 2.2

$$\mathbb{E}[\mathcal{R}_k(n; u)^2] = O\left(\frac{E_n^{2-k}}{\mathcal{N}_n^2}\right), \tag{2.13}$$

the constant involved in the O-notation only depending on k.

In particular, Theorem 2.4 (whose proof will be given in §4) shows that the "first order approximation" of $\mathcal{L}_k(n; u)$ for any k can be written as a simple explicit function (depending on k) of the threshold parameter u times the centered norm of f_n. Indeed,

$$\|f_n\|^2_{L^2(\mathbb{T})} - \mathbb{E}\left[\|f_n\|^2_{L^2(\mathbb{T})}\right] = \frac{1}{\mathcal{N}_n} \sum_{\lambda\in\Lambda_n} (|a_\lambda|^2 - 1) = \frac{1}{\mathcal{N}_n/2} \sum_{\lambda\in\Lambda_n^+} (|a_\lambda|^2 - 1), \tag{2.14}$$

cf. (2.11). This has several important consequences, as discussed just below and in §3. Moreover, for any $u \in \mathbb{R}$, $k = 0, 1, 2$, as $n \to +\infty$ such that $\mathcal{N}_n \to +\infty$,

$$\mathrm{Var}(\mathcal{L}_k(n; u)) = \frac{c_k(u)^2}{2^{1-k}} \frac{E_n^{2-k}}{\mathcal{N}_n} + O\left(\frac{E_n^{2-k}}{\mathcal{N}_n^2}\right), \tag{2.15}$$

where the constant involved in the O-notation only depends on u and k. Let us now define $\mathcal{U}_k := \{0\}$ for $k = 1, 2$ and $\mathcal{U}_0 := \{-1, 0, 1\}$; note that $c_k(u)$ defined in (2.12) vanishes if and only if $u \in \mathcal{U}_k$. An easy by-product of Theorem 2.4 is the following quantitative Central Limit Theorem in Wasserstein distance[3] (written d_W).

[3]Given X, Y integrable random variables, $d_W(X, Y) := \sup_{h\in Lip(1)} |\mathbb{E}[h(X)] - \mathbb{E}[h(Y)]|$, where $Lip(1)$ denotes the space of Lipschitz functions $h : \mathbb{R} \to \mathbb{R}$ whose Lipschitz constant is ≤ 1.

Corollary 2.5. *As $n \to \infty$ such that $\mathcal{N}_n \to +\infty$, for $k = 0, 1, 2$, $u \notin \mathcal{U}_k$ under Condition 2.2*

$$d_W\left(\widetilde{\mathcal{L}}_k(n; u), Z\right) = O\left(\frac{1}{\sqrt{\mathcal{N}_n}}\right)$$

where

$$\widetilde{\mathcal{L}}_k(n; u) := \frac{\overline{\mathcal{L}}_k(n; u)}{\sqrt{\operatorname{Var}(\mathcal{L}_k(n; u))}},$$

$Z \sim \mathcal{N}(0, 1)$, and the constant involved in the O-notation only depends on k.

The proof of Corollary 2.5 will be given in §4. Theorem 2.4 also allows to deduce Moderate Deviation estimates [DZ98, §1.2] for Lipschitz-Killing curvatures evaluated at excursion sets of Arithmetic Random Waves, see also [MRT20, Remark 1.9].

Corollary 2.6. *For $k = 0, 1, 2$, $n \in S$, $u \notin \mathcal{U}_k$, let $\{s^{(k)}_{n;u}\}_{n \in S}$ be any sequence of positive numbers such that as $\mathcal{N}_n \to +\infty$*

$$s^{(k)}_{n;u} \to +\infty, \qquad \frac{s^{(k)}_{n;u}}{\sqrt{\log \mathcal{N}_n}} \to 0. \tag{2.16}$$

Under Condition 2.2 the sequence of random variables

$$\left\{\widetilde{\mathcal{L}}_k(n; u)/s^{(k)}_{n;u}\right\}_{n \in S}$$

satisfies a Moderate Deviation principle with speed $(s^{(k)}_{n;u})^2$ and rate function $\mathcal{I}(x) := x^2/2$, $x \in \mathbb{R}$, i.e. for any Borelian set $B \subset \mathbb{R}$

$$\begin{aligned} -\inf_{x \in \mathring{B}} \mathcal{I}(x) &\le \liminf_{\mathcal{N}_n \to +\infty} \frac{1}{(s^{(k)}_{n;u})^2} \log \mathbb{P}\left(\frac{\widetilde{\mathcal{L}}_k(n; u)}{s^{(k)}_{n;u}} \in B\right) \\ &\le \limsup_{\mathcal{N}_n \to +\infty} \frac{1}{(s^{(k)}_{n;u})^2} \log \mathbb{P}\left(\frac{\widetilde{\mathcal{L}}_k(n; u)}{s^{(k)}_{n;u}} \in B\right) \le -\inf_{x \in \bar{B}} \mathcal{I}(x), \end{aligned}$$

where $\mathring{B}$ (resp. $\bar{B}$) denotes the interior (resp. the closure) of B.

Corollary 2.6 is a refinement of the Central Limit Theorem in Corollary 2.5, its proof will be given in §4. A further obvious consequence of Theorem 2.4 is the following asymptotic full correlation result.

Corollary 2.7. *Let $k_1, k_2 \in \{0, 1, 2\}$ and $u_1, u_2 \notin \mathcal{U}_0$. As $n \to \infty$ such that $\mathcal{N}_n \to +\infty$, under Condition 2.2,*

$$\operatorname{Corr}\left(\mathcal{L}_{k_1}(n; u_1), \mathcal{L}_{k_2}(n; u_2)\right) = 1 + O\left(\frac{1}{\sqrt{\mathcal{N}_n}}\right),$$

where the constant involved in the O-notation only depends on k_1 and k_2.

In words, Corollary 2.7 (whose proof will be given in §4) entails that in the "nondegenerate" points where the leading term in the asymptotic variance (2.15) does not vanish knowledge of one of the three Lipschitz-Killing curvatures at some level allows the derivation of the other two at any level, up to a term which is lower order in the $L^2(\mathbb{P})$-sense.

See §3 for further comments on our main result, its consequences and the comparison with the spherical case.

Remark 2.8 (Nodal case)**.** The geometry of Arithmetic Random Waves was initially investigated in [ORW08, RW08] and subsequently in several works with a focus on the nodal case which corresponds to the level $u = 0$. Concerning the (half) nodal length, the asymptotic variance was addressed and fully solved in [KKW13]: as $\mathcal{N}_n \to +\infty$,

$$\mathrm{Var}(\mathcal{L}_1(n;0)) = \frac{1}{4} \cdot \frac{1+\hat{\mu}_n(4)^2}{512} \frac{E_n}{\mathcal{N}_n^2} \left(1+o(1)\right), \tag{2.17}$$

where $\hat{\mu}_n(4)$ has been defined in (1.5). It is well-known that for any $\mu \in [-1,1]$, there exists a sequence of energy levels such that the corresponding sequence of fourth Fourier coefficients converges to μ. The second order fluctuations of the nodal length were investigated in [MPRW16]: as $\mathcal{N}_n \to +\infty$ and $\hat{\mu}_n(4) \to \mu$

$$\widetilde{\mathcal{L}}_1(n;0) := \frac{\overline{\mathcal{L}}_1(n;0)}{\sqrt{\mathrm{Var}(\mathcal{L}_1(n;0))}} \stackrel{d}{\to} \frac{1}{2\sqrt{1+\mu^2}}(2-(1+\mu)Z_1^2+(1-\mu)Z_2^2), \tag{2.18}$$

where Z_1, Z_2 are i.i.d. standard Gaussian random variables. A quantitative Limit Theorem in Wasserstein distance is given in [PR18].

The "signed area" of Arithmetic Random Waves restricted to shrinking balls of radius above the Planck scale has been recently investigated in [KWY20], the Euler-Poincaré characteristic of the excursion set at level zero is currently under investigation.

3 Outline of the paper

3.1 On the proofs

Our approach to proving the results of this paper stated in §2 is broadly analogous to what was used earlier to evaluate the Lipschitz-Killing Curvatures of excursion sets for random eigenfunctions on the sphere or in Euclidean settings, see e.g. [KL01, MW11, MW14, MR15, EL16, CM18, NPR19, DNPR19, PV20] and the references therein. The starting point is to derive the so-called chaotic decomposition of our geometric functionals, that is, for $k = 0,1,2$, $n \in S$ and $u \in \mathbb{R}$, a series expansion in $L^2(\mathbb{P})$ of the form

$$\mathcal{L}_k(n;u) = \sum_{q=0}^{\infty} \mathrm{Proj}[\mathcal{L}_k(n;u)|q], \tag{3.19}$$

where $\mathrm{Proj}[\mathcal{L}_k(n;u)|q]$ denotes the orthogonal projection of $\mathcal{L}_k(n;u)$ on the space spanned by *multivariate* Hermite polynomials of order q to be computed on f_n in (1.2) and their derivatives up to order two, and the random variables $\mathrm{Proj}[\mathcal{L}_k(n;u)|q]$ and $\mathrm{Proj}[\mathcal{L}_k(n;u)|q']$ are orthogonal whenever $q \neq q'$. Recall that Hermite polynomials $\{H_q\}_{q\in\mathbb{N}}$ are defined as

$$H_0 \equiv 1, \qquad H_q(t) := (-1)^q \phi^{-1}(t) \frac{d^q}{dt^q}\phi(t),\ t \in \mathbb{R},\ q \geq 1, \tag{3.20}$$

where ϕ still denotes the standard Gaussian probability function. A more complete discussion on Wiener chaos is given in §4.1, see also [NP12, §2] for background and details.

The zero-th order term just amounts to the expected value of Lipschitz-Killing curvatures given in Lemma 2.1, whereas the first order projection is easily seen to vanish identically due to the oscillating properties of these random waves f_n.

Lemma 3.1. *For $k = 0, 1, 2$, $n \in S$ and $u \in \mathbb{R}$*

$$\mathrm{Proj}[\mathcal{L}_k(n;u)|0] = m_k(u) \left(\sqrt{\frac{E_n}{2}} \right)^{2-k}, \tag{3.21}$$

where $m_k(u)$ are as in (2.9), moreover for $n \geq 1$

$$\mathrm{Proj}[\mathcal{L}_k(n;u)|1] = 0. \tag{3.22}$$

It becomes then crucial to investigate the behaviour of $\mathrm{Proj}[\mathcal{L}_k(n;u)|2]$.

Proposition 3.2. *For $k = 0, 1, 2$, $n \in S$ and $u \in \mathbb{R}$ it holds that*

$$\mathrm{Proj}[\mathcal{L}_k(n;u)|2] = \frac{c_k(u)}{\sqrt{\mathcal{N}_n/2}} \left(\sqrt{\frac{E_n}{2}} \right)^{2-k} \frac{1}{\sqrt{\mathcal{N}_n/2}} \sum_{\lambda \in \Lambda_n^+} (|a_\lambda|^2 - 1), \tag{3.23}$$

where, as in (2.12),

$$c_2(u) = \frac{1}{2} u \phi(u), \quad c_1(u) = \frac{1}{2} \sqrt{\frac{\pi}{8}} u^2 \phi(u), \quad c_0(u) = \frac{1}{2}(u^2 - 1) u \phi(u) \frac{1}{2\pi}.$$

From (3.23) it is clear that whenever $c_k(u) \neq 0$ the variance of the second order chaotic projection of $\mathcal{L}_k(n;u)$ is of order $E_n^{2-k}/\mathcal{N}_n$, otherwise $\mathrm{Proj}[\mathcal{L}_k(n;u)|2] = 0$. A careful investigation of higher order chaotic components yields the following.

Proposition 3.3. *For $k = 0, 1, 2$, $n \in S$, $u \in \mathbb{R}$, as $n \to +\infty$ such that $\mathcal{N}_n \to +\infty$, under Condition 2.2*

$$\mathrm{Var}\left(\sum_{q=3}^{+\infty} \mathrm{Proj}[\mathcal{L}_k(A_u(f_n;\mathbb{T}))|q] \right) = O\left(\frac{E_n^{2-k}}{\mathcal{N}_n^2} \right), \tag{3.24}$$

where the constant involved in the O-notation only depends on k.

Proposition 3.3 together with (3.23) proves Theorem 2.4, see §4, once setting

$$\mathcal{R}_k(n;u) := \sum_{q=3}^{+\infty} \mathrm{Proj}[\mathcal{L}_k(A_u(f_n;\mathbb{T}))|q].$$

The proofs of Corollary 2.5, Corollary 2.6 and Corollary 2.7 (postponed to §4) heavily rely on (2.11), i.e. on the fact that, at least at "non-degenerate" levels, all Lipschitz-Killing Curvatures for Arithmetic Random Waves behave (in the high-energy limit) as an element of a fixed order Wiener chaos, in particular as a sum of i.i.d. random variables. Equation (2.13) quantifies the error made when replacing $\overline{\mathcal{L}}_k(n;u)$ with the empirical mean of centered squared Fourier coefficients $\{a_\lambda\}_{\lambda \in \Lambda_n}$, allowing to get the quantitative estimates stated in these corollaries.

3.2 Discussion

The first few Hermite polynomials (3.20)

$$H_0(u) = 1, \quad H_1(u) = u, \quad H_2(u) = u^2 - 1, \qquad u \in \mathbb{R}$$

will play a crucial role in the arguments to follow. For the sake of notational simplicity, let us set

$$H_{-1}(u) := 1 - \Phi(u), \qquad u \in \mathbb{R}.$$

For the moment, let us observe that we can rewrite the empirical mean on the right hand side of (3.23) as

$$\frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} (|a_\lambda|^2 - 1) = \int_{\mathbb{T}} H_2(f_n(x))\, dx. \tag{3.25}$$

Of course,

$$1 = \int_{\mathbb{T}} H_0(f_n(x))\, dx. \tag{3.26}$$

From (3.26), bearing in mind Lemma 3.1, we can rewrite (2.8) as

$$\begin{gathered} \mathrm{Proj}[\mathcal{L}_k(n;u)|0] = m_k(u) \left(\sqrt{\frac{E_n}{2}} \right)^{2-k} \int_{\mathbb{T}} H_0(f_n(x))\, dx, \\ m_2(u) = H_{-1}(u), \quad m_1(u) = \sqrt{\frac{\pi}{8}} H_0(u)\phi(u), \quad m_0(u) = \frac{1}{2\pi} H_1(u)\phi(u), \end{gathered} \tag{3.27}$$

for every $n \in S$, $k = 0, 1, 2$ and $u \in \mathbb{R}$. Analogously, from (3.25), we can rewrite (3.23) as

$$\begin{gathered} \mathrm{Proj}[\mathcal{L}_k(n;u)|2] = c_k(u) \left(\sqrt{\frac{E_n}{2}} \right)^{2-k} \int_{\mathbb{T}} H_2(f_n(x))\, dx, \\ c_2(u) = \frac{1}{2} H_1(u)\phi(u), \quad c_1(u) = \frac{1}{2}\sqrt{\frac{\pi}{8}} H_1(u)^2\phi(u), \quad c_0(u) = \frac{1}{2} H_1(u)H_2(u)\phi(u)\frac{1}{2\pi}, \end{gathered} \tag{3.28}$$

for every $k = 0, 1, 2$, $n \in S$ and $u \in \mathbb{R}$. In the next subsection, we will show that (3.27) and (3.28) are in *perfect* analogy with the spherical case, suggesting the existence of a "second order" Gaussian Kinematic formula à la Adler and Taylor [AT07]. The generalization of these expressions to arbitrary dimensions are currently under investigation.

3.2.1 Random Spherical Harmonics: previous work

Random Spherical Harmonics are Gaussian eigenfunctions of the spherical Laplacian operator, that is, the sequence $\{f_\ell\}_{\ell \in \mathbb{N}}$ of zero mean, unit variance isotropic Gaussian fields on the unit round sphere $\mathbb{S}^2$ which satisfies the Helmholtz equation

$$\Delta_{\mathbb{S}^2} f_\ell = -\lambda_\ell f_\ell, \qquad \lambda_\ell = \ell(\ell+1),\ \ell \in \mathbb{N}.$$

Here, $\Delta_{\mathbb{S}^2}$ is the spherical Laplacian operator. For the excursion sets of these fields

$$A_u(f_\ell; \mathbb{S}^2) := \{x \in \mathbb{S}^2 : f_\ell(x) \geq u\}, \qquad u \in \mathbb{R},$$

the following results hold[4] (see [CM18] and the references therein):

$$\mathrm{Proj}[\mathcal{L}_k(A_u(f_\ell;\mathbb{S}^2))|0] = m_k(u)\left(\sqrt{\frac{\lambda_\ell}{2}}\right)^{2-k}\int_{\mathbb{S}^2} H_0(f_\ell(x))\,dx + 2H_{-1}(u)\cdot\delta_k^0,$$

where the coefficients $m_k(u)$ are as in (3.27), and

$$\mathrm{Proj}[\mathcal{L}_k(A_u(f_\ell;\mathbb{S}^2))|2] = c_k(u)\left(\sqrt{\frac{\lambda_\ell}{2}}\right)^{2-k}\int_{\mathbb{S}^2} H_2(f_\ell(x))\,dx + O_{L^2(\mathbb{P})}(1)\cdot\delta_k^0, \tag{3.29}$$

where $c_k(u)$ are as in (3.28). Here, δ_i^j is the Kronecker delta, and $O_{L^2(\mathbb{P})}(1)$ stands for a sequence of random variables bounded in $L^2(\mathbb{P})$.

3.2.2 Some more remarks

Remark 3.4 (On Berry's Cancellation)**.** It should be noted that for all three Lipschitz-Killing Curvatures the second-order chaos term disappears in the *nodal* case $u = 0$, see (3.28) and (3.29). As a result, the asymptotic variance of these geometric functionals is of smaller order for this value, thus providing an interpretation of the Berry's cancellation phenomenon first noted (for the case of boundary lengths of planar random eigenfunctions) in [Ber02] and then discussed by [Wig10, MPRW16, CM18] and others.

Remark 3.5 (On Universality)**.** It was shown in [KKW13] and later in [MPRW16], see also Remark 2.8, that some geometric features for the excursion sets of Arithmetic Random Waves are not universal, in the sense that they do not share a unique limit as the eigenvalues n diverge. In particular, for the case of nodal length it turns out that both the variance (2.17) and the limiting distributions (2.18) can vary quite substantially along subsequences characterized by different limiting values for the coefficients $\widehat{\mu}_n(4), n \in S$ introduced in (1.5). This is not the case for Lipschitz-Killing Curvatures computed on excursion sets corresponding to a non-vanishing second-order chaotic component, see §2: their expected values, their asymptotic variances and their (Gaussian) limiting distributions are universal (in other words, they are invariant under subsequences of energy levels). The sequence of parameters $\widehat{\mu}_n(4), n \in S$ appears ubiquitously in the proofs that will be presented in the following section.

Remark 3.6 (On Correlation)**.** Corollary 2.7 is, heuristically, a consequence of the fact that the fluctuations of all three Lipschitz Killing Curvatures are actually dominated by their centered norm, see Theorem 2.4, (2.14) and (3.28),

$$\|f_n\|^2_{L^2(\mathbb{T})} - \mathbb{E}\left[\|f_n\|^2_{L^2(\mathbb{T})}\right] = \int_{\mathbb{T}} H_2(f_n(x))\,dx = \frac{1}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}(|a_\lambda|^2 - 1).$$

Again, an analogous phenomenon occurs for random eigenfunctions on the sphere, see §3.2.1: in the case of Arithmetic Random Waves, however, this is slightly more surprising, because isotropy does not hold and hence one could expect the magnitude of the single random coefficients $\{|a_\lambda|^2\}_{\lambda\in\Lambda_n}$ to play a more relevant role.

[4] Note that $\int_{\mathbb{S}^2} H_0(f_\ell(x))\,dx = 4\pi$.

3.3 Plan

In §4.1 we will recall basic facts on Wiener chaos, the chaotic expansion (3.19) for Lispchitz-Killing Curvatures will be stated in §4.2 while the proofs of our main results will be given in §4.3. In particular, in §5 we will establish the chaotic expansion for Euler-Poincaré characteristic of Arithmetic Random Waves. The second chaotic components of these geometric functionals will be analyzed in §6 leading to the proof of Proposition 3.2. We will investigate higher order chaotic components in §7 proving Proposition 3.3 along the way. Some technicalities and several tedious computations will be collected in the four Appendixes A–D.

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4 Proofs of the main results

4.1 Wiener chaos

Let us recall some basic facts on Wiener chaos, restricting ourselves to the toral setting. Bear in mind the definition of Hermite polynomials in (3.20). The family $\mathbb{H} := \{H_k/\sqrt{k!}\}_{k\in\mathbb{N}}$ is a complete orthonormal system in $L^2(\mathbb{R}, \mathscr{B}(\mathbb{R}), \phi(t)dt) =: L^2(\phi)$, where ϕ still denotes the standard Gaussian density on the real line.

The Arithmetic Random Waves (1.2) considered in this work are a by-product of a family of complex-valued Gaussian random variables $\{a_\lambda\}_{\lambda\in\mathbb{Z}^2}$, defined on $(\Omega, \mathscr{F}, \mathbb{P})$ and satisfying properties (1) and (2) in (1.2). Let us define the space $\mathcal{A}$ to be the closure in $L^2(\mathbb{P})$ of all real finite linear combinations of random variables ξ of the form

$$\xi = z\, a_\lambda + \overline{z}\, a_{-\lambda},$$

where $\lambda \in \mathbb{Z}^2$ and $z \in \mathbb{C}$. The space $\mathcal{A}$ is a real centered Gaussian Hilbert subspace of $L^2(\mathbb{P})$.

Definition 4.1. *For $q \in \mathbb{N}$, the q-th Wiener chaos associated with $\mathcal{A}$, written C_q, is the closure in $L^2(\mathbb{P})$ of all real finite linear combinations of random variables of the form*

$$H_{p_1}(\xi_1) \cdot H_{p_2}(\xi_2) \cdots H_{p_k}(\xi_k)$$

for $k \in \mathbb{N}_{\geq 1}$, where $p_1, ..., p_k \in \mathbb{N}$ satisfy $p_1 + \cdots + p_k = q$, and $(\xi_1, ..., \xi_k)$ is a standard real Gaussian vector extracted from $\mathcal{A}$ (in particular, $C_0 = \mathbb{R}$).

Using the orthonormality and completeness of $\mathbb{H}$ in $L^2(\phi)$, together with a standard monotone class argument (see e.g. [NP12, Theorem 2.2.4]), it is easy to show that C_q and C_m are orthogonal in the sense of $L^2(\mathbb{P})$ for every $q \neq m$, and moreover

$$L^2_{\mathcal{A}}(\mathbb{P}) := L^2(\Omega, \sigma(\mathcal{A}), \mathbb{P}) = \bigoplus_{q=0}^{\infty} C_q;$$

that is, every real-valued functional F of $\mathcal{A}$ can be (uniquely) represented in the form

$$F = \sum_{q=0}^{\infty} \mathrm{Proj}[F|q] \tag{4.30}$$

where $\mathrm{Proj}[F|q]$ stands for the projection of F onto C_q, and the series converges in $L^2(\mathbb{P})$. Plainly, $\mathrm{Proj}[F|0] = \mathbb{E}[F]$.

From (1.3), for $j, \ell = 1, \dots, d$ the random fields $f_n, \partial_j f_n, \partial^2_{j\ell} f_n$ viewed as collections of Gaussian random variables indexed by $x \in \mathbb{T}^d$ are all lying in $\mathcal{A}$, i.e. for every $x \in \mathbb{T}^d$ we have

$$f_n(x),\ \partial_j f_n(x),\ \partial^2_{j\ell} f_n(x) \in \mathcal{A}.$$

4.2 Chaotic expansions of Lipschitz-Killing curvatures

The three geometric functionals of our interest are finite-variance functionals of $\mathcal{A}$, hence applying (4.30) we get the series expansion in (3.19). Let us be more precise.

4.2.1 Excursion area

For the second Lipschitz-Killing curvature we have the following integral representation

$$\mathcal{L}_2(n; u) = \int_{\mathbb{T}} \mathbf{1}_{\{f_n(x) \geq u\}}\, dx \tag{4.31}$$

entailing that $\mathcal{L}_2(n; u) \in L^2_{\mathcal{A}}(\mathbb{P})$. The proof of the following result is simple (see also §3 in [MW11]) and hence omitted.

Lemma 4.2. *For every $n \in S$ and $u \in \mathbb{R}$, the chaotic decomposition of $\mathcal{L}_2(n; u)$ is*

$$\mathcal{L}_2(n; u) = \sum_{q=0}^{+\infty} \frac{\gamma_q(u)}{q!} \int_{\mathbb{T}} H_q(f_n(x))\, dx, \tag{4.32}$$

where $\gamma_q(u) := H_{q-1}(u)\phi(u)$, and the convergence of the series is in $L^2(\mathbb{P})$.

4.2.2 Boundary length

For the first Lipschitz-Killing curvature we have the following *formal* integral representation

$$\mathcal{L}_1(n; u) = \frac{1}{2} \int_{\mathbb{T}} \delta_u(f_n(x)) \|\nabla f_n(x)\|\, dx, \tag{4.33}$$

where δ_u is the Dirac mass in u, and ∇f_n is the gradient of f_n. For $\epsilon > 0$, let us consider the ϵ-approximating random variable

$$\mathcal{L}_1^{\epsilon}(n; u) := \frac{1}{2} \int_{\mathbb{T}} \frac{1}{2\epsilon} \mathbf{1}_{[u-\epsilon, u+\epsilon]}(f_n(x)) \|\nabla f_n(x)\|\, dx.$$

Lemma 4.3. *For every $n \in S$ and $u \in \mathbb{R}$ it holds that, as $\epsilon \to 0$,*

$$\mathcal{L}_1^\epsilon(n;u) \to \mathcal{L}_1(n;u), \tag{4.34}$$

where the convergence holds both a.s. and in $L^2_{\mathcal{A}}(\mathbb{P})$.

The proof of Lemma 4.3 is analogous to the proof of the $L^2(\mathbb{P})$-approximation result for the nodal length of Random Spherical Harmonics in [MRW20, Appendix B] and hence omitted. In order to state the next result we need to introduce two collections of coefficients $\{\alpha_{2n,2m} : n,m \geq 1\}$ and $\{\beta_l(u) : l \geq 0\}$, that are related to the Hermite expansion of the norm $\|\cdot\|$ in $\mathbb{R}^2$ and the (formal) Hermite expansion of the Dirac mass $\delta_u(\cdot)$ respectively, cf. (4.33). These are given by

$$\beta_l(u) := H_l(u)\phi(u), \tag{4.35}$$

where H_l still denotes the l-th Hermite polynomial (3.20), and

$$\alpha_{2n,2m} := \sqrt{\frac{\pi}{2}} \frac{(2n)!(2m)!}{n!m!} \frac{1}{2^{n+m}} p_{n+m}\left(\frac{1}{4}\right), \tag{4.36}$$

where for $N \in \mathbb{N}$ and $x \in \mathbb{R}$

$$p_N(x) := \sum_{j=0}^{N} (-1)^j \cdot (-1)^N \binom{N}{j} \; \frac{(2j+1)!}{(j!)^2} x^j,$$

the ratio $\frac{(2j+1)!}{(j!)^2}$ being the so-called swinging factorial restricted to odd indices. The proof of the following lemma is analogous to the proof of Proposition 3.2 in [MPRW16] and hence omitted for the sake of brevity.

Lemma 4.4. *For every $n \in S$ and $u \in \mathbb{R}$ the chaotic expansion of $\mathcal{L}_1(n;u)$ is*

$$\begin{aligned}\mathcal{L}_1(n;u) \;&=\; \frac{1}{2}\sqrt{\frac{E_n}{2}} \sum_{q=0}^{+\infty} \sum_{u=0}^{q} \sum_{k=0}^{u} \frac{\alpha_{2k,2u-2k}\beta_{q-2u}(u)}{(2k)!(2u-2k)!(q-2u)!} \times \\ &\quad \times \int_{\mathbb{T}} H_{q-2u}(f_n(x)) H_{2k}(f_{n,1}(x)) H_{2u-2k}(f_{n,2}(x))\, dx,\end{aligned}$$

where the convergence of the series is in $L^2(\mathbb{P})$, and $f_{n,\ell}$ denotes normalized first derivatives defined in (1.4).

4.2.3 Euler-Poincaré characteristic

The zero-th Lipschitz-Killing curvature has the following *formal* representation

$$\mathcal{L}_0(n;u) = \int_{\mathbb{T}} \det(\nabla^2 f_n(x)) \mathbf{1}_{\{f_n(x)\geq u\}} \delta_0(\nabla f_n(x))\, dx, \tag{4.37}$$

where $\nabla^2 f_n$ is the Hessian matrix of f_n, and abusing notation δ_0 denotes the Dirac mass in $(0,0)$. For $\epsilon > 0$, let us consider the ϵ-approximating random variable

$$\mathcal{L}_0^\epsilon(n;u) = \int_{\mathbb{T}} \det(\nabla^2 f_n(x)) \mathbf{1}_{\{f_n(x)\geq u\}} \frac{1}{(2\epsilon)^2} \mathbf{1}_{[-\epsilon,\epsilon]^2}(\nabla f_n(x))\, dx. \tag{4.38}$$

Lemma 4.5. *For every $n \in S$, $\epsilon > 0$ and $u \in \mathbb{R}$*

$$|\mathcal{L}_0^\epsilon(u;n)| \le 4E_n. \tag{4.39}$$

The proof of Lemma 4.5 is postponed to the Appendix A.

Lemma 4.6. *For $n \in S$ and $u \in \mathbb{R}$ it holds that, as $\epsilon \to 0$,*

$$\mathcal{L}_0^\epsilon(n;u) \to \mathcal{L}_0(n;u),$$

where the convergence is a.s. and in $L^2_{\mathcal{A}}(\mathbb{P})$.

Equation (4.37) is justified by Lemma 4.6 whose proof is postponed to the Appendix A. In view of Lemma 4.6, by letting ϵ tend to zero in (4.39) we find

$$|\mathcal{L}_0(n;u)| \le 4E_n$$

for every $n \in S$ e $u \in \mathbb{R}$, in particular $\mathcal{L}_0(n;u)$ belongs to $L^2_{\mathcal{A}}(\mathbb{P})$. Obviously once the a.s. convergence is proven, it suffices to apply Lemma 4.5 to get the convergence in $L^2(\mathbb{P})$.

The next result (whose proof will be given in §5) concerns the chaotic expansion of $\mathcal{L}_0(n;u)$: we will not need explicit expressions for chaotic coefficients but those corresponding to the zero-th and second Wiener chaoses, see §4.3 and §6.2 respectively.

Lemma 4.7. *For $n \in S$ and $u \in \mathbb{R}$, the chaotic expansion of $\mathcal{L}_0(n;u)$ is*

$$\begin{aligned}\mathcal{L}_0(n;u) =& 2E_n \sum_{q=0}^{+\infty} \sum_{a+b+c+2d+2e=q} \frac{\eta^{(n)}_{a,b,c}(u)}{a!b!c!} \frac{\beta_{2d}\beta_{2e}}{(2d)!(2e)!} \int_{\mathbb{T}} H_a\left(\frac{\partial_{11} f_n(x)}{k_3}\right) \\ &\times H_b\left(\frac{\partial_{12} f_n(x)}{k_4}\right) H_c\left(\frac{\partial_{22} f_n(x)}{k_5} - \frac{k_2}{k_5 k_3}\partial_{11} f_n(x)\right) H_{2d}\left(\frac{\partial_1 f_n(x)}{k_1}\right) \\ &\times H_{2e}\left(\frac{\partial_2 f_n(x)}{k_1}\right) dx,\end{aligned} \tag{4.40}$$

for some coefficients $\eta^{(n)}_{a,b,c}(u) \in \mathbb{R}, a,b,c \in \mathbb{N}$, where the series converges in $L^2(\mathbb{P})$,

$$\beta_q := \beta_q(0) = \phi(0)H_q(0) \tag{4.41}$$

as defined in (4.35), and $k_1, \dots, k_5$ are defined in (5.53).

4.3 Proofs

Proof of Lemma 3.1. From Lemma 4.2 we have

$$\mathrm{Proj}[\mathcal{L}_2(n;u)|0] = \frac{\gamma_0(u)}{0!} \int_{\mathbb{T}} H_0(f_n(x))\, dx = 1 - \Phi(u),$$

that coincides with $\mathbb{E}[\mathcal{L}_0(n;u)]$ in Lemma 2.1. From Lemma 4.4, (4.35) and (4.36)

$$\begin{aligned}\mathrm{Proj}[\mathcal{L}_1(n;u)|0] &= \frac{1}{2}\sqrt{\frac{E_n}{2}} \frac{\alpha_{0,0}\beta_0(u)}{0!0!0!} \int_{\mathbb{T}} H_0(f_n(x)) H_0(f_{n,1}(x)) H_0(f_{n,2}(x))\, dx \\ &= \frac{1}{2}\sqrt{\frac{E_n}{2}} \sqrt{\frac{\pi}{2}} \phi(u)\end{aligned}$$

that is $\mathbb{E}[\mathcal{L}_1(n;u)]$ given in Lemma 2.1. Let us now focus on $\mathcal{L}_0(n;u)$. Exploiting the proof of Lemma 2.1 in Appendix C we have that for every $n \in S$ and $u \in \mathbb{R}$

$$\eta^{(n)}_{0,0,0}(u) = \frac{1}{4}u\phi(u), \tag{4.42}$$

hence from Lemma 4.7,

$$\begin{aligned}\mathrm{Proj}[\mathcal{L}_0(n;u)|0] &= 2E_n \cdot \eta^{(n)}_{0,0,0}(u)\beta_0^2 \\ &= \frac{1}{2\pi}u\phi(u)\cdot\frac{E_n}{2}\end{aligned}$$

that is equal to $\mathbb{E}[\mathcal{L}_0(n;u)]$ in Lemma 2.1. From Lemma 4.2 and $n \in S$, $n \geq 1$, the first order chaotic component of $\mathcal{L}_2(n;u)$ is

$$\begin{aligned}\mathrm{Proj}[\mathcal{L}_2(n;u)|1] &= \frac{\gamma_1(u)}{1!}\int_{\mathbb{T}} H_1(f_n(x))\,dx = \phi(u)\int_{\mathbb{T}} f_n(x)\,dx \\ &= \frac{\phi(u)}{\sqrt{\mathcal{N}_n}}\sum_{\lambda\in\Lambda_n}\underbrace{\int_{\mathbb{T}} e(\langle\lambda,x\rangle)\,dx}_{=\delta_\lambda^0} = 0.\end{aligned} \tag{4.43}$$

The proof of (3.22) for $k = 0, 1$ is analogous to (4.43), hence we omit the details.

□

The proofs of Proposition 3.2 and Proposition 3.3 are long and technical, hence postponed to §6 and §7 respectively.

Proof of Theorem 2.4 assuming Proposition 3.2 and Proposition 3.3. From Lemma 3.1 and (3.19) we can write

$$\overline{\mathcal{L}}_k(n;u) = \sum_{q=2}^{\infty}\mathrm{Proj}[\mathcal{L}_k(n;u)|q], \tag{4.44}$$

where the convergence of the (orthogonal) series is in $L^2(\mathbb{P})$. Proposition 3.2 ensures that we can rewrite (4.44) as

$$\overline{\mathcal{L}}_k(n;u) = \frac{c_k(u)}{\sqrt{\mathcal{N}_n/2}}\left(\sqrt{\frac{E_n}{2}}\right)^{2-k}\frac{1}{\sqrt{\mathcal{N}_n/2}}\sum_{\lambda\in\Lambda_n^+}(|a_\lambda|^2-1) + \sum_{q=3}^{\infty}\mathrm{Proj}[\mathcal{L}_k(n;u)|q],$$

where $c_k(u)$ are given in (2.12). In order to conclude the proof it suffices to set

$$\mathcal{R}_k(n;u) := \sum_{q=3}^{\infty}\mathrm{Proj}[\mathcal{L}_k(n;u)|q] \tag{4.45}$$

and recall Proposition 3.3.

□

Proof of Corollary 2.5. Let us bear in mind Theorem 2.4 and (2.15). For $n \in S$, $k = 0, 1, 2$ and $u \notin \mathcal{U}_k$, by triangle inequality,

$$d_W\left(\widetilde{\mathcal{L}}_k(n;u), Z\right) \leq d_W\left(\widetilde{\mathcal{L}}_k(n;u), \frac{\mathrm{Proj}[\mathcal{L}_k(n;u)|2]}{\sqrt{\mathrm{Var}(\mathcal{L}_k(n;u))}}\right) + d_W\left(\frac{\mathrm{Proj}[\mathcal{L}_k(n;u)|2]}{\sqrt{\mathrm{Var}(\mathcal{L}_k(n;u))}}, Z\right). \tag{4.46}$$

For the first term on the right-hand side of (4.46) it suffices to note that, by definition of Wasserstein distance and (3.19),

$$\begin{aligned} d_W\left(\widetilde{\mathcal{L}}_k(n;u), \frac{\mathrm{Proj}[\mathcal{L}_k(n;u)|2]}{\sqrt{\mathrm{Var}(\mathcal{L}_k(n;u))}}\right) &\le \sqrt{\mathbb{E}\left[\left(\frac{\mathcal{R}_k(n;u)}{\sqrt{\mathrm{Var}(\mathcal{L}_k(n;u))}}\right)^2\right]} \\ &= O\left(\sqrt{\frac{1}{\mathcal{N}_n}}\right), \end{aligned} \tag{4.47}$$

where the last estimate follows from (2.13) in Theorem 2.4 and (2.15). The second term on the right hand side of (4.46) can be controlled by Berry-Esseen's bounds (see e.g. [Ess42]), indeed the second order chaotic projection is a sum of i.i.d. random variables, see Proposition 3.2, recall also (2.15):

$$\begin{aligned} d_W\left(\frac{\mathrm{Proj}[\mathcal{L}_k(n;u)|2]}{\sqrt{\mathrm{Var}(\mathcal{L}_k(n;u))}}, Z\right) &\le d_W\left(\frac{\mathrm{Proj}[\mathcal{L}_k(n;u)|2]}{\sqrt{\mathrm{Var}(\mathrm{Proj}[\mathcal{L}_k(n;u)|2])}}, Z\right) \\ &\quad + d_W\left(\frac{\mathrm{Proj}[\mathcal{L}_k(n;u)|2]}{\sqrt{\mathrm{Var}(\mathrm{Proj}[\mathcal{L}_k(n;u)|2])}}, \frac{\mathrm{Proj}[\mathcal{L}_k(n;u)|2]}{\sqrt{\mathrm{Var}(\mathcal{L}_k(n;u))}}\right) \\ &\le d_W\left(\frac{1}{\sqrt{\mathcal{N}_n/2}}\sum_{\lambda\in\Lambda_n^+}(|a_\lambda|^2-1), Z\right) \\ &\quad + \frac{\left|\sqrt{\mathrm{Var}\left(\mathcal{L}_k(n;u)\right)} - \sqrt{\mathrm{Var}\left(\mathrm{Proj}[\mathcal{L}_k(n;u)|2]\right)}\right|}{\sqrt{\mathrm{Var}\left(\mathcal{L}_k(n;u)\right)}} \\ &= O\left(\frac{1}{\sqrt{\mathcal{N}_n}}\right). \end{aligned} \tag{4.48}$$

Plugging (4.47) and (4.48) into (4.46) we conclude the proof.

□

Proof of Corollary 2.6. The proof is analogous to the proof of Theorem 1.7 in [MRT20]. Applying standard Large Deviation results [DZ98] for sums of i.i.d. random variables we have that under (2.16) the sequence

$$\{\widetilde{\mathrm{Proj}}[\mathcal{L}_k(n;u)|2]/s^{(k)}_{n;u}\}_{n\in S}$$

satisfies a Moderate Deviation principle as $\mathcal{N}_n \to +\infty$ with speed $(s^{(k)}_{n;u})^2$ and rate function $\mathcal{I}$, where

$$\widetilde{\mathrm{Proj}}[\mathcal{L}_k(n;u)|2] := \frac{\mathrm{Proj}[\mathcal{L}_k(n;u)|2]}{\sqrt{\mathrm{Var}(\mathrm{Proj}[\mathcal{L}_k(n;u)|2])}}.$$

Moreover, for every $\delta > 0$, under (2.16),

$$\limsup_{\mathcal{N}_n\to+\infty} \frac{1}{(s^{(k)}_{n;u})^2} \log \mathbb{P}\left(\left|\widetilde{\mathcal{L}}_k(n;u)/s^{(k)}_{n;u} - \widetilde{\mathrm{Proj}}[\mathcal{L}_k(n;u)|2]/s^{(k)}_{n;u}\right| > \delta\right) = -\infty,$$

i.e. the two sequence of random variables $\{\widetilde{\mathrm{Proj}}[\mathcal{L}_k(n;u)|2]/s^{(k)}_{n;u}\}_{n\in S}$ and $\{\widetilde{\mathcal{L}}_k(n;u)/s^{(k)}_{n;u}\}_{n\in S}$ are exponentially equivalent [DZ98, Definition 4.2.10] along subsequences of energy levels such that

$\mathcal{N}_n \to +\infty$. Theorem 4.2.13 in [DZ98] then ensures that $\{\widetilde{\mathcal{L}}_k(n;u)/s^{(k)}_{n;u}\}_{n\in S}$ satisfies a Moderate Deviation principle with the same speed and rate function as the sequence $\{\widetilde{\text{Proj}}[\mathcal{L}_k(n;u)|2]/s^{(k)}_{n;u}\}_{n\in S}$. □

Proof of Corollary 2.7. From Theorem 2.4 and (4.45) we can write

$$\begin{aligned}
&\text{Corr}\left(\mathcal{L}_{k_1}(n;u_1),\mathcal{L}_{k_2}(n;u_2)\right) = \frac{\text{Cov }\left(\mathcal{L}_{k_1}(n;u_1),\mathcal{L}_{k_2}(n;u_2)\right)}{\sqrt{\text{Var}(\mathcal{L}_{k_1}(n;u_1))\text{Var}(\mathcal{L}_{k_2}(n;u_2))}}\\
&= \frac{\text{Cov }\left(\text{Proj}[\mathcal{L}_{k_1}(n;u_1)|2]+\mathcal{R}_{k_1}(n;u_1),\text{Proj}[\mathcal{L}_{k_2}(n;u_2)|2]+\mathcal{R}_{k_2}(n;u_2)\right)}{\sqrt{\text{Var}(\mathcal{L}_{k_1}(n;u_1))\text{Var}(\mathcal{L}_{k_2}(n;u_2))}}\\
&= \frac{\text{Cov }\left(\text{Proj}[\mathcal{L}_{k_1}(n;u_1)|2]),\text{Proj}[\mathcal{L}_{k_2}(n;u_2)|2]\right)}{\sqrt{\text{Var}(\mathcal{L}_{k_1}(n;u_1))\text{Var}(\mathcal{L}_{k_2}(n;u_2))}}\\
&+\frac{\text{Cov }\left(\mathcal{R}_{k_1}(n;u_1),\mathcal{R}_{k_2}(n;u_2)\right)}{\sqrt{\text{Var}(\mathcal{L}_{k_1}(n;u_1))\text{Var}(\mathcal{L}_{k_2}(n;u_2))}}\\
&= \frac{\text{Cov }\left(\text{Proj}[\mathcal{L}_{k_1}(n;u_1)|2]),\text{Proj}[\mathcal{L}_{k_2}(n;u_2)|2]\right)}{\sqrt{\text{Var}(\text{Proj}[\mathcal{L}_{k_1}(n;u_1)|2])\text{Var}(\text{Proj}[\mathcal{L}_{k_2}(n;u_2)|2])}} + O\left(\frac{1}{\mathcal{N}_n}\right)\\
&+\frac{\text{Cov }\left(\mathcal{R}_{k_1}(n;u_1),\mathcal{R}_{k_2}(n;u_2)\right)}{\sqrt{\text{Var}(\mathcal{L}_{k_1}(n;u_1))\text{Var}(\mathcal{L}_{k_2}(n;u_2))}},
\end{aligned} \tag{4.49}$$

where for the last equality we used (2.15). From Proposition 3.2 we get

$$\frac{\text{Cov }\left(\text{Proj}[\mathcal{L}_{k_1}(n;u_1)|2]),\text{Proj}[\mathcal{L}_{k_2}(n;u_2)|2]\right)}{\sqrt{\text{Var}(\text{Proj}[\mathcal{L}_{k_1}(n;u_1)|2])\text{Var}(\text{Proj}[\mathcal{L}_{k_2}(n;u_2)|2])}} = 1 \tag{4.50}$$

and still from Theorem 2.4 and (2.15), and Cauchy-Schwatz inequality, we get

$$\begin{aligned}
\frac{|\text{Cov }\left(\mathcal{R}_{k_1}(n;u_1),\mathcal{R}_{k_2}(n;u_2)\right)|}{\sqrt{\text{Var}(\mathcal{L}_{k_1}(n;u_1))\text{Var}(\mathcal{L}_{k_2}(n;u_2))}} \leq& \frac{\sqrt{\mathbb{E}[\mathcal{R}_{k_1}(n;u_1)^2]\mathbb{E}[\mathcal{R}_{k_2}(n;u_2)^2]}}{\sqrt{\text{Var}(\mathcal{L}_{k_1}(n;u_1))\text{Var}(\mathcal{L}_{k_2}(n;u_2))}}\\
=&O\left(\frac{1}{\mathcal{N}_n}\right).
\end{aligned} \tag{4.51}$$

Plugging (4.50) and (4.51) into (4.49) we conclude the proof.

□

5 EPC: chaotic expansion

5.1 Cholesky decomposition

Recall the definition of the ϵ-approximating random variable (4.38) and Lemma 4.6. In order to prove Lemma 4.7 we first derive the chaotic expansion of $\mathcal{L}^\epsilon_0(n;u)$ and then let ϵ go to zero. The integrand function

$$F^\epsilon_n(x) := \det(\nabla^2 f_n(x))\mathbf{1}_{\{\Delta f_n(x)\leq -E_n u\}}\frac{1}{(2\epsilon)^2}\mathbf{1}_{[-\epsilon,\epsilon]^2}(\nabla f_n(x)), \qquad x\in\mathbb{T}, \tag{5.52}$$

defining the ϵ-approximating random variable (4.38) is a functional of $\partial_1 f_n$, $\partial_2 f_n$, $\partial_{11} f_n$, $\partial_{12} f_n$, $\partial_{22} f_n$ which are not point-wise independent random fields. For the sake of simplicity we first express $F^\epsilon(x)$

in terms of independent random variables. Let us write $\sigma_n = \sigma_n(x)$ for the 5×5 covariance matrix (see §B.1) of the Gaussian random vector

$$(\partial_1 f_n(x), \partial_2 f_n(x), \partial_{11} f_n(x), \partial_{12} f_n(x), \partial_{22} f_n(x)).$$

We write it in the partitioned form

$$\sigma_n = \sigma_n(x)_{5\times 5} = \begin{pmatrix} a_n & b_n \\ b_n^t & c_n \end{pmatrix},$$

where the superscript t denotes transposition, and (see Appendix B.1)

$$a_n = a_n(x) = \frac{4\pi^2}{\mathcal{N}_n} \sum_\lambda \lambda\lambda^t = 2\pi^2 n\, I_2,$$

$$b_n = b_n(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

since for stationarity for random fields second derivatives and first derivatives at every fixed point are uncorrelated [see [AT07], page 114], and (recall (1.5))

$$c_n = c_n(x) = 2\pi^4 n^2 \begin{pmatrix} 3 + \hat{\mu}_n(4) & 0 & 1 - \hat{\mu}_n(4) \\ 0 & 1 - \hat{\mu}_n(4) & 0 \\ 1 - \hat{\mu}_n(4) & 0 & 3 + \hat{\mu}_n(4) \end{pmatrix}.$$

Via Cholesky decomposition we can write the Hermitian positive-definite matrix σ_n in the form $\sigma_n = K_n K_n^t$, where K_n is a lower triangular matrix with real and positive diagonal entries, and K_n^t denotes the conjugate transpose of K_n. By an explicit computation, it is possible to show that the Cholesky decomposition of σ_n takes the form $\sigma_n = K_n K_n^t$, where

$$K_n = \begin{pmatrix} \sqrt{2n}\,\pi & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2n}\,\pi & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2}\pi^2 n\sqrt{3+\hat{\mu}_n(4)} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}\pi^2 n\sqrt{1-\hat{\mu}_n(4)} & 0 \\ 0 & 0 & \sqrt{2}\pi^2 n\frac{1-\hat{\mu}_n(4)}{\sqrt{3+\hat{\mu}_n(4)}} & 0 & 4\pi^2 n\frac{\sqrt{1+\hat{\mu}_n(4)}}{\sqrt{3+\hat{\mu}_n(4)}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{E_n}}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{E_n}}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & \frac{E_n}{2\sqrt{2}}\sqrt{3+\hat{\mu}_n(4)} & 0 & 0 \\ 0 & 0 & 0 & \frac{E_n}{2\sqrt{2}}\sqrt{1-\hat{\mu}_n(4)} & 0 \\ 0 & 0 & \frac{E_n}{2\sqrt{2}}\frac{1-\hat{\mu}_n(4)}{\sqrt{3+\hat{\mu}_n(4)}} & 0 & E_n\frac{\sqrt{1+\hat{\mu}_n(4)}}{\sqrt{3+\hat{\mu}_n(4)}} \end{pmatrix}$$

$$=: \begin{pmatrix} k_1 & 0 & 0 & 0 & 0 \\ 0 & k_1 & 0 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 \\ 0 & 0 & 0 & k_4 & 0 \\ 0 & 0 & k_2 & 0 & k_5 \end{pmatrix}. \tag{5.53}$$

We can hence introduce a 5-dimensional standard Gaussian vector

$$Y(x) = (Y_1(x), Y_2(x), Y_3(x), Y_4(x), Y_5(x))$$

with independent components such that

$$\begin{aligned}&(\partial_1 f_n(x), \partial_2 f_n(x), \partial_{11} f_n(x), \partial_{12} f_n(x), \partial_{22} f_n(x)) = K_n Y(x)\\ &= (k_1 Y_1(x), k_1 Y_2(x), k_3 Y_3(x), k_4 Y_4(x), k_5 Y_5(x) + k_2 Y_3(x)).\end{aligned}$$

Hence the expression (5.52) can be rewritten as

$$\begin{aligned}F_n^\epsilon(x) &= [k_3 Y_3(x)(k_5 Y_5(x) + k_2 Y_3(x)) - (k_4 Y_4(x))^2]\, \mathbf{1}_{\{\frac{k_3}{E_n} Y_3(x) + \frac{k_5}{E_n} Y_5(x) + \frac{k_2}{E_n} Y_3(x) \le -u\}}\\ &\quad \times \frac{1}{(2\epsilon)^2} \mathbf{1}_{[-\epsilon,\epsilon]^2}(k_1 Y_1(x), k_1 Y_2(x))\\ &= [k_3 Y_3(x)(k_5 Y_5(x) + k_2 Y_3(x)) - (k_4 Y_4(x))^2]\, \mathbf{1}_{\{\frac{k_3}{E_n} Y_3(x) + \frac{k_5}{E_n} Y_5(x) + \frac{k_2}{E_n} Y_3(x) \le -u\}}\\ &\quad \times \frac{1}{k_1^2} \frac{1}{\left(\frac{2\epsilon}{k_1}\right)^2} \mathbf{1}_{\left[-\frac{\epsilon}{k_1}, \frac{\epsilon}{k_1}\right]^2}(Y_1(x), Y_2(x)).\end{aligned} \tag{5.54}$$

Let us now set

$$\widetilde{k}_i := \frac{k_i}{E_n}, \qquad i = 2, 3, 4, 5,$$

and note that $\max_{i=2,\dots,5} \widetilde{k}_i = O(1)$, where the constant involved in the O-notation is absolute. From (5.54) we can rewrite (4.38) as

$$\begin{aligned}\mathcal{L}_0^\epsilon(n; u) &= \int_{\mathbb{T}} F_n^\epsilon(x)\, dx\\ &= \frac{2E_n}{\left(\frac{2\epsilon}{k_1}\right)^2} \int_{\mathbb{T}^2} \left(\widetilde{k}_3 Y_3(x)(\widetilde{k}_5 Y_5(x) + \widetilde{k}_2 Y_3(x)) - \widetilde{k}_4^2 Y_4(x)^2\right)\\ &\quad \times \mathbf{1}_{\{(\widetilde{k}_3 + \widetilde{k}_2) Y_3(x) + \widetilde{k}_5 Y_5(x) \le -u\}} \mathbf{1}_{\left[-\frac{\epsilon}{k_1}, \frac{\epsilon}{k_1}\right]^2} (Y_1(x), Y_2(x))\, dx.\end{aligned} \tag{5.55}$$

5.2 Proof of Lemma 4.7

We will need the following preliminary results.

Lemma 5.1. *For every $\epsilon > 0$, $n \in S$ and $x \in \mathbb{T}$, the chaotic expansion of*

$$\delta^{\epsilon/k_1}(Y_1(x), Y_2(x)) := \frac{1}{\left(\frac{2\epsilon}{k_1}\right)^2} \mathbf{1}_{\left[-\frac{\epsilon}{k_1}, \frac{\epsilon}{k_1}\right]^2}(Y_1(x), Y_2(x))$$

is the following series

$$\delta^{\epsilon/k_1}(Y_1(x), Y_2(x)) = \sum_{q=0}^{+\infty} \sum_{q'=0}^{q} \frac{\beta_{2q'}^{\epsilon/k_1}}{(2q')!} \frac{\beta_{2q-2q'}^{\epsilon/k_1}}{(2q-2q')!} H_{2q'}(Y_1(x)) H_{2q-2q'}(Y_2(x)), \tag{5.56}$$

where the convergence is in the $L^2(\mathbb{P})$-sense, and for $q \in \mathbb{N}_{\ge 1}$,

$$\beta_0^{\epsilon/k_1} = \frac{1}{\frac{2\epsilon}{k_1}} \int_{-\epsilon/k_1}^{\epsilon/k_1} \phi(t)\, dt, \qquad \beta_{2q}^{\epsilon/k_1} := \frac{1}{\frac{2\epsilon}{k_1}} \int_{-\epsilon/k_1}^{\epsilon/k_1} H_{2q}(t) \phi(t)\, dt. \tag{5.57}$$

Lemma 5.2. *For every $n \in S$, $x \in \mathbb{T}$ and $u \in \mathbb{R}$, the chaotic expansion of*

$$p_n(Y_3(x), Y_4(x), Y_5(x))\mathbf{1}_{\{(\widetilde{k}_3+\widetilde{k}_2)Y_3(x)+\widetilde{k}_5 Y_5(x)\le -u\}},$$

where $p_n(Y_3(x), Y_4(x), Y_5(x)) := \widetilde{k}_3 Y_3(x)(\widetilde{k}_5 Y_5(x) + \widetilde{k}_2 Y_3(x)) - \widetilde{k}_4^2 Y_4(x)^2$, is

$$\begin{aligned} &p_n(Y_3(x), Y_4(x), Y_5(x))\mathbf{1}_{\{(\widetilde{k}_3+\widetilde{k}_2)Y_3(x)+\widetilde{k}_5 Y_5(x)\le -u\}} \\ &= \sum_{q=0}^{+\infty} \sum_{a+b+c=q} \frac{\eta^{(n)}_{a,b,c}(u)}{a!b!c!} H_a(Y_3(x))H_a(Y_4(x))H_c(Y_5(x)), \end{aligned} \tag{5.58}$$

where the series converges in $L^2(\mathbb{P})$.

We omit the proofs of Lemma 5.1 and Lemma 5.2. We are now in a position to establish Lemma 4.7.

Proof of Lemma 4.7. Let us first find the chaotic decomposition of $F_n^\epsilon(x)$ in (5.52). Lemma 5.1 together with Lemma 5.2 gives

$$\begin{aligned} F_n^\epsilon(x) &= 2E_n \sum_{q=0}^{+\infty} \sum_{a+b+c+2d+2e=q} \frac{\eta^{(n)}_{a,b,c}(u)}{a!b!c!} \frac{\beta_{2d}^{\epsilon/k_1}}{(2d)!} \frac{\beta_{2e}^{\epsilon/k_1}}{(2e)!} \\ &\quad \times H_{2d}(Y_1(x))H_{2e}(Y_2(x))H_a(Y_3(x))H_b(Y_4(x))H_c(Y_5(x)) \end{aligned}$$

entailing that the chaotic expansion of $\mathcal{L}_0(n; u)$ in (5.55) is

$$\begin{aligned} \mathcal{L}_0^\epsilon(n; u) = \quad & 2E_n \sum_{q=0}^{+\infty} \sum_{a+b+c+2d+2e=q} \frac{\eta^{(n)}_{a,b,c}(u)}{a!b!c!} \frac{\beta_{2d}^{\epsilon/k_1}}{(2d)!} \frac{\beta_{2e}^{\epsilon/k_1}}{(2e)!} \\ & \times \int_{\mathbb{T}} H_{2d}(Y_1(x))H_{2e}(Y_2(x))H_a(Y_3(x))H_b(Y_4(x))H_c(Y_5(x))\, dx. \end{aligned} \tag{5.59}$$

Now note that, as $\epsilon \to 0$, for every $q \in \mathbb{N}$

$$\beta_q^{\epsilon/k_1} \to \beta_q$$

as defined in (4.41). Hence, bearing in mind the Cholesky decomposition in §5.1, Lemma 4.6 together with (5.59) allows to get (4.40) thus concluding the proof.

□

6 Second chaotic components

6.1 EPC: preliminary results

We compute now the projection coefficients of $\mathcal{L}_0(n; u)$ on second Wiener chaos. From (4.40) in Lemma 4.7 we can write more compactly

$$\mathrm{Proj}[\mathcal{L}_0(n; u)|2] = 2E_n \sum_{a+b+c+2d+2e=2} \frac{\eta^{(n)}_{a,b,c}(u)}{a!b!c!} \frac{\beta_{2d}\beta_{2e}}{(2d)!(2e)!} \int_{\mathbb{T}} H_a\left(\frac{\partial_{11} f_n(x)}{k_3}\right)$$

$$
\begin{aligned}
&\times H_b\left(\frac{\partial_{12} f_n(x)}{k_4}\right) H_c\left(\frac{\partial_{22} f_n(x)}{k_5}-\frac{k_2}{k_5 k_3}\partial_{11} f_n(x)\right) H_{2d}\left(\frac{\partial_1 f_n(x)}{k_1}\right)\\
&\times H_{2e}\left(\frac{\partial_2 f_n(x)}{k_1}\right)\, dx \qquad (6.60)\\
&=\sum_{i=1}^{5}\sum_{j=1}^{i-1} h_{ij}(u;n)\int_{\mathbb{T}} Y_i(x)Y_j(x)dx+\frac{1}{2}\sum_{i=1}^{5} h_i(u;n)\int_{\mathbb{T}} H_2(Y_i(x))dx,
\end{aligned}
$$

where for $i,j=1,\dots 5$, $i\neq j$

$$
h_{ij}(u;n):=\lim_{\varepsilon\to 0}\mathbb{E}\left[k_3Y_3(k_5Y_5+k_2Y_3)-(k_4Y_4)^2\right]\mathbb{1}_{\left\{\frac{k_2+k_3}{E}Y_3+\frac{k_5}{E}Y_5\leq -u\right\}}\,\delta_\varepsilon(k_1Y_1,k_1Y_2)Y_iY_j\Big];
$$

on the other hand, for $i=1,\dots 5$,

$$
h_i(u;n):=\lim_{\varepsilon\to 0}\mathbb{E}\left[k_3Y_3(k_5Y_5+k_2Y_3)-(k_4Y_4)^2\right]\mathbb{1}_{\left\{\frac{k_2+k_3}{E}Y_3+\frac{k_5}{E}Y_5\leq -u\right\}}\,\delta_\varepsilon(k_1Y_1,k_1Y_2)H_2(Y_i)\Big].
$$

The following proposition provides analytic expressions for the coefficients h_{ij} and h_i.

Proposition 6.1. *It holds that $h_{ij}(u;n)=0$ for all $(i,j)\neq(3,5)$ and*

$$
h_{35}(u;n) = \frac{E_n}{2\sqrt{2}\pi}\sqrt{1+\hat{\mu}_n(4)}\frac{u\phi(u)(1+u^2)+(3+\hat{\mu}_n(4))\Phi(-u)}{3+\hat{\mu}_n(4)}.
$$

Moreover

$$
\begin{aligned}
h_1(u;n) &= h_2(u;n)=-\frac{E_n}{4\pi}u\,\phi(u),\\
h_3(u;n) &= \frac{E_n}{4\pi}\left[\frac{2u(1+u^2)\phi(u)}{3+\hat{\mu}_n(4)}+\Phi(-u)(1-\hat{\mu}_n(4))\right],\\
h_4(u;n) &= -\frac{E_n}{4\pi}(1-\hat{\mu}_n(4))\Phi(-u),\\
h_5(u;n) &= \frac{E_n}{4\pi}\frac{u(1+u^2)(1+\hat{\mu}_n(4))\phi(u)}{3+\hat{\mu}_n(4)}.
\end{aligned}
$$

The proof of Proposition 6.1 is postponed to the Appendix D.1. From equation (6.60) and Proposition 6.1 it is then immediate to obtain the following expression:

$$
\mathrm{Proj}[\mathcal{L}_0(n;u)|2]=h_{35}(u;n)A_{35}(n)+\frac{1}{2}\sum_{i=1}^{5}h_i(u;n)B_i(n); \qquad (6.61)
$$

where

$$
A_{ij}(n)=\int_{\mathbb{T}} Y_i(x)Y_j(x)dx,\qquad B_i(n)=\int_{\mathbb{T}} H_2(Y_i(x))dx. \qquad (6.62)
$$

Our next step is then to investigate the behaviour of the integrals of stochastic processes in (6.62).

Proposition 6.2. *We have that*

$$
A_{35}(n) = \frac{1}{\mathcal{N}_n}\sum_{\lambda}|a_\lambda|^2\Big[\frac{2\sqrt{2}}{n\sqrt{1+\hat{\mu}_n(4)}}\frac{5-\hat{\mu}_n(4)}{3+\hat{\mu}_n(4)}\lambda_2^2-\frac{1-\hat{\mu}_n(4)}{3+\hat{\mu}_n(4)}\frac{2\sqrt{2}}{\sqrt{1+\hat{\mu}_n(4)}}
$$

$$-\frac{\sqrt{2}}{n^2\sqrt{1+\hat{\mu}_n(4)}}\frac{8}{3+\hat{\mu}_n(4)}\lambda_2^4\Big],$$

and

$$B_1(n) = \frac{1}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\frac{2}{n}\lambda_1^2 - 1, B_2(n) = \frac{1}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\frac{2}{n}\lambda_2^2 - 1,$$

$$B_3(n) = \frac{1}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\left[\frac{8}{3+\hat{\mu}_n(4)} + \frac{8}{n^2(3+\hat{\mu}_n(4))}\lambda_2^4 - \frac{16}{n(3+\hat{\mu}_n(4))}\lambda_2^2\right] - 1,$$

$$B_4(n) = \frac{1}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\frac{8}{n^2(1-\hat{\mu}_n(4))}\lambda_1^2\lambda_2^2 - 1,$$

$$\begin{aligned} B_5(n) &= \frac{1}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\Big[\frac{16}{n^2(1+\hat{\mu}_n(4))(3+\hat{\mu}_n(4))}\lambda_2^4 + \frac{(1-\hat{\mu}_n(4))^2}{(3+\hat{\mu}_n(4))(1+\hat{\mu}_n(4))} \\ &\quad -\frac{1-\hat{\mu}_n(4)}{n(3+\hat{\mu}_n(4))(1+\hat{\mu}_n(4))}8\lambda_2^2\Big] - 1. \end{aligned}$$

The proof of Proposition 6.2 is technical hence postponed to the Appendix D.2.

6.2 Proof of Proposition 3.2

Proof. From Lemma 4.2 and (3.20) for $q = 2$ we have

$$\begin{aligned} \mathrm{Proj}[\mathcal{L}_2(n;u)|2] &= \frac{\gamma_2(u)}{2!}\int_{\mathbb{T}} H_2(f_n(x))\,dx \\ &= \frac{1}{2}u\phi(u)\frac{1}{\mathcal{N}_n/2}\sum_{\lambda\in\Lambda_n^+}(|a_\lambda|^2-1) \end{aligned}$$

which is (3.2) for $k = 2$. Proposition 3.2 for $k = 1$ has been proved in the spherical case in [Ros15, §7.3] via an application of Green's formula, the proof for Arithmetic Random Waves is analogous hence omitted for the sake of brevity. Let us now focus on the Euler-Poincaré characteristic: plugging the results of Proposition 6.1 and Proposition 6.2 into (6.61) some straightforward computations give Proposition 3.2 for $k = 0$.

□

7 Higher order chaotic components

In this section we investigate higher order chaotic components of $\mathcal{L}_k(n;u)$, $k = 0, 1, 2$. Let us start studying the variance of the projection onto the third Wiener chaos.

Lemma 7.1. *For $k = 0, 1, 2$, as $\mathcal{N}_n \to +\infty$,*

$$\mathrm{Var}\left(\mathcal{L}_k(n;u)[3]\right) = O\left(\frac{E_n^{2-k}}{\mathcal{N}_n^2}\right), \tag{7.63}$$

where the constant involved in the 'O'-notation does not depend on n.

Proof. From Lemma 4.2 and properties of Hermite polynomials (see e.g. [NP12, Proposition 2.2.1]) we have

$$\begin{aligned}
\mathrm{Var}(\mathcal{L}_2(n;u)[3]) &= \frac{\gamma_3(u)^2}{3!}\int_{\mathbb{T}} r_n(x)^3\,dx \\
&= \frac{\gamma_3(u)^2}{3!}\cdot\frac{1}{\mathcal{N}_n^3}\sum_{\lambda,\lambda_1,\lambda_2\in\Lambda_n}\int_{\mathbb{T}} e_{\lambda+\lambda_1+\lambda_2}(x)\,dx \\
&= \frac{\gamma_3(u)^2}{3!}\cdot\frac{|S_3(n)|}{\mathcal{N}_n^3}, \qquad (7.64)
\end{aligned}$$

where

$$S_3(n) := \{(\lambda,\lambda_1,\lambda_2)\in\Lambda_n^3 : \lambda+\lambda_1+\lambda_2=0\} \qquad (7.65)$$

is the length−3 spectral correlation set. Reasoning as in [KKW13, p.31] it is immediate to check that

$$|S_3(n)| = O(\mathcal{N}_n), \qquad (7.66)$$

hence from (7.64) we deduce that

$$\mathrm{Var}(\mathcal{L}_2(n;u)[3]) = O\left(\frac{1}{\mathcal{N}_n^2}\right) \qquad (7.67)$$

which is (7.63) for $k=2$. From Lemma 4.4 we have

$$\begin{aligned}
\mathrm{Var}(\mathcal{L}_1(n;u)[3]) &= \frac{1}{4}\frac{E_n}{2}\sum_{\substack{a+2b+2c=3\\a'+2b'+2c'=3}}\frac{\beta_a(u)\alpha_{2b,2c}}{a!(2b)!(2c)!}\frac{\beta_{a'}(u)\alpha_{2b',2c'}}{(a')!(2b')!(2c')!} \\
&\quad\times\int_{\mathbb{T}}\mathbb{E}\big[H_a(f_n(x))H_{2b}(f_{n,1}(x))H_{2c}(f_{n,2}(x)) \\
&\quad\times H_{a'}(f_n(y))H_{2b'}(f_{n,1}(y))H_{2c'}(f_{n,2}(y))\big]\,dxdy. \qquad (7.68)
\end{aligned}$$

From Lemma 4.7 we have

$$\begin{aligned}
\mathrm{Var}(\mathcal{L}_0(n;u)[3]) &= 4E_n^2\sum_{\substack{a+b+c+2d+2e=3\\a'+b'+c'+2d'+2e'=3}}\frac{\eta_{a,b,c}(u)}{a!b!c!}\frac{\beta_{2d}\beta_{2e}}{(2d)!(2e)!}\frac{\eta_{a',b',c'}(u)}{a'!b'!c'!}\frac{\beta_{2d'}\beta_{2e'}}{(2d')!(2e')!} \qquad (7.69)\\
&\quad\times\iint_{\mathbb{T}\times\mathbb{T}}\mathbb{E}\big[H_a(Y_3(x))H_b(Y_5(x))H_c(Y_4(x))H_{2d}(Y_1(x))H_{2e}(Y_2(x)) \qquad (7.70)\\
&\quad\times H_{a'}(Y_3(y))H_{b'}(Y_5(y))H_{c'}(Y_4(y))H_{2d'}(Y_1(y))H_{2e'}(Y_2(y))\big]\,dxdy. \qquad (7.71)
\end{aligned}$$

Consider the term for $a=a'=3$, $b=b'=c=c'=d=d'=e=e'=0$. We have

$$\begin{aligned}
&4E_n^2\frac{\eta_{3,0,0}(u)^2}{(3!)^2}\beta_0^4\iint_{\mathbb{T}\times\mathbb{T}}\mathbb{E}[Y_3(x)Y_3(y)]^3\,dxdy \\
&= 4E_n^2\frac{\eta_{3,0,0}(u)^2}{(3!)^2}\beta_0^4\iint_{\mathbb{T}\times\mathbb{T}}\frac{1}{k_3^6}\mathbb{E}\left[\partial_{11}f_n(x)\partial_{11}f_n(y)\right]^3\,dxdy \\
&= 4E_n^2\frac{\eta_{3,0,0}(u)^2}{(3!)^2}\beta_0^4\iint_{\mathbb{T}\times\mathbb{T}}\frac{1}{k_3^6}\left(\frac{16\pi^4}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}\lambda_1^4\,\mathrm{e}^{i2\pi\langle\lambda,x-y\rangle}\right)^3\,dxdy
\end{aligned}$$

$$
\begin{aligned}
&= 4E_n^2 \frac{\eta_{3,0,0}(u)^2}{(3!)^2} \beta_0^4 \iint_{\mathbb{T}\times\mathbb{T}} \frac{1}{k_3^6} \frac{(16\pi^4)^3}{\mathcal{N}_n^3} \sum_{\lambda,\lambda',\lambda''\in\Lambda_n} \lambda_1^4 (\lambda_1')^4 (\lambda_1'')^4 \, \mathrm{e}^{i2\pi\langle\lambda+\lambda'+\lambda'', x-y\rangle} \, dxdy \\
&= 4E_n^2 \frac{\eta_{3,0,0}(u)^2}{(3!)^2} \beta_0^4 \frac{1}{k_3^6} \frac{(16\pi^4)^3}{\mathcal{N}_n^3} \sum_{(\lambda,\lambda',\lambda'')\in S_3(n)} \lambda_1^4 (\lambda_1')^4 (\lambda_1'')^4 \\
&= 4E_n^2 \frac{\eta_{3,0,0}(u)^2}{(3!)^2} \beta_0^4 \underbrace{\left(\frac{n}{k_3}\right)^6}_{\le 1} \frac{(16\pi^4)^3}{\mathcal{N}_n^3} \sum_{(\lambda,\lambda',\lambda'')\in S_3(n)} \underbrace{\left(\frac{\lambda_1}{\sqrt{n}}\right)^4 \left(\frac{\lambda_1'}{\sqrt{n}}\right)^4 \left(\frac{\lambda_1''}{\sqrt{n}}\right)^4}_{\le 1} \\
&\le 4(16\pi^4)^3 \frac{\eta_{3,0,0}(u)^2 \beta_0^4}{(3!)^2} E_n^2 \frac{|S_3(n)|}{\mathcal{N}_n^3}. \qquad (7.72)
\end{aligned}
$$

Plugging (7.66) into (7.72) and repeating a similar argument for the other summands on the right hand side of (7.69) we get (7.63) for $k = 0$. The proof for (7.63) for $k = 1$ is similar. hence we omit the details.

□

Lemma 7.1 ensures that $\mathcal{L}_k(n;u)[3]$ is asymptotically negligible with respect to $\mathcal{L}_k(n;u)[2]$ for any $k \in \{0,1,2\}$ (see Proposition 3.2), as happens for the remaining chaotic projections. All the results to follow hold for every n when $k = 1, 2$, and for $n \in S'$ for $k = 0$. For brevity's sake we avoid to repeat these conditions in the statements below.

Lemma 7.2. *As $\mathcal{N}_n \to +\infty$ under Condition 2.2*

$$
\operatorname{Var}\left(\sum_{q=4}^{+\infty} \mathcal{L}_k(n;u)[q]\right) = O\left(\frac{E_n^{2-k}}{\mathcal{N}_n^2}\right), \qquad (7.73)
$$

where the constant involved in the $'O'$-notation does not depend on n.

The proof of Proposition 7.2 for $k = 2$ is simple, the proof for $k = 1$ can be treated analogously as the proof of Lemma 2 in [PR18] hence we will omit both of them.

The proof of Proposition 7.2 for $k = 0$ is inspired by the proofs of Proposition 2.3 in [DNPR19, §5] and Lemma 2 in [PR18]. Let us first decompose $\mathbb{T}$ as a *disjoint* union of squares Q_k, $k \in \mathbb{Z}^2$, each of side length $1/M$, where

$$
M = \lceil d\sqrt{E_n} \rceil, \qquad d \in \mathbb{R}_{>0} \text{ to be chosen later}, \qquad (7.74)
$$

obtained by translating along directions k/M, $k \in \mathbb{Z}^2$, the square

$$
Q_0 := [0, 1/M) \times [0, 1/M). \qquad (7.75)
$$

In what follows we will often drop the dependence of k from Q_k.

7.1 Singular squares

This part is inspired by [ORW08, §6.1] and [RW16, §4.3]. Let us fix $0 < \epsilon \ll 1$ and choose d in (7.74) such that $d \ge c/\epsilon$; write $r_n(x-y) = r(x-y) = \mathbb{E}[f_n(x) f_n(y)]$, $r_{n,1}(x-y) = r_1(x-y) = \mathbb{E}[\partial_1 f_n(x) \partial_1 f_n(y)]$, $r_{n,2}(x-y) = r_2(x-y) = \mathbb{E}[\partial_2 f_n(x) \partial_2 f_n(y)]$, and analogously for second-order derivatives. A pair of points $(x,y) \in \mathbb{T}\times\mathbb{T}$ is said to be *singular* if either $|r(x-y)| > \epsilon$ or $|r_1(x-y)| > \epsilon\sqrt{n}$ or $|r_2(x-y)| > \epsilon\sqrt{n}$ or $|r_{11}(x-y)| > \epsilon\, n$ or $|r_{12}(x-y)| > \epsilon\, n$ or $|r_{22}(x-y)| > \epsilon\, n$.

Definition 7.3. *A pair of squares (Q, Q') is said to be singular if there exists a singular pair of points $(x, y) \in Q \times Q'$.*

For instance, (Q_0, Q_0) is a singular pair of squares. The following is Lemma 5.2 in [DNPR19].

Lemma 7.4. *Let (Q, Q') be a singular pair of squares, then for every $(x, y) \in (Q, Q')$ either $|r(x-y)| > \frac{1}{2}\epsilon$ or $|r_1(x-y)| > \frac{1}{2}\epsilon\sqrt{n}$ or $|r_2(x-y)| > \frac{1}{2}\epsilon\sqrt{n}$ or $|r_{11}(x-y)| > \frac{1}{2}\epsilon\, n$ or $|r_{12}(x-y)| > \frac{1}{2}\epsilon\, n$ or $|r_{22}(x-y)| > \frac{1}{2}\epsilon\, n$.*

Let us now denote by S_Q the union of all squares Q' such that (Q, Q') is a singular pair of squares. The number of such squares Q' is $M^2 \cdot \text{area}(S_Q)$, indeed the area of each square is $1/M^2$. The following result is similar to Lemma 5.3 in [DNPR19] hence we omit the proof.

Lemma 7.5. *We have*

$$\text{area}(S_Q) \ll \int_{\mathbb{T}} r(x)^4\, dx. \tag{7.76}$$

Recall that

$$\int_{\mathbb{T}} r(x)^4\, dx = \frac{|S_4(n)|}{\mathcal{N}_n^4}, \tag{7.77}$$

where $S_4(n) := \{(\lambda, \lambda', \lambda'', \lambda''') \in \Lambda_n^4 : \lambda + \lambda' + \lambda'' + \lambda''' = 0\}$ is the length-4 spectral correlation set, and [MPRW16, Lemma 5.1] (see also [KKW13, p. 31])

$$|S_4(n)| = 3\mathcal{N}_n(\mathcal{N}_n - 1). \tag{7.78}$$

From Lemma 7.5 and (7.78) we immediately have

$$\text{area}(S_Q) \ll \frac{1}{\mathcal{N}_n^2}, \tag{7.79}$$

thus the number of squares Q' such that (Q, Q') is singular is $\ll E_n/\mathcal{N}_n^2$.

7.2 Variance and squares

Let us denote by $\mathcal{L}_0(n; u, Q)$ the Euler-Poincaré characteristic restricted to Q. Since the squares Q are disjoint we can write

$$\mathcal{L}_0(n; u) = \sum_Q \mathcal{L}_0(n; u, Q) \tag{7.80}$$

yielding

$$\text{proj}(\mathcal{L}_0(n; u)|C_{\geq 4}) = \sum_Q \text{proj}(\mathcal{L}_0(n; u, Q)|C_{\geq 4}). \tag{7.81}$$

From (7.81) we deduce

$$\begin{aligned} \text{Var}(\text{proj}(\mathcal{L}_0(n; u)|C_{\geq 4}) &= \sum_{Q,Q'} \text{Cov}(\text{proj}(\mathcal{L}_0(n; u, Q)|C_{\geq 4}), \text{proj}(\mathcal{L}_0(n; u, Q')|C_{\geq 4})) \qquad (7.82) \\ &= \sum_{(Q,Q')\text{ sing.}} \text{Cov}(\text{proj}(\mathcal{L}_0(n; u, Q)|C_{\geq 4}), \text{proj}(\mathcal{L}_0(n; u, Q')|C_{\geq 4})) \\ &\quad + \sum_{(Q,Q')\text{ non-sing.}} \text{Cov}(\text{proj}(\mathcal{L}_0(n; u, Q)|C_{\geq 4}), \text{proj}(\mathcal{L}_0(n; u, Q')|C_{\geq 4})); \end{aligned}$$

we are going to separately study the contribution of the singular part and the contribution of the non-singular part, i.e. the two summands on the right-hand-side of (7.82).

7.3 The contribution of the singular part

By Cauchy-Schwartz inequality and stationarity of the model we have

$$\begin{aligned}
&\left| \sum_{(Q,Q') \text{ sing.}} \mathrm{Cov}(\mathrm{proj}(\mathcal{L}_0(n;u,Q)|C_{\geq 4}), \mathrm{proj}(\mathcal{L}_0(n;u,Q')|C_{\geq 4})) \right| \\
&\leq \sum_{(Q,Q') \text{ sing.}} \mathrm{Var}(\mathrm{proj}(\mathcal{L}_0(u;Q_0)|C_{\geq 4})) \\
&\ll M^2 \frac{E_n}{\mathcal{N}_n^2} \mathrm{Var}(\mathrm{proj}(\mathcal{L}_0(u;Q_0)|C_{\geq 4})) \ll \frac{E_n^2}{\mathcal{N}_n^2} \mathrm{Var}(\mathrm{proj}(\mathcal{L}_0(u;Q_0)|C_{\geq 4})),
\end{aligned} \tag{7.83}$$

where for the last two estimates we used (7.79) and (7.74) respectively. Now write

$$\begin{aligned}
\mathrm{Var}(\mathrm{proj}(\mathcal{L}_0(u;Q_0)|C_{\geq 4})) &\leq \mathbb{E}\left[\mathcal{L}_0(u;Q_0)^2\right] \\
&= \mathbb{E}\left[\mathcal{L}_0(u;Q_0)(\mathcal{L}_0(u;Q_0)-1)\right] + \mathbb{E}\left[\mathcal{L}_0(u;Q_0)\right].
\end{aligned} \tag{7.84}$$

Lemma 7.6. *For every $n \in S$ and $u \in \mathbb{R}$*

$$\mathbb{E}\left[\mathcal{L}_0(u;Q_0)\right] = O(1),$$

where the constant involved in the O-notation is absolute.

The proof of Lemma 7.6 follows from the stationarity of the model and the fact that $\mathcal{L}_0(n;u)$ is bounded from above by E_n. Lemma 7.6 together with (2.10) entail that the right hand side of (7.84) is $O(1)$ hence

$$\mathrm{Var}(\mathrm{proj}(\mathcal{L}_0(u;Q_0)|C_{\geq 4})) = O(1), \tag{7.85}$$

where the constant involved in the O-notation is absolute, that together with (7.83) proves the following.

Lemma 7.7. *As $\mathcal{N}_n \to +\infty$ under Condition 2.2*

$$\left| \sum_{(Q,Q') \text{ sing.}} \mathrm{Cov}(\mathrm{proj}(\mathcal{L}_0(u;Q)|C_{\geq 4}), \mathrm{proj}(\mathcal{L}_0(u;Q')|C_{\geq 4})) \right| = O\left(\frac{E_n^2}{\mathcal{N}_n^2}\right), \tag{7.86}$$

where the constant involved in the O-notation is absolute.

7.4 The contribution of the non-singular part

In this part we prove the following.

Lemma 7.8. *As $\mathcal{N}_n \to +\infty$*

$$\left| \sum_{(Q,Q') \text{ non-sing.}} \mathrm{Cov}(\mathrm{proj}(\mathcal{L}_0(u;Q)|C_{\geq 4}), \mathrm{proj}(\mathcal{L}_0(u;Q')|C_{\geq 4})) \right| = O\left(\frac{E_n^2}{\mathcal{N}_n^2}\right), \tag{7.87}$$

where the constant involved in the O-notation is absolute.

Proof. As in the proof of Lemma 3.5 in [DNPR19] we can write

$$\begin{aligned}&\left|\sum_{(Q,Q')\text{ non-sing.}} \mathrm{Cov}(\mathrm{proj}(\mathcal{L}_0(u;Q)|C_{\geq 4}), \mathrm{proj}(\mathcal{L}_0(u;Q')|C_{\geq 4}))\right| \\ &\leq 4E_n^2 \sum_{q\geq 4} \sum_{\substack{a+b+c+2d+2e=q\\ a'+b'+c'+2d'+2e'=q}} \left| \frac{\eta^{(n)}_{a,b,c}(u)}{a!b!c!} \frac{\beta_{2d}^{\epsilon/k_1}\beta_{2e}^{\epsilon/k_1}}{(2d)!(2e)!} \frac{\eta^{(n)}_{a',b',c'}(u)}{a'!b'!c'!} \frac{\beta_{2d'}^{\epsilon/k_1}\beta_{2e'}^{\epsilon/k_1}}{(2d')!(2e')!}\right| \\ &\times |V(a,b,c,d,e,a',b',c',d',e')|,\end{aligned} \tag{7.88}$$

where $V(a,b,c,d,e,a',b',c',d',e')$ is the sum of no more than $q!$ terms of the type

$$v = \sum_{(Q,Q')\text{ non-sing.}} \iint_{Q\times Q'} \prod_{u=1}^{q} R_{l_u,k_u}(x-y)\,dxdy, \tag{7.89}$$

with $l_u, k_u \in \{0,1,2\}$ and where for $l,k = 0,1,2$ and $x,y \in \mathbb{T}$ we set

$$R_{l,k}(x-y) := \mathbb{E}[Y_l(x)Y_k(y)]. \tag{7.90}$$

For every integer $q \geq 4$

$$\begin{aligned}|v| &\leq \sum_{(Q,Q')\text{ non-sing.}} \iint_{Q\times Q'} |r_n(x-y)|^q\,dxdy \\ &\leq \epsilon^{q-4} \sum_{(Q,Q')\text{ non-sing.}} \iint_{Q\times Q'} r_n(x-y)^4\,dxdy \\ &\leq \epsilon^{q-4} \int_{\mathbb{T}} r_n(x)^4\,dx.\end{aligned} \tag{7.91}$$

From (7.91) we deduce that

$$\begin{aligned}|V(a,b,c,d,e,a',b',c',d',e')| &\leq q!\frac{\int_{\mathbb{T}} r_n(x)^4\,dx}{\epsilon^4}\epsilon^q \\ &= q!\frac{\int_{\mathbb{T}} r_n(x)^4\,dx}{\epsilon^4}\,(\sqrt{\epsilon})^{a+b+c+2d+2e}(\sqrt{\epsilon})^{a'+b'+c'+2d'+2e'}.\end{aligned} \tag{7.92}$$

Plugging (7.92) into (7.88) we get

$$\begin{aligned}&\left|\sum_{(Q,Q')\text{ non-sing.}} \mathrm{Cov}(\mathrm{proj}(\mathcal{L}_0(u;Q)|C_{\geq 4}), \mathrm{proj}(\mathcal{L}_0(u;Q')|C_{\geq 4}))\right| \\ &\leq 4E_n^2 \frac{\int_{\mathbb{T}} r_n(x)^4\,dx}{\epsilon^4} \sum_{q\geq 4} q! \sum_{\substack{a+b+c+2d+2e=q\\ a'+b'+c'+2d'+2e'=q}} \left| \frac{\eta^{(n)}_{a,b,c}(u)}{a!b!c!} \frac{\beta_{2d}^{\epsilon/k_1}\beta_{2e}^{\epsilon/k_1}}{(2d)!(2e)!} \frac{\eta^{(n)}_{a',b',c'}(u)}{a'!b'!c'!} \frac{\beta_{2d'}^{\epsilon/k_1}\beta_{2e'}^{\epsilon/k_1}}{(2d')!(2e')!}\right| \\ &\times\ (\sqrt{\epsilon})^{a+b+c+2d+2e}(\sqrt{\epsilon})^{a'+b'+c'+2d'+2e'};\end{aligned} \tag{7.93}$$

reasoning as in the proof of Lemma 3.5 in [DNPR19], for $5\sqrt{\epsilon} < 1$ we obtain

$$\begin{aligned} Z &:= \sum_{q\geq 4} q! \sum_{\substack{a+b+c+2d+2e=q \\ a'+b'+c'+2d'+2e'=q}} \left| \frac{\eta^{(n)}_{a,b,c}(u)}{a!b!c!} \frac{\beta^{\epsilon/k_1}_{2d}\beta^{\epsilon/k_1}_{2e}}{(2d)!(2e)!} \frac{\eta^{(n)}_{a',b',c'}(u)}{a'!b'!c'!} \frac{\beta^{\epsilon/k_1}_{2d'}\beta^{\epsilon/k_1}_{2e'}}{(2d')!(2e')!} \right| \\ &\times (\sqrt{\epsilon})^{a+b+c+2d+2e} (\sqrt{\epsilon})^{a'+b'+c'+2d'+2e'} \\ &\leq \sum_{a,b,c,d,e,a',b',c',d',e'} \left| \frac{\eta^{(n)}_{a,b,c}(u)}{a!b!c!} \frac{\beta^{\epsilon/k_1}_{d}\beta^{\epsilon/k_1}_{e}}{d!e!} \right|^2 (a+b+c+d+e)! \\ &\times (\sqrt{\epsilon})^{a+b+c+d+e+a'+b'+c'+d'+e'} \\ &\leq \sum_{a,b,c,d,e,a',b',c',d',e'} \frac{\left|\eta^{(n)}_{a,b,c}(u)\beta^{\epsilon/k_1}_{d}\beta^{\epsilon/k_1}_{e}\right|^2}{a!b!c!d!e!} 5^{a+b+c+d+e} (\sqrt{\epsilon_1})^{a+b+c+d+e+a'+b'+c'+d'+e'}. \end{aligned} \tag{7.94}$$

Let us now prove that

$$(a,b,c,d,e) \mapsto \frac{\left|\eta^{(n)}_{a,b,c}(u)\beta^{\epsilon/k_1}_{d}\beta^{\epsilon/k_1}_{e}\right|^2}{a!b!c!d!e!} \tag{7.95}$$

is uniformly bounded over ϵ and n. From (5.57), recalling that there exists $C > 0$ such that for every $q \in \mathbb{N}$ and $u \in \mathbb{R}$

$$|H_q(u)|\phi(u) \leq C\sqrt{q!},$$

we have for every $\epsilon > 0$, $n \in S$ and $d \in \mathbb{N}$,

$$\frac{\left|\beta^{\epsilon/k_1}_{d}\right|^2}{d!} \leq C^2. \tag{7.96}$$

Moreover from Lemma 5.2 we have

$$\begin{aligned} \sum_{q=1}^{+\infty} \sum_{a+b+c=q} \frac{\left|\eta^{(n)}_{a,b,c}(u)\right|^2}{a!b!c!} &= \mathbb{E}\left[p_n(Y_3(x), Y_4(x), Y_5(x)^2 \mathbf{1}_{(\widetilde{k}_3+\widetilde{k}_2)Y_3(x)+\widetilde{k}_5 Y_5(x)\leq -u}\right] \\ &\underset{(7.97)}{\leq} \mathbb{E}\left[p_n(Y_3(x), Y_4(x), Y_5(x)^2\right] = O(1), \end{aligned}$$

where the constant involved in the O-notation is absolute. Equation 7.95 together with (7.93), (7.94) and (7.78) allows to conclude the proof of Lemma 7.8.

□

Proof of Lemma 7.2. The proof follows from Lemma 7.7, Lemma 7.8 and (7.82).

□

7.5 Proof of Proposition 3.3

Proof. It suffices to combine Lemma 7.1 and Lemma 7.2 to get

$$\mathrm{Var}(\mathrm{proj}(\mathcal{L}_0(n;u)|C_{\geq 4}) = O(1), \tag{7.98}$$

where the constant involved in the O-notation is absolute.

□

A EPC: technical lemmas

By stationarity of the model the law of $\nabla f_n(x)$ is independent of $x \in \mathbb{T}$, it is centered Gaussian with covariance matrix given by a_n in (B.103). Since $\det(a_n) \neq 0$, Proposition 6.5 in [AW09] (with $Z = \nabla f_n$) ensures that for every $z \in \mathbb{R}$,

$$\mathbb{P}(\exists x \in \mathbb{T} : \nabla f_n(x) = z, \det(\nabla^2 f_n(x)) = 0) = 0. \tag{A.99}$$

In particular for $z = 0$, via a standard application of the inverse function theorem [AT07, p.136], we have that the set of critical points of f_n a.s. consists of a finite number of isolated points. By Bézout Theorem we deduce that the number of critical points of f_n is bounded from above by $4E_n$, and the Euler-Poincaré characteristic of any excursion set of f_n so (see the Morse representation formula below).

Now, in order to apply area formula as in [AW09, Proposition 6.1] (with $f = \nabla f_n$) we need to be sure that the set of critical *values* of ∇f_n a.s. has zero Lebeasgue measure. This follows from Sard's Lemma applied to ∇f_n.

Proof of Lemma 4.5. From (4.38) we have

$$|\mathcal{L}_0^\epsilon(u;n)| \leq \int_{\mathbb{T}} |\det(\nabla^2 f_n(x))| \frac{1}{(2\epsilon)^2} \mathbf{1}_{[-\epsilon,\epsilon]^2}(\nabla f_n(x))\, dx,$$

where the random variable on the right hand side approximates the number of critical points of f_n. Thanks to the previous discussion, we can apply the area formula [AW09, Proposition 6.1] obtaining

$$\int_{\mathbb{T}} |\det(\nabla^2 f_n(x))| \frac{1}{(2\epsilon)^2} \mathbf{1}_{[-\epsilon,\epsilon]^2}(\nabla f_n(x))\, dx = \frac{1}{(2\epsilon)^2} \int_{[-\epsilon,\epsilon]^2} \#\{x \in \mathbb{T} : \nabla f_n(x) = z\}\, dz. \tag{A.100}$$

By Bézout Theorem we have for every z

$$\#\{x \in \mathbb{T} : \nabla f_n(x) = z\} \leq 4E_n. \tag{A.101}$$

Substituting (A.101) into (A.100) we obtain the desired result.

□

By the Morse representation fomula [AT07, §9.3, §9.4] we obtain

$$\mathcal{L}_0(n;u) = \sum_{j=0}^{2} (-1)^j \mu_j \left(A_u(f_n;\mathbb{T}),\, f_n|_{A_u(f_n;\mathbb{T})} \right), \tag{A.102}$$

where

$$\begin{aligned} \mu_j \left(A_u(f_n;\mathbb{T}),\, f_n|_{A_u(f_n;\mathbb{T})} \right) &= \#\{x \in \mathbb{T} : f_n(x) \geq u, \nabla f_n(x) = 0, \mathrm{Ind}(-\nabla^2 f_n(x)) = j\} \\ &= \#\{x \in \mathbb{T} : \Delta f_n(x) \leq -E_n\, u, \nabla f_n(x) = 0, \mathrm{Ind}(-\nabla^2 f_n(x)) = j\}, \end{aligned}$$

(note that in the last equality we used the fact that $\Delta f_n = -E_n f_n$) $\mathrm{Ind}(M)$ denoting the number of negative eigenvalues of a square matrix M. More specifically, μ_0 is the number of maxima, μ_1 the number of saddles, and μ_2 the number of minima in the excursion region $A_u(f_n;\mathbb{T})$. Hence we can formally write

$$\mathcal{L}_0(n;u) = \sum_{j=0}^{2} (-1)^j \int_{\mathbb{T}} |\det(\nabla^2 f_n(x))| \mathbf{1}_{\{\Delta f_n(x) \leq -E_n\, u\}} \mathbf{1}_{\{\mathrm{Ind}(-\nabla^2 f_n(x)) = j\}} \delta_0(\nabla f_n(x))\, dx$$

which is (4.37).

Proof of Lemma 4.6. Thanks to Morse representation formula and then Theorem 11.2.3 in [AT07] (whose assumptions are satisfied in particular thanks to (A.99) for $z = 0$) we have a.s.

$$\begin{aligned}\lim_{\epsilon\to 0}\mathcal{L}_0^\epsilon(n;u) &= \lim_{\epsilon\to 0}\sum_{j=0}^{2}\frac{(-1)^j}{(2\epsilon)^2}\int_{\mathbb{T}}|\det(\nabla^2 f_n(x))|\mathbf{1}_{\{\Delta f_n(x)\le -E_n u\}}\mathbf{1}_{\{\mathrm{Ind}(-\nabla^2 f_n(x))=j\}}\mathbf{1}_{[-\epsilon,\epsilon]^2}(\nabla f_n(x))\,dx\\ &= \sum_{j=0}^{2}(-1)^j\mu_j\left(A_u(f_n;\mathbb{T}),\, f_n|_{A_u(f_n;\mathbb{T})}\right) = \mathcal{L}_0(n;u).\end{aligned}$$

The latter together with Lemma 4.5 immediately establish the $L^2(\mathbb{P})$-convergence thus concluding the proof.

□

B Computation of covariance matrices

Let $x, y \in \mathbb{T}$, and consider the Gaussian vector

$$(\partial_1 f_n(x), \partial_2 f_n(x), \partial_1 f_n(y), \partial_2 f_n(y), \partial_{11} f_n(x), \partial_{12} f_n(x), \partial_{22} f_n(x), \partial_{11} f_n(y), \partial_{12} f_n(y), \partial_{22} f_n(y)).$$

It is convenient to write its covariance matrix in block-diagonal form, i.e.

$$\Sigma_n(x,y) = \begin{pmatrix} A_n(x,y) & B_n(x,y) \\ B_n^t(x,y) & C_n(x,y) \end{pmatrix}.$$

In particular the A_n component collects the variances of the gradient terms, and it is given by

$$A_n(x,y) = \begin{pmatrix} a_n(x,x) & a_n(x,y) \\ a_n(y,x) & a_n(y,y) \end{pmatrix}, \qquad a_n(x,y) = \begin{pmatrix} r_{1,1}(x,y) & r_{1,2}(x,y) \\ r_{1,2}(x,y) & r_{2,2}(x,y) \end{pmatrix}.$$

It is easy to check that (cf. [KKW13, MPRW16]), for $i = 1, 2$,

$$r_{i,i}(x,y) = \frac{4\pi^2}{\mathcal{N}_n}\sum_{\lambda}\lambda_{(i)}^2 e(\langle\lambda, x-y\rangle),$$

while for $i \neq j$, $i, j = 1, 2$

$$r_{i,j}(x,y) = \mathbb{E}[\partial_i f_n(x)\partial_j f_n(y)] = \frac{4\pi^2}{\mathcal{N}_n}\sum_{\lambda}\lambda_{(i)}\lambda_{(j)} e(\langle\lambda, x-y\rangle) = r_{j,i}(x,y).$$

The matrix B_n collects the covariances between first and second order derivatives, and is given by

$$B_n(x,y) = \begin{pmatrix} 0 & b_n(x,y) \\ b_n(y,x) & 0 \end{pmatrix},$$

where

$$b_n(x,y) = \begin{pmatrix} r_{1,11}(x,y) & r_{1,12}(x,y) & r_{1,22}(x,y) \\ r_{2,11}(x,y) & r_{2,12}(x,y) & r_{2,22}(x,y) \end{pmatrix} = -b_n(y,x).$$

and

$$r_{1,11}(x,y) = \mathbb{E}[\partial_1 f_n(x)\partial_{11} f_n(y)] = \mathbb{E}[-\frac{2^2\pi^2}{\sqrt{\mathcal{N}_n}}\sum_{\lambda\in\Lambda_n} a_\lambda e(\langle\lambda,y\rangle)\lambda_{(1)}^2 \times \frac{2\pi i}{\sqrt{\mathcal{N}_n}}\sum_{\lambda\in\Lambda_n} a_\lambda e(\langle\lambda,x\rangle)\lambda_{(1)}]$$
$$= -\frac{8\pi^3 i}{\mathcal{N}_n}\mathbb{E}[\sum_{\lambda\in\Lambda_n} a_\lambda e(\langle\lambda,y\rangle)\lambda_{(1)}^2 \times \sum_{\lambda\in\Lambda_n} a_\lambda e(\langle\lambda,x\rangle)\lambda_{(1)}]$$
$$= -\frac{8\pi^3 i}{\mathcal{N}_n}\sum_{\lambda,\lambda'}\mathbb{E}[a_\lambda a_{\lambda'}]\ e(\langle\lambda'^2_{(1)}\ \lambda'_{(1)}$$
$$= -\frac{8\pi^3 i}{\mathcal{N}_n}\sum_{\lambda}\mathbb{E}[a_\lambda a_{-\lambda}]\ e(\langle\lambda,x\rangle)\ e(\langle-\lambda,y\rangle)\ (-\lambda_{(1)})^2\ \lambda_{(1)}$$
$$= -\frac{8\pi^3 i}{\mathcal{N}_n}\ \sum_{\lambda}\lambda_{(1)}^3 e(\langle\lambda,x-y\rangle),$$

so that for $i=1,2$

$$r_{i,ii}(x,y) = \mathbb{E}[\partial_i f_n(x)\partial_{ii} f_n(y)] = -\frac{8\pi^3 i}{\mathcal{N}_n}\ \sum_{\lambda}\lambda_{(i)}^3 e(\langle\lambda,x-y\rangle),$$

and we notice that

$$r_{i,ii}(y,x) = \mathbb{E}[\partial_i f_n(y)\partial_{ii} f_n(x)] = \mathbb{E}[-\frac{2^2\pi^2}{\sqrt{\mathcal{N}_n}}\sum_{\lambda\in\Lambda_n} a_\lambda e(\langle\lambda,x\rangle)\lambda_{(i)}^2 \times \frac{2\pi i}{\sqrt{\mathcal{N}_n}}\sum_{\lambda\in\Lambda_n} a_\lambda e(\langle\lambda,y\rangle)\lambda_{(i)}]$$
$$= -\frac{8\pi^3 i}{\mathcal{N}_n}\mathbb{E}[\sum_{\lambda\in\Lambda_n} a_\lambda e(\langle\lambda,x\rangle)\lambda_{(i)}^2 \times \sum_{\lambda\in\Lambda_n} a_\lambda e(\langle\lambda,y\rangle)\lambda_{(i)}]$$
$$= -\frac{8\pi^3 i}{\mathcal{N}_n}\sum_{\lambda,\lambda'}\mathbb{E}[a_\lambda a_{\lambda'}]\ e(\langle\lambda,x\rangle)\ e(\langle\lambda'^2_{(i)}\ \lambda'_{(i)}$$
$$= -\frac{8\pi^3 i}{\mathcal{N}_n}\sum_{\lambda}\mathbb{E}[a_\lambda a_{-\lambda}]\ e(\langle\lambda,x\rangle)\ e(\langle-\lambda,y\rangle)\ \lambda_{(i)}^2\ (-\lambda_{(i)})$$
$$= \frac{8\pi^3 i}{\mathcal{N}_n}\ \sum_{\lambda}\lambda_{(1)}^3 e(\langle\lambda,x-y\rangle),$$

and in general for $i,j,k=1,2$

$$r_{i,jk}(x,y) = -\frac{8\pi^3 i}{\mathcal{N}_n}\ \sum_{\lambda}\lambda_{(i)}\lambda_{(j)}\lambda_{(k)} e(\langle\lambda,x-y\rangle) = -r_{i,jk}(y,x),$$

so that

$$b_n(x,y) = \begin{pmatrix} r_{1,11}(x-y) & r_{1,12}(x-y) & r_{1,22}(x-y) \\ r_{1,12}(x-y) & r_{1,22}(x-y) & r_{2,22}(x-y) \end{pmatrix}.$$

Finally, for the matrix $C_n(x,y)$, we have

$$C_n(x,y) = \begin{pmatrix} c_n(x,x) & c_n(x,y) \\ c_n(y,x) & c_n(y,y) \end{pmatrix},$$

where of course $c_n(x,x) = c_n(y,y)$,

$$c_n(x,y) = \begin{pmatrix} r_{11,11}(x,y) & r_{11,12}(x,y) & r_{11,22}(x,y) \\ r_{12,11}(x,y) & r_{12,12}(x,y) & r_{12,22}(x,y) \\ r_{22,11}(x,y) & r_{22,12}(x,y) & r_{22,22}(x,y) \end{pmatrix}$$

and

$$\begin{aligned} r_{11,11}(x,y) = \mathbb{E}[\partial_{11} f_n(x) \partial_{11} f_n(y)] &= \mathbb{E}[-\frac{2^2\pi^2}{\sqrt{\mathcal{N}_n}} \sum_{\lambda\in\Lambda_n} a_\lambda e(\langle \lambda, x\rangle)\lambda_{(1)}^2 \times -\frac{2^2\pi^2}{\sqrt{\mathcal{N}_n}} \sum_{\lambda\in\Lambda_n} a_\lambda e(\langle \lambda, y\rangle)\lambda_{(1)}^2] \\ &= \frac{2^4\pi^4}{\mathcal{N}_n} \sum_\lambda \lambda_{(1)}^4 e(\langle \lambda, x-y\rangle), \end{aligned}$$

or, more generally

$$r_{ij,kl}(x,y) = \frac{2^4\pi^4}{\mathcal{N}_n} \sum_\lambda \lambda_{(i)}\lambda_{(j)}\lambda_{(k)}\lambda_{(l)} e(\langle \lambda, x-y\rangle),$$

so that

$$c_n(x,y) = \begin{pmatrix} r_{11,11}(x-y) & r_{11,12}(x-y) & r_{11,22}(x-y) \\ r_{11,12}(x-y) & r_{11,22}(x-y) & r_{12,22}(x-y) \\ r_{11,22}(x-y) & r_{12,22}(x-y) & r_{22,22}(x-y) \end{pmatrix} = c_n(y,x).$$

To sum up, we have:

$$\Sigma_n(x,y) = \Sigma_n(x-y) = \begin{pmatrix} A_n(x-y) & B_n(x-y) \\ B_n^t(x-y) & C_n(x-y) \end{pmatrix}.$$

where

$$A_n(x-y) = \begin{pmatrix} a_n & a_n(x-y) \\ a_n(x-y) & a_n \end{pmatrix},$$

with

$$a_n = \frac{E_n}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a_n(x-y) = \begin{pmatrix} r_{1,1}(x-y) & r_{1,2}(x-y) \\ r_{1,2}(x-y) & r_{1,1}(x-y) \end{pmatrix} \tag{B.103}$$

$$r_{i,j}(x,y) = \frac{4\pi^2}{\mathcal{N}_n} \sum_\lambda \lambda_{(i)}\lambda_{(j)} e(\langle \lambda, x-y\rangle).$$

Similarly

$$B_n(x-y) = \begin{pmatrix} 0 & b_n(x-y) \\ -b_n(x-y) & 0 \end{pmatrix},$$

where

$$b_n(x-y) = \begin{pmatrix} r_{1,11}(x,y) & r_{1,12}(x,y) & r_{1,12}(x,y) \\ r_{1,12}(x,y) & r_{1,12}(x,y) & r_{1,11}(x,y) \end{pmatrix},$$

$$r_{i,jk}(x,y) = -\frac{8\pi^3 i}{\mathcal{N}_n} \sum_\lambda \lambda_{(i)}\lambda_{(j)}\lambda_{(k)} e(\langle \lambda, x-y\rangle).$$

Likewise

$$C_n(x-y) = \begin{pmatrix} c_n & c_n(x-y) \\ c_n(x-y) & c_n \end{pmatrix},$$

where

$$c_n = \frac{E_n^2}{8} \begin{pmatrix} 3+\hat{\mu}_n(4) & 0 & 1-\hat{\mu}_n(4) \\ 0 & 1-\hat{\mu}_n(4) & 0 \\ 1-\hat{\mu}_n(4) & 0 & 3+\hat{\mu}_n(4) \end{pmatrix}, \tag{B.104}$$

$$c_n(x-y) = \begin{pmatrix} r_{11,11}(x-y) & r_{11,12}(x-y) & r_{11,22}(x-y) \\ r_{11,12}(x-y) & r_{11,22}(x-y) & r_{11,12}(x-y) \\ r_{11,22}(x-y) & r_{11,12}(x-y) & r_{11,11}(x-y) \end{pmatrix},$$

and

$$r_{ij,kl}(x,y) = \frac{2^4\pi^4}{\mathcal{N}_n} \sum_\lambda \lambda_{(i)}\lambda_{(j)}\lambda_{(k)}\lambda_{(l)} e(\langle \lambda, x-y\rangle).$$

B.1 The special case $x = y$

The previous expressions are greatly simplified for $x = y$; we apply here the following lemma from [MPRW16, §4.1].

Lemma B.1. *For every $n \in S$, we have*

$$\frac{1}{n^2\mathcal{N}_n}\sum_{\lambda\in\Lambda_n} \lambda_{(1)}^4 = \frac{1}{n^2\mathcal{N}_n}\sum_{\lambda\in\Lambda_n} \lambda_{(2)}^4 = \frac{1}{8}(3+\hat{\mu}_n(4)), \ \frac{1}{n^2\mathcal{N}_n}\sum_{\lambda\in\Lambda_n} \lambda_{(1)}^2\lambda_{(2)}^2 = \frac{1}{8}\left(1-\hat{\mu}_n(4)\right).$$

It is then immediate to check that, by symmetry (see [RW08], Lemma 2.3)

$$\mathbb{E}[\partial_1 f_n(x)\ \partial_2 f_n(x)] = \frac{4\pi^2}{\mathcal{N}_n} \sum_\lambda \lambda_1\lambda_2 = 0\ ;$$

similarly we have

$$\mathbb{E}[\partial_1 f_n(x)\ \partial_1 f_n(x)] = \frac{4\pi^2}{\mathcal{N}_n}\sum_\lambda \lambda_1^2 = 4\pi^2\frac{n}{2} = 2\pi^2 n.$$

On the other hand, for second order derivatives we have:

$$\begin{aligned}\mathbb{E}[\partial_{11} f_n(x)\ \partial_{11} f_n(x)] &= \frac{2^4\pi^4}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}\lambda_{(1)}^4 = 2^4\pi^4 n^2\frac{1}{8}(3+\hat{\mu}_n(4)) \\ &= E_n^2\frac{1}{8}(3+\hat{\mu}_n(4)).\end{aligned}$$

$$\mathbb{E}[\partial_{11} f_n(x)\ \partial_{12} f_n(x)] = \frac{2^4\pi^4}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}\lambda_{(1)}^3\lambda_{(2)} = 0.$$

$$\begin{aligned}\mathbb{E}[\partial_{11} f_n(x)\ \partial_{22} f_n(x)] &= \frac{2^4\pi^4}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}\lambda_{(1)}^2\lambda_{(2)}^2 = 2^4\pi^4 n^2\frac{1}{8}(1-\hat{\mu}_n(4)) \\ &= E_n^2\frac{1}{8}(1-\hat{\mu}_n(4)).\end{aligned}$$

$$\mathbb{E}[\partial_{12} f_n(x)\ \partial_{12} f_n(x)] = \frac{2^4\pi^4}{\mathcal{N}_n} \sum_{\lambda\in\Lambda_n} \lambda_{(1)}^2 \lambda_{(2)}^2 = E_n^2 \frac{1}{8}(1-\hat{\mu}_n(4)).$$

$$\mathbb{E}[\partial_{12} f_n(x)\ \partial_{22} f_n(x)] = \frac{2^4\pi^4}{\mathcal{N}_n} \sum_{\lambda\in\Lambda_n} \lambda_{(1)} \lambda_{(2)}^3 = 0.$$

$$\mathbb{E}[\partial_{22} f_n(x)\ \partial_{22} f_n(x)] = \frac{2^4\pi^4}{\mathcal{N}_n} \sum_{\lambda\in\Lambda_n} \lambda_{(2)}^4 = E_n^2 \frac{1}{8}(3+\hat{\mu}_n(4)).$$

We have hence shown that the 5×5 covariance matrix of the vector of gradient and second derivatives is

$$\sigma_n(x) = \begin{pmatrix} a_n(x) & b_n(x) \\ b_n^t(x) & c_n(x) \end{pmatrix},$$

where

$$\begin{aligned} a_n &= a_n(x) = \frac{E_n}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b_n(x) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ c_n &= c_n(x) = \frac{E_n^2}{8} \begin{pmatrix} 3+\hat{\mu}_n(4) & 0 & 1-\hat{\mu}_n(4) \\ 0 & 1-\hat{\mu}_n(4) & 0 \\ 1-\hat{\mu}_n(4) & 0 & 3+\hat{\mu}_n(4) \end{pmatrix}. \end{aligned}$$

C Proof of Lemma 2.1

Throughout this paper, we will exploited some results in the number theory literature (see [KKW13]) that we report here for completeness. Recall once again that

$$\begin{aligned} \hat{\mu}_n(4) &:= \int_{\mathcal{S}^1} z^4 d\mu_n(z) = \frac{1}{\mathcal{N}_n} \sum_{\lambda\in\Lambda_n} \int_{\mathcal{S}^1} z^4 \delta_{\frac{\lambda}{\sqrt{n}}}(z) dz = \frac{1}{n^2\mathcal{N}_n} \sum_{\lambda\in\Lambda_n} \lambda^4 = \frac{1}{n^2\mathcal{N}_n} \sum_{\lambda\in\Lambda_n} (\lambda_1 + i\lambda_2)^4 \\ &= \frac{1}{n^2\mathcal{N}_n} \sum_{\lambda\in\Lambda_n} (\lambda_1^4 + 4i\lambda_2\lambda_1^3 - 6\lambda_1^2\lambda_2^2 - 4i\lambda_1\lambda_2^3 + \lambda_2^4)\ . \end{aligned}$$

Now, since Λ_n is invariant under the group W_2 of signed permutations, consisting of coordinate permutations and sign-change of any coordinate we have

$$\begin{aligned} \hat{\mu}_n(4) &= \frac{1}{n^2\mathcal{N}_n} \sum_{\lambda\in\Lambda_n} (\lambda_1^4 - 6\lambda_1^2\lambda_2^2 + \lambda_2^4) \\ &= \frac{1}{n^2\mathcal{N}_n} \sum_{\lambda\in\Lambda_n} (\lambda_1^2 + \lambda_2^2)^2 - \frac{8}{n^2\mathcal{N}_n} \sum_{\lambda\in\Lambda_n} \lambda_1^2\lambda_2^2 = 1 - \frac{8}{n^2\mathcal{N}_n} \sum_{\lambda\in\Lambda_n} \lambda_1^2\lambda_2^2, \end{aligned}$$

so that

$$\frac{1}{n^2\mathcal{N}_n} \sum_{\lambda\in\Lambda_n} \lambda_1^2\lambda_2^2 = \frac{1}{8}\left(1-\hat{\mu}_n(4)\right).$$

Moreover

$$\frac{1}{n^2\mathcal{N}_n} \sum_{\lambda\in\Lambda_n} \lambda_1^4 = \frac{1}{n^2\mathcal{N}_n} \sum_{\lambda\in\Lambda_n} \lambda_2^4,$$

and therefore

$$\frac{1}{n^2\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}\lambda_1^4 = \frac{1}{2n^2\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}(\lambda_1^4+\lambda_2^4) = \frac{1}{2n^2\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}(\lambda_1^2+\lambda_2^2)^2 - \frac{2}{2n^2\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}\lambda_1^2\lambda_2^2$$
$$= \frac{1}{2} - \frac{1}{8}(1-\hat{\mu}_n(4)) = \frac{1}{8}(3+\hat{\mu}_n(4)).$$

For $i,j=1,2$ with $i\neq j$, and $n,m=0,1,2,\dots$, since Λ_n is invariant under the sign-change of any coordinate, we have

$$\sum_{\lambda\in\Lambda_n}\lambda_i^{2n+1}\lambda_j^m = \sum_{\lambda\in\Lambda_n}(-\lambda_i)^{2n+1}\lambda_j^m = 0.$$

Using invariance under W_d in [RW16, Lemma 2.3] the following lemma is proved :

Lemma C.1. *For any subset $\mathcal{O}\subset\Lambda_n$ which is invariant under the group W_d, we have*

$$\sum_{\lambda\in\mathcal{O}}\lambda_{(j)}\lambda_{(k)} = |\mathcal{O}|\frac{n}{d}\delta_{j,k}. \tag{C.105}$$

We note that using the invariance of Λ_n under the group W_d, we also immediately obtain that

$$\sum_{\lambda\in\Lambda_n}\prod_{i=1}^{d}\lambda_{(i)}^{\alpha_i} = 0,$$

if at least one of the exponents α_i is odd.

It is now possible to focus on the derivation of the expected values. Actually the result for the excursion area is immediate and the result for the boundary length was given already, for instance, in [ORW08], [MPRW16]. We can then focus on the EPC.

In general, a very powerful tool for the derivation of expected values of Lipschitz-Killing Curvatures is provided by the Gaussian Kinematic Formula (see [AT07], Chapter 11), which was indeed exploited to derive the analogous result in the case of random spherical harmonics. However arithmetic random waves are not isotropic processes, which makes the application of the GKF possible but more complicated; because of this, we prefer to give here a proof from first principles.

Proof of Lemma 2.1. By Kac-Rice formula we can write

$$\mathbb{E}\left[\mathcal{L}_0(A_u(f_n;\mathbb{T}))\right] = \int_{\mathbb{T}} K_1(x;I)dx$$

where

$$K_1(x;I) = K_{1;n}(x;I) = \phi_{\nabla f_n(x)}(\mathbf{0})\,\mathbb{E}[\det H_{f_n}(x)\cdot\mathbb{I}_I(f_n(x))\mid\nabla f_n(x)=\mathbf{0}]\ ;$$

here we have

$$\phi_{\nabla f_n(x)}(\mathbf{0}) = \frac{1}{2\pi}\frac{2}{E_n},$$
$$\mathbb{E}[\det H_{f_n}(x)\cdot\mathbb{I}_I(f_n(x))\mid\nabla f_n(x)=\mathbf{0}] = \mathbb{E}[\det H_{f_n}(x)\cdot\mathbb{I}_I(f_n(x))]$$
$$= \frac{E_n^2}{8}\mathbb{E}\left[(Z_1Z_3-Z_2^2)\cdot\mathbb{I}_{\{\frac{Z_1+Z_3}{\sqrt{8}}\in I\}}\right]$$

where (Z_1, Z_2, Z_3) is a Gaussian vector with covariance matrix (see Section B)

$$\begin{pmatrix} 3+\hat{\mu}_n(4) & 0 & 1-\hat{\mu}_n(4) \\ 0 & 1-\hat{\mu}_n(4) & 0 \\ 1-\hat{\mu}_n(4) & 0 & 3+\hat{\mu}_n(4) \end{pmatrix}.$$

Now consider the transformation $W_1 = Z_1$, $W_2 = Z_2$, $W_3 = Z_1 + Z_3$, so that the vector W is given by

$$W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} Z$$

with covariance matrix

$$\Sigma_W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3+\hat{\mu}_n(4) & 0 & 1-\hat{\mu}_n(4) \\ 0 & 1-\hat{\mu}_n(4) & 0 \\ 1-\hat{\mu}_n(4) & 0 & 3+\hat{\mu}_n(4) \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3+\hat{\mu}_n(4) & 0 & 4 \\ 0 & 1-\hat{\mu}_n(4) & 0 \\ 4 & 0 & 8 \end{pmatrix}.$$

Under the obvious notation we write

$$\Sigma_{(W_1,W_2)} = \begin{pmatrix} 3+\hat{\mu}_n(4) & 0 \\ 0 & 1-\hat{\mu}_n(4) \end{pmatrix}, \qquad \Sigma_{W_3} = 8,$$

so that the conditional distribution of $(W_1, W_2)|W_3 = \sqrt{8}t$ is Gaussian with covariance matrix

$$\Sigma_{(W_1,W_2)|W_3} = \begin{pmatrix} 3+\hat{\mu}_n(4) & 0 \\ 0 & 1-\hat{\mu}_n(4) \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \end{pmatrix} \frac{1}{8} \begin{pmatrix} 4 & 0 \end{pmatrix} = \begin{pmatrix} 1+\hat{\mu}_n(4) & 0 \\ 0 & 1-\hat{\mu}_n(4) \end{pmatrix},$$

and expectation

$$\mathbb{E}[(W_1, W_2)|W_3 = \sqrt{8}t] = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \frac{1}{8}\sqrt{8}t = \begin{pmatrix} \sqrt{2}t \\ 0 \end{pmatrix}.$$

We have that

$$\mathbb{E}\left[(Z_1 Z_3 - Z_2^2) \cdot \mathbb{I}_{\{\frac{Z_1+Z_3}{\sqrt{8}} \in I\}}\right] = \mathbb{E}\left[(W_1(W_3 - W_1) - W_2^2) \cdot \mathbb{I}_{\{\frac{W_3}{\sqrt{8}} \in I\}}\right].$$

After the change of variable $\frac{w_3}{\sqrt{8}} = t$

$$\begin{aligned}
&\mathbb{E}\left[(W_1(W_3 - W_1) - W_2^2) \cdot \mathbb{I}_{\{\frac{W_3}{\sqrt{8}} \in I\}}\right] \\
&= \mathbb{E}_{(W_1,W_2)}\left[\mathbb{E}\Big[(w_1(W_3 - w_1) - w_2^2) \cdot \mathbb{I}_{\{\frac{W_3}{\sqrt{8}} \in I\}} \,|(W_1, W_2) = (w_1, w_2)\Big]\right] \\
&= \mathbb{E}_{(W_1,W_2)}\left[\int_{\mathbb{R}} (w_1(w_3 - w_1) - w_2^2) \cdot \mathbb{I}_{\{\frac{w_3}{\sqrt{8}} \in I\}} \frac{1}{\sqrt{2\pi 8}} e^{-\frac{w_3^2}{2\cdot 8}} dw_3 \,|(W_1, W_2) = (w_1, w_2)\right] \\
&= \mathbb{E}_{(W_1,W_2)}\left[\int_{\mathbb{R}} (w_1(t\sqrt{8} - w_1) - w_2^2) \cdot \mathbb{I}_{\{t \in I\}} \frac{1}{\sqrt{2\pi 8}} e^{-\frac{(t\sqrt{8})^2}{2\cdot 8}} \sqrt{8} dt \,|(W_1, W_2) = (w_1, w_2)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{8} \int_{\mathbb{R}} \mathbb{E}\left[(W_1(t\sqrt{8} - W_1) - W_2^2)\right] \cdot \mathbb{I}_{\{t\in I\}} \frac{1}{\sqrt{2\pi 8}} e^{-\frac{(t\sqrt{8})^2}{2\cdot 8}} dt \\
&= \sqrt{8} \int_{I} \mathbb{E}\left[(W_1(t\sqrt{8} - W_1) - W_2^2)\right] \frac{1}{\sqrt{2\pi 8}} e^{-\frac{(t\sqrt{8})^2}{2\cdot 8}} dt \\
&= \sqrt{8} \int_{I} \mathbb{E}\left[(W_1(t\sqrt{8} - W_1) - W_2^2)\right] \phi_{W_3}(t\sqrt{8}) dt \\
&= \sqrt{8} \int_{I} \mathbb{E}\left[(W_1(W_3 - W_1) - W_2^2) | W_3 = t\sqrt{8}\right] \phi_{W_3}(t\sqrt{8}) dt \ .
\end{aligned}
$$

Now note that

$$
\phi_{W_3}(\sqrt{8}\, t) = \frac{1}{4\sqrt{\pi}} e^{-\frac{t^2}{2}}
$$

and

$$
\begin{aligned}
&\mathbb{E}\left[(W_1(W_3 - W_1) - W_2^2)\Big| W_3 = \sqrt{8}t\right] \\
&= \mathbb{E}\left[(W_1\sqrt{8}t - W_1^2 - W_2^2)\Big| W_3 = \sqrt{8}t\right] \\
&= \mathbb{E}\left[((X_1\sqrt{1+\hat{\mu}_n(4)} + \sqrt{2}\, t)\sqrt{8}\, t - (X_1\sqrt{1+\hat{\mu}_n(4)} + \sqrt{2}\, t)^2 - X_2^2(1 - \hat{\mu}_n(4)))\right] \\
&= \mathbb{E}\left[(2t^2 - (X_1^2 + X_2^2) + \hat{\mu}_n(4)(X_2^2 - X_1^2))\right] \\
&= \mathbb{E}\left[(2t^2 - X_1^2(1 + \hat{\mu}_n(4)) - X_2^2(1 - \hat{\mu}_n(4)))\right],
\end{aligned}
$$

for X_1, X_2 standard independent Gaussian. Hence

$$
\begin{aligned}
\mathbb{E}[\mathcal{L}_0(A_I(f_n; \mathbb{T})))] &= \int_{\mathbb{T}} dx \frac{1}{\pi E_n} \frac{E_n^2}{8} \sqrt{8} \int_I \frac{1}{4\sqrt{\pi}} e^{-\frac{t^2}{2}} \mathbb{E}\left[2t^2 - (X_1^2 + X_2^2) + \hat{\mu}_n(4)(X_2^2 - X_1^2)\right] dt \\
&= \frac{E_n}{8\pi} \frac{\sqrt{8}}{4\sqrt{\pi}} \int_{\mathbb{T}} dx \int_I e^{-\frac{t^2}{2}} \mathbb{E}\left[2t^2 - (X_1^2 + X_2^2) + \hat{\mu}_n(4)(X_2^2 - X_1^2)\right] dt \ . \\
&= \frac{E_n}{8\pi} \frac{\sqrt{8}}{4\sqrt{\pi}} \int_I e^{-\frac{t^2}{2}} \mathbb{E}\left[2t^2 - (X_1^2 + X_2^2) + \hat{\mu}_n(4)(X_2^2 - X_1^2)\right] dt.
\end{aligned}
$$

where we have exploited the fact that $\text{Area}(\mathbb{T}) = 1$. Writing $\Xi = X_1^2 + X_2^2$ and $\Theta := X_1^2 - X_2^2$ we observe that $\mathbb{E}[\Xi] = 2$, $\mathbb{E}[\Theta] = 0$, so

$$
\begin{aligned}
\mathbb{E}[\mathcal{L}_0(A_I(f_n; \mathbb{T})))] &= \frac{E_n}{8\pi} \frac{\sqrt{8}}{4\sqrt{\pi}} \int_I e^{-\frac{t^2}{2}} \mathbb{E}\left[2t^2 - \Xi + \hat{\mu}_n(4)\Theta\right] dt \\
&= \frac{E_n}{8\pi} \frac{\sqrt{8}}{4\sqrt{\pi}} \int_I e^{-\frac{t^2}{2}} \left[2t^2 - 2\right] dt.
\end{aligned}
$$

For $I = (u, \infty)$

$$
\mathbb{E}[\mathcal{L}_0(A_u(f_n; \mathbb{T})))] = \frac{E_n}{8\pi} \frac{\sqrt{8}}{4\sqrt{\pi}} \int_u^{\infty} e^{-\frac{t^2}{2}} \left[2t^2 - 2\right] dt = \frac{E_n}{2\sqrt{8}\sqrt{\pi}\pi} u e^{-\frac{u^2}{2}},
$$

which completes the proof.

□

D EPC: second chaotic component

D.1 Proof of Proposition 6.1

Let

$$\alpha_n = \frac{k_2+k_3}{E} = \frac{\sqrt{2}}{\sqrt{3+\hat{\mu}_n(4)}}, \qquad \beta_n = \frac{k_5}{E} = \frac{\sqrt{1+\hat{\mu}_n(4)}}{\sqrt{3+\hat{\mu}_n(4)}},$$

note that $\alpha_n^2 = \frac{2}{3+\hat{\mu}_n(4)}$ and $\alpha_n^2+\beta_n^2=1$. Now let

$$\varphi_a = \lim_{\varepsilon\to 0}\mathbb{E}[H_a(Y)\delta_\varepsilon(\lambda_1 Y)], \qquad a=0,1,2.$$

$$\theta_{ab}(u) = \mathbb{E}\left[Y_aY_b\mathbb{1}_{\{\alpha_nY_3+\beta_nY_5\le -u\}}\right], \qquad a,b=3,4,5,$$

$$\psi_{abcd}(u) = \mathbb{E}\left[Y_aY_bY_cY_d\mathbb{1}_{\{\alpha_nY_3+\beta_nY_5\le -u\}}\right], \qquad a,b,c,d=3,4,5.$$

Note first that, as in [CM18]

$$\varphi_a(\ell) = \begin{cases} \frac{1}{\sqrt{2\pi}k_1}, & a=0,\\ 0, & a=1,\\ -\frac{1}{\sqrt{2\pi}k_1}, & a=2.\end{cases} \tag{D.106}$$

We hence immediately have $h_{1j}(u;n)=0$ for all $j>1$ and $h_{2j}(u;n)=0$ for all $j>2$ since $\varphi_1=0$. Moreover, by some straightforward but tedious manipulations we obtain

$$h_{34}(u;n) = [k_3k_5\ \psi_{3345}(u)+k_2k_3\ \psi_{3334}(u)-k_4^2\ \psi_{3444}(u)]\varphi_0^2 = 0,$$

$$\begin{aligned} h_{35}(u;n) &= [k_3k_5\ \psi_{3355}(u)+k_2k_3\ \psi_{3335}(u)-k_4^2\ \psi_{3445}(u)]\varphi_0^2\\ &= \frac{n\pi\sqrt{2}\sqrt{1+\hat{\mu}_n(4)}\,[u\phi(u)(1+u^2)+(3+\hat{\mu}_n(4))\Phi(-u)]}{3+\hat{\mu}_n(4)}, \end{aligned}$$

and moreover

$$h_{45}(u;n) = [k_3k_5\ \psi_{3455}(u)+k_2k_3\ \psi_{3345}(u)-k_4^2\ \psi_{4445}(u)]\varphi_0^2 = 0,$$

$$\begin{aligned} h_1(u;n) = h_2(u;n) &= [k_3k_5\ \theta_{35}(u)+k_2k_3\ \theta_{33}(u)-k_4^2\ \theta_{44}(u)]\varphi_0\varphi_2\\ &= -n\pi u\,\phi(u), \end{aligned}$$

$$\begin{aligned} h_3(u;n) &= [k_3k_5\ \psi_{3335}(u)+k_2k_3\ \psi_{3333}(u)-k_4^2\ \psi_{3344}(u)]\varphi_0^2\\ &\quad -[k_3k_5\ \theta_{35}(u)+k_2k_3\ \theta_{33}(u)-k_4^2\ \theta_{44}(u)]\varphi_0^2\\ &= n\pi\left[\frac{2u(1+u^2)\phi(u)}{3+\hat{\mu}_n(4)}+\Phi(-u)-\hat{\mu}_n(4)\Phi(-u)\right], \end{aligned}$$

$$
\begin{aligned}
h_4(u;n) &= [k_3k_5\ \psi_{3445}(u) + k_2k_3\ \psi_{3344}(u) - k_4^2\ \psi_{4444}(u)]\varphi_0^2 \\
&\quad - [k_3k_5\ \theta_{35}(u) + k_2k_3\ \theta_{33}(u) - k_4^2\ \theta_{44}(u)]\varphi_0^2 \\
&= -n\pi(1-\hat{\mu}_n(4))\Phi(-u),
\end{aligned}
$$

$$
\begin{aligned}
h_5(u;n) &= [k_3k_5\ \psi_{3555}(u) + k_2k_3\ \psi_{3355}(u) - k_4^2\ \psi_{4455}(u)]\varphi_0^2 \\
&\quad - [k_3k_5\ \theta_{35}(u) + k_2k_3\ \theta_{33}(u) - k_4^2\ \theta_{44}(u)]\varphi_0^2 \\
&= \frac{n\pi u(1+u^2)(1+\hat{\mu}_n(4))\phi(u)}{3+\hat{\mu}_n(4)}\ ;
\end{aligned}
$$

In the previous steps, we have used a number of auxiliary functions $\psi_{abcd}(u)$, for $a,b,c,d = 3,4,5$, whose exact expressions and derivations are given in Lemmas D.1 and D.2 below.

Lemma D.1. *We have that*

$$
\theta_{33}(u) = \Phi(-u) + u\,\phi(u)\frac{2}{3+\hat{\mu}_n(4)},\ \ \theta_{35}(u) = u\,\phi(u)\frac{\sqrt{2}\sqrt{1+\hat{\mu}_n(4)}}{3+\hat{\mu}_n(4)},\ \ \textit{and}\ \theta_{44}(u) = \Phi(-u).
$$

Proof. Let X, Y and Z be three independent standard Gaussian random variables; with the same arguments as in [CM18], Lemma 12, Lemma 13 and Lemma 14, we have

$$
\begin{aligned}
\theta_{33}(u) &= \mathbb{E}\left[Y^2 \mathbb{1}_{\{\alpha_n Y + \beta_n X \le -u\}}\right] = \int_{-\infty}^{\infty} y^2\phi(y)\Phi\left(\frac{-u-\alpha_n y}{\beta_n}\right) dy \\
&= \Phi(-u) + \alpha_n^2 u\phi(-u)
\end{aligned}
$$

$$
\begin{aligned}
\theta_{35}(u) &= \mathbb{E}\left[XY \mathbb{1}_{\{\alpha_n Y + \beta_n X \le -u\}}\right] = \int_{-\infty}^{\infty} y\phi(y)dy \int_{-\infty}^{\frac{-u-\alpha_n y}{\beta_n}} x\phi(x)dx \\
&= -\int_{-\infty}^{\infty} y\phi(y)\phi\left(\frac{-u-\alpha_n y}{\beta_n}\right) dy = \alpha_n\beta_n u\,\phi(-u)
\end{aligned}
$$

$$
\theta_{44}(u) = \mathbb{E}\left[Z^2 \mathbb{1}_{\{\alpha_n Y + \beta_n X \le -u\}}\right] = \int_{-\infty}^{\infty} \phi(y)\Phi\left(\frac{-u-\alpha_n y}{\beta_n}\right) dy = \Phi(-u).
$$

□

The next computations involve moments of four random variables and are hence a bit more involved.

Lemma D.2. *We have that*

$$
\psi_{3333}(u) = 3\Phi(-u) + 4\,u\,\phi(u)\frac{6+u^2+3\hat{\mu}_n(4)}{(3+\hat{\mu}_n(4))^2},\ \ \psi_{4444}(u) = 3\Phi(-u),
$$

$$\psi_{3355}(u) = \Phi(-u) + u\,\phi(u)\frac{3+\hat{\mu}_n(4)^2+2u^2(1+\hat{\mu}_n(4))}{(3+\hat{\mu}_n(4))^2},$$
$$\psi_{3555}(u) = u\,\phi(u)\sqrt{2}\frac{\sqrt{1+\hat{\mu}_n(4)}}{(3+\hat{\mu}_n(4))^2}(6+u^2(1+\hat{\mu}_n(4))),$$
$$\psi_{3335}(u) = u\,\phi(u)\sqrt{2}\frac{\sqrt{1+\hat{\mu}_n(4)}}{(3+\hat{\mu}_n(4))^2}(3+2u^2+3\hat{\mu}_n(4)),$$

and moreover

$$\psi_{3344}(u) = \Phi(-u) + u\,\phi(u)\frac{2}{3+\hat{\mu}_n(4)},\ \ \psi_{4455}(u) = \Phi(-u) + u\,\phi(u)\frac{1+\hat{\mu}_n(4)}{3+\hat{\mu}_n(4)},$$
$$\psi_{3445}(u) = u\,\phi(u)\sqrt{2}\frac{\sqrt{1+\hat{\mu}_n(4)}}{3+\hat{\mu}_n(4)}.$$

The following remaining terms are identically zero:

$$\psi_{3334}(u) = \psi_{3345}(u) = \psi_{3444}(u) = \psi_{3455}(u) = \psi_{4445}(u) = 0.$$

Proof. In the sequel we shall use X, Y and Z to denote three independent standard Gaussian random variables. The computations to follow are then just standard evaluations of Gaussian integrals. In particular, applying [CM18] Lemma 12, Lemma 13 and Lemma 14, we have

$$\begin{aligned}\psi_{3333}(u) &= \mathbb{E}\left[Y^4 \mathbb{1}_{\{\alpha_n Y+\beta_n X\le -u\}}\right] = \int_{-\infty}^{\infty} y^4\phi(y)\Phi\Big(\frac{-u-\alpha_n y}{\beta_n}\Big)dy \\ &= 3\Phi(-u) + u\,\phi(-u)\left[3\alpha_n^2+3\alpha_n^4\beta_n^2+3\beta_n^4\alpha_n^2+\alpha_n^4u^2\right].\end{aligned}$$

$$\psi_{4444}(u) = \mathbb{E}\left[Z^4 \mathbb{1}_{\{\alpha_n Y+\beta_n X\le -u\}}\right] = 3\,\mathbb{E}\left[\mathbb{1}_{\{\alpha_n Y+\beta_n X\le -u\}}\right] = 3\Phi(-u).$$

Now we observe that

$$\int_{-\infty}^{q} x^2\phi(x)dx = \Phi(q) - q\,\phi(q),$$

and we obtain

$$\begin{aligned}\psi_{3355}(u) &= \mathbb{E}\left[Y^2X^2 \mathbb{1}_{\{\alpha_n Y+\beta_n X\le -u\}}\right] = \int_{-\infty}^{\infty} y^2\phi(y)dy\int_{-\infty}^{\frac{-u-\alpha_n y}{\beta_n}} x^2\phi(x)dx \\ &= \int_{-\infty}^{\infty} y^2\phi(y)\Phi\left(\frac{-u-\alpha_n y}{\beta_n}\right)dy - \left(\frac{-u-\alpha_n y}{\beta_n}\right)\int_{-\infty}^{\infty} y^2\phi(y)\phi\left(\frac{-u-\alpha_n y}{\beta_n}\right)dy \\ &= \Phi(-u) + \alpha_n^2\,u\,\phi(-u) + \beta_n^2\,u\,\phi(-u)(-2\alpha_n^4+\beta_n^4-\alpha_n^2\beta_n^2+\alpha_n^2u^2).\end{aligned}$$

Likewise

$$\begin{aligned}\psi_{3555}(u) &= \mathbb{E}\left[YX^3 \mathbb{1}_{\{\alpha_n Y+\beta_n X\}\le -u}\right] = \int_{-\infty}^{\infty} y\phi(y)dy\int_{-\infty}^{\frac{-u-\alpha_n y}{\beta_n}} x^3\phi(x)dx \\ &= -\int_{-\infty}^{\infty} y\phi(y)\phi\left(\frac{-u-\alpha_n y}{\beta_n}\right)\left\{\left(\frac{-u-\alpha_n y}{\beta_n}\right)^2+2\right\}dy = \alpha_n\beta_n\,u\,\phi(-u)(3\alpha_n^2+\beta_n^2u^2),\end{aligned}$$

$$\psi_{3335}(u) = \mathbb{E}\left[Y^3 X \mathbb{1}_{\{\alpha_n Y+\beta_n X\leq -u\}}\right] = \alpha_n\beta_n u\,\phi(-u)(3\beta_n^2+\alpha_n^2u^2).$$

Moreover

$$\psi_{3344}(u) = \mathbb{E}\left[Z^2Y^2 \mathbb{1}_{\{\alpha_n Y+\beta_n X\leq -u\}}\right] = \mathbb{E}\left[Y^2 \mathbb{1}_{\{\alpha_n Y+\beta_\ell X\leq -u\}}\right] = \theta_{33}(u),$$

$$\begin{aligned}\psi_{4455}(u) &= \mathbb{E}\left[Z^2X^2 \mathbb{1}_{\{\alpha_n Y+\beta_n X\leq -u\}}\right] = \mathbb{E}\left[X^2 \mathbb{1}_{\{\alpha_n Y+\beta_\ell X\leq -u\}}\right] = \theta_{55}(u)\\ &= \Phi(-u)+\beta_n^2\,u\,\phi(-u),\end{aligned}$$

and finally

$$\psi_{3445}(u) = \mathbb{E}\left[Z^2XY \mathbb{1}_{\{\alpha_n Y+\beta_n X\leq -u\}}\right] = \mathbb{E}\left[XY \mathbb{1}_{\{\alpha_n Y+\beta_\ell X\leq -u\}}\right] = \theta_{35}(u).$$

The fact that $\psi_{3334}(u)$, $\psi_{3345}(u)$, $\psi_{3444}(u)$, $\psi_{3455}(u)$ and $\psi_{4445}(u)$ are identically equal to zero, it is enough to note that they are all of the form

$$\mathbb{E}\left[Z^pX^qY^r \mathbb{1}_{\{\alpha_n Y+\beta_n X\leq -u\}}\right]$$

where $p=1,3$ is odd. □

D.2 Proof of Proposition 6.2

We have to deal with the following integrals of squares:

$$I_{00}(n) = \int_{\mathbb{T}} f_n^2(x)dx,\ I_{11}(n) = \int_{\mathbb{T}} \left\{e_1^x f_n(x)\right\}^2 dx,\ I_{22}(n) = \int_{\mathbb{T}} \left\{e_2^x f_n(x)\right\}^2 dx; \tag{D.107}$$

we shall also study the cross-product integral

$$I_{0,22}(n) = \int_{\mathbb{T}} f_n(x)e_2^xe_2^xf_n(x)dx,$$

and finally we shall consider

$$I_{12,12}(n) = \int_{\mathbb{T}} \left\{e_1^xe_2^xf_n(x)\right\}^2 dx,\ I_{22,22}(n) = \int_{\mathbb{T}} \{e_2^xe_2^xf_n(x)\}^2 dx.$$

We have

$$A_{35} = -\frac{E_n}{k_3k_5}\left\{1+2\frac{k_2}{k_3}\right\}I_{0,22}(n) - \frac{E_n^2k_2}{k_3^2k_5}I_{00}(n) - \frac{1}{k_3k_5}\left\{1+\frac{k_2}{k_3}\right\}I_{22,22}(n),$$

$$B_1 = \frac{1}{k_1^2}I_{11}(n)-1\ ,\ B_2 = \frac{1}{k_1^2}I_{22}(n)-1, B_3 = \frac{E_n^2}{k_3^2}I_{00}(n)+\frac{1}{k_3^2}I_{22,22}(n)+\frac{2E_n}{k_3^2}I_{0,22}(n)-1,$$

$$B_4 = \frac{1}{k_4^2}I_{12,12}(n)-1,$$

$$B_5 = \frac{1}{k_5^2}\big(1+\frac{k_2}{k_3}\big)^2I_{22,22}(n)+\frac{E_n^2k_2^2}{k_3^2k_5^2}I_{00}(n)+2\frac{E_nk_2}{k_3k_5^2}\big(1+\frac{k_2}{k_3}\big)I_{0,22}(n)-1.$$

Our next step is then to investigate the behaviour of these integrals of stochastic processes; this task is accomplished in the following Lemma.

Lemma D.3. *The following identities hold:*

$$I_{00}(n) = \frac{1}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2,\ \ I_{11}(n) = \frac{4\pi^2}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\lambda_1^2,\ \ I_{22}(n) = \frac{4\pi^2}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\lambda_2^2,$$

$$I_{0,22}(n) = -\frac{4\pi^2}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\lambda_2^2,\ \ I_{12,12}(n) = \frac{16\pi^4}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\lambda_1^2\lambda_2^2,\ \ I_{22,22}(n) = \frac{16\pi^4}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\lambda_2^4.$$

Proof. By Parseval's identity it follows that

$$I_{00}(n) = \frac{1}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2,$$

and similarly

$$\begin{aligned} I_{11}(n) &= -\frac{4\pi^2}{\mathcal{N}_n}\sum_{\lambda,\lambda'} a_\lambda a_{\lambda'}\lambda_1\lambda'_1 \int_{\mathbb{T}} e(\langle\lambda,x\rangle)e(\langle\lambda',x\rangle)dx = \frac{4\pi^2}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\lambda_1^2\ , \\ I_{22}(n) &= \int_{\mathbb{T}} \{e_2^x f_n(x)\}^2\, dx = \frac{4\pi^2}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\lambda_2^2\ . \end{aligned}$$

Likewise

$$I_{0,22}(n) = \frac{1}{\mathcal{N}_n}\sum_{\lambda,\lambda'} a_\lambda a_{\lambda'} \int_{\mathbb{T}} e(\langle\lambda,x\rangle)(2\pi i\lambda'_2)^2 e(\langle\lambda',x\rangle)dx = -\frac{4\pi^2}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\lambda_2^2\ ,$$

and finally

$$I_{12,12}(n) = \frac{16\pi^4}{\mathcal{N}_n}\sum_{\lambda,\lambda'} a_\lambda a_{\lambda'}\lambda_1\lambda_2\lambda'_1\lambda'_2 \int_{\mathbb{T}} e(\langle\lambda,x\rangle)e(\langle\lambda',x\rangle)dx = \frac{16\pi^4}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\lambda_1^2\lambda_2^2$$

$$I_{22,22}(n) = \frac{16\pi^4}{\mathcal{N}_n}\sum_{\lambda,\lambda'} a_\lambda a_{\lambda'}\lambda_2^2{\lambda'_2}^2 \int_{\mathbb{T}} e(\langle\lambda,x\rangle)e(\langle\lambda',x\rangle)dx = \frac{16\pi^4}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\lambda_2^4.$$

□

We are now in the position to complete the proof.
Proof of Proposition 6.2. Note that

$$
\begin{aligned}
A_{35}(n) &= -\frac{E_n}{k_3k_5}\left\{1+2\frac{k_2}{k_3}\right\}I_{0,22}(n) - \frac{E_n^2 k_2}{k_3^2 k_5}I_{00}(n) - \frac{1}{k_3k_5}\left\{1+\frac{k_2}{k_3}\right\}I_{22,22}(n)\\
&= -\frac{1}{\sqrt{2}\pi^2 n\sqrt{1+\hat{\mu}_n(4)}}\frac{5-\hat{\mu}_n(4)}{3+\hat{\mu}_n(4)}I_{0,22}(n) - \frac{1-\hat{\mu}_n(4)}{3+\hat{\mu}_n(4)}\frac{2\sqrt{2}}{\sqrt{1+\hat{\mu}_n(4)}}I_{00}(n)\\
&\quad - \frac{1}{\sqrt{2}\pi^4 n^2\sqrt{1+\hat{\mu}_n(4)}}\frac{1}{3+\hat{\mu}_n(4)}I_{22,22}(n)\\
&= \frac{1}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\left[\frac{4\pi^2}{\sqrt{2}\pi^2 n\sqrt{1+\hat{\mu}_n(4)}}\frac{5-\hat{\mu}_n(4)}{3+\hat{\mu}_n(4)}\lambda_2^2 - \frac{1-\hat{\mu}_n(4)}{3+\hat{\mu}_n(4)}\frac{2\sqrt{2}}{\sqrt{1+\hat{\mu}_n(4)}}\right.\\
&\quad \left. -\frac{16\pi^4}{\sqrt{2}\pi^4 n^2\sqrt{1+\hat{\mu}_n(4)}}\frac{1}{3+\hat{\mu}_n(4)}\lambda_2^4\right]\\
&= \frac{1}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\left[\frac{2\sqrt{2}}{n\sqrt{1+\hat{\mu}_n(4)}}\frac{5-\hat{\mu}_n(4)}{3+\hat{\mu}_n(4)}\lambda_2^2 - \frac{1-\hat{\mu}_n(4)}{3+\hat{\mu}_n(4)}\frac{2\sqrt{2}}{\sqrt{1+\hat{\mu}_n(4)}}\right.\\
&\quad \left. -\frac{8\sqrt{2}}{n^2\sqrt{1+\hat{\mu}_n(4)}}\frac{1}{3+\hat{\mu}_n(4)}\lambda_2^4\right].
\end{aligned}
$$

On the other hand

$$
B_1 = \frac{1}{2n\pi^2}\frac{4\pi^2}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\lambda_1^2 - 1 = \frac{1}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\frac{2}{n}\lambda_1^2 - 1
$$

$$
B_2 = \frac{1}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\frac{2}{n}\lambda_2^2 - 1
$$

$$
\begin{aligned}
B_3 &= \frac{E_n^2}{k_3^2}I_{00}(n) + \frac{1}{k_3^2}I_{22,22}(n) + \frac{2E_n}{k_3^2}I_{0,22}(n) - 1\\
&= \frac{8}{3+\hat{\mu}_n(4)}\frac{1}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2 + \frac{1}{2\pi^4 n^2(3+\hat{\mu}_n(4))}\frac{16\pi^4}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\lambda_2^4 - \frac{4}{\pi^2 n(3+\hat{\mu}_n(4))}\frac{4\pi^2}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\lambda_2^2 - 1\\
&= \frac{1}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\left[\frac{8}{3+\hat{\mu}_n(4)} + \frac{8}{n^2(3+\hat{\mu}_n(4))}\lambda_2^4 - \frac{16}{n(3+\hat{\mu}_n(4))}\lambda_2^2\right] - 1
\end{aligned}
$$

$$
\begin{aligned}
B_4 &= \frac{1}{k_4^2}I_{12,12}(n) - 1\\
&= \frac{1}{2\pi^4 n^2(1-\hat{\mu}_n(4))}\frac{16\pi^4}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\lambda_1^2\lambda_2^2 - 1\\
&= \frac{8}{n^2(1-\hat{\mu}_n(4))}\frac{1}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\lambda_1^2\lambda_2^2 - 1.
\end{aligned}
$$

$$
\begin{aligned}
B_5 &= \frac{1}{k_5^2}\big(1+\frac{k_2}{k_3}\big)^2 I_{22,22}(n) + \frac{E_n^2 k_2^2}{k_3^2 k_5^2} I_{00}(n) + 2\frac{E_n k_2}{k_3 k_5^2}\big(1+\frac{k_2}{k_3}\big) I_{0,22}(n) - 1\\
&= \frac{1}{\pi^4 n^2 (1+\hat{\mu}_n(4))(3+\hat{\mu}_n(4))}\frac{16\pi^4}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2 \lambda_2^4 + \frac{(1-\hat{\mu}_n(4))^2}{(3+\hat{\mu}_n(4))(1+\hat{\mu}_n(4))}\frac{1}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\\
&\quad - \frac{2(1-\hat{\mu}_n(4))}{\pi^2 n (3+\hat{\mu}_n(4))(1+\hat{\mu}_n(4))}\frac{4\pi^2}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\lambda_2^2 - 1\\
&= \frac{1}{\mathcal{N}_n}\sum_\lambda |a_\lambda|^2\Big[\frac{16}{n^2(1+\hat{\mu}_n(4))(3+\hat{\mu}_n(4))}\lambda_2^4 + \frac{(1-\hat{\mu}_n(4))^2}{(3+\hat{\mu}_n(4))(1+\hat{\mu}_n(4))}\\
&\quad - \frac{2(1-\hat{\mu}_n(4))}{n(3+\hat{\mu}_n(4))(1+\hat{\mu}_n(4))}4\lambda_2^2\Big] - 1.
\end{aligned}
$$

which concludes the proof.

□

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