GROUP RINGS WITH METABELIAN UNIT GROUPS

Tibor JUHÁSZ - Ernesto SPINELLI [1]

ABSTRACT. Let F be a field of odd characteristic and G a group. In 1991 Shalev established necessary and sufficient conditions so that the unit group of the group ring FG is metabelian when G is finite. Here, in the modular case, we do the same without restrictions on G. In particular, new cases emerge when G contains elements of infinite order.

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1. INTRODUCTION

Let FG be the group ring of a non-abelian group G over a field F of characteristic $p > 0$. A classical problem of interest in the study of group rings is to classify the groups G such that the unit group, $\mathcal{U}(FG)$, of FG satisfies certain identities. In this direction, the conditions under which $\mathcal{U}(FG)$ is solvable were determined in a series of papers over many years, beginning with Bateman [1] (dealing with finite groups) and concluding with A. Bovdi [2], thanks to the key contribution of Liu and Passman [9] characterizing when $\mathcal{U}(FG)$ satisfies a group identity. For a discussion of this and related results, we refer to Chapters 1 and 6 of [7].

On the assumption that $\mathcal{U}(FG)$ is solvable, it is natural to ask about its derived length, $\mathrm{dl}(\mathcal{U}(FG))$, but the picture is not as clear here. Indeed, it seems quite difficult to give a general formula, and just a few results have been proved. Most of them concern the investigation of the structure of FG for small values of $\mathrm{dl}(\mathcal{U}(FG))$. In this setting, one of the first results was due to Shalev [12], who classified the finite groups G such that $\mathcal{U}(FG)$ is metabelian, when $p > 2$. Namely he proved that this occurs if, and only if, $p = 3$ and the commutator subgroup, G', of G is central of order p. Some years later, Catino and Spinelli [3] extended this statement to torsion groups.

In the same paper they provided a lower bound for $\mathrm{dl}(\mathcal{U}(FG))$ in terms of p when G is a torsion nilpotent group, and characterized those groups for which it is attained. Very recently in [6] the same problem was investigated when G contains elements of infinite order, but in this setting the situation seems to be more involved and other considerations are in order (as one can expect from the fact that the conditions for $\mathcal{U}(FG)$ to be solvable are different here). More precisely, when FG is modular, that is, G has an element of order p, the lower bound established for $\mathrm{dl}(\mathcal{U}(FG))$ is in some

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cases different from that of the torsion case. As noted again in [6], it is not really sensible to ask the same question when FG is not modular: in fact, we can have any derived length we want, regardless of the characteristic.

The aim of this note is to generalize Shalev's result classifying modular group rings whose unit group is metabelian without further restrictions on G. This provides also the list of non-commutative group rings whose unit group has the smallest possible derived length when $p \in \{3, 5\}$. We stress that new cases emerge when G contains elements of infinite order, more specifically when $G' = \mathrm{Syl}_p(G)$. In order to clarify this point, we recall that a Sylow p-subgroup of a possibly infinite group G is defined to be a maximal p-subgroup of G, and it is unique if G is nilpotent. In this case, let us denote it by $\mathrm{Syl}_p(G)$. Under the extra assumption that G is torsion, G is the direct product of its Sylow subgroups. If $G' = \mathrm{Syl}_p(G)$, then all of the other Sylow subgroups are abelian, thus $\mathrm{Syl}_p(G)' = \mathrm{Syl}_p(G)$. But for a nilpotent group, this implies that $\mathrm{Syl}_p(G) = \{1\}$, so then G is abelian. The final outcome of these deductions is that for a non-abelian nilpotent group G the condition $G' = \mathrm{Syl}_p(G)$ is satisfied only if G is non-torsion. The main result we prove is the following

Theorem 1.1. *Let F be a field of characteristic $p \geq 3$ and G a non-abelian group such that FG is modular. Then $\mathcal{U}(FG)$ is metabelian if, and only if, G is nilpotent of class 2 and either*

(a) *$p = 3$ and G' has order p,*
(b) *$p = 3$ and $G' = Syl_p(G)$ is elementary abelian of order p^2, or*
(c) *$p = 5$ and $G' = Syl_p(G)$ has order p.*

2. Preliminaries

The aim of this section is to establish some notation and discuss some results necessary for the proof of Theorem 1.1. In any group G, we let $(g, h) = g^{-1}h^{-1}gh$. We also write $\gamma_n(G)$ and $\delta_n(G)$ for the terms of the lower central and derived series of G, respectively, and $\xi(G)$ for its center. If S, T are subsets of G, set $(S, T) = \langle (s, t) \mid s \in S,\ t \in T \rangle$ and, if n is a positive integer, $S^n = \langle s^n \mid s \in S \rangle$. Finally, let us denote by C_n the cyclic group of order n. In any ring, let $[a_1, a_2] = a_1a_2 - a_2a_1$.

Throughout the paper, unless otherwise stated, F is a field of characteristic $p > 2$ and G a group. When G is torsion, $\mathcal{U}(FG)$ is solvable if, and only if, G' is a finite p-group, provided $|F| > 3$ (see Theorem 6.2.8 of [7]). The characterization for non-torsion group rings is more involved and we confine ourselves to report here a partial result one can deduce from Lemma 6.2.11 and Theorem 6.2.13 of [7].

Lemma 2.1. *Let G be a non-abelian non-torsion group. If $\mathcal{U}(FG)$ is solvable, then the p-elements of G form a (normal) subgroup P of G. Furthemore, if P is infinite, $\mathcal{U}(FG)$ is solvable if, and only if, G' is a finite p-group.*

The following statement forms the starting point of our investigation. It was originally proved by Shalev for finite groups (Theorems A and B of [12]), then extended to torsion groups in Proposition 7 of [3].

Theorem 2.2. *Let G be a non-abelian torsion group. Then $\mathcal{U}(FG)$ is metabelian if, and only if, $p = 3$ and G' is central of order p.*

We stress that, if $p = 3$ and G has central commutator subgroup of order 3, then $\mathcal{U}(FG)$ is metabelian without further restrictions on G. This is part of the content of Corollary 2.2 of [12] (even if there the group G was assumed to be finite, the hypothesis on the order of G did not matter for the proofs of the results of Section 2).

Evidently, if $\mathcal{U}(FG)$ is metabelian, then so is $\mathcal{U}(FH)$ for any subgroup H of G. Furthermore, by Lemma 2 of [8], this is still true for $\mathcal{U}(F(G/N))$, where N is any finite normal subgroup of G which is either a p-group or a p'-group. In what follows, we shall freely use these facts and replace G with H or G/N when this is convenient.

Assume in the sequel that $\mathcal{U}(FG)$ is solvable. We need to know some computational aspects of the derived length of $\mathcal{U}(FG)$. In particular, in the Introduction of [3], it was observed that

$$\mathrm{dl}(\mathcal{U}(FG)) \geq \lceil \log_2(p+1) \rceil,$$

provided G is a non-abelian torsion nilpotent group. The result is different when G contains elements of infinite order, as presented in the following

Lemma 2.3. *Let G be a non-abelian nilpotent group such that FG is modular. If G has an element of infinite order, then*

(a) $\mathrm{dl}(\mathcal{U}(FG)) > \lceil log_2(p+1) \rceil$ *if G' is not a finite p-group, and*
(b) $\mathrm{dl}(\mathcal{U}(FG)) \geq \lceil log_2(\frac{2}{3}(p+1)) \rceil$ *otherwise. Furthermore,* $\mathrm{dl}(\mathcal{U}(FG)) > \lceil log_2(\frac{2}{3}(p+1)) \rceil$ *if $p > 3$ and $|G'| = p^n$ for some $n > 1$.*

Proof. The statement (a) is Theorem 1 of [6], whereas (b) follows combining Theorems 2 and 3 of [6]. □

If N is a normal subgroup of G, let $\Delta(G, N)$ be the kernel of the natural homomorphism $FG \to F(G/N)$, and $\Delta(G) = \Delta(G, G)$. We also write $\Delta^k(G, N)$ for the k-th power of $\Delta(G, N)$. When G is a finite p-group, it is well-known (see, for instance, Lemma 1.1.1 of [7]) that $\Delta(G)$ is nilpotent and denote its nilpotency index by $t(G)$. Under the extra assumption that G is also abelian, let us say $G \cong C_{p^{n_1}} \times C_{p^{n_2}} \times \cdots \times C_{p^{n_m}}$, by Jennings' theory [4] one has that

$$t(G) = 1 + \sum_{i=1}^{m} (p^{n_i} - 1).$$

When necessary, we shall use this formula without further reference.

In [5], Juhász studied $\mathrm{dl}(\mathcal{U}(FG))$ for special classes of groups, in particular for those whose commutator subgroup is cyclic.

Lemma 2.4. *Let G be a non-abelian nilpotent group such that G' is a finite abelian p-group.*

(a) *If* G' *is cyclic, then* $\mathrm{dl}(\mathcal{U}(FG)) \geq \lceil log_2(\frac{2}{3}(t(G')+1))\rceil$*;*
(b) *If* $G' = Syl_p(G)$ *and* $\gamma_3(G) \subseteq (G')^p$*, then* $\mathrm{dl}(\mathcal{U}(FG)) \leq \lceil log_2(\frac{2}{3}(t(G')+1))\rceil$.

In particular, if $G' = Syl_p(G)$ *is cyclic, then* $\mathrm{dl}(\mathcal{U}(FG)) = \lceil log_2(\frac{2}{3}(t(G')+1))\rceil$.

Proof. See Lemma 5 and Theorem 1 of [5], respectively. □

Part of the proof of our main result involves group and Lie commutator computations. We shall freely use in the sequel that, if a, b, c are elements of a ring,

$$[ab, c] = a[b, c] + [a, c]b,$$

and, if a, b are units,

$$(a, b) = 1 + a^{-1}b^{-1}[a, b] \qquad \text{and} \qquad [a, b] = ba((a, b) - 1).$$

We report now a couple of technical results. The first of them holds for arbitrary group rings. It was proved in Lemma 2 of [6].

Lemma 2.5. *Let* N *be a normal subgroup of* G*. If* $n \in N$*, then for any positive integer* i*,* $n^i - 1 \equiv i(n-1) \pmod{\Delta^2(G, N)}$.

For the second one we need a restriction on the commutator subgroup of G (it is not necessary here to assume that $\mathcal{U}(FG)$ is solvable).

Lemma 2.6. *Let* G *be a group whose commutator subgroup is abelian with* $\gamma_3(G) \subseteq (G')^p$*. Then*
(a) $[\Delta^m(G, G'), \Delta^k(G, G')] \subseteq \Delta^{m+k+1}(G, G')$ *for all* $m, k \geq 1$*;*
(b) *if* $a \in G$ *is a* p*-element, then, for all* $b \in G$*,*

$$[(1+a)^{-1}a, b]$$
$$\equiv (1+a)^{-p} b \left(\sum_{i=1}^{p-1} (-1)^{i+1} i a^i \right) ((a, b) - 1) \pmod{\Delta^2(G, G')}.$$

Proof. (a) This statement is Lemma 1 (iv) of [5].
(b) Set $x = 1 + a$. Evidently, x is a unit and $x^{-1}a = x^{-1}(1+a) - x^{-1} = 1 - x^{-1}$. For any $b \in G$ one has that

$$(1) \quad [x^{-1}a, b] = -[x^{-1}, b] = -[x^{-p}x^{p-1}, b] = -x^{-p}[x^{p-1}, b] - [x^{-p}, b]x^{p-1}$$

and

$$(2) \quad [x^{-p}, b] = -x^{-p}[x^p, b]x^{-p} = -x^{-p}[a^p, b]x^{-p} = -x^{-p}ba^p((a^p, b) - 1)x^{-p}.$$

Using the group commutator identity $(uv, w) = (u, w)(u, w, v)(v, w)$, an easy induction argument shows that, for any positive integer i, $(a^i, b) = (a, b)^i d$ for some $d \in \gamma_3(G)$. Consequently,

$$(a^i, b) - 1 = (a, b)^i d - 1 = ((a, b)^i - 1)(d - 1) + ((a, b)^i - 1) + (d - 1).$$

The fact that $\gamma_3(G) \subseteq (G')^p$ implies that $d - 1 \in \Delta(G, (G')^p) \subseteq \Delta^p(G, G') \subseteq \Delta^2(G, G')$, and from Lemma 2.5 it follows that

$$(3) \qquad (a^i, b) - 1 \equiv (a, b)^i - 1 \equiv i((a, b) - 1) \pmod{\Delta^2(G, G')}.$$

In particular, $(a^p,b)-1\in\Delta^2(G,G')$ so, by (2), $[x^{-p},b]\in\Delta^2(G,G')$ and, by (1),

$$[x^{-1}a,b]\equiv -x^{-p}[x^{p-1},b] \pmod{\Delta^2(G,G')}.$$

Taking into account that $\binom{p-1}{i}\equiv(-1)^i \pmod p$, we get

$$\begin{aligned}[x^{-1}a,b] &\equiv -x^{-p}[(1+a)^{p-1},b]\equiv -x^{-p}\sum_{i=0}^{p-1}(-1)^i[a^i,b]\\ &\equiv x^{-p}b\sum_{i=1}^{p-1}(-1)^{i+1}a^i((a^i,b)-1) \pmod{\Delta^2(G,G')},\end{aligned}$$

and, invoking (3), the desired conclusion holds. □

For any group G, let us consider the sequence of dimension subgroups of FG, whose n-th term (which is a normal subgroup of G) is defined as

$$D_n(G)=\{g\mid g\in G,\ \ g-1\in\Delta^n(G)\}.$$

Obviously, $D_1(G)=G$ and, according to Theorem 11.1.19 of [10], recursively

$$D_n(G)=(D_{n-1}(G),G)\cdot D_i(G)^p, \tag{4}$$

where i is the smallest integer satisfying $ip\geq n$. This sequence has been extensively studied in literature and we refer to Chapter 3 of [10] for a summary of its main properties.

For our aims, we need the following result, which easily follows from Theorem 3.3.7 and Lemma 3.3.5 of [10].

Lemma 2.7. *Let G be a finite p-group, $g\in G$ and m the largest integer such that $g\in D_m(G)$. Then, for any $1\leq\alpha<p$, one has that $(g-1)^\alpha\in\Delta^{\alpha m}(G)\setminus\Delta^{\alpha m+1}(G)$.*

3. Proof of the Main Theorem

The aim of this section is to present a proof of Theorem 1.1. As a first step, we shall establish the necessary conditions of the statement under the extra assumption that G is a nilpotent group.

To this end, according to Theorem 2.2 and Lemma 2.3, one has to consider only when F has characteristic $p\in\{3,5\}$ and G' is a finite p-group. To attack the more delicate $p=3$ case, we need a couple of very easy group theoretical lemmas, which could be of independent interest.

Lemma 3.1. *Let G be a nilpotent group of class 2 with $G'\cong C_p\times C_p$. Then there exist $a,b,c\in G$ such that $G'=\langle(a,b),(b,c)\rangle$.*

Proof. Let $s,t,w,z\in G$ such that $G'=\langle(s,t),(w,z)\rangle$. Assume first that $(t,z)\neq1$. If $(t,z)\in\langle(s,t)\rangle$, then $G'=\langle(t,z),(w,z)\rangle=\langle(t,z),(z,w)\rangle$, otherwise $G'=\langle(s,t),(t,z)\rangle$. Obviously the same considerations apply when $(s,w)\neq1$.

Finally, suppose that $(t,z)=(s,w)=1$. In this case, it is easily seen that $G'=\langle(s,tw),(tw,z)\rangle$. □

Lemma 3.2. *Let G be a nilpotent group such that $G'\cong C_3\times C_3$. Then G is 2-Engel if, and only if, G' is central.*

Proof. Assume that G is 2-Engel and, if possible, that $\gamma_3(G) \neq \langle 1 \rangle$. Then there exist $a, b, c \in G$ such that $((a,b),c) \neq 1$ and $G' \setminus \gamma_3(G) = x\gamma_3(G) \cup x^2\gamma_3(G)$, where $x = (a,b)$. By [11, 12.3.6], one has that $((a,b),c) = ((c,a),b)$. Therefore $((c,a),b) \neq 1$, and hence $(c,a) \in G' \setminus \gamma_3(G)$. Thus (c,a) can be written as xy or x^2y for some $y \in \gamma_3(G) \subseteq \zeta(G)$. If $(c,a) = xy$, then

$$1 \neq ((c,a),b) = (xy,b) = (x,b) = ((a,b),b).$$

When $(c,a) = x^2y$, one similarly gets that $((a,b),b)^2 \neq 1$. But both these conclusions contradict the hypothesis on G.

Therefore, G' has to be central and, as the converse is trivial, the proof is done. □

We are now in a position to establish when $\mathcal{U}(FG)$ is metabelian in characteristic 3 for small values of the order of G'.

Lemma 3.3. *Let $p = 3$, and G a non-abelian nilpotent group such that G' is a finite 3-group with $|G'| \leq 9$. If $\mathcal{U}(FG)$ is metabelian, then G' is central and either $|G'| = 3$, or $Syl_3(G) = G' \cong C_3 \times C_3$.*

Proof. By the assumptions one has that $|G'| \in \{3, 9\}$. If $|G'| = 3$, we are done. Hence suppose that $|G'| = 9$. By virtue of Lemma 2.4 (a), G' cannot be cyclic. Therefore $G' \cong C_3 \times C_3$.

Assume first that $G' \subseteq \zeta(G)$, but $G' \neq Syl_3(G)$. Our goal is to exhibit in this case a non-trivial element of $\delta_2(\mathcal{U}(FG))$. According to Lemma 3.1, we can suppose that $G' = \langle (a,b), (b,c) \rangle$ for some $a, b, c \in G$. Set $x = (a,b)$ and $y = (b,c)$.

Assume first that $a \in Syl_3(G)$. Then $1 + a \in \mathcal{U}(FG)$ and

$$u = (1+a, b) = 1 + (1+a)^{-1}b^{-1}[a,b] = 1 + (1+a)^{-1}a(x-1).$$

Take $v = (1 + b(y-1), c)$. Using the fact that y is central, one has

$$\begin{aligned} v =& 1 + (1 + b(y-1))^{-1}c^{-1}[b(y-1), c] \\ &= 1 + (1 + b(y-1))^{-1}c^{-1}[b,c](y-1) \\ &= 1 + (1 + b(y-1))^{-1}b(y-1)^2. \end{aligned}$$

Since $1 + b(y-1) \in 1 + \Delta(G, G')$, the element $(1 + b(y-1))^{-1} \in 1 + \Delta(G, G')$ and, hence,

$$v \equiv 1 + b(y-1)^2 \pmod{\Delta^3(G, G')}.$$

Consequently

$$\begin{aligned} (u,v) &= 1 + u^{-1}v^{-1}[u,v] \\ &= 1 + u^{-1}v^{-1}[(1+a)^{-1}a(x-1), b(y-1)^2 + \omega], \end{aligned}$$

for some $\omega \in \Delta^3(G, G')$. From Lemma 2.6 (a) it follows that

$$[(1+a)^{-1}a(x-1), \omega] \in \Delta^5(G, G') = \{0\}$$

and hence, as $x, y \in \zeta(G)$,

$$(u,v) = 1 + u^{-1}v^{-1}[(1+a)^{-1}a, b](x-1)(y-1)^2.$$

Furthermore Lemma 2.6 (b) yields

$$[(1+a)^{-1}a, b] \equiv (1+a)^{-3}b(a+a^2)(x-1) \pmod{\Delta^2(G, G')}.$$

Since u and v are in $1+\Delta(G, G')$, so are their inverses. Thus, using the fact that $ab \equiv ba \pmod{\Delta(G, G')}$ and again that $\Delta^5(G, G') = \{0\}$, we get

$$(u, v) = 1 + (1+a)^{-2}ba(x-1)^2(y-1)^2.$$

As $(1+a)^{-2}ba$ is a unit and $(x-1)^2(y-1)^2 \neq 0$, one has that $1 \neq (u, v) \in \delta_2(\mathcal{U}(FG))$, and the desired conclusion holds.

Therefore suppose that a cannot be chosen from $Syl_3(G)$, that is $(g, b) \in \langle y \rangle$ for any $g \in Syl_3(G)$. Take $f \in Syl_3(G) \setminus G'$ such that $f^3 \in G'$. Then $w = 1 + (f-1)^2 a$ is a unit with $w^{-1} \equiv 1 - (f-1)^2 a \pmod{\Delta(G, G')}$, and

$$\begin{aligned}[(f-1)^2 a, b] &= (f-1)^2[a, b] + [(f-1)^2, b]a \\ &= (f-1)^2 ba(x-1) + \theta(y-1)\end{aligned}$$

for some $\theta \in FG$. Now, since $(f-1)^3 \in \Delta(G, G')$ and $af \equiv fa$ and $bf \equiv fb$ $\pmod{\Delta(G, G')}$,

$$\begin{aligned}u = (w, b) &= 1 + w^{-1}b^{-1}[w, b] = 1 + w^{-1}b^{-1}[(f-1)^2 a, b] \\ &= 1 + w^{-1}b^{-1}(f-1)^2 ba(x-1) + \theta'(y-1) \\ &\equiv 1 + (1-(f-1)^2 a)(f-1)^2 a(x-1) + \theta'(y-1) \\ &\equiv 1 + (f-1)^2 a(x-1) + \theta'(y-1) \pmod{\Delta^2(G, G')},\end{aligned} \tag{5}$$

for some $\theta' \in FG$. If v is as above, combining Lemma 2.6 (a) with the fact that $(y-1)^3 = 0$ (and the already used arguments on u^{-1}, v^{-1} and $t(G')$), one has

$$\begin{aligned}(u, v) &= 1 + u^{-1}v^{-1}[(f-1)^2 a(x-1), b(y-1)^2] \\ &= 1 + u^{-1}v^{-1}[(f-1)^2 a, b](x-1)(y-1)^2 \\ &= 1 + u^{-1}v^{-1}(f-1)^2 ba(x-1)^2(y-1)^2 \\ &= 1 + (f-1)^2 ba(x-1)^2(y-1)^2.\end{aligned}$$

By Lemma 3.1.2 of [10], the annihilator of $\sum_{g \in G'} g = (x-1)^2(y-1)^2$ is $\Delta(G, G')$. But $(f-1)^2 ba \notin \Delta(G, G')$, thus (u, v) is a non-trivial element of $\delta_2(\mathcal{U}(FG))$.

It remains to deal with the case in which G' is not central. By virtue of Lemma 3.2, we know that G is not 2-Engel. Hence we can choose $a, b \in G$ such that $t = ((a, b), b) \neq 1$. Evidently, t is central and $1 + b(t-1)$ is a unit of order 3 with

$$(1 + b(t-1))^{-1} = (1 + b(t-1))^2 = 1 - b(t-1) + b^2(t-1)^2.$$

Set again $x = (a, b)$, then

$$\begin{aligned}u = (1 + b(t-1), a) &= 1 + (1 + b(t-1))^{-1}a^{-1}[b, a](t-1) \\ &= 1 + (1 + b(t-1))^{-1}b(x^{-1}-1)(t-1) \\ &= 1 + b(x^{-1}-1)(t-1) - b^2(x^{-1}-1)(t-1)^2.\end{aligned}$$

Since $u^{-1} \in 1 + \Delta(G, \langle t \rangle)$ and $\Delta^3(G, \langle t \rangle) = \{0\}$,

$$\begin{aligned}(u, x) &= 1 + u^{-1}x^{-1}[u, x] \\ &= 1 + u^{-1}x^{-1}[b(x^{-1} - 1)(t - 1) - b^2(x^{-1} - 1)(t - 1)^2, x] \\ &= 1 + u^{-1}x^{-1}[b, x](x^{-1} - 1)(t - 1) - u^{-1}x^{-1}[b^2, x](x^{-1} - 1)(t - 1)^2 \\ &= 1 + u^{-1}b(t^{-1} - 1)(x^{-1} - 1)(t - 1) \\ &= 1 + b(t^{-1} - 1)(x^{-1} - 1)(t - 1)\end{aligned}$$

is a non-trivial element of $\delta_2(\mathcal{U}(FG))$, which concludes the proof. □

Let us show that no other cases arise in characteristic 3.

Lemma 3.4. *Let $p = 3$ and G a nilpotent group whose commutator subgroup has order 3^n with $n \geq 3$. Then $\mathcal{U}(FG)$ is not metabelian.*

Proof. Assume, if possible, that $\mathcal{U}(FG)$ is metabelian. Since G is nilpotent, G' has a central element z of order 3. We factor out $\langle z \rangle$ and, replacing G with $G/\langle z \rangle$, repeat until $|G'| = 27$. Factorizing once more in this manner, from Lemma 3.3 one has that $(G/\langle z \rangle)'$ is isomorphic to $C_3 \times C_3$ and central. Therefore, $|\gamma_3(G)| \leq 3$.

Suppose first that $G' \cong C_9 \times C_3$. Pick elements $a, b \in G$ such that $x = (a, b)$ has order 9 and set $H = \langle a, b \rangle$. If $|H'| = 9$, then H' is cyclic and, by virtue of Lemma 2.4 (a), $\mathcal{U}(FH)$ is not metabelian, which is not allowed. Therefore we must have $|H'| = 27$. Hence one of the commutators (x, a) and (x, b) does not belong to $\langle x \rangle$. Assume, for instance, that $y = ((a, b), b)$ is a non-trivial central element of $\gamma_3(H)$ (analogous arguments apply in the other case). Proceeding as in the last case of the proof of Lemma 3.3, it is easily shown that

$$1 \neq ((1 + b(y - 1), a), x) \in \delta_2(\mathcal{U}(FG)).$$

Consequently, G' has to be elementary abelian. Let z be a central element of G such that $G/\langle z \rangle$ has central commutator subgroup isomorphic to $C_3 \times C_3$ (it is enough to choose a non-trivial element of G' if this is central, and a generator of $\gamma_3(G)$ otherwise). By Lemma 3.1, we can assume that $(G/\langle z \rangle)' = \langle (a, b)\langle z \rangle, (b, c)\langle z \rangle \rangle$. That is, there exist $a, b, c \in G$ such that $x = (a, b) \notin \langle z \rangle$ and $y = (b, c) \notin \langle z \rangle$. This means that $G' = \langle x \rangle \times \langle y \rangle \times \langle z \rangle$. Now,

$$\begin{aligned}u = (1 + a(z - 1), b) &= 1 + (1 + a(z - 1))^{-1}b^{-1}[a, b](z - 1) \\ &= 1 + (1 + a(z - 1))^{-1}a(x - 1)(z - 1) \\ &\equiv 1 + a(x - 1)(z - 1) \pmod{\Delta^2(G, \langle z \rangle)}\end{aligned}$$

and, similarly,

$$v = (1 + b(z - 1), c) \equiv 1 + b(y - 1)(z - 1) \pmod{\Delta^2(G, \langle z \rangle)}.$$

From the fact that $[x, b], [a, y] \in \Delta(G, \langle z \rangle)$ it follows that

$$\begin{aligned}(u, v) &= 1 + u^{-1}v^{-1}[u, v] = 1 + u^{-1}v^{-1}[a, b](x - 1)(y - 1)(z - 1)^2 \\ &= 1 + ba(x - 1)^2(y - 1)(z - 1)^2\end{aligned}$$

is a non-trivial element of $\delta_2(\mathcal{U}(FG))$, which is still not allowed. $\square$

We dispense now with the characteristic 5 case.

Lemma 3.5. *Let $p = 5$, and G a non-abelian nilpotent group such that G' is a finite 5-group. If $\mathcal{U}(FG)$ is metabelian, then $Syl_5(G) = G' \cong C_5$.*

Proof. According to Lemma 2.3 (b), $|G'| = 5$.

Suppose that $G' \neq Syl_5(G)$ and $(Syl_5(G), G) \neq \langle 1 \rangle$. Then we can pick elements $a \in Syl_5(G) \setminus G'$ and $b \in G$ such that $x = (a, b)$ has order 5. Then $1 + a$ is a unit and

$$u = (1 + a, b) = 1 + (1 + a)^{-1}a(x - 1).$$

Furthermore Lemma 2.5 yields

$$\begin{aligned} v &= (1 + b(x - 1), a) = 1 + (1 + b(x - 1))^{-1}a^{-1}[b, a](x - 1) \\ &= 1 + (1 + b(x - 1))^{-1}b(x^{-1} - 1)(x - 1) \\ &\equiv 1 - b(x - 1)^2 \pmod{\Delta^3(G, G')}. \end{aligned}$$

Applying Lemma 2.6 (a) and (b), one has that

$$\begin{aligned} (u, v) &= 1 - u^{-1}v^{-1}[(1 + a)^{-1}a, b](x - 1)^3 \\ &= 1 - (1 + a)^{-5}b(a - 2a^2 + 3a^3 - 4a^4)(x - 1)^4, \end{aligned}$$

which is different from 1, in view of Lemma 3.1.2 of [10]. Therefore in this case $\mathcal{U}(FG)$ is not metabelian.

Finally, assume that $G' \neq Syl_5(G) \subseteq \xi(G)$. Then there exists $f \in Syl_5(G) \setminus G'$ such that $f^5 \in G'$. As above, let us pick $a, b \in G$ such that $x = (a, b)$ has order 5. Clearly, $w = 1 + (f - 1)^4 a$ is a unit and $w^{-1} \equiv 1 - (f - 1)^4 a \pmod{\Delta(G, G')}$. From computations similar to those in (5) it follows that

$$t = (w, b) \equiv 1 + (f - 1)^4 a(x - 1) \pmod{\Delta^2(G, G')}.$$

If v is as before, then

$$\begin{aligned} (t, v) &= 1 + t^{-1}v^{-1}[t, v] = 1 - t^{-1}v^{-1}[(f - 1)^4 a(x - 1), b(x - 1)^2] \\ &= 1 - t^{-1}v^{-1}(f - 1)^4[a, b](x - 1)^3 = 1 - (f - 1)^4 ba(x - 1)^4 \neq 1, \end{aligned}$$

and this completes the proof. $\square$

We are now in a position to prove the main result of the paper.

Proof of Theorem 1.1. Let us start by looking at the sufficient conditions. Assume that G is nilpotent with central commutator subgroup. If $|G'| = p = 3$, we have already observed after Theorem 2.2 that $\mathcal{U}(FG)$ is metabelian. When G is as in (b) or (c), the result is a consequence of Lemma 2.4 (b).

For the converse, let us claim that, if $\mathcal{U}(FG)$ is metabelian, then G is nilpotent. To this end, assume, if possible, that G is not. By virtue of Theorem 2.2, G contains an element of infinite order. Obviously G' is abelian.

Suppose first that $H = \gamma_3(G)$ is a finite p-group. According to (4), $D_1(H) = H$ and $D_n(H) = H^{p^k}$ for all positive integers k and n such that $p^{k-1} < n \leq p^k$. We notice that there exists an integer α such that

$(H^{p^\alpha}, G) \nsubseteq H^{p^{\alpha+1}}$. In fact, if this is not the case, then $\gamma_4(G) = (H, G) \subseteq H^p$ and, inductively, $\gamma_k(G) \subseteq H^{p^{k-3}}$ for any $k \geq 4$. Since H is a finite p-group, G should be nilpotent, which contradicts the original assumption. Therefore, we can choose $x \in H^{p^\alpha}$ and $a \in G$, such that $(x, a) \in H^{p^\alpha} \setminus H^{p^{\alpha+1}}$. Hence $x - 1, (x, a) - 1 \in \Delta^{p^\alpha}(H) \setminus \Delta^{p^\alpha+1}(H)$ and

$$
\begin{aligned}
u &= (1 + a(x-1), a) = 1 + (1 + a(x-1))^{-1}a^{-1}[a(x-1), a] \\
&= 1 + (1 + a(x-1))^{-1}[x, a] \\
&= 1 + (1 + a(x-1))^{-1}ax((x, a) - 1) \\
&\equiv 1 + a((x, a) - 1) \pmod{\Delta^{p^\alpha+1}(G, H)}.
\end{aligned}
$$

Standard computations yield

$$
\begin{aligned}
(x, u) &= 1 + x^{-1}u^{-1}[x, u] \equiv 1 + x^{-1}u^{-1}[x, a]((x, a) - 1) \\
&\equiv 1 + x^{-1}u^{-1}ax((x, a) - 1)^2 \\
&\equiv 1 + a((x, a) - 1)^2 \pmod{\Delta^{2p^\alpha+1}(G, H)}.
\end{aligned}
$$

By virtue of Lemma 2.7, $((x, a) - 1)^2 \notin \Delta^{2p^\alpha+1}(H)$. Consequently, it does not belong to $\Delta^{2p^\alpha+1}(G, H)$. This means that $(u, x) \neq 1$, which is not allowed.

Therefore assume that $\gamma_3(G)$ is not a finite p-group. From Lemma 2.1 we know that the p-elements of G form a finite normal subgroup P of G. Clearly there exist $x \in G' \setminus P$ and $g \in G$ such that $(x, g) \notin P$ (otherwise $\gamma_3(G)$ would be a finite p-group). Furthermore we can pick an element $h \in P \setminus \langle 1 \rangle$ such that $(h, x) = 1$. In fact, if $G' \cap P \neq \langle 1 \rangle$, it is sufficient to choose an arbitrary element of that set, otherwise $(G, P) \subseteq G' \cap P = \langle 1 \rangle$, thus P is central and any $1 \neq h \in P$ satisfies the condition. Let i be the largest integer such that $h \in D_i(P)$. Now, $1 + (h-1)g$ is a unit lying in $1 + \Delta^i(G, P)$, and so does its inverse. Consequently,

$$
\begin{aligned}
u &= (1 + (h-1)g, x) = 1 + (1 + (h-1)g)^{-1}x^{-1}[(h-1)g, x] \\
&= 1 + (1 + (h-1)g)^{-1}x^{-1}(h-1)xg((g, x) - 1) \\
&\equiv 1 + (h-1)g((g, x) - 1) \pmod{\Delta^{i+1}(G, P)}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
(u, x) &\equiv 1 + u^{-1}x^{-1}[(h-1)g((g, x) - 1), x] \\
&= 1 + u^{-1}x^{-1}(h-1)[g, x]((g, x) - 1) \\
&\equiv 1 + (h-1)g((g, x) - 1)^2 \pmod{\Delta^{i+1}(G, P)}
\end{aligned}
$$

is a non-trivial element of $\delta_2(\mathcal{U}(FG))$, and this proves the claim.

Since G is nilpotent, as observed at the beginning of this section, we can apply Theorem 2.2 and Lemma 2.3 and conclude that $p \in \{3, 5\}$ and G' is a finite p-group. At this stage, the proof is done invoking Lemmas 3.3, 3.4 and 3.5. $\square$

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TIBOR JUHÁSZ
Institute of Mathematics and Informatics
Eszterházy Károly University
Leányka u. 4., Eger 3300 (Hungary)
e-mail: juhasz.tibor@uni-eszterhazy.hu

ERNESTO SPINELLI
Dipartimento di Matematica “G. Castelnuovo”
Università degli Studi di Roma “La Sapienza”
Piazzale Aldo Moro 5
00185-ROMA (Italy)
e-mail: spinelli@mat.uniroma1.it