

MasterPVA and WAlg: Mathematica packages for Poisson vertex algebras and classical affine \mathcal{W} -algebras

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Abstract We give an introduction to the Mathematica packages *MasterPVA* and *MasterPVAmulti* used to compute λ -brackets in Poisson vertex algebras, which play an important role in the theory of infinite-dimensional Hamiltonian systems. As an application, we give an introduction to the Mathematica package *WAlg* aimed to compute the λ -brackets among the generators of classical affine \mathcal{W} -algebras. The use of these packages is shown by providing some explicit examples.

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1 Introduction

Poisson vertex algebras (PVA) arise as the quasi-classical limit of a family of vertex algebras [14] in the same way as Poisson algebras arise as the quasi-classical limit of a family of associative algebras.

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Note also that a PVA is a local counterpart of a Coisson (= chiral Poisson) algebra defined in [3]. Moreover, a PVA can be obtained as a formal Fourier transform of a local Poisson bracket [1], which plays an important role in the theory of infinite-dimensional integrable Hamiltonian systems. In fact, as demonstrated in [1], the language of λ -brackets [11, 34] in the framework of Poisson vertex algebras is often more convenient and transparent than the equivalent languages of local Poisson brackets, used in the book [30], or of Hamiltonian operators, used in the book [26] (and references therein).

Hence, the theory of PVA has been extensively used in order to get a better understanding of generalized Drinfeld–Sokolov hierarchies for classical affine \mathcal{W} -algebras [18, 19, 24], Adler–Gelfand–Dickey hierarchies [21] and, more generally, Lax type integrable Hamiltonian equations [23], and the Lenard–Magri scheme of integrability [16, 17]. Furthermore, the notion of a PVA has been extended in [5, 7] to deal with Hamiltonian operators, or, equivalently, local Poisson brackets, for multidimensional systems of PDEs (namely, PDEs for functions depending on several spatial variables). The notion of multidimensional PVA has been used for studying the theory of symmetries and deformations of the so-called Poisson brackets of hydrodynamic type [28], as well as for the local nonlinear brackets associated with 2D Euler’s equation [6].

One of the most remarkable accomplishments of the theory of PVA has been the derivation of an explicit formula for the bi-Hamiltonian structure underlying classical \mathcal{W} -algebras [22]. Classical affine \mathcal{W} -algebras are associated to a pair (\mathfrak{g}, f) consisting of a simple Lie algebra \mathfrak{g} and a nilpotent element $f \in \mathfrak{g}$. For a principal nilpotent element $f \in \mathfrak{g}$, they appeared in the seminal paper by Drinfeld and Sokolov [27]. They were introduced as Poisson algebras of functions on an infinite dimensional Poisson manifold, and they were used to study KdV-type integrable bi-Hamiltonian hierarchies of PDE’s, nowadays known as Drinfeld–Sokolov hierarchies. Subsequently, in the 1990s, there was an extensive literature extending the Drinfeld–Sokolov construction of classical \mathcal{W} -algebras and the corresponding generalized Drinfeld–Sokolov hierarchies to other nilpotent elements [4, 12, 13, 31–33]. Recently [18], classical affine \mathcal{W} -algebras were described as PVAs. The powerful tool of the language of λ -bracket has been then used to get an explicit formula for the bi-Hamiltonian structure describing them and to give a rigorous definition, and to compute explicitly, generalized Drinfeld–Sokolov hierarchies [19, 24]. These results may find interesting applications in studying the relations of Drinfeld–Sokolov hierarchies with Kac–Wakimoto hierarchies [35] and computation of the corresponding tau-functions, and in the problem of quantization of classical integrable systems [2] and applications to CFT.

The most powerful tool in the PVA theory is the so-called Master Formula (2.8). It allows to rephrase the relevant questions in the theory of infinite-dimensional Hamiltonian systems in terms of the λ -bracket language, thus providing a completely algebraic computational technique, which replaces all the manipulations used in [26, 30] in the setting of the formal calculus of variations.

Note that, in the λ -bracket language the computations are not necessarily hard to perform by hand, but their numbers increase dramatically with the growing number of spatial dimensions.

The package *MasterPVA* and its generalization to the multidimensional case *MasterPVAmulti* have been written to exploit a Computer Algebra System, like Mathematica, to automatically compute the Master Formula for PVA. The choice of Mathematica is motivated by the pre-existing package *Lambda*, by Ekstrand [29], aimed to compute operator product expansions in conformal field theory using the λ -bracket language within the framework of vertex algebras [14, 34].

These packages have been used in [7]—with some preliminary results published in [5]—in order to compute up to second dispersive order the Casimir functions, the symmetries and the compatible deformations of the bidimensional Poisson brackets of hydrodynamic type. They have been proved effective also when working with scalar structures, whenever explicit computations are needed [9].

The Mathematica package *WAlg* provides the implementation of the results about the structure theory of classical affine \mathcal{W} -algebras obtained in [22]. It can be used to compute all λ -brackets between generators of the classical affine \mathcal{W} -algebras $\mathcal{W}(\mathfrak{g}, f)$, where \mathfrak{g} is a simple Lie algebra of type A, B, C, D and G , and $f \in \mathfrak{g}$ is an arbitrary nilpotent element. Thus we can obtain explicit expressions for the generalized Drinfeld–Sokolov hierarchies and their bi-Hamiltonian structure by combining the programs *WAlg* and *MasterPVA*.

The paper is organized as follows.

In Sect. 2 we review the definition of PVA following [1] and its multidimensional generalization given in [5]. In particular we introduce the notion of an algebra of differential functions and the Master Formula (2.8) used to perform λ -brackets computations on it, and we show that a PVA is equivalent to the notion of an Hamiltonian operator (differently from [26] we call this Hamiltonian operator a Poisson structure). We also recall the connection with infinite-dimensional Hamiltonian systems.

In Sect. 3 we review the definition of classical affine \mathcal{W} -algebras using the language of PVA, following [18]. The main results are Theorems 3.4 and 3.5 which give an explicit description of, respectively, the differential algebra structure and the Poisson structure of classical affine \mathcal{W} -algebras, see also [22].

In Sect. 4 we explain how to use the packages *MasterPVA* and *MasterPVAmulti* by giving some explicit examples. We show the well-known compatibility between GFZ and Virasoro–Magri PVA, we derive the Dubrovin–Novikov conditions for a bidimensional Poisson structure of hydrodynamic type [28], and, finally, we reprove the Mokhov’s classification for the $N = 1$ multidimensional structures of hydrodynamic type [37].

In Sect. 5 we explain how to use the package *WAlg* by giving some explicit examples. First, we consider the case of a principal nilpotent element f in the Lie algebra $\mathfrak{g} = \mathfrak{o}_7$ and we show how to compute a basis $\{q_j\}_{j \in J}$ of \mathfrak{g}^f and the corresponding set of generators $\{w(q_j)\}_{j \in J}$ of the classical affine \mathcal{W} -algebra $\mathcal{W}(\mathfrak{g}, f)$ given by Theorem 3.4. Then, we consider the case of a minimal nilpotent element in the Lie algebra \mathfrak{sp}_4 and we show how to compute the λ -brackets among the generators of the corresponding classical affine \mathcal{W} -algebra using Theorem 3.5. Finally, we compute explicitly all classical affine \mathcal{W} -algebras $\mathcal{W}(\mathfrak{g}, f)$ corresponding to a simple Lie algebra \mathfrak{g} of rank 2 and its principal nilpotent element f and we compare our results with the ones in [25].

The complete list of commands provided by the packages *MasterPVA* and *MasterPVA-multi* (respectively *WAlg*) is given in Sect. 6.1 (respectively Sect. 6.2).

Acquiring the packages The packages have been developed with Mathematica 9.0 and can be downloaded from http://www.theatron.it/daniele/MasterPVA_files.tar.gz where it is also possible to find the related libraries and the examples provided in this paper.

2 Poisson vertex algebras and Hamiltonian equations

In this section we review the connection between Poisson vertex algebras and the theory of Hamiltonian equations as laid down in [1].

2.1 Poisson vertex algebras

Let \mathcal{V} be a *differential algebra*, namely a unital commutative associative algebra over a field \mathbb{F} of characteristic 0, with a derivation $\partial: \mathcal{V} \rightarrow \mathcal{V}$.

Definition 2.1 (a) A λ -*bracket* on \mathcal{V} is an \mathbb{F} -linear map $\mathcal{V} \otimes \mathcal{V} \rightarrow \mathbb{F}[\lambda] \otimes \mathcal{V}$, denoted by $f \otimes g \rightarrow \{f_\lambda g\}$, satisfying *sesquilinearity* ($f, g \in \mathcal{V}$):

$$\{\partial f_\lambda g\} = -\lambda\{f_\lambda g\}, \quad \{f_\lambda \partial g\} = (\lambda + \partial)\{f_\lambda g\}, \tag{2.1}$$

and the *left and right Leibniz rules* ($f, g, h \in \mathcal{V}$):

$$\{f_\lambda gh\} = \{f_\lambda g\}h + \{f_\lambda h\}g, \tag{2.2}$$

$$\{f h_\lambda g\} = \{f_{\lambda+\partial} g\} \rightarrow h + \{h_{\lambda+\partial} g\} \rightarrow f, \tag{2.3}$$

where we use the following notation: if $\{f_\lambda g\} = \sum_{n \in \mathbb{Z}_+} \lambda^n c_n$, then $\{f_{\lambda+\partial} g\} \rightarrow h = \sum_{n \in \mathbb{Z}_+} c_n (\lambda + \partial)^n h$.

(b) We say that the λ -bracket is *skew-symmetric* if

$$\{g_\lambda f\} + \{f_{-\lambda-\partial} g\} = 0, \tag{2.4}$$

where, now, $\{f_{-\lambda-\partial} g\} = \sum_{n \in \mathbb{Z}_+} (-\lambda - \partial)^n c_n$ (if there is no arrow we move ∂ to the left).

(c) A *Poisson vertex algebra* (PVA) is a differential algebra \mathcal{V} endowed with a λ -bracket which is skew-symmetric and satisfies the following *Jacobi identity* in $\mathcal{V}[\lambda, \mu]$ ($f, g, h \in \mathcal{V}$):

$$\{f_\lambda \{g_\mu h\}\} - \{\{f_\lambda g\}_{\lambda+\mu} h\} - \{g_\mu \{f_\lambda h\}\} = 0. \tag{2.5}$$

Example 2.2 Let \mathfrak{g} be a Lie algebra over \mathbb{F} with a symmetric invariant bilinear form κ , and let s be an element of \mathfrak{g} . The *affine PVA* associated to the triple $(\mathfrak{g}, \kappa, s)$, is the algebra of differential polynomials $\mathcal{V} = S(\mathbb{F}[\partial]\mathfrak{g})$ (where $\mathbb{F}[\partial]\mathfrak{g}$ is the free $\mathbb{F}[\partial]$ -module generated by \mathfrak{g} and $S(R)$ denotes the symmetric algebra over the \mathbb{F} -vector space R) together with the λ -bracket given by

$$\{a_\lambda b\} = [a, b] + \kappa(s)[a, b] + \kappa(a|b)\lambda \quad \text{for } a, b \in \mathfrak{g}, \tag{2.6}$$

and extended to \mathcal{V} by sesquilinearity and the left and right Leibniz rules.

2.2 Poisson vertex algebra structures on algebras of differential functions

The basic examples of differential algebras are the *algebras of differential polynomials* in the variables u_1, \dots, u_ℓ :

$$\mathcal{R}_\ell = \mathbb{F}[u_i^{(n)} \mid i \in I = \{1, \dots, \ell\}, n \in \mathbb{Z}_+],$$

where ∂ is the derivation defined by $\partial(u_i^{(n)}) = u_i^{(n+1)}$, $i \in I, n \in \mathbb{Z}_+$. Note that we have in \mathcal{V} the following commutation relations:

$$\left[\frac{\partial}{\partial u_i^{(n)}}, \partial \right] = \frac{\partial}{\partial u_i^{(n-1)}}, \tag{2.7}$$

where the RHS is considered to be zero if $n = 0$.

An algebra of differential functions in the variables u_1, \dots, u_ℓ is a differential algebra extension \mathcal{V} of \mathcal{R}_ℓ , endowed with commuting derivations

$$\frac{\partial}{\partial u_i^{(n)}} : \mathcal{V} \rightarrow \mathcal{V}, \quad i \in I, n \in \mathbb{Z}_+,$$

extending the usual partial derivatives on \mathcal{R}_ℓ , such that only a finite number of $\frac{\partial f}{\partial u_i^{(n)}}$ are non-zero for each $f \in \mathcal{V}$, and such that the commutation relations (2.7) hold on \mathcal{V} .

The variational derivative of $f \in \mathcal{V}$ with respect to u_i is, by definition,

$$\frac{\delta f}{\delta u_i} = \sum_{n \in \mathbb{Z}_+} (-\partial)^n \frac{\partial f}{\partial u_i^{(n)}}.$$

The following result explains how to extend an arbitrary λ -bracket on a set of variables $\{u_i\}_{i \in I}$, with value in some algebra of differential functions \mathcal{V} , to a PVA structure on \mathcal{V} .

Theorem 2.3 [1, Theorem 1.15] *Let \mathcal{V} be an algebra of differential functions in the variables $\{u_i\}_{i \in I}$, and let $H_{ij}(\lambda) \in \mathbb{F}[\lambda] \otimes \mathcal{V}$, $i, j \in I$.*

(a) *The Master Formula*

$$\{f_\lambda g\} = \sum_{\substack{i, j \in I \\ m, n \in \mathbb{Z}_+}} \frac{\partial g}{\partial u_j^{(n)}} (\lambda + \partial)^n H_{ji}(\lambda + \partial) (-\lambda - \partial)^m \frac{\partial f}{\partial u_i^{(m)}} \tag{2.8}$$

defines a λ -bracket on \mathcal{V} with given $\{u_i \lambda u_j\} = H_{ji}(\lambda)$, $i, j \in I$.

(b) *The λ -bracket (2.8) on \mathcal{V} satisfies the skew-symmetry condition (2.4) provided that the same holds on generators ($i, j \in I$):*

$$\{u_i \lambda u_j\} + \{u_j -\lambda -\partial u_i\} = 0. \tag{2.9}$$

(c) *Assuming that the skew-symmetry condition (2.9) holds, the λ -bracket (2.8) satisfies the Jacobi identity (2.5), thus making \mathcal{V} a PVA, provided that the Jacobi identity holds on any triple of generators ($i, j, k \in I$):*

$$\{u_i \lambda \{u_j \mu u_k\}\} - \{u_j \mu \{u_i \lambda u_k\}\} - \{\{u_i \lambda u_j\} \lambda + \mu u_k\} = 0. \tag{2.10}$$

By Theorem 2.3(a), if \mathcal{V} is an algebra of differential functions in the variables $\{u_i\}_{i \in I}$, there is a bijective correspondence between $\ell \times \ell$ -matrices $H(\lambda) = (H_{ij}(\lambda))_{i, j \in I} \in \text{Mat}_{\ell \times \ell} \mathcal{V}[\lambda]$ and the λ -brackets $\{\cdot \lambda \cdot\}_H$ on \mathcal{V} defined by the Master Formula (2.8).

Definition 2.4 A Poisson structure on \mathcal{V} is a matrix $H \in \text{Mat}_{\ell \times \ell} \mathcal{V}[\lambda]$ such that the corresponding λ -bracket $\{\cdot \lambda \cdot\}_H$ defines a PVA structure on \mathcal{V} .

Example 2.5 Consider the affine PVA defined in Example 2.2. Let $\{u_i\}_{i \in I}$ be a basis of \mathfrak{g} . The corresponding Poisson structure $H = (H_{ij}(\lambda)) \in \text{Mat}_{\ell \times \ell} \mathcal{V}[\lambda]$ to the λ -bracket defined in (2.6) is given by

$$H_{ij}(\lambda) = \{u_j \lambda u_i\} = [u_j, u_i] + \kappa(s[[u_j, u_i]] + \kappa(u_i|u_j)\lambda).$$

2.3 Poisson structures and Hamiltonian equations

The relation between PVAs and Hamiltonian equations associated to a Poisson structure is based on the following simple observation.

Proposition 2.6 *Let \mathcal{V} be a PVA. The 0-th product on \mathcal{V} induces a well defined Lie algebra bracket on the quotient space $\mathcal{V}/\partial\mathcal{V}$:*

$$\{ \int f, \int g \} = \int \{ f_\lambda g \} |_{\lambda=0}, \tag{2.11}$$

where $\int : \mathcal{V} \rightarrow \mathcal{V}/\partial\mathcal{V}$ is the canonical quotient map. Moreover, we have a well defined Lie algebra action of $\mathcal{V}/\partial\mathcal{V}$ on \mathcal{V} by derivations of the commutative associative product on \mathcal{V} , commuting with ∂ , given by

$$\{ \int f, g \} = \{ f_\lambda g \} |_{\lambda=0}.$$

In the special case when \mathcal{V} is an algebra of differential functions in ℓ variables $\{u_i\}_{i \in I}$ and the PVA λ -bracket on \mathcal{V} is associated to the Poisson structure $H \in \text{Mat}_{\ell \times \ell} \mathcal{V}[\lambda]$, the Lie bracket (2.11) on $\mathcal{V}/\partial\mathcal{V}$ takes the form (cf. (2.8)):

$$\{ \int f, \int g \} = \sum_{i,j \in I} \int \frac{\delta g}{\delta u_j} H_{ji}(\partial) \frac{\delta f}{\delta u_i}. \tag{2.12}$$

Definition 2.7 Let \mathcal{V} be an algebra of differential functions with a Poisson structure H .

- (a) Elements of $\mathcal{V}/\partial\mathcal{V}$ are called *local functionals*.
- (b) Given a local functional $\int h \in \mathcal{V}/\partial\mathcal{V}$, the corresponding *Hamiltonian equation* is

$$\frac{du}{dt} = \{ \int h, u \}_H \quad \left(\text{equivalently, } \frac{du_i}{dt} = \sum_{j \in I} H_{ij}(\partial) \frac{\delta h}{\delta u_j}, i \in I \right). \tag{2.13}$$

- (c) A local functional $\int f \in \mathcal{V}/\partial\mathcal{V}$ is called an *integral of motion* of Eq. (2.13) if $\frac{df}{dt} = 0 \pmod{\partial\mathcal{V}}$ in virtue of (2.13), or, equivalently, if $\int h$ and $\int f$ are in *involution*:

$$\{ \int h, \int f \}_H = 0.$$

Namely, $\int f$ lies in the centralizer of $\int h$ in the Lie algebra $\mathcal{V}/\partial\mathcal{V}$ with Lie bracket (2.12).

- (d) Equation (2.13) is called *integrable* if there exists an infinite sequence $\int f_0 = \int h, \int f_1, \int f_2, \dots$, of linearly independent integrals of motion in involution. The corresponding *integrable hierarchy of Hamiltonian equations* is

$$\frac{du}{dt_n} = \{ \int f_n, u \}_H, \quad n \in \mathbb{Z}_+ \tag{2.14}$$

(equivalently, $\frac{du_i}{dt_n} = \sum_{j \in I} H_{ij}(\partial) \frac{\delta f_n}{\delta u_j}, n \in \mathbb{Z}_+, i \in I$).

2.4 Multidimensional Poisson vertex algebras

The definition of a PVA has been extended in [5] in order to study Hamiltonian evolutionary PDEs with several spatial dimensions.

A *D-dimensional differential algebra* is a unital commutative associative algebra \mathcal{V} over a field \mathbb{F} of characteristic 0, endowed with D commuting derivations $\partial_\alpha : \mathcal{V} \rightarrow \mathcal{V}, \alpha = 1 \dots, D$.

Definition 2.8 (a) A D -dimensional λ -bracket on \mathcal{V} is an \mathbb{F} -linear map $\mathcal{V} \otimes \mathcal{V} \rightarrow \mathbb{F}[\lambda_1, \dots, \lambda_D] \otimes \mathcal{V}$, denoted by $f \otimes g \rightarrow \{f_\lambda g\}$, satisfying *sesquilinearity* ($f, g \in \mathcal{V}$, $\alpha = 1, \dots, D$):

$$\{\partial_\alpha f_\lambda g\} = -\lambda_\alpha \{f_\lambda g\}, \quad \{f_\lambda \partial_\alpha g\} = (\lambda_\alpha + \partial_\alpha) \{f_\lambda g\}, \tag{2.15}$$

and the *left and right Leibniz rules* (2.2) and (2.3).

(b) We say that the λ -bracket is *skew-symmetric* if Eq. (2.4) is satisfied.

(c) A D -dimensional PVA is a D -dimensional differential algebra \mathcal{V} endowed with a λ -bracket which is skew-symmetric and satisfies the *Jacobi identity* (2.5) in $\mathcal{V}[\lambda_1, \dots, \lambda_D, \mu_1, \dots, \mu_D]$.

The definition of an algebra of differential functions given in Sect. 2.2 can be generalized to the D -dimensional case and the analogous result to Theorem 2.3 can be obtained (see [5, Theorem 1]).

3 Classical affine \mathcal{W} -algebras

In this section we recall the definition of classical affine \mathcal{W} -algebras $\mathcal{W}(\mathfrak{g}, f)$ in the language of Poisson vertex algebras, following [18] (which is a development of [27]).

3.1 Setup and notation

Let \mathfrak{g} be a simple Lie algebra with a non-degenerate symmetric invariant bilinear form $(\cdot | \cdot)$, and let $\{f, 2x, e\} \subset \mathfrak{g}$ be an \mathfrak{sl}_2 -triple in \mathfrak{g} . We have the corresponding $\text{ad } x$ -eigenspace decomposition

$$\mathfrak{g} = \bigoplus_{k \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_k \quad \text{where } \mathfrak{g}_k = \{a \in \mathfrak{g} \mid [x, a] = ka\}.$$

Clearly, $f \in \mathfrak{g}_{-1}$, $x \in \mathfrak{g}_0$ and $e \in \mathfrak{g}_1$. We let d be the *depth* of the grading, i.e. the maximal eigenvalue of $\text{ad } x$.

By representation theory of \mathfrak{sl}_2 , the Lie algebra \mathfrak{g} admits the direct sum decompositions

$$\mathfrak{g} = \mathfrak{g}^f \oplus [e, \mathfrak{g}] = \mathfrak{g}^e \oplus [f, \mathfrak{g}]. \tag{3.1}$$

They are dual to each other, in the sense that $\mathfrak{g}^f \perp [f, \mathfrak{g}]$ and $[e, \mathfrak{g}] \perp \mathfrak{g}^e$. For $a \in \mathfrak{g}$, we denote by $a^\sharp = \pi_{\mathfrak{g}^f}(a) \in \mathfrak{g}^f$ its component in \mathfrak{g}^f with respect to the first decomposition in (3.1). Note that, since $[e, \mathfrak{g}]$ is orthogonal to \mathfrak{g}^e , the spaces \mathfrak{g}^f and \mathfrak{g}^e are non-degenerately paired by $(\cdot | \cdot)$.

Next, we choose a basis of \mathfrak{g} as follows. Let $\{q_j\}_{j \in J^f}$ be a basis of \mathfrak{g}^f consisting of $\text{ad } x$ -eigenvectors, and let $\{q^j\}_{j \in J^f}$ be the dual basis of \mathfrak{g}^e . For $j \in J^f$, we let $\delta(j) \in \frac{1}{2}\mathbb{Z}$ be the $\text{ad } x$ -eigenvalue of q^j , so that

$$[x, q_j] = -\delta(j)q_j, \quad [x, q^j] = \delta(j)q^j. \tag{3.2}$$

For $k \in \frac{1}{2}\mathbb{Z}_+$ we also let $J_{-k}^f = \{i \in J^f \mid \delta(i) = k\} \subset J^f$, so that $\{q_j\}_{j \in J_{-k}^f}$ is a basis of \mathfrak{g}_{-k}^f , and $\{q^j\}_{j \in J_{-k}^f}$ is the dual basis of \mathfrak{g}_k^e . By representation theory of \mathfrak{sl}_2 , we get a basis of \mathfrak{g} consisting of the following elements:

$$q_n^j = (\text{ad } f)^n q^j \quad \text{where } n \in \{0, \dots, 2\delta(j)\}, \quad j \in J^f. \tag{3.3}$$

This basis consists of ad x -eigenvectors, and, for $k \in \frac{1}{2}\mathbb{Z}$ such that $-d \leq k \leq d$, the corresponding basis of $\mathfrak{g}_k \subset \mathfrak{g}$ is $\{q_n^j\}_{(j,n) \in J_{-k}}$, where J_{-k} is the following index set

$$J_{-k} = \{(j, n) \in J^f \times \mathbb{Z}_+ \mid \delta(j) - |k| \in \mathbb{Z}_+, n = \delta(j) - k\}. \tag{3.4}$$

The union of all these index sets is the index set for the basis of \mathfrak{g} :

$$J = \bigsqcup_{h \in \frac{1}{2}\mathbb{Z}} J_h = \{(j, n) \mid j \in J^f, n \in \{0, \dots, 2\delta(j)\}\}. \tag{3.5}$$

By [22, Lemma 2.5], the corresponding basis of \mathfrak{g} dual to (3.3) is given by $((j, n) \in J)$:

$$q_j^n = \frac{(-1)^n}{(n!)^2 \binom{2\delta(j)}{n}} (\text{ad } e)^n q_j. \tag{3.6}$$

Clearly, the bases (3.3) and (3.6) are compatible with the direct sum decompositions (3.1). In fact, we can write the corresponding projections $\pi_{\mathfrak{g}^f}, \pi_{[e, \mathfrak{g}]} = 1 - \pi_{\mathfrak{g}^f}, \pi_{\mathfrak{g}^e}$, and $\pi_{[f, \mathfrak{g}]} = 1 - \pi_{\mathfrak{g}^e}$, in terms of these bases:

$$\begin{aligned} a^\sharp &= \pi_{\mathfrak{g}^f}(a) = \sum_{j \in J^f} (a|q^j)q_j, & \pi_{[e, \mathfrak{g}]}(a) &= \sum_{j \in J^f} \sum_{n=1}^{2\delta(j)} (a|q_n^j)q_j^n, \\ \pi_{\mathfrak{g}^e}(a) &= \sum_{j \in J^f} (a|q_j)q^j, & \pi_{[f, \mathfrak{g}]}(a) &= \sum_{j \in J^f} \sum_{n=1}^{2\delta(j)} (a|q_n^j)q_n^j. \end{aligned} \tag{3.7}$$

Note that when $\delta(j) = 0$, then the sums over n in (3.7) become empty sums.

3.2 Construction of the classical affine \mathcal{W} -algebra

Recall from Example 2.2 that given an element $s \in \mathfrak{g}$, we have a PVA structure on the algebra of differential polynomials $\mathcal{V}(\mathfrak{g}) = S(\mathbb{F}[\partial]\mathfrak{g})$, with λ -bracket given on generators by

$$\{a_\lambda b\}_z = [a, b] + (a|b)\lambda + z(s|[a, b]), \quad a, b \in \mathfrak{g}, \tag{3.8}$$

and extended to $\mathcal{V}(\mathfrak{g})$ by the sesquilinearity axioms and the Leibniz rules. Here z is an element of the field \mathbb{F} .

We shall assume that s lies in \mathfrak{g}_d . In this case the $\mathbb{F}[\partial]$ -submodule $\mathbb{F}[\partial]\mathfrak{g}_{\geq \frac{1}{2}} \subset \mathcal{V}(\mathfrak{g})$ is a Lie conformal subalgebra (see [34] for the definition) with the λ -bracket $\{a_\lambda b\}_z = [a, b], a, b \in \mathfrak{g}_{\geq \frac{1}{2}}$ (it is independent of z , since s commutes with $\mathfrak{g}_{\geq \frac{1}{2}}$). Consider the differential subalgebra $\mathcal{V}(\mathfrak{g}_{\leq \frac{1}{2}}) = S(\mathbb{F}[\partial]\mathfrak{g}_{\leq \frac{1}{2}})$ of $\mathcal{V}(\mathfrak{g})$, and denote by $\rho: \mathcal{V}(\mathfrak{g}) \rightarrow \mathcal{V}(\mathfrak{g}_{\leq \frac{1}{2}})$, the differential algebra homomorphism defined on generators by

$$\rho(a) = \pi_{\leq \frac{1}{2}}(a) + (f|a), \quad a \in \mathfrak{g}, \tag{3.9}$$

where $\pi_{\leq \frac{1}{2}}: \mathfrak{g} \rightarrow \mathfrak{g}_{\leq \frac{1}{2}}$ denotes the projection with kernel $\mathfrak{g}_{\geq 1}$. Recall from [18] that we have a representation of the Lie conformal algebra $\mathbb{F}[\partial]\mathfrak{g}_{\geq \frac{1}{2}}$ on the differential subalgebra $\mathcal{V}(\mathfrak{g}_{\leq \frac{1}{2}}) \subset \mathcal{V}(\mathfrak{g})$ given by $(a \in \mathfrak{g}_{\geq \frac{1}{2}}, g \in \mathcal{V}(\mathfrak{g}_{\leq \frac{1}{2}}))$:

$$a \overset{\rho}{\underset{\lambda}{|}} g = \rho\{a_\lambda g\}_z \tag{3.10}$$

(note that the RHS is independent of z since, by assumption, $s \in Z(\mathfrak{g}_{\geq \frac{1}{2}})$).

The classical affine \mathcal{W} -algebra is, by definition, the differential algebra

$$\mathcal{W} = \mathcal{W}(\mathfrak{g}, f) = \{g \in \mathcal{V}(\mathfrak{g}_{\leq \frac{1}{2}}) \mid a_\lambda^\rho g = 0 \text{ for all } a \in \mathfrak{g}_{\geq \frac{1}{2}}\}, \tag{3.11}$$

endowed with the following PVA λ -bracket

$$\{g_\lambda h\}_{z,\rho} = \rho\{g_\lambda h\}_z, \quad g, h \in \mathcal{W}. \tag{3.12}$$

Remark 3.1 Thinking of z as a formal parameter, Eq. (3.12) gives a 1-parameter family of PVA structures on \mathcal{W} , or, equivalently, a bi-Poisson structure. Indeed, we can write $\{g_\lambda h\}_{z,\rho} = \{g_\lambda h\}_{1,\rho} + z\{g_\lambda h\}_{0,\rho}$, for every $g, h \in \mathcal{W}$. The λ -bracket $\{\cdot_\lambda \cdot\}_{1,\rho}$ does not depend on the choice of $s \in \mathfrak{g}_d$, while $\{\cdot_\lambda \cdot\}_{0,\rho}$ does.

Generalizing the results in [27] it has been shown in [18], using the Lenard–Magri scheme of integrability [36], that it is possible to construct an integrable hierarchy of bi-Hamiltonian equations for \mathcal{W} , known as generalized Drinfeld–Sokolov hierarchy, under the assumption that $f + s \in \mathfrak{g}$ is a semisimple element.

Recently, generalized Drinfeld–Sokolov hierarchies for any nilpotent element $f \in \mathfrak{gl}_N$ and non-zero $s \in \mathfrak{g}_d$ have been constructed in [24] using the theory of Adler type pseudodifferential operators [23].

3.3 Structure theorem for classical affine \mathcal{W} -algebras

In the algebra of differential polynomials $\mathcal{V}(\mathfrak{g}_{\leq \frac{1}{2}})$ we introduce the grading by conformal weight, denoted by $\Delta \in \frac{1}{2}\mathbb{Z}$, defined as follows. For $a \in \mathfrak{g}$ such that $[x, a] = \delta(a)a$, we let $\Delta(a) = 1 - \delta(a)$. For a monomial $g = a_1^{(m_1)} \cdots a_s^{(m_s)}$, product of derivatives of ad x eigenvectors $a_i \in \mathfrak{g}_{\leq \frac{1}{2}}$, we define its conformal weight as

$$\Delta(g) = \Delta(a_1) + \cdots + \Delta(a_s) + m_1 + \cdots + m_s. \tag{3.13}$$

Thus we get the conformal weight space decomposition

$$\mathcal{V}(\mathfrak{g}_{\leq \frac{1}{2}}) = \bigoplus_{\Delta \in \frac{1}{2}\mathbb{Z}_+} \mathcal{V}(\mathfrak{g}_{\leq \frac{1}{2}})\{\Delta\}.$$

For example $\mathcal{V}(\mathfrak{g}_{\leq \frac{1}{2}})\{0\} = \mathbb{F}$, $\mathcal{V}(\mathfrak{g}_{\leq \frac{1}{2}})\{\frac{1}{2}\} = \mathfrak{g}_{\frac{1}{2}}$, and $\mathcal{V}(\mathfrak{g}_{\leq \frac{1}{2}})\{1\} = \mathfrak{g}_0 \oplus S^2\mathfrak{g}_{\frac{1}{2}}$.

Theorem 3.2 [18] Consider the PVA $\mathcal{W} = \mathcal{W}(\mathfrak{g}, f)$ with the λ -bracket $\{\cdot_\lambda \cdot\}_{z,\rho}$ defined by Eq. (3.12).

- (a) For every element $q \in \mathfrak{g}_{1-\Delta}^f$ there exists a (not necessarily unique) element $w \in \mathcal{W}\{\Delta\} = \mathcal{W} \cap \mathcal{V}(\mathfrak{g}_{\leq \frac{1}{2}})\{\Delta\}$ of the form $w = q + g$, where

$$g = \sum b_1^{(m_1)} \cdots b_s^{(m_s)} \in \mathcal{V}(\mathfrak{g}_{\leq \frac{1}{2}})\{\Delta\}, \tag{3.14}$$

is a sum of products of derivatives of ad x -eigenvectors $b_i \in \mathfrak{g}_{1-\Delta_i} \subset \mathfrak{g}_{\leq \frac{1}{2}}$, such that

$$\Delta_1 + \cdots + \Delta_s + m_1 + \cdots + m_s = \Delta \quad \text{and} \quad s + m_1 + \cdots + m_s > 1.$$

- (b) Let $\{w_j = q_j + g_j\}_{j \in J^f}$ be any collection of elements in \mathcal{W} as in part (a). (Recall, from Sect. 3.1, that $\{q_j\}_{j \in J^f}$ is a basis of \mathfrak{g}^f consisting of ad x -eigenvectors.) Then the differential subalgebra $\mathcal{W} \subset \mathcal{V}(\mathfrak{g}_{\leq \frac{1}{2}})$ is the algebra of differential polynomials in the variables $\{w_j\}_{j \in J^f}$. The algebra \mathcal{W} is a graded associative algebra, graded by the conformal weights defined in (3.13): $\mathcal{W} = \mathbb{F} \oplus \mathcal{W}\{1\} \oplus \mathcal{W}\{\frac{3}{2}\} \oplus \mathcal{W}\{2\} \oplus \mathcal{W}\{\frac{5}{2}\} \oplus \cdots$

Recall the first of the direct sum decompositions (3.1). By assumption, the elements $q_j^0 = q_j, j \in J^f$, form a basis of \mathfrak{g}^f , and by construction the elements $q_j^n, (j, n) \in J$, with $n \geq 1$, form a basis of $[e, \mathfrak{g}]$ (here we are using the notation from Sect. 3.1). Since $\mathfrak{g}^f \subset \mathfrak{g}_{\leq \frac{1}{2}}$, we have the corresponding direct sum decomposition

$$\mathfrak{g}_{\leq \frac{1}{2}} = \mathfrak{g}^f \oplus [e, \mathfrak{g}_{\leq -\frac{1}{2}}]. \tag{3.15}$$

It follows that the algebra of differential polynomials $\mathcal{V}(\mathfrak{g}_{\leq \frac{1}{2}})$ admits the following decomposition in a direct sum of subspaces

$$\mathcal{V}(\mathfrak{g}_{\leq \frac{1}{2}}) = \mathcal{V}(\mathfrak{g}^f) \oplus \left\langle [e, \mathfrak{g}_{\leq -\frac{1}{2}}] \right\rangle_{\mathcal{V}(\mathfrak{g}_{\leq \frac{1}{2}})}, \tag{3.16}$$

where $\mathcal{V}(\mathfrak{g}^f)$ is the algebra of differential polynomials over \mathfrak{g}^f , and $\left\langle [e, \mathfrak{g}_{\leq -\frac{1}{2}}] \right\rangle_{\mathcal{V}(\mathfrak{g}_{\leq \frac{1}{2}})}$ is the differential ideal of $\mathcal{V}(\mathfrak{g}_{\leq \frac{1}{2}})$ generated by $[e, \mathfrak{g}_{\leq -\frac{1}{2}}]$.

Theorem 3.2 implies the following result.

Corollary 3.3 [22] *For every $q \in \mathfrak{g}^f$ there exists a unique element $w = w(q) \in \mathcal{W}$ of the form $w = q + r$, where $r \in \left\langle [e, \mathfrak{g}_{\leq -\frac{1}{2}}] \right\rangle_{\mathcal{V}(\mathfrak{g}_{\leq \frac{1}{2}})}$. Moreover, if $q \in \mathfrak{g}_{1-\Delta}$, then $w(q)$ lies in $\mathcal{W}\{\Delta\}$ and r is of the form (3.14). Consequently, \mathcal{W} coincides with the algebra of differential polynomials in the variables $w_j = w(q_j), j \in J^f$.*

As an immediate consequence of Theorem 3.2 and Corollary 3.3 we get the following:

Theorem 3.4 *The map $\pi_{\mathfrak{g}^f}$ restricts to a differential algebra isomorphism*

$$\pi := \pi_{\mathfrak{g}^f}|_{\mathcal{W}}: \mathcal{W} \xrightarrow{\sim} \mathcal{V}(\mathfrak{g}^f),$$

hence we have the inverse differential algebra isomorphism

$$w =: \mathcal{V}(\mathfrak{g}^f) \xrightarrow{\sim} \mathcal{W},$$

which associates to every element $q \in \mathfrak{g}^f$ the (unique) element $w(q) \in \mathcal{W}$ of the form $w(q) = q + r$, with $r \in \left\langle [e, \mathfrak{g}_{\leq -\frac{1}{2}}] \right\rangle_{\mathcal{V}(\mathfrak{g}_{\leq \frac{1}{2}})}$.

3.4 Poisson structure of the classical affine \mathcal{W} -algebra

Let $\ell = \dim \mathfrak{g}^f$. By Corollary 3.3 the Poisson structure $H = (H_{ij}(\lambda))_{i,j \in J^f} \in \text{Mat}_{\ell \times \ell} \mathcal{W}[\lambda]$ associated to the classical affine \mathcal{W} -algebra \mathcal{W} defined by Eqs. (3.11) and (3.12) is given by $(i, j \in J^f)$

$$H_{ji}(\lambda) = \{w(q_i)_\lambda w(q_j)\}_{z,\rho}. \tag{3.17}$$

For $h, k \in \frac{1}{2}\mathbb{Z}$, we introduce the notation

$$h < k \quad \text{if and only if} \quad h \leq k - 1. \tag{3.18}$$

Also, for $t \geq 1$, we denote $\vec{k} = (k_1, k_2, \dots, k_t) \in (\frac{1}{2}\mathbb{Z})^t$, and $J_{-\vec{k}} := J_{-k_1} \times \dots \times J_{-k_t}$. Therefore, an element $(\vec{j}, \vec{n}) \in J_{-\vec{k}}$ is an t -tuple with

$$(j_1, n_1) \in J_{-k_1}, \dots, (j_t, n_t) \in J_{-k_t}. \tag{3.19}$$

The explicit expression of the Poisson structure H defined by Eq. (3.17) can be obtained by the following result.

Theorem 3.5 [22, Theorem 5.3] *For $a \in \mathfrak{g}_{-h}^f$ and $b \in \mathfrak{g}_{-k}^f$, we have*

$$\begin{aligned} \{w(a)_\lambda w(b)\}_{z,\rho} &= w([a, b]) + (a|b)\lambda + z(s|[a, b]) \\ &- \sum_{t=1}^\infty \sum_{-h+1 \leq k_t < \dots < k_1 \leq k} \sum_{(\vec{j}, \vec{n}) \in J_{-\vec{k}}} \left(w\left(\left[b, q_{n_1}^{j_1}\right]^\sharp\right) - (b|q_{n_1}^{j_1})(\lambda + \partial) + z\left(s\left[b, q_{n_1}^{j_1}\right]\right) \right) \\ &\times \left(w\left(\left[q_{j_1}^{n_1+1}, q_{n_2}^{j_2}\right]^\sharp\right) - (q_{j_1}^{n_1+1}|q_{n_2}^{j_2})(\lambda + \partial) + z\left(s\left[q_{j_1}^{n_1+1}, q_{n_2}^{j_2}\right]\right) \right) \\ &\times \dots \times \left(w\left(\left[q_{j_{t-1}}^{n_{t-1}+1}, q_{n_t}^{j_t}\right]^\sharp\right) - (q_{j_{t-1}}^{n_{t-1}+1}|q_{n_t}^{j_t})(\lambda + \partial) + z\left(s\left[q_{j_{t-1}}^{n_{t-1}+1}, q_{n_t}^{j_t}\right]\right) \right) \\ &\times \left(w\left(\left[q_{j_t}^{n_t+1}, a\right]^\sharp\right) - (q_{j_t}^{n_t+1}|a)\lambda + z\left(s\left[q_{j_t}^{n_t+1}, a\right]\right) \right). \end{aligned} \tag{3.20}$$

Note that in each summand of (3.20) the z term can be non-zero at most in one factor. In fact, z may occur in the first factor only for $k_1 \leq 0$, in the second factor only for $k_1 \geq 1$ and $k_2 \leq -1$, in the third factor only for $k_2 \geq 1$ and $k_3 \leq -1$, and so on, and it may occur in the last factor only for $k_t \geq 1$. Since these conditions are mutually exclusive, the expression in the RHS of (3.20) is linear in z .

Some special cases and applications of Eq. (3.20) are summarized in the next result.

Proposition 3.6 [22]

(a) *If either a or b lies in \mathfrak{g}_0^f , we have*

$$\{w(a)_\lambda w(b)\}_{z,\rho} = w([a, b]) + (a|b)\lambda + z(s|[a, b]). \tag{3.21}$$

(b) *If $a, b \in \mathfrak{g}_{-\frac{1}{2}}^f$ we have*

$$\begin{aligned} \{w(a)_\lambda w(b)\}_{z,\rho} &= w([a, b]) + (\partial + 2\lambda)w([a, [e, b]]^\sharp) - (e|[a, b])\lambda^2 \\ &+ \sum_{(j,n) \in J_{-\frac{1}{2}}} w([a, q_n^j]^\sharp)w([q_j^{n+1}, b]^\sharp) + z(s|[a, b]). \end{aligned} \tag{3.22}$$

(c) *Consider the element $L_0 = \frac{1}{2} \sum_{j \in J_0^f} w(q_j)w(q^j) \in \mathcal{W}\{2\}$. Then, the element $L = w(f) + L_0 \in \mathcal{W}\{2\}$ is a Virasoro element of \mathcal{W} , and we have*

$$\{L_\lambda L\}_{z,\rho} = (\partial + 2\lambda)L - (x|x)\lambda^3 + 2z(s|f)\lambda. \tag{3.23}$$

For $a \in \mathfrak{g}_{1-\Delta}^f$ we have

$$\{L_\lambda w(a)\}_{z,\rho} = (\partial + \Delta\lambda)w(a) - \frac{(e|a)}{2}\lambda^3 + z\Delta(s|a)\lambda. \tag{3.24}$$

In particular, for $z = 0$, all the generators $w(a)$, $a \in \mathfrak{g}^f$, of \mathcal{W} are primary elements for L , provided that $(e|a) = 0$. In other words, for $z = 0$, \mathcal{W} is an algebra of differential polynomials generated by L and $\ell - 1$ primary elements with respect to L . So, \mathcal{W} is a PVA of CFT type (cf. [25]).

Remark 3.7 Equations (3.21) and (3.22) and the definition of the Virasoro element L in Proposition 3.6(c) are compatible with the analogous in [19] where the classical affine \mathcal{W} -algebra for minimal nilpotent elements has been explicitly described.

4 The package MasterPVA

In this section we show how to use the package *MasterPVA*, both in its one- and multi-dimensional versions. As a few examples, we prove the compatibility between GFZ and Virasoro–Magri PVA (case $N = D = 1$), we derive the Dubrovin–Novikov conditions for a bidimensional Poisson structure of hydrodynamic type (case $D = 1, N = 2$) and we obtain the Mokhov’s classification for the $N = 1$ multidimensional structures of hydrodynamic type [37].

The packages `MasterPVA.m` and `MasterPVAmulti.m` must be in a directory where Mathematica can find them. This can be achieved, for example, by using the command `SetDirectory`. After this, we can load the packages. The two packages cannot be loaded in the same session, because of the conflicting functions and properties definition. However, `MasterPVAmulti` can effectively deal with $D = 1$ PVA, despite using a heavier notation. This is the reason why we provide a package specifically devoted to standard monodimensional PVAs, although the same input works with *MasterPVAmulti*.

```

In[2]:= << "MasterPVA.m"
MasterPVA: a Mathematica package for computing
the lambda bracket of a Poisson Vertex Algebra of differential functions
Second version (2016). M. Casati & D. Valeri

```

4.1 GFZ and Virasoro–Magri Poisson vertex algebras

Let \mathcal{V} be an algebra of differential functions extending $R_1 = \mathbb{C}[u, u', u'', \dots]$. We recall that the Gardner–Faddeev–Zacharov (GFZ) PVA structure on \mathcal{V} is defined by

$$\{u_\lambda u\}_1 = \lambda \quad (4.1)$$

while the Virasoro–Magri PVA with central charge $c \in \mathbb{C}$ is defined by

$$\{u_\lambda u\}_0 = (\partial + 2\lambda)u + c\lambda^3. \quad (4.2)$$

We will show the well-known fact that these two structures are compatible, namely that the λ -bracket $\{\cdot_\lambda \cdot\}_z = \{\cdot_\lambda \cdot\}_0 + z\{\cdot_\lambda \cdot\}_1$ defines a PVA structure on \mathcal{V} for all $z \in \mathbb{C}$.

After loading the package, it is necessary to set the number of generators, the name for the generators, for the independent variable with respect to which the derivation ∂ acts, and for the formal indeterminate used in the definition of the λ -brackets, say λ . The syntax for these commands is

```

In[3]:= SetN[1];
SetGenName[u];
SetVarName[x];
SetFormalParameter[lambda];

```

The list of generators, written as functions of the independent variables, is called `gen` throughout the program. The λ -brackets between the generators must be provided in form of a $N \times N$ table, whose entries are polynomials in the previously declared formal indeterminate. In this example $N = 1$ and we have H_0 given by Eq. (4.2) and H_1 given by Eq. (4.1). We denote by H their linear combination.

```

In[7]:= H0 = {{2 λ gen[[1]] + D[gen[[1]], x] + c λ ^ 3}};
        H1 = {{λ}};
        H = H0 + z H1

Out[9]= {{z λ + c λ ^ 3 + 2 λ u[x] + u'[x]}}
    
```

The skewsymmetry and Jacobi identity on generators (see Theorem 2.3) can be checked by using the functions `PVASKew[]` and `JacobiCheck[]`. Indeed the output of `PVASKew[]` (respectively `JacobiCheck[]`) is the LHS of Eq. (2.9) (respectively (2.10)). We get

```

In[10]:= PVASKew[H]
Out[10]= {{0}}

In[11]:= JacobiCheck[H]
Out[11]= {{{0}}}
    
```

thus showing that H_0 and H_1 define two compatible PVA structures on \mathcal{V} .

Let us define $h_1 = \frac{1}{2}u^2 \in \mathcal{V}$. It is well known that the corresponding Hamiltonian equation (2.13) corresponding to the Hamiltonian functional $\int h_1$ and the Poisson structure H_0 is the Korteweg–de Vries (KdV) equation. Moreover, let us also define $h_2 = \frac{1}{2}u^3 + \frac{c}{2}uu'' \in \mathcal{V}$. The KdV equation is also the Hamiltonian equation corresponding to the Hamiltonian functional $\int h_2$ and the Poisson structure H_1 .

```

In[12]:= h1 = 1 / 2 gen[[1]] ^ 2;
        h2 = 1 / 2 gen[[1]] ^ 3 + c / 2 gen[[1]] D[gen[[1]], {x, 2}];
        KdVeq = LambdaB[h1, gen[[1]], H0, λ] /. λ -> 0 // Expand

Out[14]= 3 u[x] u'[x] + c u^{(3)}[x]

In[15]:= KdVeq = LambdaB[h2, gen[[1]], H1, λ] /. λ -> 0 // Expand
Out[15]= True
    
```

In fact, the KdV equation is a bi-Hamiltonian equation and its integrability can be proved using the Lenard–Magri scheme of integrability [36].

4.2 Poisson structures of hydrodynamic type

Let \mathcal{V} be an algebra of differential functions extending R_N . A Poisson structure of hydrodynamic type [28] on \mathcal{V} is defined by the following λ -bracket on generators ($i, j, k = 1, \dots, N$):

$$\{u_i \lambda u_j\} = g_{ji} \lambda + b_{ji}^k u'_k, \tag{4.3}$$

where repeated indices are summed according to Einstein’s rule and $\frac{\partial g_{ji}}{\partial u_h^{(n)}} = \frac{\partial b_{ji}^k}{\partial u_h^{(n)}} = 0$, for every $h = 1, \dots, N$ and $n \geq 1$.

The geometric interpretation of the functions g_{ij} and b_{ij}^k is well known: the λ -bracket defined in (4.3) defines a PVA structure on \mathcal{V} if and only g_{ij} are the components of a flat contravariant metric on a manifold with local coordinates (u_1, \dots, u_N) and b_{ij}^k are the contravariant Christoffel symbols of the associated Levi-Civita connection. Using *MasterPVA* we will derive the explicit form of these properties in the case $N = 2$.

After loading the package, we initialize the package settings.

```

In[3]:= n = SetN[2];
SetGenName[u];
SetVarName[x];
SetFormalParameter[λ];
SetMaxO[3];
    
```

We define the matrices g_{ij} and b_{ij}^k and use them to write the λ -bracket (4.3).

```

In[8]:= gg = Table[Subscript[g, i, j][gen], {i, n}, {j, n}];
bb = Table[Superscript[Subscript[b, i, j, k][gen], {i, n}, {j, n}, {k, n}];

In[10]:= H = Table[gg[[j, i]] λ + Sum[bb[[j, i, k]] D[gen[[k]], x], {k, n}], {i, n}, {j, n}
Out[10]= {{λ g_{1,1}[{u_1[x], u_2[x]}] + b_{1,1}^1[{u_1[x], u_2[x]}] u_1'[x] + b_{1,1}^2[{u_1[x], u_2[x]}] u_2'[x],
λ g_{2,1}[{u_1[x], u_2[x]}] + b_{2,1}^1[{u_1[x], u_2[x]}] u_1'[x] + b_{2,1}^2[{u_1[x], u_2[x]}] u_2'[x]},
{λ g_{1,2}[{u_1[x], u_2[x]}] + b_{1,2}^1[{u_1[x], u_2[x]}] u_1'[x] + b_{1,2}^2[{u_1[x], u_2[x]}] u_2'[x],
λ g_{2,2}[{u_1[x], u_2[x]}] + b_{2,2}^1[{u_1[x], u_2[x]}] u_1'[x] + b_{2,2}^2[{u_1[x], u_2[x]}] u_2'[x]}}
    
```

By equating to zero the coefficient of λ and the constant term (in λ) in the equations given by `PVASKew[P]` we get the conditions that g_{ij} and b_{ij}^k should satisfy in order to get a skewsymmetric λ -bracket.

```

In[11]:= condSkew = Simplify[LogicalExpand[Collect[PVASKew[H], Flatten[{λ, D[gen, x]}]] = 0]]
Out[11]= u_2'[x] (2 b_{1,1}^2[{u_1[x], u_2[x]}] - g_{1,1}^{(0,1)}[{u_1[x], u_2[x]}]) +
u_1'[x] (2 b_{1,1}^1[{u_1[x], u_2[x]}] - g_{1,1}^{(1,0)}[{u_1[x], u_2[x]}]) = 0 &&
λ (-g_{1,2}[{u_1[x], u_2[x]}] + g_{2,1}[{u_1[x], u_2[x]}]) +
u_2'[x] (b_{1,2}^2[{u_1[x], u_2[x]}] + b_{2,1}^2[{u_1[x], u_2[x]}] - g_{1,2}^{(0,1)}[{u_1[x], u_2[x]}]) +
u_1'[x] (b_{1,2}^1[{u_1[x], u_2[x]}] + b_{2,1}^1[{u_1[x], u_2[x]}] - g_{1,2}^{(1,0)}[{u_1[x], u_2[x]}]) = 0 &&
λ (g_{1,2}[{u_1[x], u_2[x]}] - g_{2,1}[{u_1[x], u_2[x]}]) +
u_2'[x] (b_{1,2}^2[{u_1[x], u_2[x]}] + b_{2,1}^2[{u_1[x], u_2[x]}] - g_{2,1}^{(0,1)}[{u_1[x], u_2[x]}]) +
u_1'[x] (b_{1,2}^1[{u_1[x], u_2[x]}] + b_{2,1}^1[{u_1[x], u_2[x]}] - g_{2,1}^{(1,0)}[{u_1[x], u_2[x]}]) = 0 &&
u_2'[x] (2 b_{2,2}^2[{u_1[x], u_2[x]}] - g_{2,2}^{(0,1)}[{u_1[x], u_2[x]}]) +
u_1'[x] (2 b_{2,2}^1[{u_1[x], u_2[x]}] - g_{2,2}^{(1,0)}[{u_1[x], u_2[x]}]) = 0
    
```

These conditions can be summarized by the equations

$$g_{ij} = g_{ji}, \quad b_{ij}^k + b_{ji}^k = \frac{\partial g_{ij}}{\partial u_k}. \tag{4.4}$$

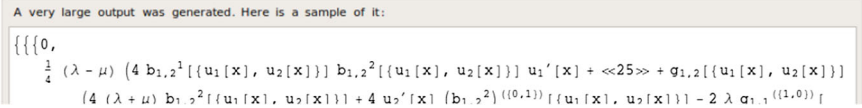
We redefine the functions g_{ij} and b_{ij}^k and the Poisson structure H in order to ensure the validity of Eq. (4.4).

```

In[12]:= gg[[2, 1]] = gg[[1, 2]];
For[i = 1, i <= n, i++, For[k = 1, k <= n, k++, bb[[i, i, k]] = 1/2 D[gg[[i, i]], gen[[k]]];];
For[k = 1, k <= n, k++, bb[[2, 1, k]] = D[gg[[1, 2]], gen[[k]]] - bb[[1, 2, k]];
H = Table[gg[[j, i]] λ + Sum[bb[[j, i, k]] D[gen[[k]], x], {k, n}], {i, n}, {j, n}
    
```

The further properties that must be satisfied to grant the Jacobi identity can be found using `JacobiCheck[P]`. Notice that, when the result of `JacobiCheck[]` is not identically vanishing, the output uses internal variables whose name starts with `MasterPVA`Private``: to make the output clearer it is advisable to replace them with the “external” names, as it is demonstrated in the following picture. However, reading the conditions for the Jacobi identity is usually much more cumbersome than inspecting the ones for the skewsymmetry.

```
In[16]:= JacobiFull = JacobiCheck[H] //. {MasterPVA`Private`λ → λ, MasterPVA`Private`μ → μ}
```



Nevertheless, we can check that the vanishing of the coefficient of λ^2 in the Jacobi identity is equivalent to the torsion-free condition for the Levi-Civita connection:

$$g_{ia} b_{kj}^a - g_{ja} b_{ki}^a = 0. \tag{4.5}$$

```
In[17]:= Expand[Simplify[LogicalExpand[Coefficient[JacobiFull, λ^2] = 0]]]
```

```
Out[17]= 2 g_{1,1} [\{u_1[x], u_2[x]\}] b_{1,2}^1 [\{u_1[x], u_2[x]\}] + 2 g_{1,2} [\{u_1[x], u_2[x]\}] b_{1,2}^2 [\{u_1[x], u_2[x]\}] - g_{1,2} [\{u_1[x], u_2[x]\}] g_{1,1}^{(1,0)} [\{u_1[x], u_2[x]\}] = g_{2,2} [\{u_1[x], u_2[x]\}] g_{1,1}^{(0,1)} [\{u_1[x], u_2[x]\}] && 2 g_{1,2} [\{u_1[x], u_2[x]\}] b_{1,2}^1 [\{u_1[x], u_2[x]\}] + 2 g_{2,2} [\{u_1[x], u_2[x]\}] b_{1,2}^2 [\{u_1[x], u_2[x]\}] - 2 g_{2,2} [\{u_1[x], u_2[x]\}] g_{1,2}^{(0,1)} [\{u_1[x], u_2[x]\}] + g_{1,2} [\{u_1[x], u_2[x]\}] g_{2,2}^{(0,1)} [\{u_1[x], u_2[x]\}] - 2 g_{1,2} [\{u_1[x], u_2[x]\}] g_{1,2}^{(1,0)} [\{u_1[x], u_2[x]\}] + g_{1,1} [\{u_1[x], u_2[x]\}] g_{2,2}^{(1,0)} [\{u_1[x], u_2[x]\}] = 0
```

```
In[18]:= Expand[Simplify[LogicalExpand[Table[Sum[gσ[[i, a]] bb[[k, j, a]] - gσ[[j, a]] bb[[k, i, a]], {a, n}], {i, n}, {j, n}, {k, n} = 0]]]
```

```
Out[18]= 2 g_{1,1} [\{u_1[x], u_2[x]\}] b_{1,2}^1 [\{u_1[x], u_2[x]\}] + 2 g_{1,2} [\{u_1[x], u_2[x]\}] b_{1,2}^2 [\{u_1[x], u_2[x]\}] - g_{1,2} [\{u_1[x], u_2[x]\}] g_{1,1}^{(1,0)} [\{u_1[x], u_2[x]\}] = g_{2,2} [\{u_1[x], u_2[x]\}] g_{1,1}^{(0,1)} [\{u_1[x], u_2[x]\}] && 2 g_{1,2} [\{u_1[x], u_2[x]\}] b_{1,2}^1 [\{u_1[x], u_2[x]\}] + 2 g_{2,2} [\{u_1[x], u_2[x]\}] b_{1,2}^2 [\{u_1[x], u_2[x]\}] - 2 g_{2,2} [\{u_1[x], u_2[x]\}] g_{1,2}^{(0,1)} [\{u_1[x], u_2[x]\}] + g_{1,2} [\{u_1[x], u_2[x]\}] g_{2,2}^{(0,1)} [\{u_1[x], u_2[x]\}] - 2 g_{1,2} [\{u_1[x], u_2[x]\}] g_{1,2}^{(1,0)} [\{u_1[x], u_2[x]\}] + g_{1,1} [\{u_1[x], u_2[x]\}] g_{2,2}^{(1,0)} [\{u_1[x], u_2[x]\}] = 0
```

```
In[19]:= Out[17] = Out[18]
```

```
Out[19]= True
```

4.3 Multidimensional scalar PVAs of hydrodynamic type

The package *MasterPVAMulti* must be used when dealing with multidimensional PVA defined in Sect. 2.4. Here, we use it to classify multidimensional Poisson structures of hydrodynamic type for the case $N = 1, D = 3$. This is a special case of a classification theorem proved by Mokhov [37].

We recall that a multidimensional scalar λ -bracket of hydrodynamic type has the form

$$\{u_\lambda u\} = \sum_{\alpha=1}^D (a_\alpha \lambda_\alpha + b_\alpha u_\alpha), \tag{4.6}$$

where we set $u_\alpha = \partial_\alpha u$ and a_α and b_α are such that $\frac{\partial a_\alpha}{\partial \partial_\beta^n u} = \frac{\partial b_\alpha}{\partial \partial_\beta^n u} = 0$, for every $\beta = 1, \dots, D$ and $n \geq 1$.

Mokhov's theorem states that the λ -bracket (4.6) defines a PVA structure if and only if it is of the form

$$\{u_\lambda u\} = \sum_{\alpha=1}^D c_\alpha \left(2g \lambda_\alpha + u_\alpha \frac{\partial g}{\partial u} \right), \tag{4.7}$$

for some $c_\alpha \in \mathbb{C}$ and a function g such that $\frac{\partial g}{\partial \partial_\beta^n u} = 0$, for every $\beta = 1, \dots, D$ and $n \geq 1$.

After loading the package we initialize the variables similarly to what we did at the beginning of Sect. 4.1, but in this case we should specify the spatial dimension D .

```

In[2]:= << "MasterPVAmulti.m"

MasterPVAmulti: a Mathematica package for computing the
lambda bracket of a Poisson Vertex Algebra of arbitrary rank
Second version (2016). M. Casati & D. Valeri

In[3]:= d = SetD[3];
SetN[1];
SetGenName[u];
SetVarName[x];
SetFormalParameter[λ];
SetMaxO[3];
    
```

We define the λ -bracket as in equation (4.6) assuming $N = 1$ and $D = 3$. The formal parameter, for which we chose the symbol λ in the initialization, is $\{\lambda_1, \lambda_2, \lambda_3\}$.

```

In[9]:= H = {Sum[a_i[gen[[1]] λ_i + b_i[gen[[1]]] D[gen[[1]], var[[i]], {i, 1, d}]]
Out[9]= {{λ_1 a_1[u[{x_1, x_2, x_3}]] + λ_2 a_2[u[{x_1, x_2, x_3}]] +
λ_3 a_3[u[{x_1, x_2, x_3}]] + b_3[u[{x_1, x_2, x_3}]] u^{(0,0,1)}[{x_1, x_2, x_3}] +
b_2[u[{x_1, x_2, x_3}]] u^{(0,1,0)}[{x_1, x_2, x_3}] + b_1[u[{x_1, x_2, x_3}]] u^{(1,0,0)}[{x_1, x_2, x_3}]]}

In[10]:= λλ = {λ_1, λ_2, λ_3};
μμ = {μ_1, μ_2, μ_3};
uprime = {D[gen[[1]], var[[1]], D[gen[[1]], var[[2]], D[gen[[1]], var[[3]]]}
    
```

We use `PVAskew[P]` to find the conditions that the functions a_α and b_α should satisfy in order to have a skewsymmetric λ -bracket. We get

$$2b_\alpha = \frac{\partial a_\alpha}{\partial u}. \tag{4.8}$$

Hence, we define a new λ -bracket, called `Hskew`, where the functions a_α and b_α satisfy Eq. (4.8). Note that `Hskew` only depends on the functions a_α now.

```

In[13]:= Collect[PVAskew[H], Join[λλ, uprime]]
Out[13]= {{(2 b_3[u[{x_1, x_2, x_3}]] - a_3'[u[{x_1, x_2, x_3}]] u^{(0,0,1)}[{x_1, x_2, x_3}] +
(2 b_2[u[{x_1, x_2, x_3}]] - a_2'[u[{x_1, x_2, x_3}]] u^{(0,1,0)}[{x_1, x_2, x_3}] +
(2 b_1[u[{x_1, x_2, x_3}]] - a_1'[u[{x_1, x_2, x_3}]] u^{(1,0,0)}[{x_1, x_2, x_3}]))}

In[14]:= RuleSkew = b_i[gen[[1]]] :> 1/2 D[a_i[gen[[1]], gen[[1]]]
Out[14]= b_i_ [u[{x_1, x_2, x_3}]] => 1/2 ∂_{gen[[1]]} a_i [gen[[1]]]

In[15]:= Hskew = H //. RuleSkew
Out[15]= {{λ_1 a_1[u[{x_1, x_2, x_3}]] + λ_2 a_2[u[{x_1, x_2, x_3}]] + λ_3 a_3[u[{x_1, x_2, x_3}]] +
1/2 a_3'[u[{x_1, x_2, x_3}]] u^{(0,0,1)}[{x_1, x_2, x_3}] + 1/2 a_2'[u[{x_1, x_2, x_3}]]
u^{(0,1,0)}[{x_1, x_2, x_3}] + 1/2 a_1'[u[{x_1, x_2, x_3}]] u^{(1,0,0)}[{x_1, x_2, x_3}]]}

In[16]:= PVAskew[Hskew]
Out[16]= {{0}}
    
```

We use `JacobiCheck[Pskew]` to write the conditions that must be satisfied by the functions a_α in order to get the validity of (2.10). We denote the LHS of (2.10) by `JacobiCond`.

In particular, by equating to zero the coefficient of $\lambda_\alpha \mu_\beta$ we get a system of ODEs for the functions a_α .

```

In[17]:= JacobiCond =
  Simplify[JacobiCheck[Hskew] //.
    {MasterPVAMulti`Private`λi => λi, MasterPVAMulti`Private`μi => μi}]
  Simplify[
    LogicalExpand[
      Table[Coefficient[JacobiCond[[1, 1, 1]], λλ[[i]], μμ[[j]], {i, d}, {j, d}] = 0]]
  a2[u[{x1, x2, x3}}] a1'[u[{x1, x2, x3}}] = a1[u[{x1, x2, x3}}] a2'[u[{x1, x2, x3}}] &&
  a3[u[{x1, x2, x3}}] a1'[u[{x1, x2, x3}}] = a1[u[{x1, x2, x3}}] a3'[u[{x1, x2, x3}}] &&
  a3[u[{x1, x2, x3}}] a2'[u[{x1, x2, x3}}] = a2[u[{x1, x2, x3}}] a3'[u[{x1, x2, x3}}]

```

A solution to this system is given by

$$a_\alpha = c_\alpha g \quad \text{for some function } g \text{ and } c_\alpha \in \mathbb{C}. \tag{4.9}$$

Then, we can substitute Eq. (4.9) in JacobiCond.

```

In[19]:= RuleJac = {ai[gen[[1]]] => ci g[gen[[1]]],
  D[ai[gen[[1]]], gen[[1]]] => D[ci g[gen[[1]]], gen[[1]]],
  D[ai[gen[[1]]], {gen[[1], 2}] => D[ci g[gen[[1]]], {gen[[1], 2}]]
Out[19]= {ai[u[{x1, x2, x3}}] => ci g[gen[[1]]],
  ai'[u[{x1, x2, x3}}] => ∂gen[[1]] (ci g[gen[[1]]]),
  ai''[u[{x1, x2, x3}}] => ∂{gen[[1], 2]} (ci g[gen[[1]]])}
In[21]:= Simplify[JacobiCond //. RuleJac]
Out[21]= {{0}}

```

Hence, Jacobi identity (2.10) holds, thus showing that (4.8) and (4.9) are the sufficient and necessary conditions for the λ -bracket (4.6) to define a PVA structure, as Mokhov’s theorem states.

5 The package WAlg

In this section we show how to use the package *WAlg*. Its main function is to compute the λ -brackets between the generators of the classical affine \mathcal{W} -algebra $\mathcal{W}(\mathfrak{g}, f)$ defined in Sect. 3.2, where \mathfrak{g} is a simple Lie algebra of type A, B, C, D and G , and $f \in \mathfrak{g}$ is a nilpotent element. Thus, we can use this result to compute the λ -brackets between arbitrary elements of the classical \mathcal{W} -algebra and the corresponding generalized Drinfeld–Sokolov hierarchies using the package *MasterPVA*.

In order to perform computations with *WAlg* we need to realize the simple Lie algebras of type A, B, C, D and G as subalgebras of \mathfrak{gl}_N . (We emphasize that we can do the same for simple Lie algebras of type E and F . Unfortunately the dimension of such a representation can be big, as for the case of E_8 , where the minimal $N = 248$.)

Given an element $A = (A_{ij})_{i,j=1}^N \in \mathfrak{gl}_N$, we denote by $A^{\text{at}} = ((A^{\text{at}})_{ij})_{i,j=1}^N$ its transpose with respect to the antidiagonal, namely $(A^{\text{at}})_{ij} = A_{N+1-j, N+1-i}$. Then, we realize the classical Lie algebras as in [27]:

- (A) Type A_n : $\mathfrak{g} = \mathfrak{sl}_n = \{A \in \mathfrak{gl}_{n+1} \mid \text{Tr}(A) = 0\}$.
- (B) Type B_n : let $S = \sum_{k=1}^{2n+1} (-1)^{k+1} E_{kk}$, then $\mathfrak{g} = \mathfrak{o}_{2n+1} = \{A \in \mathfrak{gl}_{2n+1} \mid A = -SA^{\text{at}}S\}$.

- (C) Type C_n : let $S = \sum_{k=1}^{2n} (-1)^{k+1} E_{kk}$, then $\mathfrak{g} = \mathfrak{sp}_{2n} = \{A \in \mathfrak{gl}_{2n} \mid A = -SA^{\text{at}}S\}$.
- (D) Type D_n : let $S = \sum_{k=1}^n (-1)^{k+1} (E_{kk} + E_{2n+1-k, 2n+1-k})$, then $\mathfrak{g} = \mathfrak{o}_{2n} = \{A \in \mathfrak{gl}_{2n} \mid A = -SA^{\text{at}}S\}$.

In the sequel, given a matrix $A \in \mathfrak{gl}_N$, we denote by $\sigma(A) = -SA^{\text{at}}S$, where S can be any of the matrix appearing in the definition of the classical Lie algebras of type B, C and D . Clearly, $A + \sigma(A)$ belongs to the corresponding classical Lie algebra, since $\sigma^2 = \mathbb{1}_{\mathfrak{gl}_N}$.

We realize G_2 as a subalgebra of D_4 as follows. Note that the group of automorphisms of the Dynkin diagram of D_4 is isomorphic to S_3 , the group of permutations on three elements. Then, we can consider the induced action by Lie algebra automorphisms of this group on \mathfrak{o}_8 . Then, it is easy to check that:

- (G) Type G_2 : $\mathfrak{g} = \{A \in \mathfrak{o}_8 \mid \tau(A) = A, \text{ for every } \tau \in S_3\}$.

In particular, we used the following choice of Chevalley generators for \mathfrak{g} :

$$\begin{aligned} e_1 &= E_{23} + E_{67}, & e_2 &= E_{12} + E_{34} + E_{56} + E_{78} + \frac{1}{2} (E_{35} + E_{46}), \\ h_1 &= E_{22} - E_{33} + E_{55} - E_{66}, & h_2 &= E_{11} - E_{22} + 2E_{33} - 2E_{66} + E_{77} - E_{88}, \\ f_1 &= E_{32} + E_{76}, & f_2 &= E_{21} + E_{43} + E_{65} + E_{87} + 2 (E_{53} + E_{64}). \end{aligned}$$

After the choice of the simple Lie algebra \mathfrak{g} we need to choose a nilpotent element $f \in \mathfrak{g}$. Since the construction of classical affine \mathcal{W} -algebras does not depend on the nilpotent element itself, but only on its nilpotent orbit (see [18]), we assume that the nilpotent element is given in input as a strictly lower triangular matrix. In fact, when giving in input a nilpotent element we can use the classification of nilpotent orbits given in [10]. Then the program computes an \mathfrak{sl}_2 -triple $\{f, h = 2x, e\} \subset \mathfrak{g}$ such that x is a diagonal matrix and e is strictly upper triangular.

Finally, we always assume that the nondegenerate symmetric invariant bilinear form on \mathfrak{g} is a multiple of the trace form on matrices $(a, b \in \mathfrak{g})$:

$$(a|b) = c \text{Tr}(ab), \quad c \in \mathbb{C}^*.$$

5.1 The algebraic setup

The package `WAlg` requires the use of the default library `listK_6.txt`. Hence the files `WAlg.m` and `listK_6.txt` must be in a folder where Mathematica can find them. It is also possible to use a different library as described in Sect. 6.3 to which we refer for the technical details. To select the working folder of Mathematica, where it will look for them and the potential output files will be saved, one may use the command `SetDirectory["path"]`. An alternative method to load the package, different from the one shown in Sect. 4, is using the command `Needs[]`.

Let us use the package `WAlg` to get the explicit set of generators of the classical affine \mathcal{W} -algebra $\mathcal{W}(\mathfrak{o}_7, f)$, where f is the principal nilpotent element [27]. We load the package and use the command `InitializeWAlg[]`. Recall that \mathfrak{o}_7 is a classical Lie algebra of type B_3 .

```

In[2]:= Needs["WAlg`"]
WAlg: defining classical affine W algebras with Mathematica
2016. M. Casati & D. Valeri

In[3]:= InitializeWAlg["B", 3]
Algebra correctly initialized
    
```

The dimension of the matrix representing f is obtained with the command `GetDim[]`. We define the principal nilpotent f , and we can also check that it belongs to \mathfrak{o}_7 . The command

`ComputeWAlg[]` takes the nilpotent element f as argument and computes a basis of \mathfrak{g}^f given by $\text{ad } x$ -eigenvectors. The warning notice is not a problem.

```

In[4]:= n = GetDim[]
Out[4]= 7

In[5]:= f = Sum[M[i + 1, i], {i, n - 1}];
CheckAlg[f]
f // MatrixForm

Out[6]= True

Out[7]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$


In[8]:= ComputeWAlg[f]
Solve::svars: Equations may not give solutions for all "solve" variables. >>
Computation succeeded

```

The basis computed for \mathfrak{g}^f can be recovered with the command `GetWBasis[]`. We denote it as `listQ`. The corresponding dual basis (with respect to the trace form) of \mathfrak{g}^e can be computed using the command `GetWBasisDual[]`. We denote it by `listQ`. Finally, we can also use the command `GetWEigen[]` to recover all the $\text{ad } x$ -eigenvalues (with multiplicities) and put them in a list which we call `list δ` .

The command `w[]` works as follows: it takes in input an element of \mathfrak{o}_7 , then it applies $\pi_{\mathfrak{g}^f}$ and the map w defined in Theorem 3.4 to this element. The result is a linear combination of the generators of the classical affine \mathcal{W} -algebra. Hence, by Corollary 3.3, `w[listQ[[i]]]` gives the i -th generator of the classical affine \mathcal{W} -algebra. (Note that, by an abuse of notation, these generators are denoted by q_i in Mathematica, the same letter used to denote the corresponding element of \mathfrak{g}^f to which they are attached through the map w . In fact, the notation $w(q_i)$ is used in Corollary 3.3)

```

In[10]:= listQ = GetWBasis[];
listQ = GetWBasisDual[];
list $\delta$  = GetWEigen[];

In[13]:= generators = Table[w[listQ[[i]]], {i, Length[listQ]}]
Out[13]= {Q1[Y], Q2[Y], Q3[Y]}

```

In the following example, we apply the command `w[]` to a random element of \mathfrak{o}_7 . We construct it as follows: first we define a random element $A \in \mathfrak{gl}_7$. Then, using the function `Sigma[]` (which, given A as input gives $\sigma(A) = -SA^{\text{at}}S$ as result) we get the element $A + \sigma(A) \in \mathfrak{o}_7$ (we can also check it with the command `CheckAlg[]`).

```

In[14]:= A =
  Normal[SparseArray[{{ {2, 1} → 2, {3, 2} → 4, {5, 2} → 3,
    {6, 3} → -1, {7, 2} → 5}, {n, n}}];
A = A + Sigma[A];
CheckAlg[A]
A // MatrixForm

```

Out[16]= True

Out[17]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 2 & 0 & 4 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

The command `w[]` gives the corresponding linear combination of the generators of the classical affine \mathcal{W} -algebra, see Theorem 3.4.

```

In[18]:= w[A]
Out[18]=  $\frac{13}{7} q_1[y] + \frac{4}{3} q_2[y] + 5 q_3[y]$ 

```

Note that, apart from computing explicitly the generators of the classical affine \mathcal{W} -algebra, the command `w[]` is heavily used to implement Eq. (3.20).

5.2 Computation of λ -brackets among generators

One of the most useful features of *WAlg* is the implementation of formula (3.20) for the computation of the Poisson structure H , defined by Eq. (3.17), associated to the classical affine \mathcal{W} -algebra. After we compute H , we can use the package *MasterPVA* to compute the λ -brackets between any elements of the classical affine \mathcal{W} -algebra.

Let us how to proceed in the concrete example of the Lie algebra \mathfrak{sp}_4 and its minimal nilpotent element f [19]. Recall that \mathfrak{sp}_4 is a classical Lie algebra of type C_2 .

```

In[2]:= Needs["WAlg`"]
WAlg: defining classical affine W algebras with Mathematica
2016. M. Casati & D. Valeri

In[3]:= Needs["MasterPVA`"]
MasterPVA: a Mathematica package for computing the lambda
bracket of a Poisson Vertex Algebra of differential functions
Second version (2016). M. Casati & D. Valeri

In[4]:= InitializeWAlg["C", 2]
Algebra correctly initialized

In[5]:= n = GetDim[];
f = M[n, 1];

```

```

In[8]:= ComputeWAlg[f]
Solve::svars : Equations may not give solutions for all "solve" variables. >>
Computation succeeded

In[9]:= listq = GetWBasis[];
listQ = GetWBasisDual[];
listδ = GetWEigen[];
r = Length[listq]

Out[12]= 6

```

The number of generators of $\mathcal{W}(\mathfrak{sp}_d, f)$ is the same as the dimension of \mathfrak{g}^f , which in this case is 6. The command `SetS[]` allows us to set the element in $s \in \mathfrak{g}_d$, recall that d is the maximal eigenvalue of $\text{ad } x$, which is used in formula (3.20). If this command is left without argument it automatically choose a generic s . Note that in this example $\mathfrak{g}_d = \mathbb{C}e$, so the choice is unique up to a constant. Finally, the command `GenerateH[]` gives the Poisson structure H associated to the classical affine \mathcal{W} -algebra by Eq. (3.17) implementing the formula (3.20). The optional parameter in the command is the formal parameter used in the definition of the λ -bracket, whose default value is β .

```

In[13]:= SetS[];
H = GenerateH[];

Started computation for the upper triangular component
Started computation for the lower triangular component
Computation completed

```

The next step is to allow the package *MasterPVA* to use the output of `GenerateH[]`. In order to do that, we need to set the number of variables, use q_i as the name of the generators, use y as independent variable, and use β as the formal parameter.

```

In[15]:= SetN[r];
SetGenName[q];
SetVarName[y];
SetFormalParameter[β];

```

Now the commands of *MasterPVA* can be used. For example, we can check that H is indeed a Poisson structure.

```

In[19]:= PVASkew[H]

Out[19]= {{0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
          {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}}

In[27]:= JacobiCheck[H]

Out[27]= {{{0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
           {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}},
          {{0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
           {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}},
          {{0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
           {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}}},
          {{0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
           {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}}},
          {{0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
           {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}}},
          {{0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
           {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}}}

```

We use our program to check identity (3.23). The Virasoro element L defined in Proposition 3.6(c) is computed with the command `GetVirasoro[]`, whose argument is the nilpotent element f .

```

In[20]:= L = GetVirasoro[f]
Out[20]:=  $\frac{q_1[y]^2}{4c} - \frac{q_2[y]q_3[y]}{c} - q_6[y]$ 

In[21]:= lhs = Simplify[LambdaB[L, L, H, λ]];
x = GetX[];
s = GetS[];
rhs = Simplify[2 λ L + D[L, y] - Prod[x, x] λ^3 + 2 z Prod[f, s] λ];

In[25]:= Simplify[lhs = rhs]
Out[25]:= True
    
```

Finally, we can use our program to compute the first few equations of the corresponding generalized Drinfeld–Sokolov hierarchies. We define g_0 and g_1 according to [19, Section 6.2] and we compute the Hamiltonian equation (2.13). We get

```

In[26]:= g0 = w[f]
HamEq0 = Table[LambdaB[g0, w[listq[{i}]], H, λ] /. {λ -> 0, z -> 0}, {i, 1, r}] // Expand
Out[26]:= q6[y]
Out[27]:= {0, 0, 0, - $\frac{q_1[y]q_4[y]}{2c} - \frac{q_2[y]q_5[y]}{c} - q_4'[y]$ , - $\frac{q_3[y]q_4[y]}{c} - \frac{q_1[y]q_5[y]}{2c} - q_5'[y]$ , q6'[y]}
    
```

and

```

In[28]:= g1 =
-1 / (8 Prod[x, x]) w[f]^2 -
1 / 2 Sum[w[Comm[f, listQ[{k}]]] D[w[listq[{k}]], y], {k, 4, 5}] -
1 / 2 Sum[w[listq[{i}]] w[Comm[listQ[{i}], Comm[f, listQ[{k}]]]] w[listq[{k}]],
{i, 1, 3}, {k, 4, 5}] // Expand
HamEq1 = Table[LambdaB[g1, w[listq[{i}]], H, λ] /. {λ -> 0, z -> 0}, {i, 1, r}] // Expand
Out[28]:=  $\frac{q_3[y]q_4[y]^2}{4c^2} - \frac{q_1[y]q_4[y]q_5[y]}{4c^2} - \frac{q_2[y]q_5[y]^2}{4c^2} - \frac{q_6[y]^2}{4c} - \frac{q_5[y]q_4'[y]}{4c} - \frac{q_4[y]q_5'[y]}{4c}$ 
Out[29]:= {0, 0, 0, - $\frac{q_1[y]^3q_4[y]}{8c^3} - \frac{q_1[y]q_2[y]q_3[y]q_4[y]}{2c^3} - \frac{q_1[y]^2q_2[y]q_5[y]}{4c^3} - \frac{q_2[y]^2q_3[y]q_5[y]}{c^3} -$ 
 $\frac{3q_1[y]q_4[y]q_6[y]}{4c^2} - \frac{3q_2[y]q_5[y]q_6[y]}{2c^2} - \frac{3q_1[y]q_4[y]q_1'[y]}{4c^2} - \frac{q_2[y]q_5[y]q_1'[y]}{2c^2} -$ 
 $\frac{q_3[y]q_4[y]q_2'[y]}{c^2} - \frac{q_1[y]q_4[y]q_2'[y]}{2c^2} - \frac{2q_2[y]q_4[y]q_3'[y]}{c^2} - \frac{3q_1[y]^2q_4'[y]}{4c^2} -$ 
 $\frac{3q_2[y]q_3[y]q_4'[y]}{c^2} - \frac{3q_6[y]q_4'[y]}{2c} - \frac{3q_1'[y]q_4'[y]}{2c} - \frac{3q_2'[y]q_5'[y]}{c} - \frac{3q_4[y]q_6'[y]}{4c} -$ 
 $\frac{q_4[y]q_1''[y]}{2c} - \frac{q_5[y]q_2''[y]}{c} - \frac{3q_1[y]q_4''[y]}{2c} - \frac{3q_2[y]q_5''[y]}{c} - q_6^{(3)}[y]$ ,
- $\frac{q_1[y]^2q_3[y]q_4[y]}{4c^3} - \frac{q_2[y]q_3[y]^2q_4[y]}{c^3} - \frac{q_1[y]^3q_5[y]}{8c^3} - \frac{q_1[y]q_2[y]q_3[y]q_5[y]}{2c^3} -$ 
 $\frac{3q_3[y]q_4[y]q_6[y]}{2c^2} - \frac{3q_1[y]q_5[y]q_6[y]}{4c^2} - \frac{q_3[y]q_4[y]q_1'[y]}{2c^2} - \frac{3q_1[y]q_5[y]q_1'[y]}{4c^2} -$ 
 $\frac{2q_3[y]q_5[y]q_2'[y]}{c^2} - \frac{q_1[y]q_4[y]q_3'[y]}{2c^2} - \frac{q_2[y]q_5[y]q_3'[y]}{c^2} - \frac{3q_3'[y]q_4'[y]}{c} -$ 
 $\frac{3q_1[y]^2q_5'[y]}{4c^2} - \frac{3q_2[y]q_3[y]q_5'[y]}{2c} - \frac{3q_6[y]q_5'[y]}{2c} - \frac{3q_1'[y]q_5'[y]}{2c} - \frac{3q_3[y]q_6'[y]}{4c} -$ 
 $\frac{q_5[y]q_1''[y]}{2c} - \frac{q_4[y]q_3''[y]}{c} - \frac{3q_3[y]q_4''[y]}{c} - \frac{3q_1[y]q_5''[y]}{2c} - q_6^{(3)}[y]$ ,
- $\frac{3q_4[y]q_5[y]q_1'[y]}{4c^2} - \frac{3q_5[y]^2q_2'[y]}{4c^2} - \frac{3q_4[y]^2q_3'[y]}{4c^2} - \frac{3q_3[y]q_4[y]q_4'[y]}{2c^2} -$ 
 $\frac{3q_1[y]q_5[y]q_4'[y]}{4c^2} - \frac{3q_1[y]q_4[y]q_5'[y]}{4c^2} - \frac{3q_2[y]q_3[y]q_5'[y]}{2c^2} -$ 
 $\frac{3q_6[y]q_6'[y]}{2c} - \frac{3q_5[y]q_4''[y]}{4c} - \frac{3q_4[y]q_5''[y]}{4c} - \frac{1}{4}q_6^{(3)}[y]}$ }
    
```

The above equations agree with equations (6.19) and (6.20) in [19]. After a Dirac reduction (since the generators $w(a)$, where $a \in \mathfrak{g}_0^f$, do not evolve in time), we get simpler equations

```
In[32]:= HamEq0 /. {gen[[1]] -> 0, gen[[2]] -> 0, gen[[3]] -> 0}
Out[32]:= {0, 0, 0, q4'[y], q5'[y], q6'[y]}

In[33]:= HamEq1 /. {gen[[1]] -> 0, gen[[2]] -> 0, gen[[3]] -> 0, D[gen[[1]], y] -> 0,
D[gen[[2]], y] -> 0, D[gen[[3]], y] -> 0, D[gen[[1]], y, y] -> 0, D[gen[[2]], y, y] -> 0,
D[gen[[3]], y, y] -> 0}
Out[33]:= {0, 0, 0, - (3 q6[y] q4'[y]) / (2 c) - (3 q4[y] q6'[y]) / (4 c) + q4^{(2)}[y],
- (3 q6[y] q5'[y]) / (2 c) - (3 q5[y] q6'[y]) / (4 c) - q5^{(2)}[y],
- (3 q6[y] q6'[y]) / (2 c) - (3 q5[y] q4''[y]) / (4 c) - (3 q4[y] q5''[y]) / (4 c) - (1/4) q6^{(2)}[y]}
```

The above results agree with equations (6.21) and (6.22) in [19]. The latter is a higher symmetry of the Yajima–Oikawa equation [38] (see also [20]).

5.3 Classical affine \mathcal{W} -algebras associated to simple Lie algebras of rank two and principal nilpotent element

In this section we provide explicit formulas for the λ -brackets among generators for classical affine \mathcal{W} -algebras $\mathcal{W}(\mathfrak{g}, f)$, where $\mathfrak{g} = A_2, B_2$ and G_2 and f is a principal nilpotent. In this case, $\dim \mathfrak{g}^f = \text{rank } \mathfrak{g} = 2$. Hence, by Theorem 3.4, as a differential algebra we have that $\mathcal{W}(\mathfrak{g}, f) = \mathbb{C}[w_1^{(n)}, w_2^{(n)} \mid n \in \mathbb{Z}_+]$, where $w_1 = w(q_1), w_2 = w(q_2)$ and $\{q_1, q_2\}$ is a basis of \mathfrak{g}^f as in Sect. 3.1.

5.3.1 $\mathfrak{g} = \mathfrak{sl}_3$

The computations can be found in the file `A_2_principal.nb`. The result is

$$\begin{aligned} \{w_{1\lambda} w_1\}_z &= (2\lambda + \partial)w_1 - 2c\lambda^3, \\ \{w_{1\lambda} w_2\}_z &= (3\lambda + \partial)w_2 + 3cz\lambda, \\ \{w_{2\lambda} w_2\}_z &= (2\lambda + \partial) \left(\frac{1}{3c} w_1^2 - \frac{1}{16} w_1'' \right) - (2\lambda + \partial)^3 \frac{5}{2^4 3} w_1 + \frac{c}{6} \lambda^3. \end{aligned} \tag{5.1}$$

Note that after rescaling $c \mapsto -\frac{c}{2}$ and setting $L = w_1, W = 8\sqrt{-6C}w_2$, Eq. (5.1) agrees with the results in [25] (only the PVA structure corresponding to $z = 0$ is considered there).

5.3.2 $\mathfrak{g} = \mathfrak{o}_5$

The computations can be found in the file `B_2_principal.nb`. The result is

$$\begin{aligned} \{w_{1\lambda} w_1\}_z &= (2\lambda + \partial)w_1 - 10c\lambda^3, \\ \{w_{1\lambda} w_2\}_z &= (4\lambda + \partial)w_2 + 8cz\lambda, \end{aligned}$$

$$\begin{aligned}
\{w_{2\lambda} w_2\}_z &= (2\lambda + \partial) \left(\frac{2^2 3^2}{5^4 c^2} w_1^3 + \frac{7}{5^2 c} w_1 w_2 - \frac{1}{2^2 5^3 c} (w_1')^2 - \frac{29}{2 \cdot 5^3 c} w_1 w_1'' \right. \\
&\quad \left. - \frac{1}{2^2 5} w_2'' + \frac{3}{2^3 5^2} w_1^{(4)} \right) + (2\lambda + \partial)^3 \left(-\frac{7^2}{2^2 5^3 c} w_1^2 - \frac{3}{2^2 5} w_2 + \frac{7}{2^2 5^2} w_1' \right) \\
&\quad + (2\lambda + \partial)^5 \frac{7}{2^3 5^2} w_1 - \frac{2c}{5} \lambda^7 + z \left((2\lambda + \partial) \frac{2 \cdot 7}{5^2} w_1 - \frac{2^2 3 c}{5} \lambda^3 \right). \quad (5.2)
\end{aligned}$$

Note that after rescaling $c \mapsto -\frac{c}{10}$ and setting $L = w_1$, $W = -40C\sqrt{2}w_2$, Eq. (5.2) agrees with the results in [25]. Since $\mathfrak{o}_5 \cong \mathfrak{sp}_4$, the corresponding classical affine \mathcal{W} -algebras are isomorphic. In fact we can perform the same computations starting from the Lie algebra \mathfrak{sp}_4 , which can be found in the file `C_2_principal.nb`, and check that we get the same expression for the λ -brackets given by Eq. (5.2) after rescaling c by a factor $\frac{1}{2}$.

5.3.3 $\mathfrak{g} = G_2$

The computations can be found in the file `G_2_principal.nb`. The result is

$$\begin{aligned}
\{w_{1\lambda} w_1\}_z &= (2\lambda + \partial) w_1 - 28c\lambda^3, \\
\{w_{1\lambda} w_2\}_z &= (6\lambda + \partial) w_1 + 144cz\lambda, \\
\{w_{2\lambda} w_2\}_z &= \sum_{i=0}^4 (2\lambda + \partial)^{2i+1} P_{2i+1} - \frac{3c}{7} \lambda^{11} + z \left(\sum_{i=0}^1 (2\lambda + \partial)^{2i+1} Q_{2i+1} - \frac{26c}{7} \lambda^5 \right), \quad (5.3)
\end{aligned}$$

where

$$\begin{aligned}
P_1 &= \frac{3^3 5^2}{76 c^4} w_1^5 - \frac{11 \cdot 13}{2 \cdot 7^3 c^2} w_1^2 w_2 - \frac{3 \cdot 61}{2^5 7^4 c^3} w_1^2 (w_1')^2 + \frac{5}{2^3 7^2 c} w_1' w_2' - \frac{3 \cdot 769}{2 \cdot 7^5 c^3} w_1^3 w_1'' \\
&\quad + \frac{3 \cdot 29}{2^3 7^2 c} w_2 w_1'' + \frac{3^2 11 \cdot 19}{2^8 7^4 c^2} (w_1')^2 w_1'' + \frac{3^2 23 \cdot 97}{2^6 7^4 c^2} w_1 (w_1'')^2 + \frac{5}{2^3 7 c} w_1 w_2'' \\
&\quad + \frac{3 \cdot 347}{2^7 7^4 c^2} w_1 w_1' w_1''' - \frac{3^2}{2^8 7^2 c} (w_1''')^2 + \frac{3 \cdot 9551}{2^8 7^4 c^2} w_1^2 w_1^{(4)} - \frac{3^2 \cdot 607}{2^8 7^3 c} w_1' w_1^{(4)} \\
&\quad - \frac{1}{2^4 7} w_2^{(4)} - \frac{3^2 5}{2^8 7^3 c} w_1' w_1^{(5)} - \frac{3 \cdot 5 \cdot 6 \cdot 7}{2^8 7^3 c} w_1 w_1^{(6)} + \frac{3^2 5}{2^{10} 7^2} w_1^{(8)}, \\
P_3 &= -\frac{3 \cdot 11 \cdot 479}{2^8 7^4 c^3} w_1^4 + \frac{5 \cdot 31}{2^3 7^2 c} w_1 w_2 + \frac{3 \cdot 5 \cdot 11 \cdot 19}{2^8 7^4 c^2} w_1 (w_1')^2 + \frac{3 \cdot 11 \cdot 23 \cdot 89}{2^7 7^4 c^2} w_1^2 w_1'' \\
&\quad - \frac{3^3 11 \cdot 49}{2^8 7^3 c} (w_1')^2 - \frac{5}{2^3 7} w_2'' - \frac{3 \cdot 5 \cdot 11}{2^7 7^3 c} w_1' w_1''' - \frac{3 \cdot 5^2 11^2}{2^8 7^3 c} w_1 w_1^{(4)} + \frac{3 \cdot 5 \cdot 11}{2^8 7^2} w_1^{(6)}, \\
P_5 &= \frac{3 \cdot 5 \cdot 11 \cdot 139}{2^8 7^4 c^2} w_1^3 - \frac{13}{2^4 7} w_2 - \frac{3^3 11}{2^9 7^3 c} (w_1')^2 - \frac{3^3 11 \cdot 43}{2^8 7^3 c} w_1 w_1'' + \frac{3^4 11}{2^9 7^2} w_1^{(4)}, \\
P_7 &= -\frac{3^2 11 \cdot 31}{2^9 7^3 c} w_1^2 + \frac{3^3 11}{2^8 7^2} w_1'', \quad P_9 = \frac{3 \cdot 5 \cdot 11}{2^{10} 7^2} w_1, \\
Q_1 &= -\frac{11 \cdot 13}{2 \cdot 7^3 c} w_1^2 + \frac{3 \cdot 29}{2^3 7^2} w_1'', \quad Q_3 = \frac{5 \cdot 31}{2^3 7^2} w_1.
\end{aligned}$$

The bi-Poisson structure of the classical \mathcal{W} -algebra $\mathcal{W}(G_2, f)$ associated to the Lie algebra G_2 and its principal nilpotent element f , and the corresponding Drinfeld–Sokolov hierarchy

was already computed in [8]. It can be obtained from the bi-Poisson structure (5.3) by performing the change of variables

$$u_0 = \frac{w_1}{2c}, \quad u_1 = \frac{w_2}{72c} + \frac{3}{686} \left(\frac{w_1}{2c}\right)^3 - \frac{33}{1568} \left(\frac{w_1'}{2c}\right)^2 - \frac{13}{392} \frac{w_1}{2c} \frac{w_1''}{2c} + \frac{1}{57} \frac{w_1^{(4)}}{2c},$$

by choosing $c = 4$, and by substituting z with $-72z$.

6 List and explanation of commands

6.1 List of commands in MasterPVA

In this section we list the commands provided by *MasterPVA* and *MasterPVAmulti*. Most of the commands are the same for both the versions of the package, and the syntax working for the $D = 1$ case works the same also when using the multidimensional package; on the other hand, it must be modified accordingly when working with a $D > 1$ λ bracket.

`SetN[n_Integer]` declares the number N of the generators for the PVA. Its default value is 1.

`GetN[]` gives the number N of the generators.

`SetD[d_Integer]` declares the number D of the derivations (namely, of the independent variables) for the PVA. Its default value is 1. *Available only in MasterPVAmulti.*

`GetD[]` gives the number D of the derivations. *Available only in MasterPVAmulti.*

`SetMaxO[n_Integer]` declares the order of the derivatives of the generators up to which the code computes the λ brackets by the Master Formula. Default is 5, quite high for most of the applications.

`GetMaxO[]` gives the maximum order of the derivatives of the generators taken by the program.

`SetGenName[newname]` declares the name for the generators. Default is u . They will have the form $u(x)$ if $N = 1$ or $u_1(x), \dots, u_N(x)$ for $N > 1$.

`GetGenName[]` gives the name used for the generators.

`SetVarName[newname]` declares the name for the independent variable(s). Default is x .

`GetVarName[]` gives the name used for the independent variable(s).

`gen` is the list of generators for the PVA.

`var` is the list of the independent variables. *Available only in MasterPVAmulti.*

`SetFormalParameter[newname]` declares the name for the parameter to be used (and recognized by the software) in the definition of the bracket between generators. Default is β ; notice that for $D > 1$ the parameter will be a list $(\beta_1, \dots, \beta_D)$.

`GetFormalParameter[]` gives the name of the parameter used in the definition of the λ bracket.

`LambdaB[f, g, P, λ]` computes the λ bracket between the two differential polynomials f and g , with P the matrix of the brackets between the generators. The result will be a polynomial in the formal indeterminate λ (or $(\lambda_1, \dots, \lambda_D)$ for $D > 1$). The Master Formula will take into account the derivatives of the generators up to order $n = \text{GetMaxO}[]$.

`PVASKew[P]` computes the condition of skewsymmetry for a λ bracket (namely the LHS of (2.9)) and gives the result in a matrix form.

`PrintPVASKew[P]` computes the condition of skewsymmetry and gives the result as a table with each equation of the system.

`JacobiCheck[P]` computes the LHS of the Jacobi identity (2.10), and gives the result as a $N \times N \times N$ array. The entries are given as formal polynomials in the (internal) indeterminates λ and μ . It is often convenient to clean up the result using the command `%%/. {MasterPVA`Private`\lambda->\lambda, MasterPVA`Private`\mu->\mu}`.

`PrintJacobiCheck[P]` computes the LHS of Jacobi identity (2.10) and gives the result as a table of expressions that must vanish.

`EvVField[X_List, f]` applies the evolutionary vector field of characteristic X^i , $i = 1, \dots, N$ to the differential polynomial f .

`Integr[f, param_List]` transforms a polynomial in the indeterminates `param` = $\{\lambda, \mu, \dots, \psi, \omega\}$ in a polynomial in $\{\lambda, \dots, \psi\}$ substituting ω with $-\lambda - \mu - \dots - \psi - \partial$, where ∂ acts on the coefficients. For $D > 1$ case, each of the parameter must be replaced by a list of D entries. This auxiliary function is convenient in the study of the skewsymmetry, since $\{u_{i-\lambda-\partial}u_j\}$ can be obtained by `Integr[LambdaB[gen[[i]], gen[[j]], P, \mu], {\lambda, \mu}]` or for the study of the PVA cohomology (see [15]).

6.2 List of commands in WAlg

In this section we list the commands provided by *WAlg*. We discuss separately the commands that constitute the main core of the program and the ones that can have broader applications, for instance to prepare the input the program needs.

Please note that the symbols `q, y, z, \[ScriptS]` (i. e. s) and `\[Beta]` (i. e. β) are used by the program, hence they should not be used as variable names in your program.

6.2.1 Principal commands of the program

`InitializeWAlg[name_String, n_Integer]` is the first command that the program must receive after loading the package. It sets the simple Lie algebra \mathfrak{g} underlying the classical affine \mathcal{W} -algebra. If, for instance, one would like to start from A_6 , the command should be `InitializeWAlg["A", 6]`.

`SetNil[a_List]` sets the nilpotent element $f \in \mathfrak{g}$ in order to construct $\mathcal{W}(\mathfrak{g}, f)$.

`GetNil[]` gives the nilpotent element f used in the definition of the classical affine \mathcal{W} -algebra.

`GetDim[]` gives the dimension of the matrices used for the explicit representation of \mathfrak{g} .

`SetS[s_List]` sets the element $s \in \mathfrak{g}$ used in the definition of the affine PVA, as in (3.8). If the command is given without argument, it automatically chooses a generic element of \mathfrak{g} . Notice that the command must be called before computing the λ -brackets between the generators of the classical affine \mathcal{W} -algebra.

`GetS[]` gives the element s used in (3.8) after it has been set.

`ComputeWAlg[nil_List]` computes a basis for \mathfrak{g}^f made by $\text{ad } x$ -eigenvectors, where $h = 2x$ is the diagonal element of the \mathfrak{sl}_2 -triple containing $f = \text{nil}$, as well as the dual basis (with respect to the trace form) of \mathfrak{g}^e and the corresponding $\text{ad } x$ -eigenvalues (with multiplicities).

All these outputs can be displayed by using the next three commands:

`GetWBasis[]` gives the list of elements of the aforementioned basis for \mathfrak{g}^f ;

`GetWBasisDual[]` gives the list of elements of the dual basis of \mathfrak{g}^e ;

`GetWEigen[]` gives the list of $\text{ad } x$ -eigenvalues.

`GetX[]` gives the element $x = h/2$, where h is the diagonal element of the \mathfrak{sl}_2 -triple associated to f ; it can be used only after executing the command `ComputeWAlg[]`.

`w[a_List]` given an element $a \in \mathfrak{g}$, it applies the projection $\pi_{\mathfrak{g}^f}$ to it and then the map w defined in Theorem 3.4.

`GenerateH[par_]` must be run after `ComputeWAlg[]` and `SetS[]`. It computes the Poisson structure H defined by Eq. (3.17) using Theorem 3.5. It uses `par` as the formal indeterminate (the default is β).

`LoadTableIndices[filename_String]` chooses a file different from the default (`listK_6.txt`) as the source of the indices used in the formula (3.20). It is necessary to use it (after generating the suitable file) when $d > 6$, see Sect. 6.3.

`GenerateTableIndices[n_Integer]` computes a custom list of indices going up to n , and saves it in the file `listK_n.txt` for further usage. Notice that the computation is extremely time-consuming, see Sect. 6.3.

`GetVirasoro[nil_List]` provides the Virasoro element of Proposition (2.6)(c) with $f = \text{nil}$.

6.2.2 Other useful commands

`Comm[a_, b_]` computes the commutator between the two matrices a and b .

`Prod[a_, b_]` computes the value of the symmetric invariant bilinear form $\text{Tr}(ab)$.

`Proj[a_List]` applies the map $\pi_{\mathfrak{g}^f}$ to an element $a \in \mathfrak{g}$. It must be run after `ComputeWAlg`.

`M[i_Integer, j_Integer]` gives the elementary matrix (of dimensions `GetDim[]`) with 1 in the position (i, j) .

`CheckAlg[a_List]` checks whether the matrix a belongs to the Lie algebra declared in `InitializeWAlg[]`.

`Sigma[a_List]` computes $\sigma(a)$ according to the definition given in Sect. 5.

`SetDispPar[s_]` sets a dispersive parameter (default is 1, hence making it invisible) in formula (3.20), useful if we want to compute the dispersionless limit of this formula.

`GetDispPar[]` gives the aforementioned dispersive parameter.

6.3 Generation of the indices

For $a \in \mathfrak{g}_{-l}^f$ and $b \in \mathfrak{g}_{-m}^f$, formula (3.20) involves a long summation over the indices $(\vec{j}, \vec{n}) \in J_{-\vec{k}}$, where $\vec{k} = (k_1, \dots, k_t)$, $-l + 1 \leq k_1 < \dots < k_t \leq m$. For each $t \geq 1$, the list of the indices k_1, \dots, k_t is finite, and moreover given l and m the maximum value for t is the first integer \bar{t} such that $\bar{t} \geq l + m$.

The generation of the indices \vec{k} is a long process, and it dramatically slows down the execution time of the command `GenerateH[]`. To prevent this issue, a precompiled list of indices is distributed together with the package, in the file `listK_6.txt`. It contains the default data for the computation of formula (3.20), and it works for all classical affine \mathcal{W} -algebras with $d \leq 6$ (this is sufficient, for example, to compute the classical affine \mathcal{W} -algebras associated to all nilpotent orbits of $\mathfrak{o}_{\mathfrak{g}}$). If one wants to work with Lie algebras with $d > 6$, then it is necessary to generate a bigger table of indices, that may be computed before starting the computation of the Poisson structure H , and not necessarily in an interactive session. In case the user does not notice that a bigger set of indices would be needed, the command `GenerateH[]` will produce a long list of error messages.

The Mathematica kernel, without the user interface, can be usually run in a shell with the command `math`. After loading the package, one generates the table of indices with the command

GenerateTableIndices[d].

The command can take up to several hours to be completed, and generates a file `listK_d.txt` saved in the active folder. To use a previously generated table of indices, the command `LoadTableIndices[filename]` must be run before `GenerateH[]`. The file will be looked for in the active folder, unless the full path is specified.

```
in[4]:= LoadTableIndices["listK_4.txt"]
```

```
Table of indices loaded
```

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