Gibbs measures associated to the integrals of motion of the periodic derivative nonlinear Schrödinger equation

Giuseppe Genovese[1] · **Renato Lucà**[2] ·
Daniele Valeri[3]

Abstract We study the one-dimensional periodic derivative nonlinear Schrödinger equation. This is known to be a completely integrable system, in the sense that there is an infinite sequence of formal integrals of motion $\int h_k, k \in \mathbb{Z}_+$. In each $\int h_{2k}$ the term with the highest regularity involves the Sobolev norm $\dot{H}^k(\mathbb{T})$ of the solution of the DNLS equation. We show that a functional measure on $L^2(\mathbb{T})$, absolutely continuous w.r.t. the Gaussian measure with covariance $(\mathbb{I} + (-\Delta)^k)^{-1}$, is associated to each integral of motion $\int h_{2k}, k \geq 1$.

Keywords Gibbs measures · DNLS · Integrable systems

Mathematics Subject Classification 35Q30 · 35BXX · 37K05 · 37L50 · 35Q55 · 37K10 · 37K30 · 17B69 · 17B80

✉ Renato Lucà
   renato.luca@icmat.es

   Giuseppe Genovese
   giuseppe.genovese@math.uzh.ch

   Daniele Valeri
   daniele@math.tsinghua.edu.cn

1 Institut für Mathematik, Universität Zürich, 8057 Zurich, Switzerland

2 Instituto de Ciencias Matemáticas, CSIC, 28049 Madrid, Spain

3 Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, China

🦅 Birkhäuser

1 Introduction

In this paper we consider the derivative nonlinear Schrödinger equation (DNLS) in the space periodic setting:

$$\begin{cases} i\psi_t = \psi'' + i\beta \left(\psi |\psi|^2 \right)' \\ \psi(x, 0) = \psi_0(x), \end{cases} \tag{1.1}$$

where $\psi(x, t) : \mathbb{T} \times \mathbb{R} \to \mathbb{C}$, $\psi_0(x) : \mathbb{T} \to \mathbb{C}$, $\psi'(x, t)$ denotes the derivative with respect to x, and $\beta \in \mathbb{R}$ is a real parameter.

The DNLS is a dispersive nonlinear equation coming from magnetohydrodynamics. It describes the motion along the longitudinal direction of a circularly polarized wave, generated in a low-density plasma by an external magnetic field [23,31] (see also [36]). It is known to be an integrable system [20] (see also [13]) in the sense that there is an infinite sequence of linearly independent quantities (integrals of motion) which are conserved by the flow of (1.1) for sufficiently regular solutions. The integrals of motion are functionals defined on Sobolev spaces of increasing regularity.

The aim of this paper is to construct an infinite sequence of functional Gibbs measures associated to the integrals of motion. These measures turn out to be absolutely continuous with respect to the standard Gaussian measures with covariance $(\mathbb{I} + (-\Delta)^k)^{-1}$; thus, different measures are disjointly supported (see "Appendix").

The program of statistical mechanics of PDEs begins with the seminal paper by Lebowitz et al. [21]. The authors study the periodic one-dimensional NLS equations and introduce the statistical ensembles naturally associated to the Hamiltonian, as in a classical field theory. Successively, Bourgain completed this study: in [3] by proving the invariance of the Gibbs measure for the cubic periodic case and in [6] extending the results to the real line. Similar achievements have been obtained with different methods in [26] for cubic NLS, in [25] for the wave equation, and in two dimensions in [4] for defocusing cubic NLS equation, in [7] for the focusing case, in three dimensions for the Gross–Pitaevskii equation in [5].

For integrable PDEs one can carry out the same study by profiting from an infinite number of higher Hamiltonian functionals. This was originally noted by Zhidkov [47], who analyzed the Korteweg–de Vries (KdV) and cubic nonlinear Schrödinger (NLS) equation on $\mathbb{T}$. The main idea, already contained in [21], is to restrict the measure associated to a given integral of motion to the set of solutions with fixed values for all the other integrals of motion involving less regularity (in a sense that will be clarified below). The invariance of such a set of measures gives interesting informations on the long-time behavior of the regular solutions, for instance through the Poincaré Recurrence Theorem (see [8,47]). In the last years this approach has been adopted in a series of papers by Tzvetkov et al. [11,12,39–42] for the Benjamin Ono equation on $\mathbb{T}$. In this case a more careful construction of the measure is required compared to KDV and NLS. We find similar difficulties in studying the DNLS equation.

Despite the extensive investigation in the past decades on integrable PDEs, a limited attention has been given to the integrability properties of the DNLS equation. An infinite sequence of integrals of motion for this equation has been found in [20] using the inverse scattering method. More recently, another proof of the integrability of the

DNLS equation has been achieved using the so-called the Lenard–Magri scheme [22] within the framework of (non-local) Poisson vertex algebras [13].

The first few integrals of motion of the DNLS equation are:

$$\int h_0 = \frac{1}{2}\|\psi\|_{L^2}^2,$$

$$\int h_1 = \frac{i}{2}\int \psi\bar{\psi}' + \frac{\beta}{4}\|\psi\|_{L^4}^4, \tag{1.2}$$

$$\int h_2 = \frac{1}{2}\|\psi\|_{\dot{H}^1}^2 + \frac{3i}{2}\beta\int \psi^2\bar{\psi}\bar{\psi}' + \frac{\beta^2}{4}\|\psi\|_{L^6}^6,$$

$$\int h_3 = \frac{i}{2}\int \psi'\bar{\psi}'' + \frac{\beta}{4}\int \left((\psi')^2\bar{\psi}^2 + 8\psi\bar{\psi}\psi'\bar{\psi}' + \psi^2(\bar{\psi}')^2\right)$$

$$\qquad + \frac{5i}{4}\beta^2\int \psi^3\bar{\psi}^2\bar{\psi}' + \frac{5}{16}\beta^3\|\psi\|_{L^8}^8,$$

$$\int h_4 = \frac{1}{2}\|\psi\|_{\dot{H}^2}^2 + \frac{5i}{4}\beta\int \left(\psi\bar{\psi}\psi'\bar{\psi}'' - \psi\bar{\psi}\psi''\bar{\psi}'\right)$$

$$\qquad + \frac{5}{4}\beta^2\int \left(\psi\bar{\psi}^3(\psi')^2 + 5\psi^2\bar{\psi}^2\psi'\bar{\psi}' + \psi^3\bar{\psi}(\bar{\psi}')^2\right)$$

$$\qquad + \frac{35i}{16}\beta^3\int \psi^4\bar{\psi}^3\bar{\psi}' + \frac{7}{16}\beta^4\|\psi\|_{L^{10}}^{10}. \tag{1.3}$$

Here and further, we denote by $\int f = \frac{1}{2\pi}\int_{\mathbb{T}} f$. Note that, while for k even the term of highest regularity in the integral of motion $\int h_k$ is the $\dot{H}^{\frac{k}{2}}(\mathbb{T})$ norm, for odd k this term is not definite in sign. This prevents us to associate an invariant Gibbs measure to every integral of motion $\int h_k$, $k \in \mathbb{Z}_+$. The same does not occur for KdV, NLS or Benjamin–Ono equations.

The DNLS equation has been shown to be locally well posed for initial data in $H^{s\geq 1/2}$ for both periodic and non-periodic settings (see [19] and, respectively, [37]). The global well-posedness has been proven for $H^{s\geq 1/2}(\mathbb{R})$ in [24] and in $H^{s>1/2}(\mathbb{T})$ in [44]. The global results hold for initial data with small $L^2(\mathbb{T})$ norm. For instance, a standard procedure (see [19]) allows to globalize the local $H^1(\mathbb{T})$ solutions provided that $\|\psi_0\|_{L^2(\mathbb{T})} < \delta$ with δ small enough, by using the conservation law $\int h_2$ and the Gagliardo–Nirenberg inequality

$$\|u\|_{L^6(\mathbb{T})}^3 \leq \|u\|_{\dot{H}^1(\mathbb{T})}\|u\|_{L^2(\mathbb{T})}^2 + \frac{1}{2\pi}\|u\|_{L^2(\mathbb{T})}^3. \tag{1.4}$$

On the other hand, this approach does not give the best possible value for δ, which is still unknown. In particular, in the case of DNLS on $\mathbb{R}$ the sharp Gagliardo–Nirenberg inequality

$$\|u\|_{L^6(\mathbb{R})}^3 \leq \frac{2}{\pi}\|u\|_{\dot{H}^1(\mathbb{R})}\|u\|_{L^2(\mathbb{R})}^2,$$

proved in [43], gives the value $\delta = \sqrt{2\pi/|\beta|}$ for global well-posedness [17,18], which has been actually improved to $\delta = 2\sqrt{\pi/|\beta|}$ in [45,46] by different techniques. We

point out that $\delta = 2\sqrt{\pi/|\beta|}$ is also sufficient on $\mathbb{T}$ [27]. All these results are originally stated for $\beta = \pm 1$, and the case of general β can be easily deduced by using the transformation $u(t, x) \to |\beta|^{-1/2} u(t, \frac{\beta}{|\beta|}x)$.

The lack of well-posedness at low regularity makes hard to construct an invariant measure associated to the lowest order integrals of motion. For $\int h_2$ the main issue is that there is no well-posedness in $\bigcap_{\varepsilon>0} H^{\frac{1}{2}-\varepsilon}(\mathbb{T})$ which is the support of the Gaussian measure with covariance $\mathbb{I} - \Delta$. A very delicate analysis is necessary to deal with this problem. In [28] the authors constructed a functional measure in the Fourier–Lebesgue space $\mathcal{F}L^{s,r}(\mathbb{T})$, $r \in (2, 4)$, $s \in [1/2, 1 - r^{-1})$, for which there is a local existence theorem [14]. They are able to prove the invariance of this measure with respect to the DNLS flow (studying in fact the gauged DNLS equation). Then in [29] the study is completed, by proving the absolute continuity of this measure with respect to the Gibbs measure constructed in [38], which would be a more natural candidate for the invariant measure associated to the energy functional $\int h_2$. Similar results for the DNLS equation have been obtained, with different methods, in [8]. To the best of our knowledge, so far these are the sole known results for Gibbs measures associated to the DNLS equation.

1.1 Setup and main result

The main goal of this paper is to construct Gaussian measures supported on Sobolev spaces with increasing regularity, associated to the integrals of motion of the DNLS. Let us introduce now the main objects we are going to deal with.

As usual we denote by $H^k(\mathbb{T})$, $k \in \mathbb{Z}_+$, the completion of $C^\infty(\mathbb{T})$ with respect to the norm induced by the scalar product

$$(u, v)_{H^k} := \sum_{n \in \mathbb{Z}} (1 + |n|)^{2k} \, \bar{u}_n v_n.$$

where u_n are the Fourier coefficients of u. For every $k \in \mathbb{Z}_+$, $H^k(\mathbb{T})$ is a separable Hilbert space, and we note that $H^0(\mathbb{T}) = L^2(\mathbb{T})$. A function in $H^k(\mathbb{T})$ is represented as a sequence $\{u_n\}_{n \in \mathbb{Z}_+}$ such that $\sum_{|n| \leq N} (1 + |n|)^{2k} |u_n|^2$ is finite uniformly in $N \in \mathbb{Z}_+$.

We also use the homogeneous Sobolev spaces $\dot{H}^k(\mathbb{T})$, defined as the completion of $C^\infty(\mathbb{T})$ with respect to the norm induced by the homogenous scalar product

$$(u, v)_{\dot{H}^k} := \sum_{n \in \mathbb{Z}} n^{2k} \bar{u}_n v_n.$$

Now we consider the Hilbert space $L^2(\mathbb{T})$. For any $k \in \mathbb{Z}_+$, let us denote by $\mathbb{I} + (-\Delta)^k$ the closure in $L^2(\mathbb{T})$ of the operator $1 + \left(-\frac{d^2}{dx^2}\right)^k$ acting on $C^\infty(\mathbb{T})$. As it is well known this is a positive, self-adjoint operator with a trivial kernel. Therefore, its inverse $(\mathbb{I} - \Delta^k)^{-1}$ is bounded, and moreover, it can be shown that it is of trace class.

In virtue of this last property we can construct a Gaussian measure on $L^2(\mathbb{T})$ as follows. We denote by $e_n = e^{\mathrm{i}nx}$ the eigenvectors of $\mathbb{I} + (-\Delta)^k$:

$$\left(\mathbb{I} + (-\Delta)^k\right) e_n = \left(1 + n^{2k}\right) e_n.$$

Since $\mathbb{I} + (-\Delta)^k$ is self-adjoint the set of its eigenvectors spans the space $L^2(\mathbb{T})$, and so each function $u(x) \in L^2(\mathbb{T})$ can be written as

$$u(x) = \sum_{n \in \mathbb{Z}} u_n e_n,$$

that is nothing but Fourier series. We consider at first finite dimensional truncations, looking only at the components of the expansion for $|n| \leq N$. We define

$$\gamma_k^N(A) := \frac{\prod_{|n| \leq N} \sqrt{1 + n^{2k}}}{(2\pi)^{2N+1}} \int_A \mathrm{d}u_{-N} \mathrm{d}\bar{u}_{-N} \ldots \mathrm{d}u_N \mathrm{d}\bar{u}_N e^{-\frac{1}{2} \sum_{|n| \leq N} (1 + n^{2k}) |u_n|^2}$$

to be the complex Gaussian measure of a set $A \subseteq \mathbb{C}^{2N+1}$. This measure can be extended in infinite dimensions following a standard method [33,47]. For any Borel subset $B \subset \mathbb{C}^{2N+1}$ we introduce the corresponding cylindrical set in $L^2(\mathbb{T})$ as

$$M_N(B) = \left\{u \in L^2(\mathbb{T}) \mid \left[u_{-N}, \bar{u}_{-N}, \ldots, u_N, \bar{u}_N\right] \in B\right\}.$$

Since $\mathbb{I} + (-\Delta)^k$ is of trace class, we can extend the Gaussian measure γ_k^N to $L^2(\mathbb{T})$ functions by setting $\gamma_k(M_N) := \gamma_k^N(M_N)$ and then using Kolmogorov reconstruction theorem. It can be verified that this defines a countably additive measure on $L^2(\mathbb{T})$. We refer to [2,33] for a more detailed presentation (see also "Appendix" for some properties that will be used in the paper). We denote by $L_{\gamma_k}^p$ the Banach space of functionals $F : L^2(\mathbb{T}) \to \mathbb{C}$ such that

$$\int \mathrm{d}\gamma_k(\mathrm{d}\psi) |F(\psi)|^p < \infty.$$

For the ease of notation we simply denote as $\mathbb{E}[\cdot]$ (instead of $\mathbb{E}_{\gamma_k}[\cdot]$) the expectation value w.r.t. the measure γ_k. Anyway the particular γ_k considered will be always clear from the context.

For $N \geq 1$, we set $E_N = \mathrm{span}_{\mathbb{C}}\{e^{\mathrm{i}nx} \mid |n| \leq N\}$, and we denote by $P_N : L^2(\mathbb{T}) \to E_N$ the projection map onto the space E_N. Namely, for $u = \sum_{n \in \mathbb{Z}} u_n e^{\mathrm{i}nx} \in L^2(\mathbb{T})$, we have

$$P_N u := \sum_{|n| \leq N} u_n e^{\mathrm{i}nx}. \tag{1.5}$$

When there is no confusion, we simply denote

$$u_N := P_N u. \tag{1.6}$$

For $\psi \in L^2(\mathbb{T})$, we show in Sect. 2 that

$$\int h_{2k}[\psi] = \frac{1}{2}\|\psi\|_{\dot{H}^k}^2 + \int q_k[\psi], \quad k \in \mathbb{Z}_+,$$

where $\int q_k$ is a sum of terms of the form

$$\int \bar{\psi}^{(\alpha_1)} \dots \bar{\psi}^{(\alpha_l)} \psi^{(\beta_1)} \dots \psi^{(\beta_l)},$$

with $l \leq 2k + 2$, $\alpha_i, \beta_i \in \mathbb{Z}_+$ and $\sum_{i \leq l} \alpha_i + \beta_i \leq 2k - 1$.

Let now $R > 0$, and let $\chi_R : \mathbb{R} \to [0, 1]$ be a smooth function such that $\chi = 0 \in \mathbb{R}\backslash[-R, +R]$ and $\chi = 1$ in $[-R/2, +R/2]$. For $k \geq 2$, let us fix $R_m > 0$, for $m = 0, \dots, k - 1$. Thus, we can define the density

$$G_{k,N}(\psi) = \left(\prod_{m=0}^{k-1} \chi_{R_m}\left(\int h_{2m}(\psi_N)\right)\right) e^{-\int q_k(\psi_N)}. \tag{1.7}$$

The associated measure $\mathrm{d}\rho_{k,N}$ is

$$\rho_{k,N}(\mathrm{d}\psi) = G_{k,N}(\psi)\gamma_k(\mathrm{d}\psi).$$

The main result of the paper is the following:

Theorem 1.1 *Let $k \geq 2$ and $R_0 \leq \sqrt{\frac{2}{9|\beta|}}$ sufficiently small. The sequence $G_{k,N}(\psi)$ defined by Eq. (1.7) converges in measure, as $N \to \infty$, w.r.t. the measure γ_k. Denote by $G_k(\psi)$ its limit. Then, there exists $p_0(R_0, \dots, R_{k-1}, k, |\beta|) > 1$ such that, for all $1 \leq p < p_0$, $G_k(\psi) \in L^p(\gamma_k)$ and $G_{k,N}(\psi)$ converges to $G_k(\psi)$ in $L^p(\gamma_k)$.*

Remark 1.2 The best range one should expect to obtain for R_0 is the same of the global-well-posedness problem, which at the moment is $R_0 < 2\sqrt{\frac{\pi}{|\beta|}}$. The fact that we only get $R_0 \leq \sqrt{\frac{2}{9|\beta|}}$ is, as we have observed above, a limitation of the Gagliardo–Nirenberg inequality approach. Therefore, one could presumably obtain the widest range $\left[0, 2\sqrt{\frac{\pi}{|\beta|}}\right)$ by using different techniques. We point out that this could improve the absolute constant, while the behavior $\simeq 1/\sqrt{|\beta|}$ seems to be a peculiar feature of the equation.

As a consequence of Theorem 1.1, we obtain that the measures $\rho_{k,N}$ weakly converge, as $N \to \infty$, to the Gibbs measures ρ_k on $L^2(\mathbb{T})$:

$$\rho_k(\mathrm{d}\psi) = G_k(\psi)\gamma_k(\mathrm{d}\psi).$$

Since each G_k is supported on a set of positive measure w.r.t. γ_k, for every $k \geq 2$, ρ_k is non-trivial and absolutely continuous w.r.t. to γ_k. We choose the class observables associated to each of these Gibbs measure to be the functionals in $L_{\gamma_k}^\infty$.

1.2 Strategy of the proof

The first part of our proof relies on an accurate inspection of the algebraic structures of the integrals of motion of the DNLS equation. This has been done in Sect. 2. We use the Lenard–Magri scheme of integrability for non-local Poisson vertex algebras to find out the following general structure of the integrals of motion:

$$
\int h_{2k} = \frac{1}{2}\|\psi\|_{\dot{H}^k}^2 + \frac{i}{2}\beta(2k+1)\int \bar{\psi}^{(k)}\psi^{(k-1)}\bar{\psi}\psi + \text{a remainder}, \quad k \in \mathbb{Z}_+,
$$

where we consider as remainder all the terms that we can estimate with a certain power of the H^{k-1} norm. Note that this quantity is finite in the support of the Gaussian measure γ_k.

In Sect. 3 we show, under the L^2 smallness assumption, that the Sobolev norm H^k of the solutions of the DNLS equation (1.1) stays bounded by a constant depending on the values of $\int h_{2m}$, $m = 1, \ldots, k$, integrals of motion. Therefore, when we introduce the cut off functions χ in (1.7), we know that the H^s norms, $s \leq k - 1$, are bounded a.s. in the support of the Gibbs measure $\rho_{k,N}$ uniformly in N. This allows us to prove in Sect. 4 that all the remainder terms converge point-wise in the support of $\rho_{k,N}$ as $N \to \infty$, thus also in measure w.r.t. γ_k.

The terms $\int \bar{\psi}^{(k)}\psi^{(k-1)}\bar{\psi}\psi$ are estimated by the $H^k(\mathbb{T})$ norm, which is not finite in the support of γ_k. Therefore, they need to be treated separately. This is done by using a method outlined by Bourgain [4] (see also [7]), which is reminiscent of the works in quantum field theory in the '70 [15,34]. Successively this approach has been exploited by Tzvetkov and collaborators in [38] for DNLS equation and in [39,40] for the Benjamin Ono equation.

In Sect. 4 we prove the convergence in $L^2(\gamma_k)$ of these terms as $N \to \infty$, employing essentially the Wick theorem. $L^2(\gamma_k)$ convergence yields $L^p(\gamma_k)$ ($p \in [1, \infty)$) convergence by a standard hypercontractivity argument. This is enough to prove convergence in measure of the density. In Sect. 5 we ultimate our strategy showing $L^p(\gamma_k)$ boundedness of the density G_k for $p \in [1, \infty)$, provided that $\int h_0$ is sufficiently small. Here we follow the nice ideas of [39], making use of some helpful properties of the measures γ_k reviewed in "Appendix".

From the $L^p(\gamma_k)$ boundedness the convergence in $L^p(\gamma_k)$ (and so the weak convergence) of the density easily follows.

In the whole paper (except for Sect. 4) we are not concerned about the dynamics. However, the measures that we construct are naturally expected to be invariant under the flow of DNLS. To prove this result, a careful analysis is required (as, for instance, in the case of the Benjamin–Ono equation [12,41,42]) which we leave to a forthcoming work.

Throughout the paper we write $X \lesssim Y$ to denote that $X \leq CY$ for some positive constant C independent on X, Y.

2 Structure of the integrals of motion of the DNLS equation

In this section we recall briefly the theory of Poisson vertex algebras aimed at the study of the integrability properties of bi-Hamiltonian equations using the so-called Lenard–Magri scheme (see [1,13,22]). We use this formalism to describe explicitly the structure of the integrals of motion of the DNLS equation which will be used throughout the paper.

2.1 Algebras of differential polynomials

Let $\mathcal{V}$ be the algebra of differential polynomials in ℓ variables: $\mathcal{V} = \mathbb{C}[u_i^{(n)} | i \in I, n \in \mathbb{Z}_+]$, where $I = \{1, \ldots, \ell\}$. (In fact, most of the results hold in the generality of algebras of differential functions, as defined in [13].) It is a differential algebra with derivation defined by $\partial(u_i^{(n)}) = u_i^{(n+1)}$. We also let $\mathcal{K}$ be the field of fractions of $\mathcal{V}$ (it is still a differential algebra).

For $P \in \mathcal{V}^\ell$ we have the associated *evolutionary vector field*

$$X_P = \sum_{i \in I, n \in \mathbb{Z}_+} \left(\partial^n P_i\right) \frac{\partial}{\partial u_i^{(n)}}.$$

This makes $\mathcal{V}^\ell$ into a Lie algebra, with Lie bracket $[X_P, X_Q] = X_{[P,Q]}$, given by

$$[P, Q] = X_P(Q) - X_Q(P) = D_Q(\partial)P - D_P(\partial)Q,$$

where $D_P(\partial)$ and $D_Q(\partial)$ denote the Frechet derivatives of $P, Q \in \mathcal{V}^\ell$ (we refer to [1] for the definition of Frechet derivative).

For $f \in \mathcal{V}$ its *variational derivative* is $\frac{\delta f}{\delta u} = \left(\frac{\delta f}{\delta u_i}\right)_{i \in I} \in \mathcal{V}^{\oplus \ell}$, where

$$\frac{\delta f}{\delta u_i} = \sum_{n \in \mathbb{Z}_+} (-\partial)^n \frac{\partial f}{\partial u_i^{(n)}}. \tag{2.1}$$

Given an element $\xi \in \mathcal{V}^{\oplus \ell}$, the equation $\xi = \frac{\delta h}{\delta u}$ can be solved for $h \in \mathcal{V}$ if and only if $D_\xi(\partial)$ is a self-adjoint operator: $D_\xi(\partial) = D_\xi^*(\partial)$ (see [1]).

For $f \in \mathcal{V}$, we denote by $\int f = f + \partial \mathcal{V}$, where $\partial \mathcal{V} = \{\partial h \mid h \in \mathcal{V}\}$, the image of f in the quotient space $\mathcal{V}/\partial \mathcal{V}$, and we call it a *local functional*. Note that the integral symbol is motivated by the fact that $\mathcal{V}/\partial \mathcal{V}$ provides a universal space where integration by parts holds, namely

$$\int f \partial g = -\int g \partial f, \quad \text{for every } f, \ g \in \mathcal{V}.$$

It is possible to show that $\text{Ker} \frac{\delta}{\delta u} = \partial \mathcal{V} \oplus \mathbb{C}$. Hence, $\frac{\delta f}{\delta u} = \frac{\delta \int f}{\delta u} = 0$. Recall also that we have a non-degenerate pairing $(\cdot | \cdot) : \mathcal{V}^\ell \times \mathcal{V}^\ell \to \mathcal{V}/\partial \mathcal{V}$ given by $(P|\xi) = \int P \cdot \xi$ (see [1]).

Given $f \in \mathcal{V} \setminus \mathbb{C}$, we say that it has *differential order n*, and we write $\mathrm{ord}(f) = n$, if $\frac{\partial f}{\partial u_i^{(n)}} \neq 0$ for some $i \in I$ and $\frac{\partial f}{\partial u_j^{(m)}} = 0$ for all $j \in I$ and $m > n$. We also set the differential order of elements in $\mathbb{C}$ equal to $-\infty$. Let us denote by $\mathcal{V}_n$ the space of polynomials of differential order at most n. This gives an increasing sequence of subalgebras $\mathbb{C} = \mathcal{V}_{-\infty} \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}$ such that $\partial \mathcal{V}_n \subset \mathcal{V}_{n+1}$.

We extend the notion of differential order to elements in $P \in \mathcal{V}^\ell$ as follows:

$$\mathrm{ord}(P) = \max \left\{ \mathrm{ord}(P_1), \ldots, \mathrm{ord}(P_\ell) \right\}.$$

We also define two gradings on $\mathcal{V}$ in the following way. First, we let deg be the usual polynomial grading of $\mathcal{V}$ defined by

$$\deg u_i^{(n)} = 1, \quad \text{for every } i \in I, \quad n \in \mathbb{Z}_+.$$

On the other hand, we define the differential grading on $\mathcal{V}$, which we denote dd, by

$$\mathrm{dd}\, u_i^{(n)} = n, \quad \text{for every } i \in I, \quad n \in \mathbb{Z}_+.$$

This means that, given a monomial $(i_1, \ldots, i_k \in I, n_1, \ldots, n_k \in \mathbb{Z}_+)$

$$f = u_{i_1}^{(n_1)} u_{i_2}^{(n_2)} \ldots u_{i_k}^{(n_k)} \in \mathcal{V},$$

we have

$$\deg(f) = k, \quad \mathrm{dd}(f) = n_1 + \cdots + n_k.$$

Note that, for a homogeneous polynomial $f \in \mathcal{V}$, we have

$$\deg(\partial f) = \deg(f), \quad \mathrm{dd}(\partial f) = \mathrm{dd}(f) + 1. \tag{2.2}$$

2.2 Rational matrix pseudodifferential operators and the association relation

Consider the skewfield $\mathcal{K}((\partial^{-1}))$ of pseudodifferential operators with coefficients in $\mathcal{K}$, and the subalgebra $\mathcal{V}[\partial]$ of differential operators on $\mathcal{V}$.

The algebra $\mathcal{V}(\partial)$ of *rational* pseudodifferential operators consists of pseudodifferential operators $L(\partial) \in \mathcal{V}((\partial^{-1}))$ which admit a fractional decomposition $L(\partial) = A(\partial) B(\partial)^{-1}$, for some $A(\partial), B(\partial) \in \mathcal{V}[\partial]$, $B(\partial) \neq 0$. The algebra of *rational matrix pseudodifferential operators* is, by definition, $\mathrm{Mat}_{\ell \times \ell} \mathcal{V}(\partial)$ [10].

A matrix differential operator $B(\partial) \in \mathrm{Mat}_{\ell \times \ell} \mathcal{V}[\partial]$ is called *non-degenerate* if it is invertible in $\mathrm{Mat}_{\ell \times \ell} \mathcal{K}((\partial^{-1}))$. Any matrix $H(\partial) \in \mathrm{Mat}_{\ell \times \ell} \mathcal{V}(\partial)$ can be written as a ratio of two matrix differential operators: $H(\partial) = A(\partial) B^{-1}(\partial)$, with $A(\partial), B(\partial) \in \mathrm{Mat}_{\ell \times \ell} \mathcal{V}[\partial]$, and $B(\partial)$ non-degenerate.

Given $H(\partial) \in \mathrm{Mat}_{\ell \times \ell} \mathcal{V}(\partial)$, we say that $\xi \in \mathcal{V}^{\oplus l}$ and $P \in \mathcal{V}^\ell$ are *H-associated*, and denote it by

$$\xi \overset{H}{\longleftrightarrow} P, \tag{2.3}$$

if there exist a fractional decomposition $H = AB^{-1}$ with $A, B \in \mathrm{Mat}_{\ell \times \ell} \mathcal{V}[\partial]$ and B non-degenerate, and an element $F \in \mathcal{K}^{\ell}$, such that $\xi = BF$, $P = AF$ [13].

2.3 Non-local Poisson structures

A *non-local Poisson vertex algebra* is a differential algebra $\mathcal{V}$ endowed with a λ-bracket $\{\cdot_\lambda \cdot\} : \mathcal{V} \times \mathcal{V} \to \mathcal{V}((\lambda^{-1}))$, where $\mathcal{V}((\lambda^{-1}))$ denotes the space of Laurent series in λ^{-1} with coefficients in $\mathcal{V}$, satisfying sesquilinearity ($f, g \in \mathcal{V}$):

$$\{\partial f_\lambda g\} = -\lambda \{f_\lambda g\}, \quad \{f_\lambda \partial g\} = (\lambda + \partial)\{f_\lambda g\},$$

the Leibniz rule ($f, g, h \in \mathcal{V}$):

$$\{f_\lambda gh\} = \{f_\lambda g\} h + \{f_\lambda h\} g,$$

skewsymmetry ($f, g \in \mathcal{V}$):

$$\{f_\lambda g\} = -\{g_{-\lambda-\partial} f\},$$

admissibility ($f, g, h \in \mathcal{V}$):

$$\left\{ f_\lambda \{g_\mu h\} \right\} \in \mathcal{V}\left[\left[\lambda^{-1}, \mu^{-1}, (\lambda + \mu)^{-1}\right]\right][\lambda, \mu],$$

and Jacobi identity ($f, g, h \in \mathcal{V}$):

$$\left\{ f_\lambda \left\{ g_\mu h \right\} \right\} - \left\{ g_\mu \left\{ f_\lambda h \right\} \right\} = \{\{f_\lambda g\}_{\lambda+\mu} h\}.$$

We refer to [13] for the details on the notation.

To a matrix pseudodifferential operator $H = (H_{ij}(\partial))_{i,j \in I} \in \mathrm{Mat}_{\ell \times \ell} \mathcal{V}((\partial^{-1}))$ we associate a λ-bracket, $\{\cdot_\lambda \cdot\}_H : \mathcal{V} \times \mathcal{V} \to \mathcal{V}((\lambda^{-1}))$, given by the following *Master Formula* (see [13]):

$$\{f_\lambda g\}_H = \sum_{\substack{i,j \in I \\ m,n \in \mathbb{Z}_+}} \frac{\partial g}{\partial u_j^{(n)}} (\lambda + \partial)^n H_{ji}(\lambda + \partial)(-\lambda - \partial)^m \frac{\partial f}{\partial u_i^{(m)}} \in \mathcal{V}\left(\left(\lambda^{-1}\right)\right). \quad (2.4)$$

For arbitrary H, it is proved in [1,13], that the λ-bracket (2.4) satisfies sesquilinearity and the Leibniz rule. Furthermore, it has been shown that skewadjointness of H is equivalent to the skewsymmetry condition, and that, if H is a rational matrix pseudodifferential operator, then the admissibility condition holds.

Definition 2.1 A *non-local Poisson structure* on $\mathcal{V}$ is a skewadjoint rational matrix pseudodifferential operator H with coefficients in $\mathcal{V}$ such that the corresponding λ-bracket (2.4) satisfies Jacobi identity; namely, $\mathcal{V}$ endowed with the λ-bracket (2.4) is a non-local Poisson vertex algebra.

Two non-local Poisson structures $H, K \in \mathrm{Mat}_{\ell \times \ell} \mathcal{V}(\partial)$ on $\mathcal{V}$ are said to be *compatible* if any of their linear combination (or, equivalently, their sum) is a non-local Poisson structure. In this case we say that (H, K) form a *bi-Poisson structure* on $\mathcal{V}$.

2.4 Hamiltonian equations and integrability

Let $H \in \mathrm{Mat}_{\ell \times \ell} \mathcal{V}(\partial)$ be a non-local Poisson structure. An evolution equation on the variables $u = (u_i)_{i \in I}$,

$$\frac{du}{dt} = P \in \mathcal{V}^\ell, \tag{2.5}$$

is called *Hamiltonian* with respect to the non-local Poisson structure H and the Hamiltonian functional $\int h \in \mathcal{V}/\partial\mathcal{V}$ if (see Sect. 2.2)

$$\frac{\delta h}{\delta u} \overset{H}{\longleftrightarrow} P.$$

Equation (2.5) is called *bi-Hamiltonian* if there are two compatible non-local Poisson structures H and K, and two local functionals $\int h_0, \int h_1 \in \mathcal{V}/\partial\mathcal{V}$, such that

$$\frac{\delta h_0}{\delta u} \overset{H}{\longleftrightarrow} P \quad \text{and} \quad \frac{\delta h_1}{\delta u} \overset{K}{\longleftrightarrow} P. \tag{2.6}$$

An *integral of motion* for the Hamiltonian equation (2.5) is a local functional $\int f \in \mathcal{V}/\partial\mathcal{V}$ which is constant in time, namely such that $(P | \frac{\delta f}{\delta u}) = 0$. The usual requirement for *integrability* is to have sequences $\{\int h_n\}_{n \in \mathbb{Z}_+} \subset \mathcal{V}/\partial\mathcal{V}$ and $\{P_n\}_{n \in \mathbb{Z}_+} \subset \mathcal{V}^\ell$, starting with $\int h_0 = \int h$ and $P_0 = P$, such that

(C1) $\frac{\delta h_n}{\delta u} \overset{H}{\longleftrightarrow} P_n$ for every $n \in \mathbb{Z}_+$,
(C2) $[P_m, P_n] = 0$ for all $m, n \in \mathbb{Z}_+$,
(C3) $(P_m | \frac{\delta h_n}{\delta u}) = 0$ for all $m, n \in \mathbb{Z}_+$.
(C4) The elements P_n span an infinite dimensional subspace of $\mathcal{V}^\ell$.

In this case, we have an *integrable hierarchy* of Hamiltonian equations

$$\frac{du}{dt_n} = P_n, \quad n \in \mathbb{Z}_+.$$

Elements $\int h_n$'s are called *higher Hamiltonians*, the P_n's are called *higher symmetries*, and the condition $(P_m | \frac{\delta h_n}{\delta u}) = 0$ says that $\int h_m$ and $\int h_n$ are *in involution*. Note that (C4) implies that element $\frac{\delta h_n}{\delta u}$ span an infinite dimensional subspace of $\mathcal{V}^\ell$. The converse holds provided that either H or K is non-degenerate.

Suppose we have a bi-Hamiltonian equation (2.5), associated to the compatible non-local Poisson structures H, K and the Hamiltonian functionals $\int h_0, \int h_1$, in the sense of Eq. (2.6). The *Lenard–Magri scheme of integrability* consists in finding sequences

$\{\int h_n\}_{n\in\mathbb{Z}_+} \subset \mathcal{V}/\partial\mathcal{V}$ and $\{P_n\}_{n\in\mathbb{Z}_+} \subset \mathcal{V}^\ell$, starting with $P_0 = P$ and the given Hamiltonian functionals $\int h_0$, $\int h_1$, satisfying the following recursive relations:

$$\frac{\delta h_n}{\delta u} \xleftrightarrow{H} P_n \xleftrightarrow{K} \frac{\delta h_{n+1}}{\delta u} \quad \text{for all} \quad n \in \mathbb{Z}_+. \tag{2.7}$$

In this case, we have the corresponding bi-Hamiltonian hierarchy

$$\frac{du}{dt_n} = P_n \in \mathcal{V}^\ell, \quad n \in \mathbb{Z}_+, \tag{2.8}$$

all Hamiltonian functionals $\int h_n$, $n \in \mathbb{Z}_+$, are integrals of motion for all equations of the hierarchy, and they are in involution with respect to both non-local Poisson structures H and K, and all commutators $[P_m, P_n]$ are zero, provided that one of the non-local Poisson structures H or K is local (see [13, Sect. 7.4]). Hence, in this situation (2.8) is an integrable hierarchy of compatible evolution equations, provided that condition (C4) holds.

2.5 A bi-Hamiltonian structure and integrability for the DNLS equation

Let $\mathcal{V} = \mathbb{C}[a^{(n)}, b^{(n)} \mid n \in \mathbb{Z}_+]$ be the algebra of differential polynomials in two variables a and b. Sometimes we will also use the notation $a' = a^{(1)}$, $a'' = a^{(2)}$ and so on (and similarly for the $b^{(n)}$'s).

Let $H, K \in \mathrm{Mat}_{2\times 2}\,\mathcal{V}((\partial^{-1}))$ be pseudodifferential operators with coefficients in $\mathcal{V}$ defined as follows:

$$H = \begin{pmatrix} \partial & 0 \\ 0 & \partial \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 2\beta b\partial^{-1} \circ b & -1 - 2\beta b\partial^{-1} \circ a \\ 1 - 2\beta a\partial^{-1} \circ b & 2\beta a\partial^{-1} \circ a \end{pmatrix},$$

where $\beta \in \mathbb{C}$. Note that $H(\partial) \in \mathrm{Mat}_{2\times 2}\,\mathcal{V}[\partial]$ is in fact a differential operator.

The following result have been proved in [13].

Theorem 2.2 (a) *There exist* $A(\partial), B(\partial) \in \mathrm{Mat}_{2\times 2}\,\mathcal{V}[\partial]$, *with* $B(\partial)$ *non-degenerate, such that* $K = A(\partial)B(\partial)^{-1}$. *Explicitly:*

$$A(\partial) = \begin{pmatrix} -\frac{b}{a} & \frac{1}{a}\partial \circ a - 2\beta ab \\ 1 & 2\beta a^2 \end{pmatrix} \quad \text{and} \quad B(\partial) = \begin{pmatrix} 1 & 0 \\ \frac{b}{a} & \frac{1}{a}\partial \circ a \end{pmatrix}.$$

(b) (H, K) *is a bi-Poisson structure on* $\mathcal{V}$.

(c) *There exist infinite sequences* $\{\int h_n\}_{n\in\mathbb{Z}_+} \subset \mathcal{V}/\partial\mathcal{V}$ *and* $\{P_n\}_{n\in\mathbb{Z}_+} \subset \mathcal{V}^2$ *such that the Lenard–Magri recursive relations* (2.7) *hold.*

(d) $\mathrm{ord}\left(\frac{\delta h_n}{\delta u}\right) = n$, *for every* $n \in \mathbb{Z}_+$. *In particular, since* H *is non-degenerate, all the elements* $\int h_n$'s *and* P_n's *are linearly independent* (*see* Sect. 2.4).

In conclusion, by the discussion in Sect. 2.4, *we get an integrable hierarchy of bi-Hamiltonian equations* (2.8) *and all the Hamiltonian functionals* $\int h_n$, $n \in \mathbb{Z}_+$, *are integrals of motion for all equations of the hierarchy.*

The first few elements in the series of the integrals of motion are

$$\int h_0 = \frac{1}{2} \int \left(a^2 + b^2 \right), \qquad \int h_1 = \int \left(ab' + \frac{\beta}{4} \left(a^2 + b^2 \right)^2 \right). \qquad (2.9)$$

The corresponding Hamiltonian equations, given by (2.8), are

$$\begin{cases} \frac{\mathrm{d}a}{\mathrm{d}t_0} = a' \\ \frac{\mathrm{d}b}{\mathrm{d}t_0} = b' \end{cases}, \qquad \begin{cases} \frac{\mathrm{d}a}{\mathrm{d}t_1} = b'' + \beta \left(a \left(a^2 + b^2 \right) \right)' \\ \frac{\mathrm{d}b}{\mathrm{d}t_1} = -a'' + \beta \left(b \left(a^2 + b^2 \right) \right)' \end{cases}.$$

Let us write $\psi = a + ib$. Then, the first non-trivial equation of the hierarchy is the DNLS equation:

$$i \frac{\mathrm{d}\psi}{\mathrm{d}t_1} = \psi'' + i\beta \left(\psi |\psi|^2 \right)'.$$

Let us consider $\beta \in \mathbb{C}$ as a formal parameter, and let us naturally extend the notion of polynomial degree and differential degree of $\mathcal{V}$ to the field of fractions $\mathcal{K}$ and to $\mathcal{K}^2$. The following result is a consequence of the Lenard–Magri recursive relations (2.7) and the explicit form of the differential operators A and B.

Proposition 2.3 *For every $n \in \mathbb{Z}_+$, the variational derivatives $\frac{\delta h_n}{\delta u}$'s are polynomials in β (with coefficients in $\mathcal{V}^2$) of order n. Let us write*

$$\frac{\delta h_n}{\delta u} = \sum_{k=0}^{n} \left(\frac{\delta h_n}{\delta u} \right)_k \beta^k.$$

Then, for every $0 \le k \le n$, we have

$$\mathrm{ord} \left(\frac{\delta h_n}{\delta u} \right)_k = n - k.$$

Moreover, the components of $\left(\frac{\delta h_n}{\delta u} \right)_k$ are homogeneous polynomials with respect to the polynomial grading (respectively, differential grading) of degree:

$$\deg \left(\frac{\delta h_n}{\delta u} \right)_k = 2k + 1 \quad \left(\text{respectively, } \mathrm{dd} \left(\frac{\delta h_n}{\delta u} \right)_k = n - k \right).$$

Proof The fact that the variational derivatives $\frac{\delta h_n}{\delta u}$'s are polynomials in β (with coefficients in $\mathcal{V}^2$) of order n is true for $n = 0, 1$ using Eq. (2.12) and the definition of variational derivative (2.1). Let us assume that $\frac{\delta h_n}{\delta u}$ has order n as a polynomial in β, and let us write explicitly the Lenard–Magri recursion relations (2.7) using the formulas for the differential operators A and B. We get the following system of equations

$$\begin{cases} \partial(ag) = -a\partial\frac{\delta h_n}{\delta a} - b\partial\frac{\delta h_n}{\delta b} \\ \frac{\delta h_{n+1}}{\delta a} = \partial\frac{\delta h_n}{\delta b} - 2\beta a^2 g \\ \frac{\delta h_{n+1}}{\delta b} = -\partial\frac{\delta h_n}{\delta a} - 2\beta abg, \end{cases} \tag{2.10}$$

where $g \in \mathcal{K}$ and $\frac{\delta h_{n+1}}{\delta a}, \frac{\delta h_{n+1}}{\delta b} \in \mathcal{V}$ have to be determined [we know the system can be solved by Theorem 2.2(b)]. From the first equation in (2.10) and inductive assumption, it follows that g is a polynomial of order n in β. Then, by the second and third equation in (2.10), it follows that $\frac{\delta h_{n+1}}{\delta u}$ is a polynomial of order $n+1$ in β.

Moreover, by Theorem 7.15(c) in [13], we have that ord $\left(\frac{\delta h_{n+1}}{\delta u}\right) = \text{ord}(P_n)$. Recall that $P_n = H\left(\frac{\delta h_n}{\delta u}\right)$. Hence, equating the orders of the coefficients of powers of β we get

$$\text{ord}\left(\frac{\delta h_{n+1}}{\delta u}\right)_k = \text{ord}\left(H\left(\frac{\delta h_n}{\delta u}\right)_k\right) = n + 1 - k.$$

In the last equality we used the fact that $\partial \mathcal{V}_n \subset \mathcal{V}_{n+1}$. The last part of the proposition follows by a simple inductive argument using Eqs. (2.2) and (2.10). □

Remark 2.4 By the first part of Proposition 2.3, we can write h_n as a polynomial in β. By the second part, using the definition of variational derivative and Eq. (2.2), we get that

$$h_n = \sum_{k=0}^{n} h_{n,k}\beta^k, \tag{2.11}$$

where $h_{n,k} \in \mathcal{V}$ are homogeneous differential polynomials such that $\deg(h_{n,k}) = 2k + 2$ and $dd(h_{n,k}) = n - k$.

2.6 Explicit structure of the integrals of motion of the DNLS equation

Let us define a sequence $\{\xi_n\}_{n\in\mathbb{Z}_+} \subset \mathcal{V}^2$ as follows:

$$\xi_0 = \begin{pmatrix} a \\ b \end{pmatrix}, \qquad \xi_1 = \begin{pmatrix} b' + \beta a\left(a^2 + b^2\right) \\ -a' + \beta b\left(a^2 + b^2\right) \end{pmatrix}, \tag{2.12}$$

and, for $n \geq 1$, we set

$$\xi_{2n} = (-1)^n \begin{pmatrix} a^{(2n)} - (2n+1)\beta\left(a^2 + b^2\right)b^{(2n-1)} + r_{2n}^a \\ b^{(2n)} + (2n+1)\beta\left(a^2 + b^2\right)a^{(2n-1)} + r_{2n}^b \end{pmatrix},$$

$$\xi_{2n+1} = (-1)^n \begin{pmatrix} b^{(2n+1)} + 2\beta a\left(aa^{(2n)} + bb^{(2n)}\right) + (2n+1)\beta\left(a^2 + b^2\right)a^{(2n)} + r_{2n+1}^a \\ -a^{(2n+1)} + 2\beta b\left(aa^{(2n)} + bb^{(2n)}\right) + (2n+1)\beta\left(a^2 + b^2\right)b^{(2n)} + r_{2n+1}^b \end{pmatrix},$$

where $r_{2n}^x \in \mathcal{V}_{2n-2}$ and $r_{2n+1}^x \in \mathcal{V}_{2n-1}$, for $x = a$ or b.

Lemma 2.5 *Let us denote* $\xi_n = \begin{pmatrix} \xi_n^a \\ \xi_n^b \end{pmatrix} \in \mathcal{V}^2$, *for every* $n \in \mathbb{Z}_+$. *Then we have:*

(a)

$$a\xi_{2n+1}^b - b\xi_{2n+1}^a - (-1)^{n+1}\partial\left(aa^{(2n)} + bb^{(2n)}\right)$$
$$- (-1)^n(2n+1)\beta\partial\left((a^2 + b^2)(ab^{(2n-1)} - a^{(2n-1)}b)\right) \in \mathcal{V}_{2n-1}.$$

(b)

$$a\xi_{2n}^b - b\xi_{2n}^a - (-1)^n\partial\left(ab^{(2n-1)} - a^{(2n-1)}b\right)$$
$$- (-1)^n(2n+1)\beta\partial\left((a^2 + b^2)\left(aa^{(2n-2)} + bb^{(2n-2)}\right)\right) \in \mathcal{V}_{2n-2}.$$

(c)

$$\xi_{2n+1}^a - \partial\xi_{2n}^b - (-1)^n 2\beta a\left(aa^{(2n)} + bb^{(2n)}\right) \in \mathcal{V}_{2n-1},$$
$$\xi_{2n+1}^b + \partial\xi_{2n}^a - (-1)^n 2\beta b\left(aa^{(2n)} + bb^{(2n)}\right) \in \mathcal{V}_{2n-1}.$$

(d)

$$\xi_{2n}^a - \partial\xi_{2n-1}^b - (-1)^{n+1} 2\beta a\left(ab^{(2n-1)}b - a^{(2n-1)}b\right) \in \mathcal{V}_{2n-2},$$
$$\xi_{2n}^b + \partial\xi_{2n-1}^a - (-1)^{n+1} 2\beta b\left(ab^{(2n-1)} - a^{(2n-1)}b\right) \in \mathcal{V}_{2n-2}.$$

Proof Straightforward. □

Let us also define a sequence $\{P_n\}_{n\in\mathbb{Z}_+} \subset \mathcal{V}^2$ as follows:

$$P_n = H\xi_n = \begin{pmatrix} \partial\xi_n^a \\ \partial\xi_n^b \end{pmatrix}.$$

Lemma 2.6 *For every* $n \in \mathbb{Z}_+$, *there exists* $F_n \in \mathcal{K}^2$ *such that:*

(a) $P_{2n} - AF_{2n} \in \mathcal{V}_{2n-1}^2$ *and* $\xi_{2n+1} - BF_{2n} \in \mathcal{V}_{2n-1}^2$;
(b) $P_{2n+1} - AF_{2n+1} \in \mathcal{V}_{2n}^2$ *and* $\xi_{2n+2} - BF_{2n+1} \in \mathcal{V}_{2n}^2$.

Proof For every $n \in \mathbb{Z}_+$, let us consider

$$F_n = \begin{pmatrix} \xi_{n+1}^a \\ f_n + g_n \end{pmatrix} \in \mathcal{K}^2,$$

where

$$f_{2n} = (-1)^{n+1} \frac{aa^{(2n)} + bb^{(2n)}}{a} + (-1)^n (2n+1)\beta \frac{(a^2+b^2)(ab^{(2n-1)} - a^{(2n-1)}b)}{a},$$

$$f_{2n+1} = (-1)^{n+1} \frac{ab^{(2n+1)} - a^{(2n+1)}b}{a}$$
$$+ (-1)^{n+1}(2n+3)\beta \frac{(a^2+b^2)(aa^{(2n)} + bb^{(2n)})}{a},$$

and $g_n \in \mathcal{V}_{n-2}$. Then, using the definition of the differential operators A and B given by Theorem 2.2(a), it is straightforward to check that part (a) follows from Lemma 2.5(a) and (c), while part (b) follows from Lemma 2.5(b) and (d). □

Proposition 2.7 *Let* $\{\int h_n\}_{n\in\mathbb{Z}_+} \subset \mathcal{V}/\partial\mathcal{V}$ *be the sequence in* Theorem 2.2. *Then, for every* $n \in \mathbb{Z}_+$, *we have*

$$\frac{\delta h_n}{\delta u} - \xi_n \in \mathcal{V}_{n-2}^2.$$

Proof By Eq. (2.9) and the definition of variational derivative (2.1) it follows that $\frac{\delta h_n}{\delta u} = \xi_n$, for $n = 0, 1$. Hence, by Theorem 2.2(d), in order to prove the proposition we need to show that the sequence $\{\xi_n\}_{n\in\mathbb{Z}_+} \subset \mathcal{V}^2$ satisfies the Lenard–Magri recursive relations (2.7) up to elements in $\mathcal{V}_{n-2}$. This follows by definition of the association relation (2.3), the definition of the sequence $\{P_n\}_{n\in\mathbb{Z}_+} \subset \mathcal{V}^2$ and Lemma 2.6(a) and (b). □

Corollary 2.8 *For every* $n \in \mathbb{Z}_+$ *we can assume that the conserved densities* $h_{2n} \in \mathcal{V}$, *defined by* Theorem 2.2, *have the form:*

$$h_{2n} = \frac{1}{2}\left(\left(a^{(n)}\right)^2 + \left(b^{(n)}\right)^2\right) + (2n+1)\beta\left(a^2+b^2\right)a^{(n-1)}b^{(n)} + R_{2n},$$

where $R_{2n} \in \mathcal{V}_{n-1}$.

Proof It follows by Proposition 2.7 and the definition of the variational derivative (2.1), using the fact that $\partial\mathcal{V}_k \subset \mathcal{V}_{k+1}$, for every $k \in \mathbb{Z}_+$, and that the variational derivative of a total derivative is zero. □

2.7 Changing variables

Let $\mathcal{V}^{\mathbb{C}}$ be the algebra of differential polynomials in two variables ψ and $\bar{\psi}$. We have a differential algebra isomorphism $\mathcal{V} \xrightarrow{\sim} \mathcal{V}^{\mathbb{C}}$ given on generators by

$$a = \frac{\psi + \bar{\psi}}{2}, \quad b = \frac{\psi - \bar{\psi}}{2i}.$$

Clearly, the inverse map is given by $\psi = a+ib$ and $\bar{\psi} = a-ib$. (In the usual analytical language, if a and b are real functions, then we want to consider them as the real and imaginary parts of the function ψ.)

The differential order, the polynomial grading and the differential grading of $\mathcal{V}$ and $\mathcal{V}^{\mathbb{C}}$ are compatible under this isomorphism. Hence, all the results in Sect. 2.5 hold true for $\frac{\delta h_n}{\delta u} \in (V^{\mathbb{C}})^2$ (by an abuse of notation we are denoting with the same symbol an element in $\mathcal{V}$ and its image in $\mathcal{V}^{\mathbb{C}}$) Moreover, we can restate Corollary 2.8 as follows.

Corollary 2.9 *For every $n \in \mathbb{Z}_+$ we can assume that the conserved densities $h_{2n} \in \mathcal{V}^{\mathbb{C}}$, defined by* Theorem 2.2, *have the form:*

$$h_{2n} = \frac{1}{2} \psi^{(n)} \bar{\psi}^{(n)} + \frac{(2n+1)i}{2} \beta \bar{\psi}^{(n)} \psi^{(n-1)} \bar{\psi} \psi + R_{2n},$$

where $R_{2n} \in \mathcal{V}^{\mathbb{C}}_{n-1}$.

Proof Clearly, $(a^{(n)})^2 + (b^{(n)})^2 = \psi^{(n)} \bar{\psi}^{(n)}$, for every $n \in \mathbb{Z}_+$. Moreover, we have

$$\left(a^2 + b^2 \right) a^{(n-1)} b^{(n)} = \frac{i}{4} \left(\bar{\psi}^{(n)} \psi^{(n-1)} \bar{\psi} \psi - \psi^{(n)} \bar{\psi}^{(n-1)} \bar{\psi} \psi + \bar{\psi}^{(n)} \bar{\psi}^{(n-1)} \bar{\psi} \psi \right.$$
$$\left. - \psi^{(n)} \psi^{(n-1)} \bar{\psi} \psi \right). \tag{2.13}$$

Note that, integrating by parts, we have

$$\psi^{(n)} \bar{\psi}^{(n-1)} \bar{\psi} \psi = -\psi^{(n-1)} \partial \left(\bar{\psi}^{(n-1)} \bar{\psi} \psi \right) \mod \partial \mathcal{V}$$
$$= \left(-\bar{\psi}^{(n)} \psi^{(n-1)} \bar{\psi} \psi + f \right) \mod \partial \mathcal{V}, \tag{2.14}$$

where $f \in \mathcal{V}^{\mathbb{C}}_{n-1}$. Moreover, again using integration by parts, we have

$$\psi^{(n)} \psi^{(n-1)} \bar{\psi} \psi = -\psi^{(n-1)} \partial \left(\psi^{(n-1)} \bar{\psi} \psi \right) \mod \partial \mathcal{V}$$
$$= \left(-\psi^{(n)} \psi^{(n-1)} \bar{\psi} \psi + 2g \right) \mod \partial \mathcal{V},$$

with $g \in \mathcal{V}^{\mathbb{C}}_{n-1}$. Then,
$$\psi^{(n)} \psi^{(n-1)} \bar{\psi} \psi = f \mod \partial \mathcal{V}. \tag{2.15}$$

Similarly, we get that
$$\bar{\psi}^{(n)} \bar{\psi}^{(n-1)} \bar{\psi} \psi = h \mod \partial \mathcal{V}. \tag{2.16}$$

for some $h \in \mathcal{V}^{\mathbb{C}}_{n-1}$. Combining Eqs. (2.13)–(2.16) the proof is concluded. □

We want to give a description of the conserved densities $h_{2n} \in \mathcal{V}^{\mathbb{C}}$ which will be used throughout the rest of the paper.

Let $\widetilde{\mathcal{V}}$ be the algebra of differential polynomials in one variable u. Let us denote by

$$\tilde{} : \mathcal{V}^{\mathbb{C}} \to \widetilde{\mathcal{V}} \tag{2.17}$$

the differential algebra homomorphism defined as follows: Given $f \in \mathcal{V}^{\mathbb{C}}$, we denote by $\widetilde{f} \in \widetilde{\mathcal{V}}$ the differential polynomial obtained by replacing ψ and $\bar{\psi}$ by u (and their nth derivatives by $u^{(n)}$). Note that $\widetilde{\mathcal{V}}$ inherits the polynomial and differential grading of $\mathcal{V}^{\mathbb{C}}$.

Recall, by Remark 2.4, that we can write the conserved densities as in Eq. (2.11). Then, by Corollary 2.9, we have that

$$h_{2n,0} = \frac{1}{2} \psi^{(n)} \bar{\psi}^{(n)}, \tag{2.18}$$

and

$$h_{2n,1} = \frac{(2n+1)i}{2} \bar{\psi}^{(n)} \psi^{(n-1)} \bar{\psi}\psi + \sum_{p \in \widetilde{P}} c_{2n}(p)p, \tag{2.19}$$

where $c_k(p) \in \mathbb{C}$ (they can be possibly 0) and

$$\widetilde{P} = \left\{ p \in \mathcal{V}^{\mathbb{C}} \mid \widetilde{p} = u^{(n-1)}u^{(n_1)}u^{(n_2)}u^{(n_3)}, \quad n_1 + n_2 + n_3 = n, \right.$$
$$\left. 0 \le n_3 \le n_2 \le n_1 \le n-1 \right\}. \tag{2.20}$$

3 Control of the Sobolev norms

The goal of this section is to show the persistence of regularity of small solutions of DNLS equation (1.1), using the higher Hamiltonians introduced in Theorem 2.2.

For every $k \in \mathbb{Z}_+$, we denote

$$E_k = \int h_{2k}.$$

By Eqs. (2.11), (2.18), (2.19) and Corollary 2.9 it is possible to write

$$E_k = \frac{1}{2} \|\psi\|_{\dot{H}^k} + \int q_k, \tag{3.1}$$

where

$$q_k := \frac{(2k+1)i}{2} \beta \bar{\psi}^{(k)} \psi^{(k-1)} \bar{\psi}\psi + \beta \sum_{p \in \tilde{P}} c_{2k}(p)p + \sum_{m=2}^{2k} \beta^m h_{2k,m}. \tag{3.2}$$

We recall that $\mathrm{dd}(h_{k,m}) = 2k - m$ and $\widetilde{P}$ is defined in (2.20).

Remark 3.1 Note that using Proposition 2.7 [and recalling Eq. (2.11)] it is possible to write

$$\int h_{2k+1} = \frac{i}{2} \int \psi^{(k)} \bar{\psi}^{(k+1)} + \sum_{m=1}^{2k+1} \beta^m \int h_{2k+1,m}.$$

Differently from the case of $\int h_{2k}$, the constant term in β of the above equation has no definite sign and, in particular, it does not coincide with $\|\psi\|_{\dot{H}^{k/2}}$.

The main result of the section is the following.

Proposition 3.2 *Let $k \in \mathbb{Z}_+$. For every $0 \leq m \leq k$ let us fix $R_m \geq 0$, assuming $R_0 \leq \sqrt{\frac{2}{9|\beta|}}$. There exists $C = C(R_0, \ldots, R_k, k, |\beta|)$ such that if*

$$|E_m[\psi]| \leq R_m, \quad \text{for any} \quad m = 0, \ldots, k,$$

then

$$\|\psi\|_{\dot{H}^k} \leq C. \tag{3.3}$$

To prove Proposition 3.2 we need some preliminary results.

Lemma 3.3 *Let $k \geq 2$ and $u \in H^{k-1}$. For $l \geq 5$ and $\alpha_i \geq 0$ $(i = 1, \ldots, l)$ such that $\alpha_1 + \cdots + \alpha_l \leq 2k - 2$, we have*

$$\left| \int u^{(\alpha_1)} \ldots u^{(\alpha_l)} \right| \lesssim \|u\|_{H^{k-1}}^l. \tag{3.4}$$

Proof We reorder the terms in the integrand in the l.h.s. of (3.4), such that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_l$. Furthermore, using integration by parts, we may assume that

$$\alpha_1, \alpha_2 \leq k - 1, \quad \text{and} \quad \alpha_i \leq k - 2, \quad i = 3, \ldots, l. \tag{3.5}$$

By the Holder inequality and the first condition in (3.5) we get

$$\left| \int u^{(\alpha_1)} \ldots u^{(\alpha_l)} \right| \leq \|u\|_{\dot{H}^{k-1}}^2 \prod_{i=3}^{l} \|u^{(\alpha_i)}\|_{L^\infty}. \tag{3.6}$$

Using the embedding $H^1 \hookrightarrow L^\infty$ and the second condition in (3.5) we have (for all $i = 3, \ldots, l$):

$$\|u^{(\alpha_i)}\|_{L^\infty} \lesssim \|u^{(\alpha_i)}\|_{H^1} \leq \|u\|_{H^{k-1}}. \tag{3.7}$$

The inequality (3.4) follows combining the inequalities (3.6) and (3.7). $\qquad \square$

Lemma 3.4 *Let $k \geq 2$ and $u \in H^{k-1}$. Let also $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq 0$ be such that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2k - 1$. For $\alpha_1 = k - 1$ and $\alpha_2, \alpha_3, \alpha_4 \leq k - 1$, we have*

$$\left| \int u^{(k-1)} u^{(\alpha_2)} u^{(\alpha_3)} u^{(\alpha_4)} \right| \lesssim \|u\|_{H^{k-1}}^4.$$

Proof Same as the proof of Lemma 3.3. $\qquad \square$

Lemma 3.5 *Let $k \geq 2$ and let $u \in H^k$. Then*

$$\left| \int u^{(k)} u^{(k-1)} u^2 \right| \leq \varepsilon \|u\|_{\dot{H}^k}^2 + C(\varepsilon) \|u\|_{H^{k-1}}^6,$$

for all $\varepsilon > 0$.

Proof By using the Holder inequality and the embedding $H^1 \hookrightarrow L^\infty$ we get

$$\left| \int u^{(k)} u^{(k-1)} u^2 \right| \leq \|u\|_{\dot{H}^k} \|u\|_{\dot{H}^{k-1}} \|u\|_{L^\infty}^2 \lesssim \|u\|_{\dot{H}^k} \|u\|_{H^{k-1}}^3.$$

The proof is concluded by applying the Young inequality in the last expression. □

Corollary 3.6 *Let $k \geq 2$ and $c(k) > 0$ a constant depending only by k. For every $\psi \in H^k$ and $\varepsilon > 0$, we have*

$$\left| \int q_k(\psi) \right| \leq c(k)\varepsilon \|\psi\|_{\dot{H}^k}^2 + \mathcal{C}, \tag{3.8}$$

where $\mathcal{C} = \mathcal{C}(\|\psi\|_{\dot{H}^0}, \|\psi\|_{\dot{H}^{k-1}}, \varepsilon, k, |\beta|)$.

Proof Let us focus on the representation (3.2):

$$q_k(\psi) = \frac{(2k+1)i}{2} \beta \bar{\psi}^{(k)} \psi^{(k-1)} \bar{\psi} \psi + \beta \sum_{p \in \tilde{P}} c_{2k}(p) p + \sum_{m=2}^{2k} \beta^m h_{2k,m}.$$

The Lemma 3.3 and the fact that $|\psi| = |\bar{\psi}|$ allow us to bound [through the homomorphism defined in (2.17)]

$$\left| \int h_{k,m} \right| \leq \mathcal{C}(\|\psi\|_{\dot{H}^0}, \|\psi\|_{\dot{H}^{k-1}}, k), \tag{3.9}$$

for all $m = 2, \ldots, 2k$. Similarly, Lemma 3.4 implies

$$\left| \int p \right| \leq \mathcal{C}\left(\|\psi\|_{\dot{H}^0}, \|\psi\|_{\dot{H}^{k-1}}, k\right), \tag{3.10}$$

for all $p \in \tilde{P}$. Finally, Lemma 3.5 gives

$$\left| \int \bar{\psi}^{(k)} \psi^{(k-1)} \bar{\psi} \psi \right| \leq \varepsilon \|u\|_{\dot{H}^k}^2 + C(\varepsilon) \|u\|_{H^{k-1}}^6. \tag{3.11}$$

Combining Eq. (3.2), the inequalities (3.9)–(3.11), the estimate (3.8) follows. □

Lemma 3.7 *Let $\psi \in H^1$ and let us denote $R_0 = \|\psi\|_{L^2}$. Then*

$$\left| \int \bar{\psi}' \bar{\psi} \psi^2 \right| \le \frac{3}{2} \|\psi\|^2_{\dot{H}^1} R_0^2 + \frac{1}{8\pi^2} R_0^4.$$

Proof By the Hölder inequality

$$\left| \bar{\psi}' \bar{\psi} \psi^2 \right| \le \|\psi\|_{\dot{H}^1} \|\psi^3\|_{L^2} = \|\psi\|_{\dot{H}^1} \|\psi\|^3_{L^6}. \tag{3.12}$$

Using the Gagliardo–Nirenberg inequaltity (1.4) we get

$$\|\psi\|_{\dot{H}^1} \|\psi\|^3_{L^6} \le \|\psi\|^2_{\dot{H}^1} \|\psi\|^2_{L^2} + \frac{1}{2\pi} \|\psi\|_{\dot{H}^1} \|\psi\|^3_{L^2} = \|\psi\|^2_{\dot{H}^1} R_0^2 + \frac{1}{2\pi} \|\psi\|_{\dot{H}^1} R_0^3. \tag{3.13}$$

Furthermore, using the Young inequality we have

$$\frac{1}{2\pi} \|\psi\|_{\dot{H}^1} R_0^3 \le \frac{1}{2} \|\psi\|^2_{\dot{H}^1} R_0^2 + \frac{1}{8\pi^2} R_0^4. \tag{3.14}$$

Combining (3.12)–(3.14) the proof follows.   □

Now we are ready to prove Proposition 3.2.

Proof of Proposition 3.2 We prove (3.3) by induction on k. For $k = 0$, there is nothing to prove, since $E_0(\psi) = 1/2\|\psi\|^2_{L^2}$ [Eq. (1.2)].

For $k = 1$, by Eq. (1.3) we can write $E_1(\psi) = 1/2\|\psi\|^2_{\dot{H}^1} + \int q_1(\psi)$, where

$$\int q_1(\psi) = \frac{3i}{4} \beta \int \bar{\psi}' \bar{\psi} \psi^2 + \frac{\beta^2}{4} \|\psi\|^6_{L^6}.$$

Hence,

$$\frac{1}{2} \|\psi\|^2_{\dot{H}^1} = E_1(\psi) - \int q_1(\psi) \le E_1(\psi) - \frac{3i}{4} \beta \int \bar{\psi}' \bar{\psi} \psi^2. \tag{3.15}$$

By Lemma 3.7 and choosing $R_0 \le \sqrt{\frac{2}{9|\beta|}}$ we obtain

$$\left| \frac{3i}{4} \beta \int \bar{\psi}' \bar{\psi} \psi^2 \right| \le \frac{1}{4} \|\psi\|^2_{\dot{H}^1} + \frac{3}{32} R_0^4. \tag{3.16}$$

Thus, by (3.15) and (3.16), it follows that

$$\frac{1}{4} \|\psi\|^2_{\dot{H}^1} \le |E_1| + \frac{3}{32} R_0^4 =: \mathcal{C}(R_0, R_1).$$

This proves (3.3) in the case $k = 1$. Let us assume that Eq. (3.3) holds for $k \ge 2$, namely

$$\|\psi\|_{\dot{H}^k} \le \mathcal{C}(R_0, \dots, R_k, k, |\beta|),$$

and let us show that it holds for $k + 1$. By Eq. (3.1) and Corollary 3.6 we have

$$\frac{1}{2}\|\psi\|_{\dot{H}^{k+1}}^2 \leq |E_{k+1}(\psi)| - \int q_{k+1}(\psi)$$

$$\leq R_{k+1} + c(k)\varepsilon\|\psi\|_{\dot{H}^{k+1}}^2 + \mathcal{C}\left(\|\psi\|_{\dot{H}^0}, \|\psi\|_{\dot{H}^k}, \varepsilon, k, |\beta|\right). \quad (3.17)$$

On the other by the inductive assumption we have

$$\mathcal{C}\left(\|\psi\|_{\dot{H}^0}, \|\psi\|_{\dot{H}^k}, \varepsilon, k, |\beta|\right) = \mathcal{C}\left(R_0, \ldots, R_k, \varepsilon, k, |\beta|\right).$$

Hence, from (3.17), choosing $\varepsilon \leq 1/4c(k)$, we get

$$\frac{1}{4}\|\psi\|_{\dot{H}^{k+1}}^2 \leq \mathcal{C}\left(R_0, \ldots, R_k, R_{k+1}, k+1, |\beta|\right),$$

thus proving Eq. (3.3) and concluding the proof.   □

4 Convergence of the integrals of motion

In this section we study the convergence of $G_{k,N}(\psi)$ defined in (1.7) with respect to the Gaussian measure γ_k. The main result is given by the following.

Proposition 4.1 *Let $k \geq 2$ and $1 \leq m \leq k$. Then $\int q_m(\psi_N)$ converges in measure to $\int q_m(\psi)$ w.r.t. the Gaussian measure $d\gamma_k$. Furthermore, if $1 \leq m < k$, then $E_m(\psi_N)$ converges in measure to $E_m(\psi)$ w.r.t. γ_k.*

As a consequence, by composition and multiplication of continuous functions, we obtain

Corollary 4.2 *The sequence $G_{k,N}(\psi)$ converges in measure, with respect to γ_k, as $N \to \infty$, to a function which we (already) denoted $G_k(\psi)$.*

We split the proof of Proposition 4.1 in several steps.

Lemma 4.3 *Let $k \geq 2$, and let $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq 0$ be such that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2k - 1$. For $\alpha_1 = k - 1$ and $\alpha_2, \alpha_3, \alpha_4 \leq k - 1$, we have*

$$\lim_{N \to \infty} \int u_N^{(k-1)} u_N^{(\alpha_2)} u_N^{(\alpha_3)} u_N^{(\alpha_4)} = \int u^{(k-1)} u^{(\alpha_2)} u^{(\alpha_3)} u^{(\alpha_4)},$$

almost everywhere with respect to the measure γ_k.

Proof We have

$$\left|\int u_N^{(k-1)} u_N^{(\alpha_2)} u_N^{(\alpha_3)} u_N^{(\alpha_4)} - \int u^{(k-1)} u^{(\alpha_2)} u^{(\alpha_3)} u^{(\alpha_4)}\right| \leq A_1 + A_2,$$

where

$$A_1 := \left| \int (u_N^{(k-1)} - u^{(k-1)}) u_N^{(\alpha_2)} u_N^{(\alpha_3)} u_N^{(\alpha_4)} \right|,$$

$$A_2 := \left| \int u^{(k-1)} \left(u_N^{(\alpha_2)} u_N^{(\alpha_3)} u_N^{(\alpha_4)} - u^{(\alpha_2)} u^{(\alpha_3)} u^{(\alpha_4)} \right) \right|$$

by using the embedding $H^1 \hookrightarrow L^\infty$ and $u_N \to u$ in $\dot{H}^{k-1}$, γ_k-a.s. we immediately see that $A_1 \to 0$, γ_k-a.s. Then we notice that

$$A_2 \leq B_1 + B_2$$

where

$$B_1 := \left| \int u^{(k-1)} \left(u_N^{(\alpha_2)} - u^{(\alpha_2)} \right) u_N^{(\alpha_3)} u_N^{(\alpha_4)} \right|,$$

$$B_2 := \left| \int u^{(k-1)} u^{(\alpha_2)} \left(u^{(\alpha_3)} u^{(\alpha_4)} - u_N^{(\alpha_3)} u_N^{(\alpha_4)} \right) \right|$$

and as before $B_1 \to 0$, γ_k-a.s. We finally notice that

$$B_2 \leq C_1 + C_2$$

where

$$C_1 := \left| \int u^{(k-1)} u^{(\alpha_2)} \left(u^{(\alpha_3)} - u_N^{(\alpha_3)} \right) u^{(\alpha_4)} \right|,$$

$$C_2 := \left| \int u^{(k-1)} u^{(\alpha_2)} u_N^{(\alpha_3)} \left(u^{(\alpha_4)} - u_N^{(\alpha_4)} \right) \right| \tag{4.1}$$

and as before both $C_1, C_2 \to 0$, γ_k-a.s., which completes the proof. □

Lemma 4.4 *For $k \geq 2$, $l \geq 5$, and $\alpha_i \geq 0 \, (i = 1, \ldots, l)$ such that $0 \leq \alpha_1 + \cdots + \alpha_l \leq 2k - 2$, we have*

$$\lim_{N \to \infty} \int u_N^{(\alpha_1)} \ldots u_N^{(\alpha_l)} = \int u^{(\alpha_1)} \ldots u^{(\alpha_l)},$$

almost everywhere with respect to the measure γ_k.

Proof As in the proof of Lemma 3.3, by reordering and integration by parts we can reduce to the case

$$\alpha_1, \alpha_2 \leq k - 1, \quad \text{and} \quad \alpha_i \leq k - 2, \quad i = 3, \ldots, l.$$

Then the proof is the same of Lemma 4.3. □

Let $l \in \mathbb{Z}_+$. We denote by S_l the group of permutations on l elements. In the sequel we use the following version of the Wick formula (we refer to [9] or to [15,34] for more details). Let $(m_1, \ldots, m_l, n_1, \ldots, n_l) \in \mathbb{Z}^{2l}$. Then we have

$$\mathbb{E}\left[\prod_{j=1}^{l} \bar{\psi}_{m_j} \psi_{n_j}\right] = \sum_{\sigma \in S_l} \prod_{i=1}^{l} \frac{\delta_{m_i, n_{\sigma(i)}}}{\left(1 + |n_{\sigma(i)}|^k\right)^2}. \tag{4.2}$$

Let us denote by

$$f_N^k(\psi) := \int \psi_N^{(k)} \bar{\psi}_N^{(k-1)} \bar{\psi}_N \psi_N. \tag{4.3}$$

Proposition 4.5 *Let $k \geq 2$. The sequence $\{f_N^k\}_{N \in \mathbb{Z}_+}$ is a Cauchy sequence in $L_{\gamma_k}^2$, for all $s < k - 1/2$. Indeed, for all $N > M \geq 1$, we have*

$$\|f_M^k - f_N^k\|_{L_{\gamma_k}^2} \lesssim \frac{1}{\sqrt{M}}.$$

Proof By an explicit computation we get

$$f_N^k(\psi) = i \sum_{A_N} n_1^k m_1^{k-1} \bar{\psi}_{m_1} \bar{\psi}_{m_2} \psi_{n_1} \psi_{n_2},$$

where

$$A_N := \left\{(m_1, m_2, n_1, n_2) \in \mathbb{Z}^4 \mid |m_i|, \quad |n_i| \leq N, \quad n_1 + n_2 = m_1 + m_2\right\}.$$

According to our convention, the labels m_i (respectively n_i) are associated to the Fourier coefficients of $\bar{\psi}$ (respectively ψ). Moreover, we define

$$A_{N,M} := \{(m_1, m_2, n_1, n_2) \in A_N, \quad \max(|m_1|, |m_2|, |n_1|, |n_2|) > M\}.$$

Thus,

$$f_N^k(\psi) - f_M^k(\psi) = i \sum_{A_{N,M}} n_1^k m_1^{k-1} \bar{\psi}_{m_1} \bar{\psi}_{m_2} \psi_{n_1} \psi_{n_2}. \tag{4.4}$$

Taking the square of Eq. (4.4) we get

$$\left|f_N^k(\psi) - f_M^k(\psi)\right|^2 = \sum_{A_{N,M} \times A'_{N,M}} n_1^k m_1^{k-1} m_3^k n_3^{k-1} \prod_{j=1}^{4} \bar{\psi}_{m_j} \psi_{n_j},$$

where

$$A'_N := \left\{(m_3, m_4, n_3, n_4) \in \mathbb{Z}^4 \mid |m_i|, \quad |n_i| \leq N, \quad m_3 + m_4 = n_3 + n_4\right\},$$
$$A'_{N,M} := \left\{(m_3, m_4, n_3, n_4) \in A'_N \mid \max(|m_3|, |m_4|, |n_3|, |n_4|) > M\right\}.$$

By definition of the measure γ_k we have

$$\| f_M^k - f_N^k \|_{L^2_{\gamma_k}}^2 = \sum_{A_{N,M} \times A'_{N,M}} n_1^k m_1^{k-1} m_3^k n_3^{k-1} \mathbb{E} \left[\prod_{j=1}^4 \bar{\psi}_{m_j} \psi_{n_j} \right]. \tag{4.5}$$

By using the Wick formula (4.2) with $l = 4$, Eq. (4.5) becomes

$$\left\| f_M^k - f_N^k \right\|_{L^2_{\gamma_k}}^2 = \sum_{A_{N,M} \times A'_{N,M}} n_1^k m_1^{k-1} m_3^k n_3^{k-1} \sum_{\sigma \in S_4} \prod_{i=1}^4 \frac{\delta_{m_i, n_{\sigma(i)}}}{\left(1 + |n_{\sigma(i)}|^k \right)^2}. \tag{4.6}$$

Let us consider the subgroup $G = \{1, (12), (34), (12)(34)\} \subset S_4$ and its action on S_4 by left multiplication. For $X \subset S_4$, we denote by $G \cdot X = \{gx \mid g \in G, x \in X\}$ the orbit of the subset X. We have the following partition of $S_4 = W_1 \cup W_2 \cup W_3$, where $W_1 := G \cdot \{1\} = G$, $W_2 := G \cdot \{(13), (14), (23), (24)\}$ and $W_3 := G \cdot \{(13)(24)\}$. Hence, we can further rewrite Eq. (4.6) as follows:

$$\left\| f_M^k - f_N^k \right\|_{L^2_{\gamma_k}}^2 = \sum_{i=1}^3 \sum_{A^i_{N,M}} \sum_{\sigma \in W_i} \frac{n_1^k n_{\sigma(1)}^{k-1} n_3^{k-1} n_{\sigma(3)}^k}{\prod_{j=1}^4 \left(1 + |n_j|^k \right)^2}, \tag{4.7}$$

where the subsets of indices $A^i_{N,M}$ will be presented case by case.

We consider the three contributions to the sum in (4.7) separately.

First case: $i = 1$ We have

$$A^1_{N,M} = \Big\{ (n_1, n_2, n_3, n_4) \in \mathbb{Z}^4 \mid |n_i| \leq N, \quad \max(|n_1|, |n_2|) > M,$$
$$\max(|n_3|, |n_4|) > M \Big\},$$

and the contribution to the sum in (4.7) is

$$\sum_{A^1_{N,M}} \left(\frac{n_1^{2k-1} n_3^{2k-1}}{\prod_{j=1}^4 \left(1 + |n_j|^k \right)^2} + \frac{n_1^k n_2^{k-1} n_3^{2k-1}}{\prod_{j=1}^4 \left(1 + |n_j|^k \right)^2} + \frac{n_1^{2k-1} n_3^{k-1} n_4^k}{\prod_{j=1}^4 \left(1 + |n_j|^k \right)^2} \right.$$
$$\left. + \frac{n_1^k n_2^{k-1} n_3^{k-1} n_4^k}{\prod_{j=1}^4 \left(1 + |n_j|^k \right)^2} \right). \tag{4.8}$$

The sum in (4.8) is zero. In fact, all the functions involved in the sum are odd functions with respect to the transformation $n_1 \to -n_1$, $n_2 \to -n_2$ while the index set $A^1_{N,M}$ is invariant.

Second case: $i = 2$ In this case we have

$$A^2_{N,M} = \Big\{ (n_1, n_2, n_3) \in \mathbb{Z}^3 \mid |n_i| \leq N, \ \max(|n_1|, |n_2|) > M, \ \max(|n_1|, |n_3|) > M \Big\}.$$

Similarly to the previous case, the contribution in the sum (4.7) corresponding to a permutation $\sigma \in W_2$ which fixes 1 (respectively 3) is zero since the summand is odd with respect to the transformation $n_1 \to -n_1$ (respectively $n_3 \to -n_3$) while the index set $A^2_{N,M}$ is invariant. The summands corresponding to the remaining elements in W_2 have the following form

$$\sum_{A^2_{N,M}} \frac{n_1^{a_1}}{\left(1 + |n_1|^k\right)^4} \frac{n_2^{a_2}}{\left(1 + |n_2|^k\right)^2} \frac{n_3^{a_3}}{\left(1 + |n_1|^k\right)^2} \tag{4.9}$$

where $a_2, a_3 \in \{0, k-1, k\}$, $a_1 + a_2 + a_3 = 4k - 2$ (hence $2k - 2 \le a_1 \le 4k - 2$). So, by a straightforward computation, we have (we remind that we are considering $k \ge 2$)

$$(4.9) \lesssim \sum_{\substack{\max(|n_1|,|n_2|)>M, \\ \max(|n_1|,|n_3|)>M}} \frac{n_1^{a_1}}{\left(1 + |n_1|^k\right)^2} \frac{n_2^{a_2}}{\left(1 + |n_2|^k\right)^2} \frac{n_3^{a_3}}{\left(1 + |n_3|^k\right)^2} \lesssim \frac{1}{M}. \tag{4.10}$$

Third case: $i = 3$ We have

$$A^3_{N,M} = \left\{(n_2, n_3, n_4) \in \mathbb{Z}^3 \mid |n_i| \le N, \quad \max\left(|n_3 + n_4 - n_2|, |n_2|, |n_3|, |n_4|\right) > M\right\}.$$

Two summands in (4.7), corresponding to the elements (13) (24) and (1423) in W_3, have respectively the following form

$$\sum_{A^3_{N,M}} \frac{(n_3 + n_4 - n_2)^{2k}}{\left(1 + |n_3 + n_4 - n_2|^k\right)^2} \frac{1}{\left(1 + |n_2|^k\right)^2} \frac{n_3^{2(k-1)}}{\left(1 + |n_3|^k\right)^2} \frac{1}{\left(1 + |n_4|^k\right)^2}, \tag{4.11}$$

$$\sum_{A^3_{N,M}} \frac{(n_3 + n_4 - n_2)^{2k}}{\left(1 + |n_3 + n_4 - n_2|^k\right)^2} \frac{1}{\left(1 + |n_2|^k\right)^2} \frac{n_3^{k-1}}{\left(1 + |n_3|^k\right)^2} \frac{n_4^{k-1}}{\left(1 + |n_4|^k\right)^2}. \tag{4.12}$$

We can bound these terms as

$$(4.11) \lesssim \sum_{\max(|n_2|,|n_3|,|n_4|)>M/3} \frac{1}{\left(1 + |n_2|^k\right)^2} \frac{n_3^{2(k-1)}}{\left(1 + |n_3|^k\right)^2} \frac{1}{\left(1 + |n_4|^k\right)^2} \lesssim \frac{1}{M}, \tag{4.13}$$

$$(4.12) \lesssim \sum_{\max(|n_2|,|n_3|,|n_4|)>M/3} \frac{1}{\left(1 + |n_2|^k\right)^2} \frac{n_3^{k-1}}{\left(1 + |n_3|^k\right)^2} \frac{n_4^{k-1}}{\left(1 + |n_4|^k\right)^2} \lesssim \frac{1}{M^k}. \tag{4.14}$$

The other two terms correspond to the elements (14) (23) and (1324). They can be estimated respectively as

$$\sum_{A_{N,M}^3} \frac{(n_3 + n_4 - n_2)^k}{\left(1 + |n_3 + n_4 - n_2|^k\right)^2} \frac{n_2^k}{\left(1 + |n_2|^k\right)^2} \frac{n_3^{k-1}}{\left(1 + |n_3|^k\right)^2} \frac{n_4^{k-1}}{\left(1 + |n_4|^k\right)^2} \lesssim \frac{1}{M^{k-1}},$$

(4.15)

$$\sum_{A_{N,M}^3} \frac{(n_3 + n_4 - n_2)^k}{(1 + |n_3 + n_4 - n_2|^k)^2} \frac{n_2^{k-1}}{\left(1 + |n_2|^k\right)^2} \frac{n_3^{2(k-1)}}{\left(1 + |n_3|^k\right)^2} \frac{1}{\left(1 + |n_4|^k\right)^2} \lesssim \frac{1}{M^{k-1}}.$$

(4.16)

In conclusion, recollecting all the contributions given by (4.10) and (4.13–4.16) , we see immediately that, for $k \geq 2$, we have

$$\|f_N^k - f_M^k\|_{L^2_{\gamma_k}}^2 \lesssim \frac{1}{M},$$

(4.17)

thus concluding the proof. $\qquad\square$

We can extend the estimate (4.17) to all the $L^p(H^s, \gamma_k)$-norms, with $p \geq 1$. For $1 \leq p < 2$ it is trivial, since γ_k is a probability measure. For $p > 2$ we have to use the properties of the Gaussian measure. For any $r-$linear form $\Psi^r(\psi)$, a direct application of the Nelson hypercontractivity inequality [30], as shown for instance in [34, Theorem I.22], yields

$$\left\|\Psi^r\right\|_{L^p_{\gamma_k}} \leq (p - 1)^{\frac{r}{2}} \|\Psi^r\|_{L^2_{\gamma_k}}.$$

This leads us to the following.

Corollary 4.6 *For all $p \geq 2$ and $N > M \geq 1$, we have*

$$\left\|f_M^k(\psi) - f_N^k(\psi)\right\|_{L^p(H^s,\gamma_k)} \lesssim \frac{(p - 1)^2}{\sqrt{M}}.$$

(4.18)

Corollary 4.7 *Let $k \geq 2$, then $\int q_{k,2k-1}(\psi_N)$ converges in measure to $\int q_{k,2k-1}(\psi)$, w.r.t. γ_k.*

Proof It follows by the explicit form of $\int q_{k,2k-1}$ given in Corollary 2.9 and by Proposition 4.5 and Lemma 4.3. $\qquad\square$

Finally, we can prove Proposition 4.1.

Proof of Proposition 4.1 The explicit form of $\int q_k$ given by Corollary 2.9, Lemma 4.4 and Corollary 4.7 imply that $\int q_m(\psi_N)$ converges in measure to $\int q_m(\psi)$ w.r.t. γ_k, for $1 \leq m \leq k, k \geq 2$. In addition, Proposition 3.2 ensures that as long as $1 \leq m < k$ we have $\|\psi_N\|_{H^m} \leq C$ N-uniformly; thereby it converges to $\|\psi\|_{H^m}$ a.e. w.r.t. γ_k. $\qquad\square$

5 Proof of Theorem 1.1

In this section we conclude the proof of Theorem 1.1. First, we state a useful technical lemma that we borrow from [39, Proposition 4.5]. We report the proof for the sake of completeness:

Lemma 5.1 *Let* $(\Omega, \mathcal{S}, \mu)$ *a finite measure space. If there are* $C, r > 0$, *and an integer* $p_0 > 0$, *such that for every* $p \geq p_0$ *we have*

$$\|F\|_p \leq Cp^r,$$

then there exist $0 < \delta < re^{-1}$ *and a constant* $L = L(r, \delta, p_0)$ *such that*

$$\int_\Omega d\mu \exp\left[\delta\left(\frac{|F|}{C}\right)^{\frac{1}{r}}\right] \leq L. \tag{5.1}$$

Proof We expand

$$\exp\left[\delta\left(\frac{|F|}{C}\right)^{\frac{1}{r}}\right] = \sum_{n\in\mathbb{Z}_+} \frac{\delta^n}{n!}\left(\frac{|F|}{C}\right)^{n/r}.$$

Thus,

$$\int_\Omega d\mu \exp\left[\delta\left(\frac{|F|}{C}\right)^{\frac{1}{r}}\right] = \int_\Omega dx \sum_{n\in\mathbb{Z}_+} \frac{\delta^n}{n!}\left(\frac{|F|}{C}\right)^{n/r}$$

$$= \sum_{n\in\mathbb{Z}_+} \frac{\delta^n}{n!} \frac{\|F\|_{n/r}^{n/r}}{C^{n/r}}$$

$$\leq \sum_{n<p_0r} \frac{\delta^n}{n!} \frac{\|F\|_{n/r}^{n/r}}{C^{n/r}} + \sum_{n\geq p_0r} \frac{\delta^n}{n!}\left(\frac{n}{r}\right)^n$$

$$=: \sum_{n<p_0r} \frac{\delta^n}{n!} \frac{\|F\|_{n/r}^{n/r}}{C^{n/r}} + L_1(r, \delta, p_0),$$

where the constant $L_1(r, \delta)$ is finite for $\delta < re^{-1}$. For the finite sum we readily have

$$\|F\|_{n/r}^{n/r} \leq \|F\|_{p_0}^{n/r} \leq C^{n/r} p_0^n,$$

hence

$$\sum_{n<p_0r} \frac{\delta^n}{n!} \frac{\|F\|_{n/r}^{n/r}}{C^{n/r}} \leq \sum_{n<p_0r} \frac{\delta^n}{n!} p_0^n =: L_2(r, \delta, p_0).$$

The constant L_2 is always finite, so we can set $L = L_1 + L_2$ and the assert follows.
□

Remark 5.2 The exponent $1/r$ in (5.1) is optimal: The formula remains valid for each $\alpha \leq 1/r$ and fails otherwise.

By using Lemma 5.1 and Proposition 4.5 we can deduce that we have a sub-exponential tail for the convergence in probability of the Cauchy sequence f_N^k defined in Eq. (4.3).

Lemma 5.3 *Let $N > M \geq 1$ be integer numbers and f_N^k defined as in (4.3). Then for any $\lambda > 0$ and $k \geq 2$ we have*

$$\gamma_k \left(\left| f_N^k - f_M^k \right| \geq \lambda^2 \right) \lesssim \exp\left(-\frac{\sqrt{2}}{3} \lambda M^{1/4} \right). \tag{5.2}$$

Proof By formula (4.18) in Proposition 4.5 we can apply the Lemma 5.1 with $F = f_N^k - f_M^k$, $p_0 = 2$, $r = 2$, $C = 2/\sqrt{M}$ and $\delta = 2/3$. We immediately obtain

$$\int \gamma_k(d\psi) \exp\left[\frac{2}{3} \left(\frac{|f_N^k - f_M^k| \sqrt{M}}{2} \right)^{1/2} \right] < \infty.$$

Formula (5.2) follows straightforwardly from Markov inequality:

$$\gamma_k \left(\left| f_N^k - f_M^k \right| \geq \lambda^2 \right) = \gamma_k \left(\sqrt{\frac{\sqrt{M}|f_N^k - f_M^k|}{2}} \geq \frac{\lambda M^{1/4}}{\sqrt{2}} \right)$$

$$\leq \exp\left(-\frac{2}{3} \frac{\lambda M^{1/4}}{\sqrt{2}} \right) \mathbb{E}\left[\exp\left(\frac{2}{3} \sqrt{\frac{\sqrt{M}|f_N^k - f_M^k|}{2}} \right) \right]$$

$$\leq \exp\left(-\frac{2}{3} \frac{\lambda M^{1/4}}{\sqrt{2}} \right) = \exp\left(-\frac{\sqrt{2}}{3} \lambda M^{1/4} \right).$$

□

Now we come to the most important result of this section, namely the integrability of the density $G_{k,N}(\psi)$ w.r.t. the Gaussian measure γ_k. More precisely we state:

Proposition 5.4 *Let $\mathcal{C} = \mathcal{C}(R_0, \ldots, R_{k-1}, k, |\beta|)$ be the constant appearing in Proposition 3.2 and let us take $R_0 \leq \sqrt{\frac{2}{9|\beta|}}$ such that*

$$p_0 := \min\left(2 \left(3(2k+1)|\beta|\sqrt{\mathcal{C}R_0^3} \right)^{-1}, (4(2k+1)|\beta|R_0\mathcal{C})^{-1} \right) > 1.$$

Then for any $k \geq 2$, $1 \leq p < p_0$ and $N \geq \left(\frac{2k+1}{2}|\beta|\right)^2 R_0^6 C^2$ we have

$$\|G_{k,N}(\psi)\|_{L^p(\gamma_k)} \leq C < +\infty,$$

where $G_{k,N}(\psi)$ are the Gibbs densities introduced in (1.7).

The proof needs two accessory results:

Lemma 5.5 *For every $p \geq 1$ and $k \geq 2$, we have*

$$\|G_{k,N}(\psi)\|_{L^p(\gamma_k)} \leq e^{\mathcal{C}} \left\| \prod_{m=0}^{k-1} \chi_{R_m} \left(\int h_m(\psi_N) \right) e^{-f_N^k[\psi_N]} \right\|_{L^p(\gamma_k)}.$$

Proof The lemma follows as a direct consequence from Corollary 2.9, Lemmas 3.3, 3.4, 4.3, 4.4, and Proposition 3.2.   □

Lemma 5.6 *For $\lambda \geq R_0^2 \sqrt{N}$ we have*

$$\gamma_k \left(\sup_{x \in \mathbb{T}} \left| \psi_N^{(k)} \bar{\psi}_N \right| \geq \lambda \right) \lesssim N^{2+2k} e^{-\frac{\lambda}{4}}.$$

Proof The proof follows from Propositions 6.5 and 6.8 for quadratic forms in "Appendix." Expanding in Fourier series we see that

$$Q_N(x) := \left| \psi_N^{(k)}(x) \bar{\psi}_N(x) \right| = \left| \sum_{|j|,|h| \leq N} (ih)^k e^{i(h-j)x} \psi_h \bar{\psi}_j \right|$$

is a quadratic form in the Fourier coefficients of ψ and it fulfills the requirement (6.5) in Proposition 6.5, with $T_k \leq 1$. Hence, for each $x \in \mathbb{T}$ we obtain

$$\gamma_k \left(\left| \psi_N^{(k)}(x) \bar{\psi}_N(x) \right| \geq \lambda \right) \lesssim e^{-\frac{\lambda}{4}},$$

for all $\lambda > 0$. Moreover, for any $x, y \in \mathbb{T}$, by the Cauchy–Schwarz and Bernstein inequality

$$
\begin{aligned}
|Q_N(x) - Q_N(y)| = \left| \int_y^x \tilde{Q}_N(z) \mathrm{d}z \right| \\
\leq \sqrt{|x-y|} \|\tilde{Q}_N\|_{L^2} \\
\leq \sqrt{|x-y|} N \|Q_N\|_{L^2} \\
\leq \sqrt{|x-y|} N^{\frac{3}{2}+k} R_0^2.
\end{aligned}
$$

Therefore, we can apply Proposition 6.8 with $\alpha = \frac{1}{2}$ and $L_N = N^{\frac{3}{2}+k} R_0^2$ to get for any $\varepsilon > 0$ and $\lambda \geq N^{\frac{3}{2}+k} R_0^2 \sqrt{\varepsilon}$

$$\gamma_k \left(\sup_{x \in \mathbb{T}} \left| \psi_N^{(k)}(x) \bar{\psi}_N(x) \right| \geq \lambda \right) \lesssim \frac{e^{-\lambda/4}}{\varepsilon}.$$

We recover the assert by setting $\varepsilon = N^{-2-2k}$. $\qquad\qquad\square$

Now we can give the

Proof of Proposition 5.4 Let us set for brevity $\sigma := i \frac{2k+1}{2} \beta$. By Lemma 5.5 we have to estimate

$$\int_0^{+\infty} t^{p-1} \gamma_k \left(\prod_{m=0}^{k-1} \chi_{R_m} \left(\int h_{2m}(\psi_N) \right) e^{-\sigma \int \psi_N^{(k)} \bar{\psi}_N^{(k-1)} \psi_N \bar{\psi}_N} \geq t \right) dt. \qquad (5.3)$$

We use

$$\gamma_k \left(\prod_{m=0}^{k-1} \chi_{R_m} \left(\int h_{2m}(\psi_N) \right) e^{-\sigma \int \psi_N^{(k)} \bar{\psi}_N^{(k-1)} \psi_N \bar{\psi}_N} \geq t \right)$$

$$= \gamma_k \left(\prod_{m=0}^{k-1} \chi_{R_m} \left(\int h_{2m}(\psi_N) \right) e^{-\sigma \int \psi_N^{(k)} \bar{\psi}_N^{(k-1)} \psi_N \bar{\psi}_N} \geq t, \right.$$

$$\left. \left| \int h_{2m}(\psi_N) \right| \leq R_m, \quad 0 \leq m \leq k-1 \right)$$

$$\leq \gamma_k \left(\left| \int \psi_N^{(k)} \bar{\psi}_N^{(k-1)} \psi_N \bar{\psi}_N \right| \geq \frac{\ln t}{|\sigma|}, \left| \int h_{2m}(\psi_N) \right| \leq R_m, \quad 0 \leq m \leq k-1 \right).$$

It is convenient to split the integral in (5.3) into three parts:

$$(5.3) = \int_0^{\exp(\sigma^2 R_0^6 C^2)} (\cdot) + \int_{\exp(\sigma^2 R_0^6 C^2)}^{\exp(|\sigma| R_0^3 C \sqrt{N})} (\cdot) + \int_{\exp(|\sigma| R_0^3 C \sqrt{N})}^{+\infty} (\cdot). \qquad (5.4)$$

For $t \leq e^{\sigma^2 R_0^6 C^2}$ it suffices to use the trivial bound

$$\gamma_k \left(\left| \int \psi_N^{(k)} \bar{\psi}_N^{(k-1)} \psi_N \bar{\psi}_N \right| \geq \frac{\ln t}{|\sigma|}, \left| \int h_{2m}(\psi_N) \right| \leq R_m, \quad 0 \leq m \leq k-1 \right) \leq 1. \qquad (5.5)$$

In the range $\sigma^2 R_0^6 C^2 \leq \ln t \leq |\sigma| C R_0^3 \sqrt{N}$ we define

$$N^* = N^*(t) := \left\lfloor \frac{\ln t}{|\sigma| C R_0^3} \right\rfloor^2,$$

noting that $N > N^*$. We decompose

$$\gamma_k \left(\left| \int \psi_N^{(k)} \bar{\psi}_N^{(k-1)} \psi_N \bar{\psi}_N \right| \geq \frac{\ln t}{|\sigma|}, \quad \left| \int h_{2m}(\psi_N) \right| \leq R_m, \quad 0 \leq m \leq k-1 \right)$$

$$\leq \gamma_k \left(\left| \int \psi_N^{(k)} \bar{\psi}_N^{(k-1)} \psi_N \bar{\psi}_N - \int \psi_{N^*}^{(k)} \bar{\psi}_{N^*}^{(k-1)} \psi_{N^*} \bar{\psi}_{N^*} \right| \geq \frac{\ln t}{2|\sigma|} \right) \tag{5.6}$$

$$+ \gamma_k \left(\left| \int \psi_{N^*}^{(k)} \bar{\psi}_{N^*}^{(k-1)} \psi_{N^*} \bar{\psi}_{N^*} \right| \geq \frac{\ln t}{2|\sigma|} \right). \tag{5.7}$$

For the first addendum (5.6), we exploit formula (5.2) in Lemma 5.3, to obtain

$$\gamma_k \left(\left| \int \psi_N^{(k)} \bar{\psi}_N^{(k-1)} \psi_N \bar{\psi}_N - \int \psi_{N^*}^{(k)} \bar{\psi}_{N^*}^{(k-1)} \psi_{N^*} \bar{\psi}_{N^*} \right| \geq \frac{\ln t}{2|\sigma|} \right) \lesssim t^{-\left(3|\sigma|\sqrt{\mathcal{C}R_0^3} \right)^{-1}}. \tag{5.8}$$

Since in (5.7) we have $\ln t \geq |\sigma| R_0^3 \mathcal{C} \sqrt{N^*}$, we can treat this term and the third addendum in (5.4) (where we consider $\ln t \geq |\sigma| R_0^3 \mathcal{C} \sqrt{N}$) by the same method as follows. We bound

$$\left| \int \psi_N^{(k)} \bar{\psi}_N^{(k-1)} \psi_N \bar{\psi}_N \right| \leq \| \psi_N^{(k)} \bar{\psi}_N \|_\infty R_0 \mathcal{C},$$

whence

$$\gamma_k \left(\left| \int \psi_N^{(k)} \bar{\psi}_N^{(k-1)} \psi_N \bar{\psi}_N \right| \geq \frac{\ln t}{|\sigma|}, \quad \left| \int h_{2m}(\psi_N) \right| \leq R_m, \quad 0 \leq m \leq k-1 \right)$$

$$\leq \gamma_k \left(\max_{x \in \mathbb{T}} \left| \psi_N^{(k)} \bar{\psi}_N \right| R_0 \mathcal{C} \geq \frac{\ln t}{|\sigma|} \right).$$

Thus, to estimate the r.h.s. probability we use Lemma 5.6 with $\lambda = \frac{\ln t}{|\sigma| R_0 \mathcal{C}}$ to get

$$\gamma_k \left(\prod_{m=0}^{k-1} \chi_{R_m} \left(\int h_{2m}(\psi_N) \right) e^{-\sigma \int \psi_N^{(k)} \bar{\psi}_N^{(k-1)} \psi_N \bar{\psi}_N} \geq t \right) \leq N^{2+2k} e^{-\frac{\ln t}{4|\sigma| R_0 \mathcal{C}}}. \tag{5.9}$$

In particular for $N = N^*$ we have

$$\gamma_k \left(\left| \int \psi_{N^*}^{(k)} \bar{\psi}_{N^*}^{(k-1)} \psi_{N^*} \bar{\psi}_{N^*} \right| \geq \frac{\ln t}{2|\sigma|} \right) \leq (N^*)^{2+2k} e^{-\frac{\ln t}{8|\sigma| R_0 \mathcal{C}}}. \tag{5.10}$$

Now we can estimate (5.3). We first notice that (5.5) gives

$$\int_0^{e^{\sigma^2 R_0^6 \mathcal{C}^2}} t^{p-1} \gamma_k \left(\prod_{m=0}^{k-1} \chi_{R_m} \left(\int h_{2m}(\psi_N) \right) e^{-\sigma \int \psi_N^{(k)} \bar{\psi}_N^{(k-1)} \psi_N \bar{\psi}_N} \geq t \right) dt < e^{\sigma^2 R_0^6 \mathcal{C}^2}.$$

Then by using (5.10) and (5.8) we obtain

$$\int_{e^{\sigma^2 R_0^6 C^2}}^{\exp(|\sigma| R_0^3 C \sqrt{N})} t^{p-1} \gamma_k \left(\prod_{m=0}^{k-1} \chi_{R_m} \left(\int h_{2m}(\psi_N) \right) e^{-\int \psi_N^{(k)} \bar{\psi}_N^{(k-1)} \psi_N \bar{\psi}_N} \geq t \right) dt$$

$$\lesssim \int_{e^{\sigma^2 R_0^6 C^2}}^{\exp(|\sigma| R_0^3 C \sqrt{N})} t^{p-1-(8|\sigma| R_0 C)^{-1}} \ln t^{2+2k}$$

$$+ \int_{e^{\sigma^2 R_0^6 C^2}}^{\exp(|\sigma| R_0^3 C \sqrt{N})} t^{p-1-\left(3|\sigma|\sqrt{C R_0^3}\right)^{-1}}.$$

We note that as $p < \min((3|\sigma|\sqrt{C R_0^3})^{-1}, (8|\sigma| R_0 C)^{-1})$ both the functions on the r.h.s. are integrable, so we can bound both terms by an appropriate constant.

Finally, using (5.9), we have

$$\int_{\exp(|\sigma| R_0^3 C \sqrt{N})}^{+\infty} t^{p-1} \gamma_k \left(\prod_{m=0}^{k-1} \chi_{R_m} \left(\int h_{2m}(\psi_N) \right) e^{-\sigma \int \psi_N^{(k)} \bar{\psi}_N^{(k-1)} \psi_N \bar{\psi}_N} \geq t \right)$$

$$\lesssim N^{2+2k} \int_{\exp(|\sigma| R_0^3 C \sqrt{N})}^{+\infty} t^{p-1} e^{-\frac{\ln t}{4|\sigma| R_0 C}} dt$$

$$= N^{2+2k} \int_{\exp(|\sigma| R_0^3 C \sqrt{N})}^{+\infty} t^{p-1-(4|\sigma| R_0 C)^{-1}} dt$$

$$= \frac{N^{2+2k} e^{-[p-(4|\sigma| R_0 C)^{-1}]\sqrt{N}}}{|p-1-(4|\sigma| R_0 C)^{-1}|},$$

that vanishes for $N \to \infty$, provided that $p < (4|\sigma| R_0 C)^{-1}$. $\qquad \square$

We can finally proceed to complete the proof of Theorem 1.1 as follows

Proof of Theorem 1.1 The first part of the statement has been proved in Corollary 4.2. We are left to show that $G_k(\psi) \in L^p(\gamma_k)$ and that it is the $L^p(\gamma_k)$-limit of the sequence $G_{k,N}(\psi)$.

We start proving that $G_k(\psi) \in L^p(\gamma_k)$. Let $p \geq 1$ and let us choose $R_0 > 0$ such that Proposition 5.4 holds. Then there exists a subsequence $G_{k,N_m}(\psi)$, $m \in \mathbb{Z}_+$, such that $G_{k,N_m}(\psi) \to G_k(\psi)$, γ_k-a.s. Hence, by Fatou's Lemma, we have

$$\int |G_k(\psi)|^p \gamma_k(\psi) \leq \liminf_{m \to \infty} \int |G_{k,N_m}(\psi)|^p \gamma_k(\psi) < \infty,$$

thus proving that for $1 \leq p < p_0$ given by Proposition 5.4 $G_k(\psi) \in L^p(\gamma_k)$. By the uniform (for N large enough) $L^p(\gamma_k)$-boundedness of $G_{k,N}(\psi)$ we also have

$$\int |G_{k,N}(\psi) - G_k(\psi)|^p d\gamma_k(\psi) < \infty.$$

We are now ready to prove the convergence in $L^p(\gamma_k)$ for $p < p_0$. For all $\varepsilon > 0$, we define

$$A_{k,N,\varepsilon} = \{\psi \in H^k \mid |G_{k,N}(\psi) - G_k(\psi)| \leq \varepsilon\},$$

and denote by $A_{k,N,\varepsilon}^c$ its complement. Then let $p < q < p_0$

$$\int |G_{k,N}(\psi) - G_k(\psi)|^p \mathrm{d}\gamma_k(\psi) = \int_{A_{k,N,\varepsilon}} |G_{k,N}(\psi) - G_k(\psi)|^p \mathrm{d}\gamma_k(\psi)$$
$$+ \int_{A_{k,N,\varepsilon}^c} |G_{k,N}(\psi) - G_k(\psi)|^p \mathrm{d}\gamma_k(\psi)$$
$$\leq \varepsilon^p \gamma_k(A_{k,N,\varepsilon})$$
$$+ \|G_{k,N}(\psi) - G_k(\psi)\|_{L^q(\gamma_k)}^p \left(\gamma_k\left(A_{k,N,\varepsilon}^c\right)\right)^{1-p/q}.$$

Since $G_{k,N}(\psi)$ converges to $G_k(\psi)$ with respect to the measure γ_k, we have that, as $N \to \infty$,

$$\gamma_k(A_{k,N,\varepsilon}) \to 1, \quad \gamma_k\left(A_{k,N,\varepsilon}^c\right) \to 0,$$

Therefore, for a certain δ_N, vanishing for $N \to \infty$, we have the inequality

$$\left\|G_{k,N}(\psi) - G_k(\psi)\right\|_{L^p(\gamma_k)}^p \leq \varepsilon^p + \delta_N \left\|G_{k,N}(\psi) - G_k(\psi)\right\|_{L^q(\gamma_k)}^p$$

that concludes the proof. □

6 Appendix: Gaussian measures in Sobolev spaces: a toolbox

We are here interested in giving a succinct but self-contained survey on the theory of Gaussian measures in Hilbert Sobolev spaces. For a complete treatment we refer to [2,33].

6.1 Concentration of measure in $\dot{H}^k(\mathbb{T})$

Here we study the concentration property of the Gaussian measure with covariance $(\mathbb{I} + (-\Delta)^k)^{-1}$. The main feature is that the measure is concentrated on functions in $L^2(\mathbb{T})$ having slightly less then $k - \frac{1}{2}$ weak derivatives as regularity. This is stated precisely in the following.

Proposition 6.1 *For every $k \geq 0$ we have $\gamma_k(\bigcap_{\varepsilon>0} \dot{H}^{k-\frac{1}{2}-\varepsilon}) = 1$.*

We will proceed by steps. At first we prove

Lemma 6.2 $\gamma_k(\dot{H}^{k-\frac{1}{2}+\varepsilon}) = 0$ *for every* $\varepsilon \geq 0$.

Proof We take any function $\varphi \in \dot{H}^s(\mathbb{T})$ with $s \geq k - \frac{1}{2}$. We have that $\|\varphi_N\|_{\dot{H}^s}$ is finite uniformly in N, where we recall φ_N is the projection on the Fourier modes $|n| \leq N$ defined by (1.5) and (1.6). We show that for all $\lambda > 0$

$$\gamma_k\left(\|\varphi_N\|_{\dot{H}^s} \leq \lambda\right) \to 0, \quad \text{as} \quad N \to \infty.$$

To do so, we make use of the Markov inequality: For every $\mu > 0$

$$\gamma_k\left(\|\varphi_N\|_{\dot{H}^s} \leq \lambda\right) \leq e^{\frac{\mu\lambda}{2}} \int \prod_{|n| \leq N} \left(\frac{1 + n^{2k}}{\sqrt{2\pi}} d\varphi_n d\bar{\varphi}_n\right) e^{-\frac{1}{2}\sum_n (1+n^{2k})|\varphi_n|^2} e^{-\frac{\mu}{2}\sum_n n^{2s}|\varphi_n|^2}$$

$$\leq e^{\frac{\mu\lambda}{2}} \int \frac{1}{(2\pi)^{2N+1}} d\varphi_n' d\bar{\varphi}_n' e^{-\frac{1}{2}\sum_n |\varphi_n'|^2} e^{-\mu \sum_n n^{2(s-k)}|\varphi_n'|^2}$$

$$= \exp\left[\frac{\mu\lambda}{2} - \frac{1}{2} \sum_{\substack{|n| \leq N, \\ n \neq 0}} \ln\left(1 + \frac{\mu}{|n|^\kappa}\right)\right],$$

where we have performed the change of variables $\varphi_n' = \sqrt{1 + n^{2k}}\varphi_n$ and set $-2(k - s) =: \kappa$. Let us first consider negative κ. In this case

$$\sum_{\substack{|n| \leq N, \\ n \neq 0}} \ln\left(1 + \frac{\mu}{|n|^\kappa}\right) \geq 2N \ln(1 + \mu),$$

and so we have an exponential decay in N for every choice of positive μ:

$$\gamma_p\left(\|\varphi_N\|_{\dot{H}^s} \leq \lambda\right) \lesssim e^{\frac{\mu\lambda}{2}} e^{-2N \ln(1+\mu)}, \quad (s > k). \tag{6.1}$$

For $\kappa \in [0, 1)$ the series $\sum_n \ln\left(1 + \frac{\mu}{n^\kappa}\right)$ diverges as $N^{1-\kappa}$. Hence,

$$\gamma_k\left(\|\varphi_N\|_{\dot{H}^s} \leq \lambda\right) \lesssim e^{-\mu N^{1-2(k-s)}}, \quad \left(k \geq s > k - \frac{1}{2}\right). \tag{6.2}$$

Finally, for $\kappa = 1$ we have a logarithmic divergence at exponent and therefore

$$\gamma_k\left(\|\varphi_N\|_{\dot{H}^s} \leq \lambda\right) \leq \left(\frac{e^{\frac{\lambda}{2}}}{N}\right)^\mu, \quad \left(s = k - \frac{1}{2}\right), \tag{6.3}$$

for arbitrary $\mu > 0$. We obtain the statement by taking $N \to \infty$ in (6.1)–(6.3). $\qquad \square$

Remark 6.3 The same strategy can be also used to show the stronger statement

$$\gamma_k\left(\|\varphi_N\|_{\dot{H}^s}\leq\ln N\right)\to 0 \text{ as } N\to\infty, \qquad \left(s\geq k-\frac{1}{2}\right).$$

Lemma 6.4 *We have that for every $s < k-\frac{1}{2}$ and $\lambda > 0$*

$$\gamma_k\left(\|\varphi\|_{\dot{H}^s}\geq\lambda\right)\lesssim e^{-\lambda/4}. \tag{6.4}$$

Proof Let us take a function $\varphi\in\dot{H}^s$ for some $s<k-\frac{1}{2}$. We look at its truncation φ_N and again it is $\|\varphi_N\|_{\dot{H}^s}$ finite uniformly in N. We exploit the reverse Chernoff bound at finite N: For every $\mu\in(0,1)$ and $\lambda>0$, we get

$$\gamma_k\left(\|\varphi_N\|_{\dot{H}^s}\geq\lambda\right)\leq e^{-\frac{\mu\lambda}{2}}\int\prod_{|n|\leq N}\left(\frac{1+n^{2k}}{\sqrt{2\pi}}\mathrm{d}\varphi_n\mathrm{d}\bar{\varphi}_n\right)e^{-\frac{1}{2}\sum_n(1+n^{2k})|\varphi_n|^2}e^{\frac{\mu}{2}\sum_n n^{2s}|\varphi_n|^2}$$

$$\leq e^{-\frac{\mu\lambda}{2}}\int\frac{1}{(2\pi)^{2N+1}}\mathrm{d}\varphi_n'\mathrm{d}\bar{\varphi}_n'e^{-\frac{1}{2}\sum_n|\varphi_n'|^2}e^{\mu\sum_n n^{2(s-k)}|\varphi_n|^2}$$

$$=\exp\left[-\frac{\mu\lambda}{2}-\frac{1}{2}\sum_{\substack{|n|\leq N,\\n\neq 0}}\ln\left(1-\frac{\mu}{|n|^\kappa}\right)\right],$$

where again we have used the same change of variables as before. Note that now it is $\kappa>1$. Since $\frac{1}{2}\sum_n\ln(1-\frac{\mu}{n^\kappa})$ is convergent for all $\mu<1$ and $\kappa>1$, we can choose $\mu\in(0,1)$ and take the limit $N\to\infty$. We get (6.4) by setting $\mu=1/2$. $\qquad\square$

Equation (6.4) implies that $\|u\|_{H^q}$ is bounded with probability 1 for every $k<s-\frac{1}{2}$. This is sufficient to complete the proof of Proposition 6.1.

6.2 Quadratic forms

Then we present some results about quadratic forms of Gaussian random variables, used in the paper.

Proposition 6.5 *Let $k\geq 2$ and Q be a $(2N+1)\times(2N+1)$ matrix such that*

$$\sup_{l,h}\frac{|Q_{lh}|}{\sqrt{1+h^{2k}}}=:T_k<+\infty \tag{6.5}$$

Then for $\lambda>0$

$$\gamma_k\left((\varphi,Q\varphi)\geq\lambda\right)\lesssim e^{-\lambda/4T_k}. \tag{6.6}$$

Proof To begin with, we exploit the Markov inequality: For any $\mu>0$

$$\gamma_k\left((\varphi,Q\varphi)\geq\lambda\right)\leq e^{-\mu\lambda}\mathbb{E}e^{\mu(\varphi,Q\varphi)}. \tag{6.7}$$

Now we compute

$$\mathbb{E}e^{\mu(\varphi, Q\varphi)} = \int \prod_{|n| \leq N} \left(\frac{1 + n^{2k}}{\sqrt{2\pi}} d\varphi_n d\bar{\varphi}_n \right) \exp \left[-\frac{1}{2} \sum_{i,j} \bar{\varphi}_i \left((1 + j^{2k}) \delta_{ij} - 2\mu Q_{ij} \right) \varphi_j \right]$$

$$= \frac{1}{(2\pi)^N} \int d\varphi'_{-N} \ldots d\varphi'_N \bar{\varphi}'_{-N} \ldots d\bar{\varphi}'_N \exp \left[-\frac{1}{2} \sum_{i,j} \bar{\varphi}'_i \left(\delta_{ij} - 2\mu Q_{ij}(k) \right) \varphi'_j \right]$$

$$= e^{-\frac{1}{2} \ln \det(\mathbb{I} - 2\mu Q(k))}. \tag{6.8}$$

where we have performed the change of variables $\varphi'_j = \sqrt{1 + j^{2k}} \varphi_j$, $\bar{\varphi}'_j = \sqrt{1 + j^{2k}} \bar{\varphi}_j$ and we have introduced $Q_{ij}(k) := Q_{ij} / \sqrt{(1 + j^{2k})(1 + i^{2k})}$. We claim that

$$| \text{Tr}((Q_{ij}(k))^m)| \lesssim T_k^m, \quad m \in \mathbb{Z}_+, \tag{6.9}$$

so the expansion of the determinant

$$- \ln \det(\mathbb{I} - 2\mu Q(k)) = \sum_{m=1}^{+\infty} \frac{(2\mu)^m \text{Tr} \left((Q_{ij}(k))^m \right)}{m}$$

is convergent provided that $\mu < \frac{1}{2T_k}$. We choose $\mu = \frac{1}{4T_k}$, so that (6.7, 6.8) imply the desired inequality. It remains to show the (6.9).

$$\text{Tr} \left((Q(k))^m \right) = \sum_{i_1, \ldots, i_{m+1}} Q_{i_1 i_2}(k) \ldots Q_{i_m i_{m+1}}(k) \delta_{i_1 i_{m+1}}$$

$$= \sum_{i_1, \ldots, i_{m+1}} \frac{Q_{i_1 i_2} \ldots Q_{i_m i_{m+1}} \delta_{i_1 i_{m+1}}}{\sqrt{\left(1 + i_1^{2k}\right) \left(1 + i_2^{2k}\right) \left(1 + i_2^{2k}\right) \ldots \left(1 + i_m^{2k}\right) \left(1 + i_m^{2k}\right) \left(1 + i_{m+1}^{2k}\right)}}$$

$$\leq T_k^m \sum_{i_1, \ldots, i_{m+1}} \frac{\delta_{i_1, i_{m+1}}}{\sqrt{\left(1 + i_1^{2k}\right) \ldots \left(1 + i_m^{2k}\right)}}$$

$$= T_k^m \sum_{i_1, \ldots, i_m} \frac{1}{\sqrt{\left(1 + i_1^{2k}\right) \ldots \left(1 + i_m^{2k}\right)}}$$

$$= T_k^m \left(\sum_i \frac{1}{\sqrt{\left(1 + i^{2k}\right)}} \right)^m \lesssim T_k^m,$$

where we have used the assumption (6.5) in the first inequality and $k \geq 2$ in the last inequality. $\square$

Remark 6.6 We observe that we can make different assumptions on the matrix Q and obtain similar inequalities. For instance, if the trace norm of $Q(k)$ is finite uniformly

in N, we have (see, for instance, Lemma 3.3 in [35])

$$- \ln \det \left(\mathbb{I} - 2\mu Q(k) \right) \leq \| Q(k) \|_{\mathrm{Tr}}$$

and so for every N

$$\gamma_k \left((\varphi, Q\varphi) \geq \lambda \right) \lesssim e^{-\lambda / \| Q(k) \|_{\mathrm{Tr}}}, \tag{6.10}$$

by the same argument of the last proposition. If we assume the Hilbert-Schmidt norm of $Q(k)$ to be finite uniformly in N, we obtain the Hanson–Wright inequality (see [16] and more recently [32]), holding for any N

$$\gamma_k \left(\mathrm{Var}(\varphi, Q\varphi) \geq \lambda^2 \right) \lesssim e^{-c \min \left(\lambda^2 / \| Q(k) \|_{HS}, \lambda / \| Q(k) \| \right)}, \tag{6.11}$$

where $\| A \|$ denotes the operator norm of A and c is a positive constant.

Remark 6.7 For any linear operator $A\varphi := \sum_{n=1}^{N} a_i \varphi_i$, with

$$\mathcal{T}_k := \sum_{|i| \leq N} \frac{|a_i|^2}{(1 + i^{2k})} < \infty \quad \text{uniformly in } N,$$

by using $\gamma_k(|A\varphi| \geq \lambda) = \gamma_k(|A\varphi|^2 \geq \lambda^2)$ we can infer

$$\gamma_k \left(|A\varphi| \geq \lambda \right) \lesssim e^{-\lambda^2 / \mathcal{T}_k}. \tag{6.12}$$

Note that if $A\varphi = \varphi^{(s)}$ we have $\mathcal{T}_k < \infty$ uniformly in N for $s < k - \frac{1}{2}$. In this way we can improve Lemma 6.4, obtaining a sub-Gaussian decay.

Proposition 6.8 *Let $Q(x)$ be a $N \times N$ matrix as before. Moreover, we assume $Q(x)$ to be Hölder continuous w.r.t. $x \in \mathbb{T}$ with exponent α and constant L_N, i.e.,*

$$|(\varphi, Q(x)\varphi) - (\varphi, Q(y)\varphi)| \leq L_N |x - y|^{\alpha}, \quad \text{for every } \varphi \in \mathbb{R}^N. \tag{6.13}$$

Then for any $\varepsilon > 0$ and $\lambda \geq 2 L_N \varepsilon^{\alpha}$

$$\gamma_k \left(\sup_x (\varphi, Q\varphi) \geq \lambda \right) \lesssim \frac{e^{-\lambda / 4 T_k}}{\varepsilon}. \tag{6.14}$$

Proof We exploit Proposition 6.5 along with an ε-net argument. For $\varepsilon > 0$ we divide the interval $\mathbb{T}$ in $1/\varepsilon$ points at distance ε. We denote by x_j a point in the jth segment, and by x^* the point in which the maximum is attained. By Proposition 6.5 for each $x \in \mathbb{T}$ we obtain for $\lambda > 0$

$$\gamma_k \left((\varphi, Q(x)\varphi) \geq \lambda \right) \lesssim e^{-\lambda / 4 T_k}. \tag{6.15}$$

Let j_0 be such that $|x_{j_0} - x^*| \leq \varepsilon$. Therefore, it has to be

$$\left| (\varphi, Q\left(x^* \right) \varphi) - (\varphi, Q\left(x_{j_0} \right) \varphi) \right| \leq L_N \varepsilon^{\alpha}, \quad \text{for every } \varphi \in \mathbb{R}^N.$$

Then we use the union bound for the probabilities:

$$
\gamma_k \left(\left| Q \left(x^* \right) \right| \geq \lambda \right) \leq \sum_j \gamma_k \left(\left| Q \left(x^* \right) \right| \geq \lambda \,\Big|\, \left| x^* - x_j \right| \leq \varepsilon \right)
$$

$$
\leq \sum_j \gamma_k \left(\left| Q(x_j) \right| \geq \frac{\lambda}{2} \right)
$$

$$
+ \sum_j \gamma_k \left(\left| Q(x_j) - Q \left(x^* \right) \right| \geq \frac{\lambda}{2} \,\Big|\, \left| x^* - x_j \right| \leq \varepsilon \right).
$$

We immediately see by (6.13) that the second addendum in the last formula is zero as soon as $\lambda \geq 2L_N \varepsilon^\alpha$. Therefore, we bound the first addendum by the total number of terms in the sum, which is ε^{-1}, times the estimate (6.15), so obtaining (6.14). □

References

1. Barakat, A., De Sole, A., Kac, V.G.: Poisson vertex algebras in the theory of Hamiltonian equations. Jpn. J. Math. **4**, 141–252 (2009)
2. Bogachev, V.I.: Gaussian measures. In: Mathematical Surveys and Monographs, vol. 62. American Mathematical Society, Providence, RI (1998)
3. Bourgain, J.: Periodic nonlinear Schrödinger equation and invariant measures. Commun. Math. Phys. **166**(1), 1–26 (1994)
4. Bourgain, J.: Invariant measures for the 2D-defocusing nonlinear Schrödinger equation. Commun. Math. Phys. **176**, 421–445 (1996)
5. Bourgain, J.: Invariant measures for the Gross–Piatevskii equation. J. Math. Pures Appl. **76**, 649–702 (1997)
6. Bourgain, J.: Invariant measures for NLS in infinite volume. Commun. Math. Phys. **210**, 605–620 (2000)
7. Brydges, D.C., Slade, G.: Statistical mechanics of the 2-dimensional focusing nonlinear Schrödinger equation. Commun. Math. Phys. **182**, 485–504 (1996)
8. Burq, N., Thomann, L., Tzvetkov, N.: Remarks on the Gibbs Measures for Nonlinear Dispersive Equations. arXiv:1412.7499 [math.AP]
9. Caianiello, E.R.: Combinatorics and Renormalization in Quantum Field Theory, vol. 111. W.A. Benjamin, Reading (1973)
10. Carpentier, S., De Sole, A., Kac, V.G.: Rational matrix pseudodifferential operators. Sel. Math. **20**(2), 403–419 (2014)
11. Deng, Y.: Invariance of the Gibbs measure for the Benjamin-Ono equation. J. Eur. Math. Soc. (JEMS) **17**(15), 1107–1198 (2015)
12. Deng, Y., Tzvetkov, N., Visciglia, N.: Invariant measures and long time behaviour for the Benjamin–Ono equation III. Commun. Math. Phys. **339**(3), 815–857 (2015)
13. De Sole, A., Kac, V.G.: Non-local Poisson structures and applications to the theory of integrable systems. Jpn. J. Math. **8**(2), 233–347 (2013)
14. Grünrock, A., Herr, S.: Low regularity local well-posedness of the derivative nonlinear Schrödinger equation with periodic initial data. SIAM J. Math. Anal. **39**, 1890–1920 (2008)
15. Guerra, F., Rosen, L., Simon, B.: The $P(\Phi)_2$ Euclidean quantum field theory as classical statistical mechanics. Ann. Math. **101**, 111–189 (1975)
16. Hanson, D.L., Wright, F.T.: A bound on tail probabilities for quadratic forms in independent random variables. Ann. Math. Stat. **42**, 1079–1083 (1971)
17. Hayashi, N., Ozawa, T.: On the derivative nonlinear Schringer equation. Phys. D **55**(1–2), 14–36 (1992)
18. Hayashi, N., Ozawa, T.: Finite energy solutions of nonlinear Schringer equations of derivative type. SIAM J. Math. Anal. **25**(6), 1488–1503 (1994)

19. Herr, S.: On the Cauchy problem for the derivative nonlinear Schrödinger equation with periodic boundary condition. Int. Math. Res. Not. **2006**, 1–33 (2006)
20. Kaup, D.J., Newell, A.C.: An exact solution for a derivative nonlinear Schrödinger equation. J. Math. Phys. **19**(4), 798–801 (1978)
21. Lebowitz, J., Rose, R., Speer, E.: Statistical mechanics of the nonlinear Schrödinger equation. J. Stat. Phys. **50**, 657–687 (1988)
22. Magri, F.: A simple model of the integrable Hamiltonian equation. J. Math. Phys. **19**(5), 1156–1162 (1978)
23. Mjølhus, E.: On the modulation instability of hydromagnetic waves parallel to the magnetic field. J. Plasma Phys. **19**(17), 321–334 (1976)
24. Miao, C., Wu, Y., Xu, G.: Global well-posedness for Schrödinger equation with derivative. J. Differ. Equ. **251**, 2164–2195 (2011)
25. McKean, H., Vaninski, K.: Statistical mechanics of nonlinear wave equations and Brownian motion with restoring drift. Commun. Math. Phys. **160**, 615–630 (1994)
26. McKean, H.P.: Statistical mechanics of nonlinear wave equations (4): cubic Schrödinger. Commun. Math. Phys. **168**, 479–491 (1995)
27. Mosincat, R., Oh, T.: A Remark on Global Well-Posedness of the Derivative Nonlinear Schrödinger Equation on the Circle. C.R. Math. Acad. Sci. Paris **353**(9), 837–841 (2015)
28. Nahmod, A.R., Oh, T., Rey-Bellet, L., Staffilani, G.: Invariant weighted Wiener measures and almost sure global well-posedness for the periodic derivative NLS. J. Eur. Math. Soc. (JEMS) **14**(4), 1275–1330 (2012)
29. Nahmod, A., Rey-Bellet, L., Sheffield, S., Staffilani, G.: Absolute continuity of Brownian bridges under certain gauge transformations. Math. Res. Lett. **18**, 875 (2011)
30. Nelson, E.: The free Markoff field. J. Funct. Anal. **12**, 211–227 (1973)
31. Rogister, A.: Parallel propagation of nonlinear low-frequency waves in high-β plasma. Phys. Fluids **14**, 2733–2739 (1971)
32. Rudelson, M., Vershynin, R.: Hanson–Wright inequality and sub-Gaussian concentration. Electron. Commun. Probab. **18**, 1–9 (2013)
33. Skorohod, A.V.: Integration in Hilbert space. In: Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 79. Springer-Verlag Berlin, Heidelberg, New York (1974)
34. Simon, B.: The $P(\Phi)_2$ Euclidean (quantum) field theory. In: Princeton Series in Physics. Princeton University Press, Princeton, NJ (1974)
35. Simon, B.: Trace ideals and their applications. In: Mathematical Surveys and Monograph, 2nd edn., vol. 120. American Mathematical Society, Providence, RI (2005)
36. Sulem, C., Sulem, P.L.: The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse. Applied Mathematical Sciences, vol. 139. Springer-Verlag, New-York (1999)
37. Takaoka, H.: Well-posedness for the one-dimensional nonlinear Schrödinger equation with the derivative nonlinearity. Adv. Differ. Equ. **4**, 561–580 (1999)
38. Thomann, L., Tzvetkov, N.: Gibbs measure for the periodic derivative nonlinear Schrödinger equation. Nonlinearity **23**(11), 2771–2791 (2010)
39. Tzvetkov, N.: Construction of a Gibbs measure associated to the periodic Benjamin–Ono equation. Probab. Theory Rel. Fields **146**(3–4), 481 (2010)
40. Tzvetkov, N., Visciglia, N.: Gaussian measures associated to the higher order conservation laws of the Benjamin–Ono equation. Ann. Sci. Norm. Super. **46**, 249 (2013)
41. Tzvetkov, N., Visciglia, N.: Invariant measures and long time behaviour for the Benjamin–Ono equation. Int. Math. Res. Not. **17**, 4679 (2014)
42. Tzvetkov, N., Visciglia, N.: Invariant measures and long time behaviour for the Benjamin–Ono equation II. J. Math. Pures Appl. **103**, 102 (2014)
43. Weinstein, M.I.: Nonlinear Schrödinger equations and sharp interpolation estimates. Commun. Math. Phys. **87**(4), 567–576 (1982/1983)
44. Win, Y.Y.S.: Global well-posedness of the derivative nonlinear Schrödinger equations on T. Funkc. Ekvacioj **53**, 51–88 (2010)
45. Wu, Y.: Global well-posedness for the nonlinear Schrödinger equation with derivative in energy space. Anal. PDE **6**(8), 1989–2002 (2013)
46. Wu, Y.: Global Well-Posedness for the Nonlinear Schrödinger Equation Revisited. arXiv:1404.5159
47. Zhidkov, P.: KdV and Nonlinear Schrödinger Equations: Qualitative Theory. Lecture Notes in Mathematics, vol. 1756. Springer, Berlin (2001)