Kaluza-Klein theories in the framework of polymer quantum mechanics

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We provide a reanalysis of the $5D$ Kaluza-Klein theory by implementing the polymer representation of the dynamics, both on a classical and a quantum level, in order to introduce in the model information about the existence of a cut off scale. We start by showing that, in the framework of semiclassical quantum mechanics, the $5D$ Bianchi I model admits a solution in which three space directions expand isotropically, while the remaining one is static, offering in this way a very valuable scenario to implement a Kaluza-Klein paradigm, identifying in such a static dimension the compactified one. We then analyze the behavior of geodesic motion in the context of the polymer representation, as referred to a $5D$ space-time with a static dimension. We demonstrate that such a revised formulation allows overcoming one of the puzzling questions of the standard Kaluza-Klein model corresponding to the limit of the charge to mass ratio for a particle, inapplicable to any fundamental one. Indeed, here, such a ratio can be naturally attributed to particles predicted by the Standard Model and no internal contradiction of the theory arises on this level. Finally, we study the morphology of the field equation associated with a charged scalar particle, i.e., we analyze a Klein-Gordon equation, whose fifth coordinate is viewed in the polymer representation. Here we obtain the surprising result that, although the Kaluza-Klein tower has a deformed structure characterized by irregular steps, the value predicted for the particle mass can be, in principle, set within the Standard Model mass distribution. Hence, the problem of the Planckian value of such mass, typical of the standard formulation, is now overcome. However, a problem with the charge to mass ratio still survives in this quantum field formulation.

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I. INTRODUCTION

The original Kaluza-Klein idea [1–3] consists in a $5D$ space-time formulation having the aim to include the electromagnetic interaction in a geometrical picture.

The surprising formal success in providing a metric representation of the vector potential suggested, in the Seventies, to attempt for a geometrical unification [4] able to assess all the fundamental interactions into a multidimensional space-time, with particular attention to the electroweak model. The basic idea at the root of this approach consists of the possibility of reproducing the Lie algebra, characterizing the elementary particle symmetries by the isometries of the extra-dimensional space. The nontrivial result obtained by the extra-dimensional Kaluza-Klein theories relies on the emergence from the multidimensional Einstein-Hilbert–Lagrangian of the correct Yang-Mills action for the vector bosons which are the interaction carriers.

However, many nontrivial problems affected this fascinating attempt for a geometrization of nature. One of the main questions came from the difficulty of providing a geometrical version of the chirality singled out by the electroweak interaction [5], as well as the impossibility of representing the Standard Model of elementary particles in a Kaluza-Klein scenario [6]. For alternative non-Riemannian approaches to solve the chirality problem of the electroweak model see [7,8].

Finally, we observe that a full geometrical picture of nature would involve the geometrical formulation of the fermionic field; a really nontrivial perspective if supersymmetry is not considered [9].

Even the $5D$ Kaluza-Klein theory presents some important difficulties (see [10] for a review) which leaves the question open concerning the viability of this approach as a geometrization of the electromagnetic interaction.

First of all, the $5D$ metric tensor contains an additional degree of freedom besides the $4D$ metric and the vector potential, namely the fifth diagonal component. Under the necessary restriction of the coordinate transformation in order to deal with the $U(1)$ symmetry, this quantity behaves as an additional scalar field, whose presence nontrivially affects basic features of the electromagnetism; for instance, the charge conservation itself [4,11,12]. But, even fixing this scalar field to unity in the Lagrangian for the model (with the right sign of a spacelike object), the ratio between the charge and the mass of an elementary particle is

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constrained to remain too small in order to reproduce the Standard Model spectrum of masses (for a proposal to solve the charge to mass ratio problem see [11]).

Finally, studying the morphology of a five-dimensional D'Alambertian operator, we immediately recognize the emergence of huge massive modes of a boson field, as result of the compactified scale of the fifth dimension [13].

In this analysis we approach the formulation of the $5D$ Kaluza-Klein theory within the semiclassical and quantum framework of the so-called polymer quantum mechanics [14,15]. This revised formulation of quantum physics has the aim of introducing a discrete nature in the generalized coordinate (a real coordinate of a generic degree of freedom), as an effect of the emergence of cut off physics.

Indeed, the fifth compactified dimension in the standard approach is about an order of two greater than the Planck size, so it is the natural condition to be approached via the continuum limit of polymer quantum mechanics when referring to a point particle living in this dimension. Furthermore, the corresponding diagonal metric component (namely, the additional Universe scale factor) in such a dynamical regime is affected—as expected—by cut off physics effects.

The present analysis follows the scenario proposed in [13] but is revised in view of the polymer formulation.

We first show that a five-dimensional Kasner solution [16–18] (characterizing the Bianchi I Universe) admits a configuration in which three spatial directions isotropically expand, while the fourth remains static. This result is of impact in the implementation of Kaluza-Klein theory, since it removes some of the nontrivial inconvenient features of a collapsing dimension, close to a Planckian size. For a previous attempt to deal with a static compactified dimension, on the base of a physical phenomenon see [19].

Then, we analyze the geodesic motion on a generic $5D$ space-time, having a fifth steady dimension and we outline a natural solution to the charge to mass ratio problem. This result comes from the details of the semiclassical polymer formulation, adopted for the Hamiltonian dynamics of the free-falling particle. In particular, the modified expression used by the fifth momentum of the particle leads to a modified constitutive relation; that is, when passing from the momenta to the velocities the previous constraint on the charge to mass ratio allows the consideration of values which are natural in the Standard Model particles.

Finally, we study a five-dimensional Klein-Gordon equation and we clarify that by addressing the fifth coordinate via the quantum polymer prescription, the spectrum of emerging masses can fit some values of the Standard Model one and no tachyonic modes emerge, unlike the case discussed in [13].

However, it should be noted that, in this quantum field approach, a problem with the definition of the correct q/m ratio for a Standard Model particle still survives.

The present study suggests that when cut off physics is included in the Kaluza-Klein formulation, some of the puzzling features of this approach are restated into a form that can give new physical insight for their understanding and overcoming.

The manuscript presentation is structured as follows: In Section II we review the main features of ordinary Kaluza-Klein theory, from the metric tensor construction and the resulting field equations to the geodesic motion of a pointlike particle. This analysis leads to the ordinary quantization law for the electric charge, an estimate for the size L of the fifth dimension, and the aforementioned shortcoming of the charge to mass ratio of a particle.

In Section III we review polymer quantum mechanics, summarizing the construction of the relative kinematic Hilbert space via the introduction of a Weyl-Heisenberg algebra and under the assumption of the existence of a discrete spatial coordinate, and the implementation of the proper dynamics both on a quantum and semiclassical level, with particular regard to the p-polarization.

In Section IV we analyze the polymer-modified Kasner solution obtained from the introduction of the polymer framework on a semiclassical level in a $5D$ Bianchi I model, focusing on the behavior of the fifth dimension.

Finally, in Section V (based on the result of the previous section) we analyze, in a semiclassical formulation of polymer quantum mechanics, the geodesic motion of a pointlike particle and all its features. We compare all the results with the ones from ordinary theory and then carry out the study of the polymer quantum dynamics of a complex Klein-Gordon field (along the lines of [13]) discussing with particular attention the resulting electric charge distribution and mass spectrum. In Section VI brief concluding remarks follow.

II. KALUZA-KLEIN THEORY

Kaluza-Klein theory is a $5D$ extension of Einstein's theory of general relativity which aims to provide a unified description of gravitational and electromagnetic interaction in a purely geometric fashion.

In the original theory [1–3] the space-time is described by a $5D$ smooth manifold V^5, which is assumed to be the direct product $V^4 \otimes S^1$ between a generic $4D$ manifold and a circus of length L, that is a compact space.

A crucial assumption relies on the fact that all the observable physical quantities do not depend on the fifth coordinate x^5. This hypothesis is further motivated by noticing that, due to the compactness of the fifth dimension, all the observable physical quantities are periodic in x^5; hence the independence on the fifth coordinate can be regarded as zero-order cut off of a Fourier expansion of these quantities themselves—dubbed as the cylinder condition.

Once restricted the $5D$ general relativity principle to the following coordinate transformations (and their inverse)

$$\begin{cases} x^{\mu'} = \Psi(x^\mu) \\ x^{5'} = x^5 + k\Lambda(x^\mu), \end{cases} \tag{1}$$

and the $5D$ metric tensor of the expanded theory can be written as follows:

$$\tilde{g}_{ab} = \left(\begin{array}{c|c} g_{\mu\nu} + k^2\phi^2 A_\mu A_\nu & k\phi^2 A_\mu \\ \hline k\phi^2 A_\nu & \phi^2 \end{array} \right), \tag{2}$$

where $g_{\mu\nu}$ is the $4D$ metric tensor of the ordinary theory, A_μ is the electromagnetic four-potential, ϕ is a scalar field, and k is a constant to be properly determined.

A. Kaluza-Klein field equations

The field equations of the theory can be obtained from a $5D$ Einstein-Hilbert action

$$^{(5)}S \coloneqq \tilde{S} = -\frac{1}{16\pi\tilde{G}} \int_{V^4 \otimes S^1} d^5x \sqrt{-\tilde{g}}\tilde{R}, \tag{3}$$

where $\tilde{G}$, $\tilde{g}$, and $\tilde{R}$ are respectively the $5D$ gravitational constant, the metric tensor $\tilde{g}_{ab}$ determinant, and the $5D$ scalar curvature.

By performing a $(4+1)$-dimensional reduction the ordinary $4D$ Einstein-Maxwell action is surprisingly obtained

$$\tilde{S} = -\frac{c^3}{16\pi G} \int_{V^4} d^4x \sqrt{-g}\phi \left(R + \frac{1}{4}\phi^2 k^2 F_{\mu\nu}F^{\mu\nu} + \frac{2}{\phi}\nabla_\mu \partial^\mu \phi \right). \tag{4}$$

By setting $\phi = 1$ in the action—as in the original work of Kaluza [1] and Klein [2,3]—and by using the ordinary variational principle, the Einstein-Maxwell field equations can be correctly recovered once k is set equal to $2\sqrt{G}/c^2$.

B. Geodesic motion

A free pointlike particle in this theory will move along a $5D$ geodesic; hence the respective action, with signature $(-,+,+,+,+)$, will be

$$\tilde{S} = -mc \int d\tilde{s} = -mc \int \sqrt{-\tilde{g}_{ab} \frac{dx^a}{d\tilde{s}} \frac{dx^b}{d\tilde{s}}} d\tilde{s}, \tag{5}$$

where $d\tilde{s}$ is the $5D$ line element, to be distinguished from the $4D$ line element ds.

Once set—here and in further developments—$\phi = 1$ in the metric (2), and from the variational principle $5D$ geodesic equation is immediately restored

$$\tilde{u}^a \tilde{\nabla}_a \tilde{u}^b = 0. \tag{6}$$

It is essential to point out that the $5D$ velocity $\tilde{u}^a$ is different from $4D$ velocity u^a; indeed they are related as follows:

$$\tilde{u}^a = \frac{1}{\sqrt{1-u_5^2}} u^a. \tag{7}$$

From relations (6) and (7) it can be easily shown that u_5 is a constant of motion.

In order to achieve $4D$ equation of motion, the geodesic equation (6) has to be evaluated for the usual space-time variables only, which we indicate with Greek letters.

By making use of relation (7), the following result is attained,

$$u^\nu \nabla_\nu u^\mu = \frac{2\sqrt{G}}{c^2} u_5 u^\nu g^{\mu\lambda} F_{\nu\lambda}, \tag{8}$$

where $F_{\nu\lambda}$ is the antisymmetric electromagnetic tensor.

By comparison with the ordinary classical equation

$$u^\nu \nabla_\nu u^\mu = \frac{q}{mc^2} u^\nu g^{\mu\lambda} F_{\nu\lambda}, \tag{9}$$

the following fundamental identification is achieved

$$u_5 = \frac{q}{2m\sqrt{G}}. \tag{10}$$

Since $p_5 = mcu_5$ it can then be written as

$$p_5 = \frac{qc}{2\sqrt{G}}, \tag{11}$$

which establishes a fundamental relation between the particle's fifth component of momentum and its electric charge.

The compactness of the fifth dimension implies a quantization of momentum along the fifth direction

$$p_5 = \frac{2\pi\hbar}{L} n, \qquad n \in \mathbb{Z}, \tag{12}$$

where we recall that L is the length of the circus describing the fifth dimension. By a direct comparison between relations (11) and (12) a natural quantization law for the electric charge and an estimate of the size L of the fifth dimension are obtained,

$$L = 4\pi \frac{\hbar\sqrt{G}}{ec} \approx 2.37 \times 10^{-31}\ \text{cm}, \qquad q = ne, \tag{13}$$

where e is the electron charge.

Coherently the size of the fifth dimension is in agreement with its nonobservability and with its impossibility to be currently detected.

Nevertheless, despite these remarkable results, the relation (7) sets the constraint $|u_5| < 1$; by virtue of relation (10), this implies the following condition on the charge/mass ratio of a particle

$$\frac{|q|}{m} < 2\sqrt{G} \approx 5.16 \times 10^{-4}\ \text{e.s.u./g}, \tag{14}$$

which, unfortunately, has no phenomenological confirmation, either for elementary particles or for macroscopic objects—hence representing one of the puzzling shortcomings of the theory.

III. POLYMER QUANTUM MECHANICS

Polymer quantum mechanics is a nonstandard representation of nonrelativistic quantum theory, unitarily inequivalent to the Schrödinger one [14,15]. Its developments are due mainly to the exploration of background-independent theories, such as quantum gravity, of which mimics several structures [20].

Given a discrete orthonormal basis $|\mu_i\rangle$ for a space $\mathcal{H}'$, such that $\langle\mu_i|\mu_j\rangle = \delta_{ij}$, where $\mu_i \in \mathbb{R}$ and $i = 1, 2..., n$, the kinematic polymer Hilbert space $\mathcal{H}_{\text{poly}}$ is obtained as a Cauchy completion of $\mathcal{H}'$.

On this space two abstract operators can be defined,

$$\hat{\epsilon}|\mu\rangle \coloneqq \mu|\mu\rangle, \tag{15}$$

$$\hat{s}(\lambda)|\mu\rangle \coloneqq |\mu + \lambda\rangle. \tag{16}$$

The operator $\hat{\epsilon}$ is a symmetric operator and $\hat{s}(\lambda)$ defines a one-parameter family of unitary operators. In spite of this $\hat{s}(\lambda)$ is discontinuous with respect to λ; this means that no self-adjoint operator exists that can generate $\hat{s}(\lambda)$ by exponentiation. Examining a physical system with configuration space spanned by the coordinate q, which is assumed to have a discrete character, and its conjugate momentum p, the previous abstract representation can be projected and studied with respect to p-polarization. In this polarization the basis states will be

$$\psi_\mu(p) = \langle p|\mu\rangle = e^{i\mu p/\hbar}. \tag{17}$$

Following the algebraic construction method, a Weyl-Heisenberg algebra is introduced on $\mathcal{H}_{\text{poly}}$ and the action of its generators on the basis states is defined as follows:

$$\hat{\mathcal{U}}(\nu)\psi_\mu(p) = \psi_\mu(p+\nu) = e^{i\mu(p+\nu)/\hbar} \tag{18}$$

$$\hat{\mathcal{V}}(\lambda)\psi_\mu(p) = e^{i(\lambda+\mu)p/\hbar} = \psi_{\mu+\lambda}(p). \tag{19}$$

From this it can be inferred that the shifting operator $\hat{s}(\lambda)$ can be identified with the operator $\hat{\mathcal{V}}(\lambda)$, which is then discontinuous in λ; this means that the spatial translations generator, that is the momentum operator $\hat{p}$, does not exist. On the other hand, the operator $\hat{\mathcal{U}}(\nu)$ is continuous, so that the translation generator in the momentum space, i.e., the position operator $\hat{q}$, exists and it can be identified with the abstract operator $\hat{\epsilon}$.

Indeed

$$\hat{q}\psi_\mu(p) = -i\hbar\partial_p\psi_\mu(p) = \mu\psi_\mu(p). \tag{20}$$

It can be proved [15] that the kinematic polymer Hilbert space in this polarization is explicitly given by $\mathcal{H}_{\text{poly},p} = L^2(\mathbb{R}_B, d\mu_H)$, where $\mathbb{R}_B$ is the so-called Bohr compactification of the real line and $d\mu_H$ is the Haar measure.

Universally speaking, the Bohr compactification of a topological group G is a compact Hausdorff topological group H, canonically associated to G.

In particular, the Bohr compactification of an Abelian locally-compact topological group A (such as $\mathbb{R}$) is the dual group (in the sense of Pontryagin duality) of A itself, equipped with a discrete topology (see [21] for an extensive treatment on the subject).

It can be shown that the Bohr compactification $\mathbb{R}_B$ of the real line $\mathbb{R}$ is the dual group of $\mathbb{R}$, equipped with a discrete topology, in which any open set does not contain more than one point and which contains $\mathbb{R}$ itself densely.

It is interesting to notice that $\mathbb{R}_B$ is an Abelian topological group and there exists a one to one correspondence between irreducible representations of $\mathbb{R}$ and irreducible representations of $\mathbb{R}_B$, which can also be used as a definition for the Bohr compactification itself.

In the end, since the resulting configuration space of the theory is a compact group, the introduction of the Haar measure—which is unique for these kind of groups—is the most natural choice.

A similar picture is obtained in the q-polarization; the momentum operator cannot still be defined, while it is possible to show that the fundamental wave functions are Kronecker deltas and that the kinematic polymer Hilbert space is explicitly given by $\mathcal{H}_{\text{poly},x} = L^2(\mathbb{R}_d, d\mu_c)$, where $\mathbb{R}_d$ is the real line equipped with a discrete topology and $d\mu_c$ is the counting measure.

In order to build the dynamics a Hamiltonian operator $\hat{H}$ has to be defined on $\mathcal{H}_{\text{poly}}$, but since $\hat{p}$ does not exist, a direct implementation is not possible. To overcome this problem the momentum operator can be approximated by defining a regular graph $\gamma_\mu = \{q \in \mathbb{R} | q = n\mu, n \in \mathbb{Z}\}$ on the configuration space of the system, where μ is the fundamental scale introduced by the polymer representation. The basis kets $|\mu\rangle$ can now be indicated as $|\mu_n\rangle$, where $\mu_n = n\mu$ are the points belonging to the graph γ_{μ_0}. Consequently the generic states will be

$$|\psi\rangle_{\gamma_\mu} = \sum_n a_n|\mu_n\rangle, \tag{21}$$

and they will belong to the new Hilbert space $\mathcal{H}_{\gamma_\mu} \subset \mathcal{H}_{\text{poly}}$, given that they satisfy the condition $\sum_n |a_n|^2 < \infty$. Since the dynamics have to be closed in $\mathcal{H}_{\gamma_\mu}$, the shift parameter λ has to be fixed equal to μ, hence the action of $\hat{\mathcal{V}}(\lambda)$ will be

$$\hat{\mathcal{V}}(\lambda)|\mu_n\rangle = \hat{\mathcal{V}}(\mu)|\mu_n\rangle = |\mu_{n+1}\rangle. \tag{22}$$

In general, the variable p it can be written as

$$p \approx \frac{\hbar}{\mu}\sin\left(\frac{\mu}{\hbar}p\right) = \frac{\hbar}{2i\mu}(e^{i\frac{\mu}{\hbar}p} - e^{-i\frac{\mu}{\hbar}p}), \tag{23}$$

when the condition $p \ll \hbar/\mu$ holds.

Based on this approximation and visualizing the action of $\hat{\mathcal{V}}(\mu)$ in the p-polarization, it is clear that the operator $\hat{p}$ and its action can be approximated as

$$\hat{p}_\mu|\mu_n\rangle \approx \frac{\hbar}{2i\mu}[\hat{\mathcal{V}}(\mu) - \hat{\mathcal{V}}(-\mu)]|\mu_n\rangle, \tag{24}$$

where μ acts as a regulator.

To approximate the operator $\hat{p}^2$ two paths are possible

$$\hat{p}_\mu^2 \approx \frac{\hbar^2}{4\mu^2}[2 - \hat{\mathcal{V}}(2\mu) - \hat{\mathcal{V}}(-2\mu)], \tag{25}$$

based on the approximation

$$p^2 \approx \frac{\hbar^2}{\mu^2}\sin^2\left(\frac{\mu}{\hbar}p\right), \tag{26}$$

and hence defined by iterating the action of $\hat{p}$ according (24), or

$$\hat{p}_\mu^2 \approx \frac{\hbar^2}{\mu^2}[2 - \hat{\mathcal{V}}(\mu) - \hat{\mathcal{V}}(-\mu)], \tag{27}$$

by exploiting the approximation

$$p^2 \approx \frac{2\hbar^2}{\mu^2}\left(1 - \cos\left(\frac{\mu}{\hbar}p\right)\right), \tag{28}$$

which is valid as long as $p \ll \hbar/\mu$.

Hence, the well-defined, symmetric Hamiltonian operator will be

$$\hat{H}_\mu \coloneqq \frac{\hat{p}_\mu^2}{2m} + \hat{V}(\hat{q}), \tag{29}$$

where $\hat{V}(\hat{q})$ is the potential operator.

Therefore, quantizing a system according to the polymer representation, in the p-polarization, implies the use of the approximation (25) or (27) for the momentum operator, while the position operator will be the natural differential operator, whose action is expressed in (24).

In a semiclassical approach, this procedure corresponds to the proper introduction of the approximations (23), (26), and (28) on the variable p in the dynamics of the system of interest. Hence, on this level, the whole procedure can be thought of as a prescription to provide physical insight into the behavior of the quantum expectations values, according to the so-called Ehrenfest theorem.

IV. POLYMER KASNER SOLUTION

We will now apply the polymer formalism in a semiclassical framework to the study of a $5D$ Bianchi I model, whose solution in the vacuum is the well-known Kasner metric [16,17], focusing on the kinematics and dynamics of the fifth dimension.

In order to obtain the polymer Kasner cosmological solution, we need a minisuperspace Hamiltonian formulation, extended to the $5D$ case.

The $5D$ Bianchi I line element, written in the ADM formalism [22], is a straightforward generalization of the $4D$ one[1]

$$\begin{aligned} ds^2 &= -N^2(t)c^2dt^2 + {}^{(4)}h_{ij}dx^i dx^j \\ &= -N^2(t)c^2dt^2 + a^2(t)(dx^1)^2 + b^2(t)(dx^2)^2 \\ &\quad + c^2(t)(dx^3)^2 + d^2(t)(dx^5)^2, \end{aligned} \tag{30}$$

where $N(t)$ is the lapse function of the ADM formalism, ${}^{(4)}h_{ij}$ $(i,j = 1,2,3,5)$ is the metric tensor of the $4D$ manifold, which has coordinates that are all spacelike.

Having the general structure of this metric as starting point, we can build the Hamiltonian of the system

$$H_B = Ne^{-\sum_a q^a/2}\left\{\sum_a p_a^2 - \frac{1}{3}\left[\sum_b p_b\right]^2\right\}, \tag{31}$$

which, by varying with respect to $N(t)$ turns out to be a constraint for the dynamics, namely $H_B = 0$.

The couple (q^a, p_a) in (31) are the conjugate variables spanning an highly symmetric phase space, the so-called minisuperspace, and the relation between the metric factors and the q-variable is the usual one in the literature [18], extended to the $5D$ case

$$\begin{aligned} a(t) &= e^{q^1(t)/2}, \qquad b(t) = e^{q^2(t)/2}, \\ c(t) &= e^{q^3(t)/2}, \qquad d(t) = e^{q^5(t)/2}. \end{aligned} \tag{32}$$

As is well known, it is more convenient to express the obtained Hamiltonian in its diagonal form

[1]As it should be clear from the context, the metric factor $c(t)$ is not to be confused with the velocity of light c.

$$H'_B = Ne^{-\alpha}\left[-\frac{1}{3}p_\alpha^2 + p_+^2 + p_-^2 + p_\gamma^2\right], \tag{33}$$

which is the canonical form of the quadratic form associated to H_B.

The p-variables in the Hamiltonian (33) are the conjugate momenta of a set of variables α, β_+, β_-, γ, which represent the generalization of the Misner variables [23]. The relation between these variables and the previous q-variables is defined through the following linear transformation

$$\begin{cases} q^1 = \frac{1}{2}\alpha - \frac{1}{2\sqrt{3}}\beta_+ - \frac{1}{\sqrt{6}}\beta_- - \frac{1}{\sqrt{2}}\gamma \\ q^2 = \frac{1}{2}\alpha - \frac{1}{2\sqrt{3}}\beta_+ - \frac{1}{\sqrt{6}}\beta_- + \frac{1}{\sqrt{2}}\gamma \\ q^3 = \frac{1}{2}\alpha - \frac{1}{2\sqrt{3}}\beta_+ + \sqrt{\frac{2}{3}}\beta_- \\ q^5 = \frac{1}{2}\alpha + \frac{\sqrt{3}}{2}\beta_+. \end{cases} \tag{34}$$

By using the Hamilton-Jacobi method and the Hamilton equation for the variable α (which represents the universe volume) in the synchronous reference frame, the standard classical Kasner solution for the $5D$ case can be recovered,

$$ds^2 = -c^2dt^2 + (t/t_0)^{2k_1}(dx^1)^2 + (t/t_0)^{2k_2}(dx^2)^2 + (t/t_0)^{2k_3}(dx^3)^2 + (t/t_0)^{2k_5}(dx^5)^2. \tag{35}$$

The k parameters are the so-called Kasner exponents and they satisfy the following conditions

$$\sum_{i=1,2,3,5} k_i = 1, \qquad \sum_{i=1,2,3,5} k_i^2 = 1. \tag{36}$$

In particular, if we assume isotropy in the three usual spatial dimensions as observations suggest, that is, if we set $k_1 = k_2 = k_3$, the solution of the previous system becomes

$$k_1 = k_2 = k_3 = \frac{1}{2}, \qquad k_5 = -\frac{1}{2}. \tag{37}$$

This means that while the three usual spatial dimensions expand, the fifth one collapses indefinitely.

We want now to introduce polymer formalism in Bianchi I dynamics. In order to do this we choose to operate the substitutions (23) and (26) on the conjugate momentum p_γ of the Misner variable γ, connected with the metric factor of the fifth dimension.

We rewrite the action of Bianchi I Universe

$$S_B^{\mathrm{poly}} = \int dt[p_\alpha\dot{\alpha} + p_+\dot{\beta}_+ + p_-\dot{\beta}_- + p_\gamma\dot{\gamma} - H_B^{\mathrm{poly}}], \tag{38}$$

where H_B^{poly} is the new polymerized Hamiltonian:

$$H_B^{\mathrm{poly}} = Ne^{-\alpha}\left[-\frac{1}{3}p_\alpha^2 + p_+^2 + p_-^2 + \frac{\hbar^2}{\mu^2}\sin^2\left(\frac{\mu}{\hbar}p_\gamma\right)\right]. \tag{39}$$

As it is known from the Hamiltonian formulation of general relativity, the expression (33) is constrained by the dynamics to be zero.

This must happen also for the modified Hamiltonian (39)

$$H_B^{\mathrm{poly}} = 0 \to -\frac{1}{3}p_\alpha^2 + p_+^2 + p_-^2 + \frac{\hbar^2}{\mu^2}\sin^2\left(\frac{\mu}{\hbar}p_\gamma\right) = 0. \tag{40}$$

Following the literature, we explicitly solve the previous constraint for p_α, given the physical meaning of the α variable

$$H_B'^{\mathrm{poly}} \coloneqq p_\alpha = \pm\sqrt{3}\sqrt{p_+^2 + p_-^2 + \frac{\hbar^2}{\mu^2}\sin^2\left(\frac{\mu}{\hbar}p_\gamma\right)}. \tag{41}$$

To proceed in solving the dynamics we make use (as for the standard case) of the Hamilton-Jacobi method, according to which the following relations hold

$$\begin{cases} p_\alpha = \partial_\alpha S_B, \\ p_+ = \partial_{\beta_+} S_B, \\ p_- = \partial_{\beta_-} S_B, \\ p_\gamma = \partial_\gamma S_B, \end{cases} \tag{42}$$

where S_B is the action of the system.

Through the combination of the relations (40) and (42) we attain the Hamilton-Jacobi equation for the system

$$-\frac{1}{3}(\partial_\alpha S_B)^2 + (\partial_+ S_B)^2 + (\partial_- S_B)^2 + (\partial_\gamma S_B)^2 = 0. \tag{43}$$

By taking into account the fact that all the p-variables are first integrals of motion, we are able to write the solution of the above partial differential equation as

$$S_B = \pm\sqrt{3}\sqrt{p_+^2 + p_-^2 + p_\gamma^2}\,\alpha + \beta_+ p_+ + \beta_- p_- + \gamma p_\gamma, \tag{44}$$

where we have used the expression (41) for p_α.

In the Hamilton-Jacobi method, the action S_B can be seen as a generating function of a proper canonical transformation and its partial derivatives with respect to the momenta are constructed to be constant, that is,

$$\begin{cases} \partial_{p_\pm} S_B = \tilde{\beta}_\pm = \text{constant}, \\ \partial_{p_\gamma} S_B = \tilde{\gamma} = \text{constant}. \end{cases} \tag{45}$$

From the latter system we are able to write down the polymer solutions for the Misner variables as function of α

$$\beta_+(\alpha) = \mp \frac{3p_+}{H_B^{\prime\text{poly}}}\alpha + \tilde{\beta}_+, \tag{46}$$

$$\beta_-(\alpha) = \mp \frac{3p_-}{H_B^{\prime\text{poly}}}\alpha + \tilde{\beta}_-, \tag{47}$$

$$\gamma(\alpha) = \mp \frac{\hbar}{\mu}\frac{3\sin(\frac{\mu}{\hbar}p_\gamma)\cos(\frac{\mu}{\hbar}p_\gamma)}{H_B^{\prime\text{poly}}}\alpha + \tilde{\gamma}. \tag{48}$$

In this semiclassical picture, it is still possible to recover the notion of coordinate synchronous time through the Hamilton equations for the couple (α, p_α) by exploiting the Hamiltonian (39)

$$\begin{cases} \dot{\alpha} = \frac{\partial H_B^{\text{poly}}}{\partial p_\alpha} = -\frac{2}{3}Np_\alpha e^{-\alpha}, \\ \dot{p_\alpha} = -\frac{\partial H_B^{\text{poly}}}{\partial \alpha} = 0, \end{cases} \tag{49}$$

where in the second equation we have taken into account the dynamic constraint (40), which coherently confirms that p_α is a constant of motion.

Since α is linked to the volume of the Universe, the previous equations clearly show that if we want to deal with an expanding Universe we need to choose $p_\alpha < 0$.

As a consequence, this choice determines the $\pm$ sign in the functional S_B (44) and in the expressions of the Misner variables (46), respectively a minus sign and a plus sign.

The Hamilton equations (49) leads to the following solution for α as a function of the coordinate time

$$\alpha(t) = \ln\left(-\frac{2}{3}p_\alpha t\right), \tag{50}$$

where, without loss of generality, we set $N = 1$, a choice which is always possible in a synchronous reference frame.

By inserting the obtained expression in the system (46) we can write down the Misner variables as a function of the coordinate time

$$\beta_+(t) = \frac{3p_+}{H_B^{\prime\text{poly}}}\ln(t/t_0), \tag{51}$$

$$\beta_-(t) = \frac{3p_-}{H_B^{\prime\text{poly}}}\ln(t/t_0), \tag{52}$$

$$\gamma(t) = \frac{\hbar}{\mu}\frac{3\sin(\frac{\mu}{\hbar}p_\gamma)\cos(\frac{\mu}{\hbar}p_\gamma)}{H_B^{\prime\text{poly}}}\ln(t/t_0), \tag{53}$$

where we set $(-2/3p_\alpha) = 1/t_0$ and ignore the additive constants, which only leads to a new rescaling of the time constant t_0.

In order to determine the expression of cosmic metric factors which appear in the line element (30), we have to trace back all the changes of variables made so far.

First we recover the expression of the q-coordinates from the system (34)

$$\begin{cases} q^1(t) = \left[\frac{1}{2} + \frac{\sqrt{3}}{H_B^{\prime\text{poly}}}\Pi_-\right]\ln(t/t_0), \\ q^2(t) = \left[\frac{1}{2} + \frac{\sqrt{3}}{H_B^{\prime\text{poly}}}\Pi_+\right]\ln(t/t_0), \\ q^3(t) = \left[\frac{1}{2} + \frac{\sqrt{3}}{H_B^{\prime\text{poly}}}\left(-\frac{1}{2\sqrt{3}}p_+ + \sqrt{\frac{2}{3}}p_-\right)\right]\ln(t/t_0), \\ q^5(t) = \left[\frac{1}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{3}p_+}{H_B^{\prime\text{poly}}}\right]\ln(t/t_0), \end{cases} \tag{54}$$

where we have defined the quantity

$$\Pi_\pm = -\frac{1}{2\sqrt{3}}p_+ - \frac{1}{\sqrt{6}}p_- \pm \frac{1}{\sqrt{2}}\frac{\hbar}{\mu}\sin\left(\frac{\mu}{\hbar}p_\gamma\right)\cos\left(\frac{\mu}{\hbar}p_\gamma\right). \tag{55}$$

Then, recalling the relations (32), we obtain the expression for the cosmic metric factors

$$\begin{cases} a(t) = e^{q^1(t)/2} = (t/t_0)^{k_1}, \\ b(t) = e^{q^2(t)/2} = (t/t_0)^{k_2}, \\ c(t) = e^{q^3(t)/2} = (t/t_0)^{k_3}, \\ d(t) = e^{q^5(t)/2} = (t/t_0)^{k_5}, \end{cases} \tag{56}$$

where k_i are now the polymer Kasner exponents, and the explicit expression is

$$\begin{aligned} k_1 &= \frac{1}{2}\left[\frac{1}{2} + \frac{\sqrt{3}}{H_B^{\prime\text{poly}}}\Pi_-\right], \\ k_2 &= \frac{1}{2}\left[\frac{1}{2} + \frac{\sqrt{3}}{H_B^{\prime\text{poly}}}\Pi_+\right], \\ k_3 &= \frac{1}{2}\left[\frac{1}{2} + \frac{\sqrt{3}}{H_B^{\prime\text{poly}}}\left(-\frac{1}{2\sqrt{3}}p_+ + \sqrt{\frac{2}{3}}p_-\right)\right], \\ k_5 &= \frac{1}{2}\left[\frac{1}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{3}p_+}{H_B^{\prime\text{poly}}}\right]. \end{aligned} \tag{57}$$

We see that our solution is still a Kasner-like one, where the cosmic metric factors have a coordinate time power trend as in (35), but their exponents (that is the Kasner indices, due to the quantum polymer modifications) satisfy different constraints.

Indeed it is straightforward to show that the following relations hold

$$\sum_{i=1,2,3,5} k_i = 1,$$
$$\sum_{i=1,2,3,5} k_i^2 = 1 - \frac{3}{4}\frac{\hbar^2}{\mu^2} \frac{\sin^4(\frac{\mu}{\hbar} p_\gamma)}{\sqrt{p_+^2 + p_-^2 + \frac{\hbar^2}{\mu^2}\sin^2(\frac{\mu}{\hbar} p_\gamma)}}. \tag{58}$$

The second term of the right-hand side of the second condition is non-negative, so that we can restate the system as follows:

$$\sum_{i=1,2,3,5} k_i = 1,$$
$$\sum_{i=1,2,3,5} k_i^2 \leq 1. \tag{59}$$

Assuming isotropy in the three usual spatial dimensions and introducing an order between exponents (in particular setting $k_5 < k$) the system (59) has the following solution:

$$\begin{cases} 3k + k_5 = 1 \\ 3k^2 + k_5^2 \leq 1 \\ k_5 < k \end{cases} \Rightarrow \begin{cases} 1/4 < k \leq 1/2 \\ -1/2 \leq k_5 < 1/4. \end{cases} \tag{60}$$

We note that the Kasner exponents relative to the three usual spatial dimensions are bound to be positive and they can take values in a precise interval while the exponent relative to the fifth dimension has values in an interval in which it can take both positive and negative values, which is a new feature of the Kasner solution entirely due to the polymer physics.

The latter statement grows in importance when we draw our attention to the permitted values of k_5.

Indeed, even if the power trend with respect to the coordinate time variable of fifth dimension metric factor—which is ultimately responsible for the singularity at an infinite time—is not removed in this scenario; the introduction of the polymer formalism leads to a modification on the constraint of the fifth Kasner exponent k_5, where the interval of definition allows the value $k_5 = 0$.

For this particular choice of k_5, the other Kasner indices will be equal to $k = 1/3$, correctly reproducing the observed isotropic expansion in the three usual spatial dimensions, while the metric factor relative to the fifth dimension will, of course, be equal to one.

This means that the fifth dimension has no dynamics since the time dependence of the relative metric factor disappears.

We, therefore, have obtained a static solution, which somehow solves the singularity problem by remarkably removing the indefinite collapse of the dimension itself. The developments of the next sections will be based on this important result.

Finally, it is worth noticing that, once a set of Kasner indices is chosen—for example $(1/3, 1/3, 1/3, 0)$ in our case—the quadratic condition of the constraints (58) establishes a relation among the conjugate momenta of the generalized Misner variables, the polymer scale μ and the Planck constant $\hbar$. Since these momenta are constants of motion, for any fixed value of μ, the above relation represents a constraint on the initial conditions of the problem, which coherently derives from the Hamiltonian constraint of the whole theory.

In particular, this implies that any value of the polymer scale μ can account for our choice on the Kasner exponents, by properly varying the initial conditions p_+, p_-, p_γ according to (58).

V. KALUZA-KLEIN THEORY IN POLYMER QUANTUM MECHANICS FRAMEWORK

In this section we face the analysis of the Kaluza-Klein paradigm in the framework of polymer quantum mechanics, both studying the behavior of the geodesic motion and the quantum dynamics of a Klein-Gordon field. From the point of view of the geodesic discussion, the polymer framework is addressed on a semiclassical level, in the spirit of the Ehrenfest theorem. Instead, the study of the quantum scalar field is performed in a full quantum picture, as restricted to the fifth coordinate.

A. Geodesic motion

In this section polymer formalism will be now applied to the Hamiltonian formulation of the $5D$ geodesic motion, at a semiclassical level, that is introducing quantum modifications to the classical dynamics.

The $5D$ Hamiltonian of a free particle in a general Kaluza-Klein background, with $\tilde{g}_{55} = 1$ (see previous section) reads as

$$\tilde{H} = \frac{1}{2mc}\left[\tilde{p}_\mu \tilde{p}_\nu g^{\mu\nu} - \frac{4\sqrt{G}}{c^2}\tilde{p}_\mu \tilde{p}_5 A^\mu + \tilde{p}_5^2\left(1 + \frac{4G}{c^4}A_\mu A^\mu\right)\right], \tag{61}$$

where, as before, the quantities with a tilde are the $5D$ ones, that is those defined with respect to the $5D$ line element.

We want now introduce the polymer formalism only with respect to the canonical couple $(x^5, \tilde{p}_5)$, that is we assume that the coordinate x^5 has an essential discrete nature and we redefine the respective conjugate momentum $\tilde{p}_5$ by introducing a regular graph structure on $S^1_{x^5}$, i.e., on the fifth dimension, which is now equipped with a discrete topology according to the discussion in Section III. Following the polymer prescription (23), we can made the substitution

$$\tilde{p}_5 \to \frac{\hbar}{\mu}\sin\left(\frac{\mu}{\hbar}\tilde{p}_5\right). \tag{62}$$

The new Hamiltonian will be rewritten as

$$\tilde{H}_{\rm poly} = \frac{1}{2mc}\left[\tilde{p}_\mu\tilde{p}_\nu g^{\mu\nu} - \frac{4\sqrt{G}}{c^2}\frac{\hbar}{\mu}\tilde{p}_\mu\sin\left(\frac{\mu}{\hbar}\tilde{p}_5\right)A^\mu + \frac{\hbar^2}{\mu^2}\sin^2\left(\frac{\mu}{\hbar}\tilde{p}_5\right)\left(1+\frac{4G}{c^4}A_\mu A^\mu\right)\right], \tag{63}$$

from which the equations of motion (Hamilton equations) can be obtained

$$\begin{cases} \tilde{u}^\mu = \frac{\tilde{p}_\nu g^{\mu\nu}}{mc} - \frac{2\sqrt{G}}{mc^3}\frac{\hbar}{\mu}\sin\left(\frac{\mu}{\hbar}\tilde{p}_5\right)A^\mu \\ \dot{\tilde{p}}_\mu = -\frac{\tilde{p}_\rho\tilde{p}_\sigma\partial_\mu g^{\rho\sigma}}{2mc} + \frac{2\sqrt{G}}{mc^3}\tilde{p}_\rho\frac{\hbar}{\mu}\sin\left(\frac{\mu}{\hbar}\tilde{p}_5\right)\partial_\mu A^\rho - \frac{2G}{mc^5}\frac{\hbar^2}{\mu^2}\sin^2\left(\frac{\mu}{\hbar}\tilde{p}_5\right)\partial_\mu(A_\nu A^\nu), \end{cases} \tag{64}$$

$$\begin{cases} \tilde{u}^5 = \frac{1}{mc}\left[-\frac{2\sqrt{G}}{c^2}\tilde{p}_\mu A^\mu\cos\left(\frac{\mu}{\hbar}\tilde{p}_5\right) + \frac{\hbar}{\mu}\sin\left(\frac{\mu}{\hbar}\tilde{p}_5\right)\cos\left(\frac{\mu}{\hbar}\tilde{p}_5\right)\left(1+\frac{4G}{c^4}A_\nu A^\nu\right)\right] \\ \dot{\tilde{p}}_5 = 0, \end{cases} \tag{65}$$

where the dot refers to the derivative with respect to the five-dimensional line element $d\tilde{s}$.

We outline that in the last equation we made use of the cylinder condition, which implies that $\tilde{p}_5$ is a constant of motion, as expected.

As we know from the standard case, in order to make a comparison with the $4D$ equations of motion, we need to know the relation between $4D$ and $5D$ quantities, in particular the relation between five-velocities and four-velocities. These are formally the same of equation (7) also in our framework.

Nevertheless, while in the ordinary theory with the scalar field of the Kaluza-Klein metric set to be constant, u_5 is a constant of motion; in the polymer framework this is not the case.

Indeed, from the explicit expression of $\tilde{u}^5$ (65) we can obtain the corresponding expression of $\tilde{u}_5$

$$\tilde{u}_5 = \frac{2\sqrt{G}}{c^2}\cos\left(\frac{\mu}{\hbar}\tilde{p}_5\right)A_\mu\tilde{u}^\mu + \frac{\hbar}{\mu mc}\sin\left(\frac{\mu}{\hbar}\tilde{p}_5\right)\cos\left(\frac{\mu}{\hbar}\tilde{p}_5\right), \tag{66}$$

where we have made use of the first Hamilton equation for $\tilde{u}^\mu$.

The latter can be solved as a function of u_5 and two solutions are obtained

$$u_5 = \frac{1}{D(\mu,\tilde{p}_5)}\left\{16\sqrt{G}\mu^2m^2\left(1-\cos\left(\frac{\mu}{\hbar}\tilde{p}_5\right)\right)A_\mu u^\mu \pm\sqrt{2}\hbar\left|\sin\left(2\frac{\mu}{\hbar}\tilde{p}_5\right)\right|R_{\mu,\tilde{p}_5}(A_\mu,u^\mu)\right\}, \tag{67}$$

where

$$R_{\mu,\tilde{p}_5}(A_\mu,u^\mu) \coloneqq \sqrt{D(\mu,\tilde{p}_5)+K(\mu,\tilde{p}_5)(A_\mu u^\mu)^2}, \tag{68}$$

$$D(\mu,\tilde{p}_5) \coloneqq \hbar^2 + 8\mu^2m^2c^2 - \hbar^2\cos\left(4\frac{\mu}{\hbar}\tilde{p}_5\right), \tag{69}$$

$$K(\mu,\tilde{p}_5) \coloneqq -128\frac{G}{c^2}\mu^2m^2\sin^4\left(\frac{\mu}{2\hbar}\tilde{p}_5\right). \tag{70}$$

These are rather complicated nonconstant expressions, due to the dependence from the electromagnetic and the four-velocity fields, which consequently heavily modify the $\frac{ds}{d\tilde{s}}$ factor.

From the Hamilton equations (64), by exploiting the formal relation (7), we can write down the part relative to the $4D$ indices of the new generalized $5D$ geodesic as a function of the $4D$ quantities

$$u^\nu\nabla_\nu u^\mu = \frac{2\sqrt{G}\hbar}{\mu mc^3}\sqrt{1-u_5^2}\sin\left(\frac{\mu}{\hbar}\tilde{p}_5\right)u^\nu F^\mu_\nu - \frac{u_5u^\mu}{1-u_5^2}\frac{du_5}{ds}, \tag{71}$$

in which it is possible to distinguish the polymer-modified part of the standard $4D$ geodesic (9) and an extra term entirely due to the variation of u_5. It is worth pointing out that Eq. (71) correctly reduces to the standard one in the $\mu\to 0$ limit.

At this point, in order to compare the last expression with the equation of motion (9), we need to make a series expansion of u_5 by assuming $A_\mu \ll 1$. This can be perfectly legitimate in a cosmological setting since there is no relevant coherent electromagnetic field which permeates the Universe.

This leads to the following expression, to the first order in A_μ

$$u_5 \approx \pm\left\{\frac{16\sqrt{G}\mu^2 m^2(\cos(\frac{\mu}{\hbar}\tilde{p}_5)-1)}{D(\mu,\tilde{p}_5)}A_\mu u^\mu + \frac{\sqrt{2}\hbar|\sin(2\frac{\mu}{\hbar}\tilde{p}_5)|}{\sqrt{D(\mu,\tilde{p}_5)}} + O(A^2)\right\}. \tag{72}$$

Hence, by inserting this expansion in the geodesic equation (71) and ignoring the extra term—the meaning of which is beyond the purpose of this article—we obtain, keeping only the overall first order term in A_μ

$$u^\nu\nabla_\nu u^\mu \approx \frac{2\sqrt{G}\hbar}{\mu mc^3}\frac{2\sqrt{2}\mu mc\sin(\frac{\mu}{\hbar}\tilde{p}_5)}{\sqrt{D(\mu,\tilde{p}_5)}}u^\nu F^\mu_\nu, \tag{73}$$

from which we can read off the perturbative polymer relation between the electric charge q and $\tilde{p}_5$—which we recall is a constant of motion

$$q \approx \frac{2\sqrt{G}}{c}\frac{2\sqrt{2}\hbar mc}{\sqrt{D(\mu,\tilde{p}_5)}}\sin\left(\frac{\mu}{\hbar}\tilde{p}_5\right). \tag{74}$$

To correctly interpret this formula we can refer to the ordinary relation between q and $\tilde{p}_5$, which can be obtained from the expression (11). In fact, since in the standard case the following simple statement holds

$$\tilde{p}_5 = \frac{mc}{\sqrt{m^2c^2 - p_5^2}}p_5, \qquad p_5 = \frac{mc}{\sqrt{m^2c^2+\tilde{p}_5^2}}\tilde{p}_5, \tag{75}$$

the relation (11) can be rewritten as

$$q = \frac{2\sqrt{G}}{c}\frac{mc}{\sqrt{m^2c^2+\tilde{p}_5^2}}\tilde{p}_5. \tag{76}$$

This suggests that the expression (74) is the polymer generalization to the first perturbative order in the electromagnetic field of the latter relation.

This interpretation is confirmed by the $\mu\to 0$ limit of (74), from which the standard expression (76) is recovered.

Therefore, based on the relations (75), we can impose

$$\frac{\hbar}{\mu}\sin\left(\frac{\mu}{\hbar}p_5\right) \approx \frac{2\sqrt{2}\hbar mc}{\sqrt{D(\mu,\tilde{p}_5)}}\sin\left(\frac{\mu}{\hbar}\tilde{p}_5\right) \tag{77}$$

and, in the end

$$q \approx \frac{2\hbar\sqrt{G}}{\mu c}\sin\left(\frac{\mu}{\hbar}p_5\right). \tag{78}$$

This is a periodic and bounded function of p_5 with period 2π, defined in the generic periodic interval $[\pi(2k-1)\hbar/\mu, \pi(2k+1)\hbar/\mu]$, with $k\in\mathbb{Z}$. Since we want to preserve the Cauchy problem symmetry with respect to the initial values of p_5 (that is the symmetry of left-handed and right-handed particles in the fifth direction) we are led to choose the interval $[-\pi\hbar/\mu, \pi\hbar/\mu]$, i.e., the one for $k=0$, as the natural periodic interval of definition. In the $\mu\to 0$ limit, the obtained function correctly reproduces the ordinary relation inferable from (11).

As in the ordinary case, being the fifth dimension a compact space, we have to assume that p_5 is quantized; in order to determine the quantization law, we need to represent the free-particle state in the fifth coordinate as described by the wave function of the polymer free-particle. As comprehensively discussed in [15], in the position representation, this will be

$$\psi_\mu(x^5) = \frac{1}{\sqrt{L}}e^{ix^5 p_5^{(\mu)}/\hbar}, \tag{79}$$

normalized on $S^1_{x_5}$, where L is the length of the circus which characterizes the fifth dimension. Hence, by imposing $\psi_\mu(x^5)$ to be periodic in x^5, we achieve the following quantization law

$$p_5^{(\mu)} = \frac{2\pi n}{L}\hbar, \qquad n\in\mathbb{Z}, \tag{80}$$

which immediately leads, accordingly to (78), to a new quantization law for the electric charge

$$q_n \approx \frac{2\hbar\sqrt{G}}{\mu c}\sin\left(\frac{2\pi\mu}{L}n\right), \tag{81}$$

defined in the aforementioned periodic interval.

We can choose q_n to be equal to the electron electric charge e for $n=1$, as in the ordinary case, and in this way we are able to find an expression for L,

$$L \approx \frac{2\pi\mu}{\arcsin(\frac{e\mu c}{2\hbar\sqrt{G}})}, \tag{82}$$

from which we deduce a constraint for the polymer scale μ

$$0 < \mu \le \frac{2\hbar\sqrt{G}}{|e|c} \approx 3.78\times 10^{-32}\ \mathrm{cm}. \tag{83}$$

At this point we choose μ equal to the Planck length and obtain

$$L \approx 2.377 \times 10^{-31}\ \mathrm{cm}, \tag{84}$$

which almost coincides with the result of the standard theory and therefore it can account for the nonobservability of the fifth dimension.

There are basically two reasons behind this particular choice of the polymer scale:

a. it is a scale with a strong physical meaning
b. the L/μ ratio, for this value of μ, is large enough to ensure some calculations to be carried out in the polymer continuum limit, through the assumptions discussed in [14,15].

A further discussion about the function (82) is postponed to the next subsection.

The charge function (78), instead, can be rewritten, by means of the function $L(\mu)$ (82), as follows:

$$q(\mu; n) = \frac{2\hbar\sqrt{G}}{\mu c} \sin\left(n \arcsin\left(\frac{\mu e c}{2\hbar\sqrt{G}}\right)\right) \tag{85}$$

and by setting the polymer scale μ equal to the Planck length, we obtain the symmetric distribution of positive and negative charges reported in Fig. 1.

The number of modes n in the considered interval is limited by the periodic condition of the function itself and it clearly depends on μ. For our choice of μ we find that $-73 \leq n \leq 73$ (see again Fig. 1).

It is worth noting that for any fixed value of $n = n^*$ it is always possible to expand the sine function in correspondence to a suitable small cut off parameter μ, accordingly to the inequality $n^*\mu \ll 2\hbar\sqrt{G}/ec$, where we also have expanded the function $L(\mu)$.

In this limit we recover the standard expression for the charge as multiple of the elementary electron charge, at least for $n < n*$.

Finally, we want to evaluate the consequence of the condition $|u_5| < 1$ in our framework. The relation between u_5 and q can be found employing the expression (74). Hence, by retaining only the zero-order term in u_5—coherently with the overall expansion in A_μ and its derivatives we have done in the equation of motion (71)—we can write

$$|u_5| \approx \frac{|q|}{2m\sqrt{G}}\sqrt{\frac{1}{2} + \frac{2G}{q^2m^2} \pm \frac{1}{q^2\hbar}F(\mu)}, \tag{86}$$

where

$$F(\mu) \coloneqq [16G^2\hbar^2m^4 - 8G\hbar^2m^2q^2 + 4\mu^2m^2c^2q^4 + h^2q^4]^{1/2}. \tag{87}$$

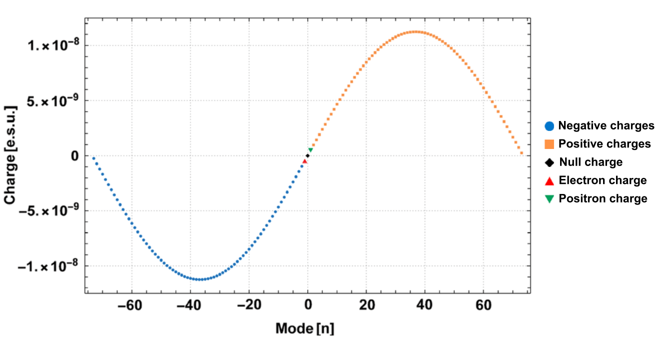

FIG. 1. Plot of charges distribution for a value of μ equal to the Planck length. It is possible to appreciate the oscillating profile of the function and the negative and positive symmetric branches.

By imposing the condition $|u_5| < 1$, the solution with a plus sign can be ruled out since it always violates the constraint, while for the solution with minus sign (which is the meaningful one) we obtain

$$\frac{|q|}{m} \leq \frac{2\hbar\sqrt{G}}{\mu m c}, \tag{88}$$

which is in agreement with the quantization law (78).

This constraint—that we stress is valid only in a perturbative regime—is different from the ordinary one (14) and it introduces a dependence in the q/m ratio from the polymer scale μ (as expected) and from the mass itself. Remarkably, the empirical q/m ratio of any known particle always respects the bound for every value of μ in the interval (83). This means that the new constraint does not set an unphysical condition and at least does not contradict the experimental evidence; thus resolving one of the shortcoming aspects of the standard theory. In other words, we are now able, in principle, to reproduce the q/m ratio of any Standard Model particle.

B. Polymer complex scalar field coupled with Kaluza-Klein metric

In this subsection we will carry out the study of a complex scalar field polymer dynamics, on a Kaluza-Klein background, obtained through a proper perturbation of the previous polymer Kasner metric.

Following the literature [13,17,18], under the isotropy assumption in the three usual spatial dimensions, we rewrite the polymer Kasner solution as

$$ds^2 = -c^2dt^2 + \left(\frac{t}{\tau}\right)^{2k}(d\vec{x})^2 + \left(\frac{t}{\tau}\right)^{2k_5}(dx^5)^2, \tag{89}$$

where τ is a time characteristic of the present age of the Universe and we have rescaled the coordinates by the same factor.

Since we are interested in the static case with respect to the fifth dimension, we choose $k_5 = 0$ and hence, as we have seen in the previous section, $k = 1/3$

$$ds^2 = -c^2dt^2 + \left(\frac{t}{\tau}\right)^{2/3}(d\vec{x})^2 + (dx^5)^2. \qquad (90)$$

Taking into account the rescaling, the coordinates will now satisfy the following conditions,

$$0 \leq |\vec{x}| < L(\tau/t_0)^k, \qquad 0 \leq x^5 < L. \qquad (91)$$

In order to include a perturbative electromagnetic field, a small perturbation $h_{\mu 5}$ ($\mu = 0, 1, 2, 3$) is added to the polymer-modified Kasner metric, properly proportional to the electromagnetic field itself

$$h_{\mu 5} = \frac{2\sqrt{G}}{c^2}A_\mu. \qquad (92)$$

As stated in the previous section this is naturally justified in a cosmological setting.

Coupled with this background we introduce a quantum complex scalar field Φ, which ordinary dynamics is described by a $5D$ Klein-Gordon equation and we set the $5D$ "mass" term equal to zero

$$^{(5)}\Box\Phi(x^a) = 0, \quad a = 0, 1, 2, 3, 5. \qquad (93)$$

As in the standard case, our analysis relies on the assumption of the cylinder hypothesis for observable physical quantities, still providing the scalar field Φ with a phase factor depending on x^5, the latter described by the polymer free-particle wave function, with periodicity condition on the fifth coordinate due to the topology of the space

$$\Phi(x^\mu, x^5) = \frac{1}{\sqrt{L}}\phi(x^\mu)e^{ix^5 2\pi n/L}, \qquad (94)$$

where the function $\phi(x^\mu)$ depends only on the variables of the $4D$ space-time.

A $(4+1)$-dimensional splitting of Eq. (93) results in the following expression

$$\partial_\mu\partial^\mu\Phi - \frac{2\sqrt{G}}{c^2}\partial_\mu A^\mu\partial_5\Phi - \frac{4\sqrt{G}}{c^2}A^\mu\partial_\mu\partial_5\Phi + \left[1 + \frac{4G}{c^4}A_\nu A^\nu\right]\partial_5^2\Phi = 0. \qquad (95)$$

We want now to study Eq. (95) by introducing the polymer quantum framework only with respect to the fifth dimension; therefore we switch to momentum representation only on the fifth coordinate and then let the operator $\hat{p}_5$ act according to the polymer prescription (24).

It is worth noting that, acting in this way, we are working in a mixed representation of position and momentum, which would not be possible universally speaking, nevertheless, since the coupling term in (95) between x^μ and x^5 is a small perturbation, we are legitimate, under this assumption, to proceed along this path according to the diagonal form of the background metric.

Finally, the following equation for the complex scalar field is achieved

$$^{(4)}\Box\phi(x^\mu) - i\frac{2\sqrt{G}}{c^2\mu}\phi(x^\mu)\partial_\mu A^\mu \sin\left(\mu\frac{2\pi n}{L}\right) - i\frac{4\sqrt{G}}{c^2\mu}\sin\left(\mu\frac{2\pi n}{L}\right)A^\mu\partial_\mu\phi(x^\mu) - \left[1 + \frac{4G}{c^4}A_\nu A^\nu\right]\frac{1}{\mu^2}\sin^2\left(\mu\frac{2\pi n}{L}\right)\phi(x^\mu) = 0. \qquad (96)$$

By comparing term by term the latter with the $4D$ equation of a massive complex scalar field, coupled with an electromagnetic one, on a curved space-time, we attain the following identifications

$$q_n = \frac{2\hbar\sqrt{G}}{\mu c}\sin\left(\frac{2\pi\mu}{L}n\right), \qquad m_n = \frac{\hbar}{\mu c}\left|\sin\left(\frac{2\pi\mu}{L}n\right)\right|. \qquad (97)$$

The first one is again a quantization law for the electric charge and it is clearly the same we obtained in (81) in the modified geodesic study.

Hence, by imposing $q_n = e$ for $n = 1$ again, we recover for the size L of the fifth dimension the function (82), with the constraint (83).

This function is a monotonically decreasing function of μ, which reaches its minimum value at the right endpoint of the interval in which is defined and it tends exactly to the value of the ordinary theory (13) as μ approaches zero (see Fig. 2). Clearly, choosing again μ around the Planck length, we find for the value of L the result (84), discussed above.

On the other hand, the second relation (97) represents a mass distribution law for the scalar field, which, by means

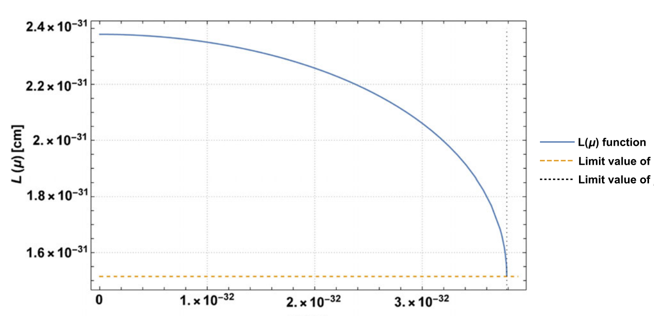

FIG. 2. Monotonically decreasing trend of the size L of the fifth dimension as a function of the polymer scale μ. The constraint (83) for μ establishes a precise domain and codomain for the function $L(\mu)$.

of the expression (82), can be rewritten as a function of the integer parameter n and of the continuous variable μ

$$m(\mu; n) = \frac{\hbar}{\mu c}\left|\sin\left(n \arcsin\left(\frac{\mu e c}{2\hbar\sqrt{G}}\right)\right)\right|. \qquad (98)$$

This function—for the same reasons stated in the previous subsection regarding the charge function—can be defined in the periodic interval $[-\pi\hbar/\mu, \pi\hbar/\mu]$, but, because of the presence of the modulus, we can restrict our attention only on the positive segment of the interval itself, that is $[0, \pi\hbar/\mu]$. For a fixed value of the polymer scale μ, the resulting sequence of masses, due to the periodicity condition, will include only a finite number of mass modes n and it clearly will be bounded and oscillating, as n changes.

This freedom in the choice of the scale leads to a crucial feature of such distribution; picking a particular mode n, through a fine tuning of μ, it is possible to fit any desired value of the mass, and therefore also Standard Model-comparable masses. In particular, the spectrum can be explored in correspondence to a Planckian-like value of the parameter μ.

The specific pattern of the distribution then will depend on the chosen reference mass to be fitted to a certain mode n, for a fixed value of μ; nevertheless, it can be shown that a qualitative general trend can be recovered, which provides the presence of the assigned mass in the minimum of the sequence as a ground level of the Kaluza-Klein tower. We stress that these regular values appear accompanied by Planckian masses, as sketched in Fig. 3.

This distribution deeply differs from the one obtained in the standard theory, reported in [13]. In fact, the latter is a linear and increasing sequence of only Planckian masses without an upper limit.

Actually, the possibility of fitting arbitrary masses, and from this obtaining various distributions can be achieved in the ordinary theory, as shown in [13] [by introducing a $5D$ mass term a in the Klein-Gordon Eq. (93)] the role of which is that of an additive parameter to be fine tuned.

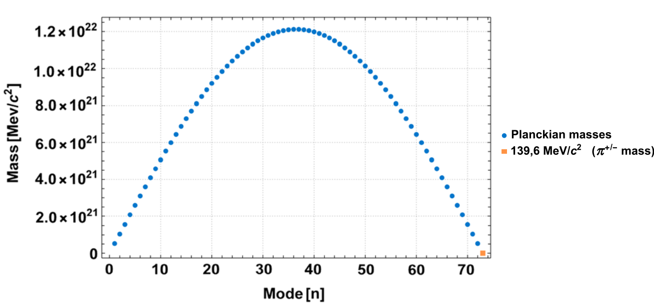

FIG. 3. Plot of masses distribution for the complex scalar field, for the value $\mu_{\text{Planck}} \approx 1.627 \times 10^{-33}$ cm. It is possible to observe the oscillating profile of the function and the fitted pion mass, placed in the minimum, as ground level of the Kaluza-Klein tower.

Nevertheless, this way of proceeding reveals several issues, which consequently are all addressed in our framework, the introduction of such an extra quantity a not being necessary. First of all, in general, the fine-tuning procedure cannot be applied to every mode, since the linear and increasing trend of the distribution is not modified at all. Secondly, the addition of this parameter eventually generates the rise of tachyonic masses in the past. This happens because of the presence of the metric coefficient $(t/\tau)^{|k_5|}$ in the standard momentum expression, as discussed in [13], which cannot be removed, since in the ordinary theory the Kasner solution does not admit the case $k_5 = 0$. This means that in our framework the tachyonic masses are ruled out from the polymer-modified distribution.

We outline that, in our analysis, even in the case $k_5 \neq 0$, i.e., in the nonstatic case, the tachyonic problem would be solved since it is the chance to set $a = 0$ which removes the unphysical masses.

Finally, the a parameter has to be introduced on purpose in the theory and its physical meaning remains ambiguous, while in our framework the fine-tuning parameter is an internal degree of freedom of the theory.

Hence, even if polymer quantum mechanics fails in removing completely the Planckian masses from the spectrum, it succeeds first in introducing a cutoff since the mass distribution is bounded. Furthermore, in the considered natural interval a finite number of variable mass values take place in the mass distribution in the considered periodic interval of definition of the mass function itself, and secondly in accounting for Standard Model-comparable masses, under a proper fine-tuned choice of the scale variable μ.

Since we are dealing with a complex scalar field, the charged pions $\pi^{\pm}$ seem to be relevant (phenomenologically) candidates in nature for the reference masses. The result is that the pion mass can be fitted at the mode $n = 73$ for a fine-tuned value of μ (up to twenty decimal figures) almost equal to the Planck length, which we will denote from now on as μ_{Planck}.

Nevertheless, for this procedure to be consistent, it is necessary to verify that the resulting mode $n = 73$ belongs to the range of n admitted in the periodic interval of definition of the masses function. It is easy to show that for $\mu = \mu_{\text{Planck}}$ the condition for positive n for belonging—in which we are interested—is $n \leq 73$. Therefore, the results coming from the fit procedure are valid and legitimate in this regard.

In Fig. 3 it is reported the whole sequence of masses resulting for $\mu = \mu_{\text{Planck}}$, where it is possible to appreciate the general behavior discussed above.

In particular, we observe that the pion mass is in the minimum of the distribution and that the maximum available mass is always

$$m_{\max} = \frac{\hbar}{\mu c},$$

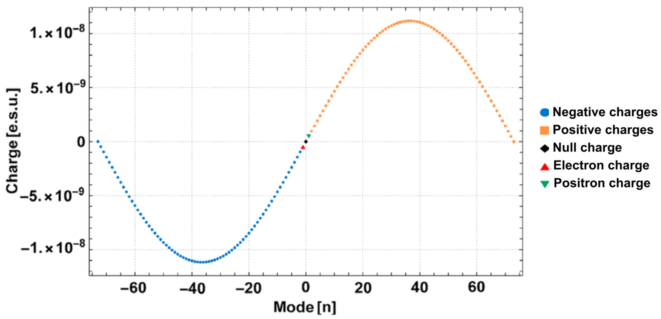

FIG. 4. Plot of charges distribution for value $\mu = \mu_{\mathrm{Planck}}$. Again, it is possible to observe the oscillating profile of the sequence and the negative and positive symmetric branches.

which is inversely proportional to μ, while the first mode of the sequence ($n = 1$) is

$$m(\mu; 1) = \frac{e}{2\sqrt{G}} \approx 9.29 \times 10^{-7}\ \mathrm{g}. \tag{99}$$

It does not depend on μ and it is almost a Planckian mass, defined only by fundamental constants, equal to the one obtained in the standard framework (without the introduction of the *ad hoc* parameter a).

In Fig. 4 instead the corresponding charge distribution [which again can be put in the form (85)] for $\mu = \mu_{\mathrm{Planck}}$ is represented.

We observe, however, that only the electron charge ($n = 1$) has a phenomenological correspondence, while the remaining points of the sequence have not a clear interpretation. In particular according to these mass and charge distributions the fundamental charge e has to be associated with a particle of Planckian mass, while the corresponding charge of the pion would be several orders of magnitude smaller than the electron charge. Clearly, this peculiar charge-mass configuration is not phenomenological consistent. Indeed, calculating the q/m ratio for the modes of the scalar field, we obtain

$$\frac{q(\mu; n)}{m(\mu; n)} = \pm 2\sqrt{G} \approx 5.16 \times 10^{-4}\ \mathrm{e.s.u./g}, \tag{100}$$

where the $\pm$ sign is due to the sign of the electric charge.

This value, which coincides with the upper limit of the q/m ratio (14) of the standard classical case, does not depend on μ or n, rather it is constant for every polymer scale and every mode and no known particle satisfies such a relation.

VI. CONCLUSION

We investigated the formulation of a five-dimensional Kaluza-Klein theory in the framework of polymer quantum mechanics, viewed both in a semiclassical and quantum approach. The polymer modifications have been implemented to the fifth coordinate only, on a semiclassical level in the spirit of the Ehrenfest theorem (the modification provides the dynamics of the quantum expectation values) and in a full quantum approach when a Klein-Gordon equation has been investigated.

We started by applying the semiclassical polymer formulation to the evolution of the Bianchi I model, by showing that the corresponding Kasner solution can be taken in a form in which three scale factors isotropically expand and the remaining one is static and, in the considered model, it coincides with the compactified extra dimension.

Then we studied the geodesic motion of a particle, starting with a Hamiltonian formulation (the only one in which the polymer formulation is viable) and then turning to a formalism based on the particle velocities. This procedure allows us, in analogy to the standard literature on this same subject, to identify the expression for the electric charge via the fifth-momentum component of the particle. The important consequence of this revised formulation consists of overcoming the problem of a too small charge to mass ratio to account in the model for any known elementary particle. In fact, the revised constraint, due to the polymer relation between the fifth-momentum component and the corresponding velocity, is in principle compatible with all the elementary particle predicted by the Standard Model.

Finally, we implemented a quantum polymer modification in the Klein-Gordon equation, by adopting a mixed representation of quantum mechanics (based on the coordinates for the usual four dimensions and the momentum for the extra one). This study aims to revise the analysis in [13] for a static (now available) extra dimension, under a polymer prescription for the compactified dimension physics.

We got the fundamental result that the tachyon mode, present in [13], is now removed from the mass spectrum and that the obtained values for the boson mass can fit the values spanned in the Standard Model. Actually, we arrived at a deformed morphology of the so-called Kaluza-Klein tower (the steps are no longer equispaced), but this revised structure allows us to avoid the only Planckian mode naturally present in the standard Kaluza-Klein formulation.

All these results suggest that some of the puzzling questions affecting the viability of the Kaluza-Klein idea must be reanalyzed phenomenologically including the notion of a cut off physics. In fact, in the case of small dimensions, living about two orders of magnitude over the Planck scale, it should be unavoidable to feel the effects of the nearby cutoff and when its presence is made manifest a new paradigm can be assessed. In other words, we argue that some limits of the geometrical unification theories are possibly due to the ultraviolet divergence that the gravitational field possesses and when they are somehow attenuated, like by the polymer scenario adopted here, the compactified dimension takes a more regular behavior, which is reflected into the solution of some inconsistencies of the underlying model.

The emergence of a static dimension in the 5D Kasner solution (which prevents the necessity to deal with unphysical tachyonic modes) undoubtedly represents the simplest elucidation of this point of view.

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