Cluster partitioning of heterogeneous multi-agent systems ⋆

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Abstract

This paper investigates the collective behaviors induced by the network interconnection of heterogeneous input-affine single-input nonlinear systems through a constant but general directed graph. In this sense, we prove that the dynamics of all agents cluster into as many subgroups as the number of cells of the almost equitable partition induced by the communication graph. All agents belonging to the same cell are equally influenced by a new mean-field dynamics which is paradigmatic of the network. The case of a network of pendula illustrates the results through simulations.

Key words: Multiconsensus; Multi-agent systems; Linear/nonlinear models.

1 Introduction

Networked systems are nowadays considered a bridging paradigm among several disciplines spanning, among many others, from physics to engineering, psychology to medicine, biology to computer science. As typical in control theory (Isidori (2017)), we refer to a network (or multi-agent) system as composed of several dynamical units (agents) interconnected through a *communication graph*: each node of the communication graph uniquely corresponds to one dynamical unit whereas edges model the exchange of information among agents. As a consequence, even for simple agents and with no issue in the network interconnection (e.g., time-delays), the network behavior is described by a complex dynamical system.
In these regards, several works have been devoted to providing methodological understanding on the collective behavior induced by the network interaction through the graph. Such a behavior might be declined into different scenarios such as flocking, randez-vous, formation control and swarming allowing to describe a huge number of control problems in a unifying framework. Most of these can be generally lead to the possibility of driving all systems composing the network toward a *consensus* behavior that might be global for all individuals or common only to some clusters uniquely identified by the graph (e.g., Jadbabaie et al. (2003); Olfati-Saber et al. (2007); Aeyels and De Smet (2010); Ren and Cao (2010); Chen et al. (2011); Egerstedt et al. (2012); Wang and Lu (2019); Gambuzza and Frasca (2020); Mattioni (2020); Cristofaro and Mattioni (2021)).
In case of integrator dynamics, the consensus of the network is dictated by the properties of the underlying graph. Starting from this and motivated by practical applications, several ad-hoc studies have been carried out for larger classes of phenomena and systems such as chaotic oscillators and circuits, mobile and autonomous robots, power systems, cyber security or opinion dynamics by assuming particular structures of the graph underlying the interconnection and, possibly, under sampling, delays or disturbance actions on the nodes (e.g., Moreau (2005); Sun and Wang (2009); Arenas et al. (2008); Dimarogonas et al. (2012); Cui et al. (2012); Chen et al. (2013); Pasqualetti et al. (2013); Zhan and Li (2013); Wen et al. (2013); Battilotti and Califano (2019); Pietrabissa and Suraci (2017); DeLellis et al. (2018); Trumpf and Trentelman (2018)). Those works, focused on single consensus, suggest that the evolutions of each unit are strongly affected by the network topology, the type and strength of the interconnection and the dynamics of each unit. However, general results allowing to understand the simultaneous influence of these three features of the network dynamics are unavailable.
In Panteley and Loría (2017) the case of heterogeneous (input-output) feedback linearizable systems under diffusing coupling and a strongly connected graph has been investigated. In particular, it is explicitly revealed that the agents' interconnection generates a network behavior (the so-called *mean-field dynamics*) which depends,

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through the underlying graph, on a suitable combination of each unit's dynamics. When agents evolve identically, it reduces to the so-called *emergent dynamics*. Those new dynamics are of paramount importance as they offer a unifying framework for dealing with several problems related to multi-agent systems. Further developments of this approach have been proposed in Lee and Shim (2020) for classes of heterogeneous multi-agent systems also embedding distributed control design. At the same time, in Monaco and Ricciardi Celsi (2019) the multiple consensus behaviors and their clustered distribution in a network of simple integrators have been precisely characterized for a general digraph exploiting the notion of almost equitable partition (Cardoso et al. (2007); Caughman and Veerman (2006)).
The aforementioned results motivate the present work whose contribution stands in providing an exact characterization of the behaviors induced by the topology over a network of heterogeneous nonlinear systems under diffusely coupling (through their outputs). No assumptions on the graph structure nor on the agent dynamics are set. Namely, it is proved that, as the intuition suggests, exploiting the properties inferred in Monaco and Ricciardi Celsi (2019), the network topology induces as many mean-field dynamics as the number of reaches of the communication graphs; each mean-field dynamics uniquely affects a suitable cluster of agents. In addition, the common component to all reaches splits into further sub-groups, each of which is driven by a suitable combination of the mean-field dynamics associated to the reaches. In this sense, the emergent and mean-field dynamics are composed of as many independent units as the number of exclusive reaches. The clusters arising in the network (and in particular within the common) are uniquely associated to the almost equitable partition associated to the communication graph $\mathcal{G}$, as in the case of simple scalar integrators. The results here developed are reminiscent of the ones in Frasca et al. (2018) dealing with to multi-clustered consensus of chaotic oscillators and circuits and the ones in Liu et al. (2012); Xiao and Wang (2006) for the case of LTI homogeneous dynamics.

The rest of the paper is organized as follows. In Section 2, preliminaries on graphs and Laplacian matrices are given and the problem is settled. The main results are in Section 3 by providing the structure of the network through the emergent and mean-field dynamics and the corresponding behavior they induce through the synchronization errors. In Section 4, convergence is briefly analized specifying the main result in Panteley and Loría (2017). The example of a network of gravity pendula is illustrated in Section 5 while Section 6 concludes the paper.

Notations. The sets $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{N}$ denote the set of complex, real and natural numbers including 0 respectively. $\mathbb{C}^+$ and $\mathbb{C}^-$ denote the right and left half of the complex plane respectively. The symbols " >0" and " <0" denote positive and negative definite functions whereas $\succ$ and $\prec$ ($\succeq$ and $\preceq$) positive and negative (semi) definite matrices. I_n denotes the identity matrix of dimension $n \geq 1$ whereas $\mathbf{0}$ is the zero-matrix of suitable dimensions. For a square matrix $A \in \mathbb{R}^{n\times n}$, $\sigma(A) \subset \mathbb{C}$ denotes its spectrum. Given m column vectors $g_j \in \mathbb{R}^n$ with $j = 1, \ldots, m$ we denote by $\text{diag}(g_1, \ldots, g_m) \in \mathbb{R}^{mn\times m}$ the block-diagonal matrix with g_j in the main diagonal whereas $\text{col}(g_1, \ldots, g_m) = (g_1^\top \ \ldots g_m^\top)^\top \in \mathbb{R}^{nm}$. Given to matrices $A \in \mathbb{R}^{n_1\times n_2}$ and $B \in \mathbb{R}^{m_1\times m_2}$, the Kronecker product is denoted by $A \otimes B$. For a given set $\mathcal{S}$, $|\mathcal{S}|$ denotes its cardinality. If $(\mathcal{X}, d)$ is a metric space, $\Gamma \subset \mathcal{X}$ and $x \in \mathcal{X}$, then $\|x\|_\Gamma = \inf_{y\in\Gamma} d(x, y)$ defines the point-to-set distance of x to Γ.

2 Preliminaries and problem statement

2.1 Recalls on graph theory

Consider a digraph (that is an unweighted directed graph) $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ with $\mathcal{V}$ being the set of vertices with cardinality $|\mathcal{V}| = N$ and $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ being the set of edges (i.e., the set of ordered pairs of node). For all pairs of distinct notes $\nu_i, \nu_j \in \mathcal{V}$ then $(\nu_i, \nu_j) \in \mathcal{E}$ if there exists an edge from ν_i to ν_j or, equivalently, ν_i is a neighbour of ν_j for all $i \neq j = 1, \ldots, N$. For all pairs of distinct nodes $\nu, \mu \in \mathcal{V}$, a directed path from ν to μ is defined as $\nu \rightsquigarrow \mu = \{(\nu_r, \nu_{r+1}) \in \mathcal{E} \text{ s.t. } \cup_{r=0}^{\ell-1} (\nu_r, \nu_{r+1}) \subseteq \mathcal{E} \text{ with } \nu_0 = \nu, \nu_\ell = \mu \text{ and } \ell > 0\}$. $\mathcal{G}$ is said to be: *weakly connected* if its undirected version is connected and there is no unreachable node (that is there exists a path between all pairs of nodes); *rooted* if it is weakly connected and contains at least one rooted out-branching; *strongly connected* if there always exists a directed path between every pair of nodes and there is no unreachable node. The reachable set from a node $\nu \in \mathcal{V}$ is defined as $R(\nu) := \{\nu\} \cup \{\mu \in \mathcal{V} \text{ s.t. } \nu \rightsquigarrow \mu\}$. A set $\mathcal{R}$ is called a reach if it is a maximal reachable set that $\mathcal{R} = R(\nu)$ for some $\nu \in \mathcal{V}$ and there is no $\mu \in \mathcal{V}$ such that $R(\nu) \subset R(\mu)$. Since $\mathcal{G}$ possesses a finite number of vertices, such maximal sets exist and are uniquely determined by the graph itself. Let $\mathcal{R}_i$ with $i = 1, \ldots, \mu$ and $\mu \leq N$ denote the (not empty) reaches of the graph $\mathcal{G}$. For each reach $\mathcal{R}_i$, $\mathcal{H}_i = \mathcal{R}_i / \cup_{j=1, j\neq i}^{\mu} \mathcal{R}_j$ with $h_i = |\mathcal{H}_i|$ defines the exclusive part while $\mathcal{C}_i = \mathcal{R}_i/\mathcal{H}_i$ is the corresponding common part whose union defines $\mathcal{C} = \cup_{i=1}^{\mu} \mathcal{C}_i$ with $\delta = |\mathcal{C}|$.
The set of neighbours associated to $\nu_i \in \mathcal{V}$ is defined as $\mathcal{N}_i = \{\nu \in \mathcal{V}/\{\nu_i\} \text{ s.t. } (\nu, \nu_i) \in \mathcal{E}\}$ with cardinality $d_i = |\mathcal{N}_i|$ being also referred to as in-degree of ν_i. Accordingly, the in-degree matrix is defined as $\mathcal{D} = \text{diag}(d_1, \ldots, d_N) \in \mathbb{R}^{N\times N}$ whereas the adjacency matrix is $\mathcal{A} = \{a_{ij}\} \in \mathbb{R}^{N\times N}$ with $a_{ii} = 0$ and $a_{ij} = 1$ if $(\nu_i, \nu_j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The Laplacian of $\mathcal{G}$ is $\mathcal{L} = \mathcal{D} - \mathcal{A}$ and possesses one eigenvalue $\lambda = 0$ with algebraic multiplicity μ being equal to the number of reaches of $\mathcal{G}$ and all other eigenvalues in the left-hand side of the complex plane (Agaev and Chebotarev

(2005); Caughman and Veerman (2006)). As a result, it has been shown in Caughman and Veerman (2006); Monshizadeh et al. (2015); Monaco and Ricciardi Celsi (2019) that, after suitably reordering the graphs nodes, the Laplacian $\mathcal{L}$ always admits the following upper triangular form

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_1 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathcal{L}_2 & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathcal{L}_\mu & \mathbf{0} \\ \mathcal{M}_1 & \mathcal{M}_2 & \dots & \mathcal{M}_\mu & \mathcal{M} \end{pmatrix} \tag{1}$$

with each $\mathcal{L}_i \in \mathbb{R}^{h_i \times h_i}$ being the Laplacian associated to the strongly connected subgraph induced by the exclusive $\mathcal{H}_i$; $\mathcal{M} \in \mathbb{R}^{\delta \times \delta}$ and $\mathcal{M}_i \in \mathbb{R}^{\delta \times h_i}$ define, respectively, the internal connections in the common part $\mathcal{C}$ of the digraph and the ones with the exclusive $\mathcal{H}_i$. As a consequence, each $\mathcal{L}_i$ possesses an eigenvalue in $\lambda = 0$ with algebraic multiplicity 1 whereas $\sigma(\mathcal{M}) \subset \mathbb{C}^+$. By this structure, it is evident that the eigenspace associated to the zero eigenvalue is hence given by $E = \text{span}\{z_1, \dots, z_\mu\}$ with

$$z_1 = \begin{pmatrix} \mathbb{1}_{h_1} \\ \vdots \\ \mathbf{0} \\ \gamma^1 \end{pmatrix}, \dots, z_\mu = \begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbb{1}_{h_\mu} \\ \gamma^\mu \end{pmatrix} \tag{2}$$

with $\mathbb{1}_p = \text{col}(1, \dots, 1) \in \mathbb{R}^p$, $\sum_{i=1}^{\mu} \gamma^i = \mathbb{1}$ and verifying $\mathcal{L}_i \mathbb{1}_{h_i} = 0$ and $\mathcal{M}_i \mathbb{1}_{h_i} + \mathcal{M}\gamma^i = 0$ for all $i = 1, \dots, \mu$. Correspondingly, the left eigenvectors associated to the zero eigenvalue of $\mathcal{L}$ read

$$\tilde{v}_1^\top = \begin{pmatrix} v_1^\top & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \end{pmatrix}, \dots, \ \tilde{v}_\mu^\top = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \dots & v_\mu^\top & \mathbf{0} \end{pmatrix} \tag{3}$$

with $v_i^\top = (v_i^1 \ \dots v_i^{h_i})$ verifying for all $i = 1, \dots, \mu$ (for $h_0 = 0$): $v_i^\top \mathcal{L}_i = 0$; $v_i^\top \mathbb{1}_{h_i} = 1$; for each $s = 1, \dots, h_i$, $v_i^s > 0$ if $\nu_{h_1 + \dots + h_{i-1} + s} \in \mathcal{H}_i$ is a root and $v_i^s = 0$ otherwise. Moreover, if $\mathcal{M}_i = \mathbf{0}$ the reach $\mathcal{H}_i$ defines a disconnected component of $\mathcal{G}$.
A partition $\pi = \{\rho_1, \dots, \rho_r\}$ of $\mathcal{V}$ is a collection of cells $\rho_i \subseteq \mathcal{V}$ verifying $\rho_i \cap \rho_j = \emptyset$ for all $i \neq j$ and $\cup_{i=1}^r \rho_i = \mathcal{V}$. A partition $\pi = \{\rho_1, \dots, \rho_r\}$ of $\mathcal{V}$ is said to be an *almost equitable partition* (AEP, in short) if each node of ρ_i has the same number of neighbors in ρ_ℓ, for all $i, \ell \in \{1, \dots, r\}$ with $i \neq \ell$. More precisely, for a node $\nu_i \in \mathcal{V}$ denote by $\mathcal{N}(\nu_i, \rho) = \{\nu \in \rho \text{ s.t } (\nu, \nu_i) \in \mathcal{E}\}$ the set of neighbors of ν_i in the cell ρ; π is an AEP of $\mathcal{G}$ if, for each $i, j \in \{1, 2, \dots, r\}$, with $i \neq j$, there exists an integer d_{ij} such that $|\mathcal{N}(\nu, \rho_j)| = d_{ij}$ for all $\nu \in \rho_i$

2.2 Problem statement

Consider N dynamical systems of the form

$$\dot{x}_i = f_i(x_i) + g_i(x_i) u_i \tag{4a}$$
$$y_i = h_i(x_i) \tag{4b}$$

with $x_i \in \mathbb{R}^n$, $y_i, u_i \in \mathbb{R}$ for all $i = 1, \dots, N$ and all vector fields and mappings being locally Lipschitz. Roughly speaking, each dynamical system is driven by a fixed linear combination of the outputs of neighbour agents defined by the so-called *communication graph* $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ (Isidori (2017)). More in details, each node $\nu_i \in \mathcal{V}$ is a dynamical system (also referred to as agent) of the form (4); edges of $\mathcal{G}$ specify the interconnections among all agents through the coupling $u_i = -\kappa \sum_{\nu_j \in \mathcal{N}_i} (y_i - y_j)$ with $\kappa > 0$ being referred to as the strength of the interconnection.

For the sake of compactness and without loss of generality, we assume the Laplacian of the form (1) so that states, outputs and couplings are partitioned as $\mathbf{x} = \text{col}(\mathbf{x}_1, \dots, \mathbf{x}_\mu, \mathbf{x}_\delta) \in \mathbb{R}^{nN}$ with $\mathbf{x}_i = \text{col}(\{x_j\}_{\nu_j \in \mathcal{H}_i}) \in \mathbb{R}^{nh_i}$ for $i = 1, \dots, \mu$ and $\mathbf{x}_\delta = \text{col}(\{x_j\}_{\nu_j \in \mathcal{C}}) \in \mathbb{R}^{n\delta}$. In a similar manner, we set $\mathbf{y} = \text{col}(\mathbf{y}_1, \dots, \mathbf{y}_\mu, \mathbf{y}_\delta) \in \mathbb{R}^N$ and $\mathbf{u} = \text{col}(\mathbf{u}_1, \dots, \mathbf{u}_\mu, \mathbf{u}_\delta) \in \mathbb{R}^N$.
In this context, we are interested in characterizing the behavior the network induces over each agent (4) under no restriction on the graph properties. Namely, we investigate the evolutions of each node arising from the output-based diffusively interconnection $\mathbf{u} = -\kappa \mathcal{L} \mathbf{y}$ defined by the Laplacian of the graph $\mathcal{G}$.
Complete results for this problem have been provided only for the case of scalar integrators. However, when all nodes are nonlinear heterogeneous agents, only partial results are available and mostly restricted to the case of $\mathcal{G}$ being a directed or undirected but strongly connected graph. Among these, when (4) possesses relative degree $r_i = 1$, the work in Panteley and Loría (2017) shows that a new (mean-field) behavior is rising and directly affects the evolutions of all agents. However, what about the effect of the network under a general graph $\mathcal{G}$?

In the following, the case of scalar integrator dynamics is recalled as a paradigm of the general scenario under investigation.

2.3 The case of scalar integrators and multi-consensus

When associating to each node $\nu_i \in \mathcal{V}$ a dynamical scalar-integrator system, the Laplacian $\mathcal{L}$ completely governs the behaviors of the overall network (Olfati-Saber et al. (2007); Li et al. (2010); Monaco and Ricciardi Celsi (2019)). Let each agent dynamics (4) specify as

$$\dot{x}_i = u_i$$

with $x_i \in \mathbb{R}$ and trivially $y_i = x_i$. Setting $\mathbf{x} = \mathrm{col}(\mathbf{x}_1, \dots, \mathbf{x}_\mu, \mathbf{x}_\delta) \in \mathbb{R}^N$ and $\mathbf{u} = -\mathcal{L}\mathbf{x}$, one gets that the networked system is described by

$$\dot{\mathbf{x}} = -\mathcal{L}\mathbf{x}. \tag{5}$$

As proved in Monaco and Ricciardi Celsi (2019), for all initial conditions $\mathbf{x}(0) \in \mathbb{R}^N$, as $t \to \infty$, agents cluster and the network reaches the multi-consensus according to the almost equitable partition

$$\pi = \{\mathcal{H}_1, \dots, \mathcal{H}_\mu, \mathcal{C}_{\mu+1}, \dots, \mathcal{C}_{\mu+p}\} \tag{6}$$

associated to $\mathcal{G}$ with $\mathcal{C} = \cup_{i=1}^{p} \mathcal{C}_{\mu+i}$. Accordingly, denoting $c_\ell = |\mathcal{C}_{\mu+\ell}|$ for $\ell = 1, \dots, p$ and regrouping nodes in $\mathcal{C}$ in such a way that $\mathbf{x}_\delta = \mathrm{col}(\mathbf{x}_{\delta_1}, \dots, \mathbf{x}_{\delta_p})$ with $\mathbf{x}_{\delta_\ell} = \mathrm{col}(\{x_j\}_{\nu_j \in \mathcal{C}_{\mu+\ell}})$, the following result is recalled.

Theorem 2.1 (Monaco and Ricciardi Celsi (2019)) *Consider the multi-agent system* (5) *with communication graph* $\mathcal{G}$ *and Laplacian* (1). *Then, nodes belong to the the same cell* $\mathcal{C}_{\mu+\ell} \subseteq \mathcal{C}$ *of the AEP (6) if and only if they share the same component of the vector* γ^i *in* (2). *In addition, as* $t \to \infty$ *nodes within the same cell of the AEP (6) converge to the same consensus steady-state; i.e., for* $t \to \infty$ *and rewriting the vectors* $\gamma^i \in \mathbb{R}^\delta$ *in (2)*

$$\gamma^i = \begin{pmatrix} \gamma_1^i \mathbb{1}_{c_1}^\top & \dots & \gamma_p^i \mathbb{1}_{c_p}^\top \end{pmatrix} \tag{7}$$

(1) for all reaches $\mathcal{H}_i$ *with* $i = 1, \dots, \mu$ *and* $v_i^\top$ *as in* (3)

$$\mathbf{x}_i(t) \to x_{s,i} \mathbb{1}_{h_i}, \quad x_{s,i} = v_i^\top \mathbf{x}_i(0);$$

(2) for all cells $\mathcal{C}_{\mu+\ell} \subseteq \mathcal{C}$ *with* $\ell = 1, \dots, p$

$$\mathbf{x}_{\delta_\ell}(t) \to x_{s,\delta_\ell} \mathbb{1}_{c_\ell}, \quad x_{s,\delta_\ell} = \sum_{i=1}^{\mu} \gamma_\ell^i x_{s,i}.$$

The network asymptotic behavior clusters: all agents in the same exclusive subgraph (i.e., $\nu_j \in \mathcal{H}_i$) converge asymptotically to the same consensus value being a mean of the initial states of the corresponding roots; agents in the common part (i.e., $\nu_j \in \mathcal{C}$) converge to different consensuses being a convex combination of the ones over the reaches.

3 The main result

Based on the arguments above, we investigate the so-called topology containment (Liu et al. (2012)) induced by the network over the agents and the possibility of defining a suitably defined set of new dynamics which are paradigmatic of the network collective behavior. In particular, resorting to the results in Panteley and Loría (2017); Lee and Shim (2020), we investigate the structure of the emergent and mean-field dynamics associated to the network and affecting the agents evolutions. In this, sense, based on the AEP (6) of the graph, we show that $\mu + p$ network behaviors emerge forcing agents to cluster into: $\mu \geq 1$ independent subgroups associated to each exclusive reach $\mathcal{H}_i$; $p \geq 0$ further subgroups that are generally influenced by all others. Accordingly, we define a *multi-consensus error* representing the offset among the behavior of each dynamical unit in the group with respect to the corresponding network dynamics. Exploiting the partition induced by the Laplacian (1), the agglomerate dynamics reads

$$\dot{\mathbf{x}}_i = \mathbf{f}_i(\mathbf{x}_i) + \mathbf{g}_i(\mathbf{x}_i)\mathbf{u}_i \tag{8a}$$
$$\dot{\mathbf{x}}_\delta = \mathbf{f}_\delta(\mathbf{x}_\delta) + \mathbf{g}_\delta(\mathbf{x}_\delta)\mathbf{u}_\delta \tag{8b}$$
$$\mathbf{y}_i = \mathbf{h}_i(\mathbf{x}_i) \tag{8c}$$
$$\mathbf{y}_\delta = \mathbf{h}_\delta(\mathbf{x}_\delta) \tag{8d}$$

for $i = 1, \dots, \mu$ and

$$\mathbf{f}_i(\cdot) = \mathrm{col}(\{f_j(\cdot)\}_{\nu_j \in \mathcal{H}_i}), \quad \mathbf{g}_i(\cdot) = \mathrm{diag}(\{g_j(\cdot)\}_{\nu_j \in \mathcal{H}_i})$$
$$\mathbf{f}_\delta(\cdot) = \mathrm{col}(\{f_j(\cdot)\}_{\nu_j \in \mathcal{C}}), \quad \mathbf{g}_\delta(\cdot) = \mathrm{diag}(\{g_j(\cdot)\}_{\nu_j \in \mathcal{C}})$$
$$\mathbf{h}_i(\cdot) = \mathrm{col}(\{h_j(\cdot)\}_{\nu_j \in \mathcal{H}_i}), \quad \mathbf{h}_\delta(\cdot) = \mathrm{col}(\{h_j(\cdot)\}_{\nu_j \in \mathcal{C}}).$$

By taking into account the general form (1) and for $i = 1, \dots, \mu$, the diffusely coupling term $\mathbf{u} = -\kappa\mathcal{L}\mathbf{y}$ reads

$$\mathbf{u}_i = -\kappa\mathcal{L}_i\mathbf{y}_i \tag{9a}$$
$$\mathbf{u}_\delta = -\kappa\Big(\sum_{j=1}^{\mu} \mathcal{M}_j\mathbf{y}_j + \mathcal{M}\mathbf{y}_\delta\Big). \tag{9b}$$

Plugging (9) into (8), the *network dynamics* is

$$\dot{\mathbf{x}}_i = \mathbf{f}_i(\mathbf{x}_i) - \kappa\mathbf{g}_i(\mathbf{x}_i)\mathcal{L}_i\mathbf{h}_i(\mathbf{x}_i) \tag{10a}$$
$$\dot{\mathbf{x}}_\delta = \mathbf{f}_\delta(\mathbf{x}_\delta) - \kappa\mathbf{g}_\delta(\mathbf{x}_\delta)\mathcal{M}\mathbf{h}_\delta(\mathbf{x}_\delta) - \kappa\mathbf{g}_\delta(\mathbf{x}_\delta)\sum_{j=1}^{\mu}\mathcal{M}_j\mathbf{h}_j(\mathbf{x}_j) \tag{10b}$$

with $i = 1, \dots, \mu$. The above equations underline that the network exhibits a cascade form: for $i = 1, \dots, \mu$, all agents corresponding to a node $\nu_j \in \mathcal{H}_i$ evolve according to the subgraph associated to $\mathcal{H}_i$ and independently of the dynamics in the other reaches and common; the agents belonging to $\mathcal{C}$ a-priori evolve under the influence of all units of the graph. It is intuitively understood that the network is partitioned into at least $\mu+1$ subnetworks composed of μ-independent networks and a further one associated to $\mathcal{C}$. The structure of (10) highlights the existence of μ mean-field independent dynamics which can be computed exploiting the result in Panteley and Loría (2017) as presented in the next section.

3.1 The exclusive mean-field dynamics

Proposition 3.1 *Consider the nonlinear dynamical agents (4) under a communication graph with Laplacian*

$\mathcal{L}$ of the form (1). Then, the network dynamics (10) exhibits $\mu \geq 1$ independent mean-field dynamics (referred to as exclusive mean-field dynamics*) of the form*

$$\begin{aligned}\dot{\mathbf{x}}_{s,i} =& \mathbf{f}_{s,i}(\mathbf{x}_{s,i}) + \tilde{\mathbf{f}}_{s,i}(\mathbf{x}_{s,i}, \mathbf{e}_i) - \kappa\tilde{\mathbf{g}}_{s,i}(\mathbf{x}_{s,i}, \mathbf{e}_i)\mathcal{L}_i\boldsymbol{\varepsilon}_i \quad &(11a)\\ \mathbf{y}_{s,i} =& \mathbf{h}_{s,i}(\mathbf{x}_{s,i}) + \tilde{\mathbf{h}}_{s,i}(\mathbf{x}_{s,i}, \mathbf{e}_i) \quad &(11b)\end{aligned}$$

with exclusive mean-field unit and consensus error as

$$\begin{aligned}\mathbf{x}_{s,i} =& (v_i^\top \otimes I_n)\mathbf{x}_i \quad &(12a)\\ \mathbf{e}_i =& \mathbf{x}_i - (\mathbb{1}_{h_i} \otimes I_n)\mathbf{x}_{s,i} \quad &(12b)\\ \boldsymbol{\varepsilon}_i =& \mathbf{y}_i - \mathbb{1}_{h_i}\mathbf{y}_{s,i} \quad &(12c)\end{aligned}$$

for $i = 1, \dots, \mu$, $h_0 = 0$ and

$$\begin{aligned}\mathbf{f}_{s,i}(\mathbf{x}_{s,i}) =& (v_i^\top \otimes I_n)\mathbf{f}_i((\mathbb{1}_{h_i} \otimes I_n)\mathbf{x}_{s,i})\\ \mathbf{h}_{s,i}(\mathbf{x}_{s,i}) =& v_i^\top \mathbf{h}_i((\mathbb{1}_{h_i} \otimes I_n)\mathbf{x}_{s,i}) \quad (13)\\ \tilde{\mathbf{f}}_{s,i}(\mathbf{x}_{s,i}, \mathbf{e}_i) =& (v_i^\top \otimes I_n)\mathbf{f}_i(\mathbf{e}_i + (\mathbb{1}_{h_i} \otimes I_n)\mathbf{x}_{s,i}) - \mathbf{f}_{s,i}(\mathbf{x}_{s,i})\\ \tilde{\mathbf{g}}_{s,i}(\mathbf{x}_{s,i}, \mathbf{e}_i) =& (v_i^\top \otimes I_n)\mathbf{g}_i(\mathbf{e}_i + (\mathbb{1}_{h_i} \otimes I_n)\mathbf{x}_{s,i})\\ \tilde{\mathbf{h}}_{s,i}(\mathbf{x}_{s,i}, \mathbf{e}_i) =& v_i^\top \mathbf{h}_i(\mathbf{e}_i + (\mathbb{1}_{h_i} \otimes I_n)\mathbf{x}_{s,i}) - \mathbf{h}_{s,i}(\mathbf{x}_{s,i}).\end{aligned}$$

In addition, the consensus error over $\mathcal{C}$ reads

$$\mathbf{e}_\delta = \mathbf{x}_\delta - \sum_{i=1}^{\mu}(\gamma^i \otimes I_n)\mathbf{x}_{s,i}. \quad (14)$$

Proof. The proof extends the one in Panteley and Loría (2017); Lee and Shim (2020) by computing a suitable network transformation. To this end, consider the Laplacian of the form (1) with the spectral properties underlined in Section 2.1. Then, one can introduce the matrix

$$Z = \begin{pmatrix} z_1 & \dots & z_\mu & Z_r \end{pmatrix}$$

with Z_r containing the eigenvectors associated to all other $N-\mu$ non-zero eigenvalues of (1) and, consequently

$$V^\top = Z^{-1} = \begin{pmatrix} \tilde{v}_1^\top \\ \vdots \\ \tilde{v}_\mu^\top \\ V_r^\top \end{pmatrix}, \quad V_r^\top = (Z_r^\top Z_r)^{-1} Z_r^\top$$

such that $J_{\mathcal{L}} = V^\top \mathcal{L} Z$ is in Jordan form. Starting from this, we introduce the mapping

$$\mathbf{x}_s = \begin{pmatrix} \mathbf{x}_{s,1} \\ \vdots \\ \mathbf{x}_{s,\mu} \end{pmatrix} = \begin{pmatrix} \tilde{v}_1^\top \otimes I_n \\ \vdots \\ \tilde{v}_\mu^\top \otimes I_n \end{pmatrix} \mathbf{x} = \begin{pmatrix} (v_1^\top \otimes I_n)\mathbf{x}_1 \\ \vdots \\ (v_\mu^\top \otimes I_n)\mathbf{x}_\mu \end{pmatrix}$$

with $x_{s,i} \in \mathbb{R}^n$ for $i = 1, \dots, \mu$ denoting the set of *mean-fields units*. Accordingly, we define

$$\mathbf{e} = (Z_r \otimes I_n)(V_r^\top \otimes I_n)\mathbf{x} = (Z_r V_r^\top \otimes I_n)\mathbf{x}.$$

Since by construction $Z_r V_r^\top = I_N - \sum_{i=1}^{\mu} z_i \tilde{v}_i^\top$ one gets

$$\mathbf{e} = ((I_N - \sum_{i=1}^{\mu} z_i \tilde{v}_i^\top) \otimes I_n)\mathbf{x} = \mathbf{x} - \sum_{i=1}^{\mu}(z_i \otimes I_n)\mathbf{x}_{s,i}$$

Accordingly, because of the structure of all right eigenvalues of the Laplacian (1), the error $\mathbf{e}$ can be partitioned as $\mathbf{e} = \mathrm{col}(\mathbf{e}_1, \dots, \mathbf{e}_\mu, \mathbf{e}_\delta)$ with $\mathbf{e}_i \in \mathbb{R}^{nh_i}$ with $i = 1, \dots, \mu$ and $\mathbf{e}_\delta \in \mathbb{R}^{n\delta}$ as in (12b) and (14). Accordingly, by differentiating each $\mathbf{x}_{s,i}$ one gets

$$\dot{\mathbf{x}}_{s,i} = (v_i^\top \otimes I_n)\big(\mathbf{f}_i(\mathbf{x}_i) - \kappa\mathbf{g}_i(\mathbf{x}_i)\mathcal{L}_i\mathbf{y}_i\big).$$

Substituting now $\mathbf{x}_i = \mathbf{e}_i + (\mathbb{1}_{h_i} \otimes I_n)\mathbf{x}_{s,i}$ and $\mathbf{y}_i = \boldsymbol{\varepsilon}_i + \mathbb{1}_{h_i}\mathbf{y}_{s,i}$ in the expression below one gets the result exploiting that $\mathcal{L}_i\boldsymbol{\varepsilon}_i = \mathcal{L}_i(\mathbf{y}_i - \mathbb{1}_{h_i}\mathbf{y}_{s,i}) = \mathcal{L}_i\mathbf{y}_i$. ■

Each exclusive mean-field unit is uniquely associated to the exclusive reach $\mathcal{H}_i$ and corresponds to a weighted average of the states of all agents in the same subnetwork $\mathcal{H}_i$. Correspondingly, the multi-consensus error $\mathbf{e}_i = \mathbf{x}_i - (\mathbb{1}_{h_i} \otimes I_n)\mathbf{x}_{s,i}$ defines the behavior of each agent in $\mathcal{H}_i$ with respect to the exclusive mean-field unit $\mathbf{x}_{s,i}$ for all $i = 1, \dots, \mu$. In what follows, we refer to (12c) as the output consensus error over $\mathcal{H}_i$ whereas, for $\mathcal{C}$, we set

$$\boldsymbol{\varepsilon}_\delta = \mathbf{y}_\delta - \sum_{i=1}^{\mu}\gamma^i\mathbf{y}_{s,i}. \quad (15)$$

Based on Proposition 3.1, the dynamics within the reach $\mathcal{H}_i$ rewrites in terms of the state consensus error as

$$\begin{aligned}\dot{\mathbf{e}}_i =& -\kappa\Pi_i\tilde{\mathbf{g}}_i(\mathbf{x}_{s,i}, \mathbf{e}_i)\mathcal{L}_i\boldsymbol{\varepsilon}_i \quad (16)\\ &+ \Pi_i\tilde{\mathbf{f}}_i(\mathbf{x}_{s,i}, \mathbf{e}_i) + \Pi_i\mathbf{f}_i((\mathbb{1}_{h_i} \otimes I_n)\mathbf{x}_{s,i}).\end{aligned}$$

with $\Pi_i = \big((I_{h_i} - \mathbb{1}_{h_i}v_i^\top) \otimes I_n\big)$ and

$$\begin{aligned}\tilde{\mathbf{f}}_i(\mathbf{x}_{s,i}, \mathbf{e}_i) &= \mathbf{f}_i(\mathbf{e}_i + (\mathbb{1}_{h_i} \otimes I_n)\mathbf{x}_{s,i}) - \mathbf{f}_i((\mathbb{1}_{h_i} \otimes I_n)\mathbf{x}_{s,i})\\ \tilde{\mathbf{g}}_i(\mathbf{x}_{s,i}, \mathbf{e}_i) &= \mathbf{g}_i(\mathbf{e}_i + (\mathbb{1}_{h_i} \otimes I_n)\mathbf{x}_{s,i}).\end{aligned}$$

In addition, (12c) defines the output consensus error. When the consensus error is indentically zero (11a) reduces to the so-called emergent-dynamics

$$\dot{\mathbf{x}}_{e,i} = \mathbf{f}_{s,i}(\mathbf{x}_{e,i}), \quad \mathbf{y}_{e,i} = \mathbf{h}_{s,i}(\mathbf{x}_{e,i}) \quad (17)$$

that is, the consensus dynamics over the reach $\mathcal{H}_i$.

3.2 The common mean-field unit

In this section, we show that further $p \geq 0$ mean-field dynamics arise within the common as suitable combinations of the exclusive ones in (11a). To this end, exploiting Proposition 2.1 and denoting in (1)

$$\mathcal{M}_i = \begin{pmatrix} \mathcal{M}_{i,1} \\ \vdots \\ \mathcal{M}_{i,p} \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} \mathcal{M}_{11} & \dots & \mathcal{M}_{1p} \\ & \ddots & \\ \mathcal{M}_{p1} & \dots & \mathcal{M}_{pp} \end{pmatrix}$$

with $\mathcal{M}_{i,\ell} \in \mathbb{R}^{c_\ell \times h_i}$, $\mathcal{M}_{\ell_1 \ell_2} \in \mathbb{R}^{c_{\ell_1} \times c_{\ell_2}}$, nodes in $\mathcal{C}$ are further regrouped so that (8b) gets the form

$$\dot{\mathbf{x}}_{\delta_\ell} = \mathbf{f}_{\delta_\ell}(\mathbf{x}_{\delta_\ell}) - \kappa \mathbf{g}_{\delta_\ell}(\mathbf{x}_{\delta_\ell}) \Big(\sum_{i=1}^{\mu} \mathcal{M}_{i,\ell} \mathbf{y}_i + \sum_{q=1}^{p} \mathcal{M}_{\ell q} \mathbf{y}_q \Big) \tag{18a}$$

$$\mathbf{y}_{\delta_\ell} = \mathbf{h}_{\delta_\ell}(\mathbf{x}_{\delta_\ell}) \tag{18b}$$

for $\ell = 1, \dots, p$ and

$$\mathbf{f}_{\delta_\ell}(\cdot) = \mathrm{col}(\{f_j(\cdot)\}_{\nu_j \in \mathcal{C}_{\mu+\ell}}), \mathbf{g}_{\delta_\ell}(\cdot) = \mathrm{diag}(\{g_j(\cdot)\}_{\nu_j \in \mathcal{C}_{\mu+\ell}})$$
$$\mathbf{h}_{\delta_\ell}(\cdot) = \mathrm{col}(\{h_j(\cdot)\}_{\nu_j \in \mathcal{C}_{\mu+\ell}}).$$

At this point, the following result can be proved.

Theorem 3.1 *Consider a network of nonlinear dynamical agents (4) coupled via (9) for $\kappa \geq 0$ and graph Laplacian of the form (1). Let (6) be the AEP associated to $\mathcal{G}$. Then, nodes in $\mathcal{C}$ cluster into $p \geq 1$ cells (with $p \geq \delta$). Nodes in each $\mathcal{C}_{\mu+\ell}$ are driven by a convex combination of the exclusive mean-field dynamics (12a); namely, for $\ell = 1, \dots, p$ nodes in the same cell $\mathcal{C}_{\mu+\ell} \subset \mathcal{C}$ evolve as*

$$\begin{aligned} \dot{\mathbf{e}}_{\delta_\ell} =& \tilde{\mathbf{f}}_{\delta_\ell}(\mathbf{x}_{s,\delta_\ell}, \mathbf{e}_{\delta_\ell}) + \mathbf{f}_{\delta_\ell}((\mathbb{1}_{c_i} \otimes I_n)\mathbf{x}_{s,\delta_\ell}) \\ & - \sum_{i=1}^{\mu} (\gamma_\ell^i \mathbb{1}_{c_\ell} \otimes I_n) \Big(\tilde{\mathbf{f}}_{s,i}(\mathbf{x}_{s,i}, \mathbf{e}_{s,i}) \\ & + \mathbf{f}_{s,i}(\mathbf{x}_{s,i}) \Big) - \kappa \tilde{\mathbf{g}}_{\delta_\ell}(\mathbf{x}_{s,\delta_\ell}, \mathbf{e}_{\delta_\ell}) \sum_{j=1}^{p} \mathcal{M}_{\ell j} \boldsymbol{\varepsilon}_{\delta_\ell} \\ & - \kappa \sum_{i=1}^{\mu} \Big(\tilde{\mathbf{g}}_{\delta_\ell}(\mathbf{x}_{s,\delta_\ell}, \mathbf{e}_{\delta_\ell}) \mathcal{M}_{\ell,i} \\ & - (\gamma_\ell^i \mathbb{1}_{c_\ell} \otimes I_n) \tilde{\mathbf{g}}_{s,i}(\mathbf{x}_{s,i}, \mathbf{e}_i) \mathcal{L}_i \Big) \boldsymbol{\varepsilon}_i \end{aligned} \tag{19}$$

with common mean-field unit *given by*

$$\mathbf{x}_{s,\delta_\ell} = \sum_{i=1}^{\mu} \gamma_\ell^i \mathbf{x}_{s,i}. \tag{20}$$

and consensus error

$$\mathbf{e}_{\delta_\ell} = \mathbf{x}_{\delta_\ell} - (\mathbb{1}_{c_\ell} \otimes I_n)\mathbf{x}_{s,\delta_\ell} \tag{21a}$$
$$\boldsymbol{\varepsilon}_{\delta_\ell} = \mathbf{y}_{\delta_\ell} - \mathbb{1}_{c_\ell} \mathbf{y}_{s,\delta_\ell} \tag{21b}$$

and

$$\begin{aligned} \tilde{\mathbf{f}}_{\delta_\ell}(\mathbf{x}_{s,\delta_\ell}, \mathbf{e}_{\delta_\ell}) =& \mathbf{f}_{\delta_\ell}(\mathbf{e}_{\delta_\ell} + (\mathbb{1}_{c_i} \otimes I_n)\mathbf{x}_{s,\delta_\ell}) \\ & - \mathbf{f}_{\delta_\ell}((\mathbb{1}_{c_i} \otimes I_n)\mathbf{x}_{s,\delta_\ell}) \\ \tilde{\mathbf{g}}_{\delta_\ell}(\mathbf{x}_{s,\delta_\ell}, \mathbf{e}_{\delta_\ell}) =& \mathbf{g}_{\delta_\ell}(\mathbf{e}_{\delta_\ell} + (\mathbb{1}_{c_i} \otimes I_n)\mathbf{x}_{s,\delta_\ell}). \end{aligned}$$

Proof. The proof follows partitioning the error defined in the proof of Proposition 3.1 according to the subcells $\mathcal{C}_{\mu+\ell} \subseteq \mathcal{C}$; namely, considering the form (7), one gets

$$\begin{aligned} \mathbf{e}_\delta =& \mathbf{x}_\delta - \sum_{i=1}^{\mu} (\gamma^i \otimes I_n)\mathbf{x}_{s,i} \\ =& \begin{pmatrix} \mathbf{x}_{\delta_1} - \sum_{i=1}^{\mu} \gamma_1^i (\mathbb{1}_{c_1} \otimes I_n)\mathbf{x}_{s,i} \\ \vdots \\ \mathbf{x}_{\delta_p} - \sum_{i=1}^{\mu} \gamma_p^i (\mathbb{1}_{c_p} \otimes I_n)\mathbf{x}_{s,i} \end{pmatrix} \\ \boldsymbol{\varepsilon}_\delta =& \mathbf{y}_\delta - \sum_{i=1}^{\mu} \gamma^i \mathbf{y}_{s,i} \\ =& \begin{pmatrix} \mathbf{y}_{\delta_1} - \sum_{i=1}^{\mu} \gamma_1^i \mathbb{1}_{c_1} \mathbf{y}_{s,i} \\ \vdots \\ \mathbf{y}_{\delta_p} - \sum_{i=1}^{\mu} \gamma_p^i \mathbb{1}_{c_p} \mathbf{y}_{s,i}. \end{pmatrix} \end{aligned}$$

Differentiating the agglomerate common conensus error $\mathbf{e}_\delta = \mathbf{x}_\delta - \sum_{i=1}^{\mu} \gamma^i \mathbf{x}_{s,i}$ one gets

$$\begin{aligned} \dot{\mathbf{e}}_\delta =& \mathbf{f}_\delta(\mathbf{x}_\delta) - \kappa \mathbf{g}_\delta(\mathbf{x}_\delta)\mathcal{M}\mathbf{y}_\delta - \kappa \mathbf{g}_\delta(\mathbf{x}_\delta) \sum_{i=1}^{\mu} \mathcal{M}_i \mathbf{y}_i \\ & - \sum_{i=1}^{\mu} (\gamma^i \otimes I_n)\big(\mathbf{f}_{s,i}(\mathbf{x}_{s,i}) + \tilde{\mathbf{f}}_{s,i}(\mathbf{x}_{s,i}, \mathbf{e}_i) \\ & - \kappa \tilde{\mathbf{g}}_{s,i}(\mathbf{x}_{s,i}, \mathbf{e}_i)\mathcal{L}_i \boldsymbol{\varepsilon}_i\big). \end{aligned}$$

At this point, because $\sum_{i=1}^{\mu} \big(\mathcal{M}\gamma^i + \mathcal{M}_i \mathbb{1}_{h_i}\big) = 0$

$$\mathcal{M}\mathbf{y}_\delta + \sum_{i=1}^{\mu} \mathcal{M}_i \mathbf{y}_i = \mathcal{M}\boldsymbol{\varepsilon}_\delta + \sum_{i=1}^{\mu} \mathcal{M}_i \boldsymbol{\varepsilon}_i$$

so that setting $\mathbf{x}_s = \mathrm{col}(\mathbf{x}_{s,1}, \dots, \mathbf{x}_{s,\mu})$ and

$$\begin{aligned} \tilde{\mathbf{f}}_\delta(\mathbf{x}_s, \mathbf{e}_\delta) =& \mathbf{f}_\delta(\mathbf{e}_\delta + \sum_{i=1}^{\mu} (\gamma^i \otimes I_n)\mathbf{x}_{s,i}) \\ \tilde{\mathbf{g}}_\delta(\mathbf{x}_s, \mathbf{e}_\delta) =& \mathbf{g}_\delta(\mathbf{e}_\delta + \sum_{i=1}^{\mu} (\gamma^i \otimes I_n)\mathbf{x}_{s,i}). \end{aligned}$$

one finally obtains for $\Pi_\delta^i = \gamma^i \otimes I_n$

$$\begin{aligned}\dot{\mathbf{e}}_\delta =& -\kappa \tilde{\mathbf{g}}_\delta(\mathbf{x}_s, \mathbf{e}_\delta)\mathcal{M}\boldsymbol{\varepsilon}_\delta - \kappa \sum_{i=1}^{\mu} \big(\tilde{\mathbf{g}}_\delta(\mathbf{x}_s, \mathbf{e}_\delta)\mathcal{M}_i \\ & - \Pi_\delta^i \tilde{\mathbf{g}}_{s,i}(\mathbf{x}_{s,i}, \mathbf{e}_i)\mathcal{L}_i\big)\boldsymbol{\varepsilon}_i + \tilde{\mathbf{f}}_\delta(\mathbf{x}_s, \mathbf{e}_\delta) \\ & - \sum_{i=1}^{\mu} \Pi_\delta^i \big(\mathbf{f}_{s,i}(\mathbf{x}_{s,i}) + \tilde{\mathbf{f}}_{s,i}(\mathbf{x}_{s,i}, \mathbf{e}_i)\big).\end{aligned}$$

Rewriting the equations above component-wise for all $\mathbf{e}_{\delta_\ell}$ as in (14) and exploiting (7) one gets the result. ■

Summarizing, the network dynamics (10) rewrites as the composition of an extra group of topology-induced dynamics (the mean-field) and with respect to which all agents evolve over time (through the consensus error dynamics). The common mean-field unit (20) represents a combination of the exclusive mean-field units underlying the influence of each exclusive reach over the corresponding cluster of nodes of $\mathcal{C}$. As expected, no independent network behavior emerges within the common.

Remark 3.1 *The network model inherits the same block-diagonal structure of the Laplacian. Each coefficient v_i^s of the left eigenvectors $v_i^\top$ of $\mathcal{L}_i$ weights the influence of the corresponding node in the network behavior. Moreover, the components γ^i of the right eigenvectors z_i weight the influence of the corresponding reach over the common $\mathcal{C}$.*

Remark 3.2 *It must be noted that properties of the mean-field and emergent dynamics are independent on the coordinate employed to represent each agent.*

Remark 3.3 *In general and even in the LTI case, the state-consensus error* (12b)-(14) *is never vanishing due to the direct influence of $\mathbf{x}_{s,i}$ over each error-dynamics (16)-(19) through the vector fields $\Pi_i \mathbf{f}_i((\mathbb{1}_{h_i} \otimes I_n)\mathbf{x}_{s,i})$ and $\mathbf{f}_{\delta_\ell}(\mathbf{x}_{s,\delta_\ell}) - \sum_{i=1}^{\mu}(\gamma_\ell^i \mathbb{1}_{c_\ell} v_i^\top \otimes I_n)\mathbf{f}_i((\mathbb{1}_{h_i} \otimes I_n)\mathbf{x}_{s,i})$.*

From (12b)-(12c) and (21a)-(21b), we observe that state consensus (i.e., $\mathbf{e} \equiv 0$) does not imply output synchronization (i.e., $\boldsymbol{\varepsilon} \equiv 0$) in general.

4 Practical output synchronization

Convergence to state and output multi-consensus can be carried out applying the same results available in the literature for single-consensus of heterogeneous systems depending on the structure of the agents and the framework (e.g., Qu (2009); Frasca et al. (2018); Panteley and Loría (2017); Lee and Shim (2020) to cite a few). As a matter of fact, asymptotic synchronization does not depend on the communication topology (i.e., on the graph) but only on the agent dynamics. To illustrate this, a brief analysis is given using the same arguments as in Panteley and Loría (2017) aimed at investigating practical stability of output multi-synchronization set

$$\mathcal{S}_\mathbf{y} = \{\mathbf{x} \in \mathbb{R}^{nN} : \mathbf{y}_i = \mathbb{1}_{h_i}\mathbf{y}_{s,i} \text{ and } \mathbf{y}_{\delta_\ell} = \sum_{i=1}^{\mu} \gamma_\ell^i \mathbf{y}_{s,i}, \\ \text{for } \mathbf{y}_{s,i} = v_i^\top \mathbf{y}_i\}.$$

Similarly, the state consensus set is given by

$$\mathcal{S}_\mathbf{x} = \{\mathbf{x} \in \mathbb{R}^{nN} : \mathbf{x}_i = \mathbb{1}_{h_i} \otimes \mathbf{x}_{s,i} \text{ and } \mathbf{x}_{\delta_\ell} = \sum_{i=1}^{\mu} \gamma_\ell^i \otimes \mathbf{x}_{s,i}, \\ \text{for } \mathbf{y}_{s,i} = v_i^\top \otimes \mathbf{x}_i\}.$$

With this in mind, we assume agents verify the following hypotheses which are borrowed from Panteley and Loría (2017).

A.1 Each agent (4) possesses relative degree one and the state-space equations are in normal form; namely, setting $x_i = \mathrm{col}(y_i, \eta_i) \in \mathbb{R} \times \mathbb{R}^{n-1}$ one gets

$$\dot{y}_i = b_i(y_i, \eta_i) + u_i \tag{22a}$$
$$\dot{\eta}_i = q_i(y_i, \eta_i) \tag{22b}$$

with the functions $b_i : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}$ and $q_i : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ locally Lipschitz.

A.2 The link $u_i \mapsto y_i$ is strictly semi-passive for each agent (4) with continuously differentiable and proper storage functions $V_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$; i.e., there exists constant $\rho_i > 0$, continuous functions $H_i : \mathbb{R}^n \to \mathbb{R}$ and $\psi_i : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that $H_i(x_i) \geq \psi_i(\|x_i\|)$ for all $\|x_i\| \geq \rho_i$ and

$$\dot{V}_i(x_i) \leq u_i y_i - H_i(x_i). \tag{23}$$

A.3 For each agent $\nu_i \in \mathcal{V}$, there exist compact sets $\mathbb{B}_\eta \subset \mathbb{R}^{n-1}$ and $\mathbb{B}_y \subset \mathbb{R}$, a continuously differentiable positive definite function $V_{oi} : \mathbb{B}_\eta \to \mathbb{R}_{\geq 0}$, functions $\gamma_{1i}, \gamma_{2i} \in \mathcal{K}_\infty$ and constants $\bar{\alpha}_i, \beta_i > 0$ such that for all $\eta_i, \eta' \in \mathbb{B}_\eta$ and $y_i \in \mathbb{B}_y$

$$\gamma_{i1}(\|\eta_i)\|) \leq V_{oi}(\eta_i) \leq \gamma_{i2}(\|\eta_i)\|)$$
$$\frac{\partial V_{oi}}{\partial z_i}(\eta_i - \eta')\big(q_i(y_i, \eta_i) - q_i(y_i, \eta')\big) \leq -\bar{\alpha}\|\eta_i - \eta'\|^2 + \beta_i.$$

Remark 4.1 *Assumption **A.2** allows to establish ultimate boundedness of the network dynamics* (10) *with ultimate bound B_x, independent on κ, (Panteley and Loría, 2017, Proposition 2).*

Remark 4.2 *By Lipschitz continuity, for all B_x and $\|\mathbf{x}\| \leq B_x$, there exist constant $C_1, C_2 > 0$ such that for* *all* $i = 1, \ldots, \mu$

$$
\begin{aligned}
&\|\Pi_i \tilde{f}_i(\mathbf{x}_{s,i}, \mathbf{e}_i)\| \leq C_1(B_x), \ \|\Pi_i f_{s,i}(\mathbf{x}_{s,i})\| \leq C_2(B_x) \\
&\|\sum_{i=1}^{\mu}(\gamma^i \otimes I_n)\tilde{f}_{s,i}(\mathbf{x}_{s,i}, \mathbf{e}_i)\| \leq C_1(B_x) \qquad (24) \\
&\|\sum_{i=1}^{\mu}(\gamma^i \otimes I_n) f_{s,i}(\mathbf{x}_{s,i})\| \leq C_2(B_x)
\end{aligned}
$$

with $\Pi_i = \left(I - \mathbb{1}_{h_i} v_i^\top\right) \otimes I_n$.

As the decomposition (10) suggests, output synchronization over each reach component follows with exactly the same arguments as in (Panteley and Loría, 2017, Theorem 1) by investigating stability of the solely $S_{\mathbf{y}_i}$. The result is stated for the general case here below and with the proof omitted as identical to the one reported in Panteley and Loría (2017).

Theorem 4.1 (Output synchronization) *Consider the network dynamics* (10) *under the Assumptions* ***A.1*** *and* ***A.3****. Then, the following holds.*

(1) The set $\mathcal{S}_{\mathbf{y}}$ *is uniformly globally practically asymptotically stable; for all* $R > 0$ *and* $\varepsilon > 0$ *there exists* $T(R, \varepsilon) > 0$ *such that, for all* $t \geq T$ *and* $\mathbf{x}(0) \in B_R = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq R\}$

$$
\|\varepsilon(t)\| \leq \frac{C_1 + C_2}{\kappa}\|Z_r \otimes I_n\| + \varepsilon
$$

with $C_1, C_2 > 0$ *as in* (24).

(2) if $\kappa > 0$ *is sufficiently large and* ***A.2*** *holds with* $\mathbb{B}_\eta = \{\eta \in \mathbb{R}^{n-1} : \|\eta\| \leq B_x\}$ *and* $\mathbb{B}_y = \{y \in \mathbb{R} : \|y\| \leq B_x\}$ *then, the set*

$$
\left\{\mathbf{x} \in \mathbb{R}^{nN} : \|\mathbf{x}\|_{\mathcal{S}_x} \leq c\right\}
$$

with, for $\Delta'_f > 0$,

$$
\begin{aligned}
c &= \frac{1}{\alpha_m N}\sqrt{N^2\Delta'_f + \frac{C_1 + C_2}{2} + \bar{\beta}} \\
\alpha &= min_{1 \leq k \leq N}\bar{\alpha}_k, \quad \beta = N\sum_{k=1}^{N}\beta_k
\end{aligned}
$$

is uniformly globally asymptotically stable.

5 An example: networks of gravity pendula

In the simulations we consider a network of $N = 7$ agents with communication graph $\mathcal{G}$ with Laplacian

$$
\mathcal{L} = \begin{pmatrix}
2 & -1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 2 & 0 \\
0 & 0 & -1 & 0 & -1 & -1 & 3
\end{pmatrix} \qquad (25)
$$

possessing two zero eigenvalues with corresponding left and right eigenvectors

$$
\begin{aligned}
v_1^\top &= \begin{pmatrix}\frac{1}{6} & \frac{1}{3} & \frac{1}{2} & \mathbf{0}^\top & \mathbf{0}^\top\end{pmatrix}, \quad v_2^\top = \begin{pmatrix}\mathbf{0}^\top & \frac{1}{2} & \frac{1}{2} & \mathbf{0}^\top\end{pmatrix} \\
z_1^\top &= \begin{pmatrix}1 & 1 & 1 & \mathbf{0}^\top & \frac{1}{2} & \frac{1}{2}\end{pmatrix}, \quad z_2^\top = \begin{pmatrix}\mathbf{0}^\top & 1 & 1 & \frac{1}{2} & \frac{1}{2}\end{pmatrix}.
\end{aligned}
$$

Accordingly, nodes can be partitioned as $\mathcal{H}_1 = \{\nu_1, \nu_2, \nu_3\}$, $\mathcal{H}_2 = \{\nu_4, \nu_5\}$ and $\mathcal{C} = \{\nu_6, \nu_7\}$ so that, here, $\mu = 2$ with $h_1 = 3$, $h_2 = 2$ and $\delta = 2$. The graph admits the almost equitable partition (6) with $p = 1$ and $\rho_3 = \mathcal{C}$. According to Monaco and Ricciardi Celsi (2019), when dealing with simple integrators $\dot{x}_i = u_i$ with $i = 1, \ldots, 7$, three consensuses are expected (as the number of cells of π) depending only on the initial conditions of nodes in the exclusive reaches: one common value for nodes in the same $\mathcal{H}_i$ ($i = 1, 2$) and one distinct values for the nodes in $\mathcal{C}$. The digraph (25) induces two main-field behaviors $\mathbf{x}_{s,i}$ with $i = 1, 2$ each governing the nodes in $\mathcal{H}_i$ and the ones in $\mathcal{C}$.

We consider now the case in which agents are gravity-pendula described by the Lagrangian equation

$$
\ddot{q}_i = -\frac{g}{l_i}\sin q_i + \frac{1}{m_i l_i}u_i, \quad y_i = \dot{q}_i
$$

with $l_i > 0$ and $m_i > 0$ for $i = 1, \ldots, 7$ and $g = 9.81$. Denoting $x_i = (q_i \ \dot{q}_i)^\top$ and assuming an interconnection of the form (9) with Laplacian (25), by Proposition 3.1 two exclusive mean-field units arise

$$
\mathbf{x}_{s,1} = \frac{1}{6}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3, \quad \mathbf{x}_{s,2} = \frac{1}{2}(x_4 + x_6)
$$

with corresponding outputs

$$
\mathbf{y}_{s,1} = \frac{1}{6}\dot{q}_1 + \frac{1}{3}\dot{q}_2 + \frac{1}{2}\dot{q}_3, \quad \mathbf{y}_{s,2} = \frac{1}{2}(\dot{q}_4 + \dot{q}_6).
$$

Accordingly, the emergent dynamics (17) is provided by the Lagrangian system

$$\ddot{\mathbf{q}}_{e,1} = -g\Big(\frac{l_1}{6} + \frac{l_2}{3} + \frac{l_3}{2}\Big)^{-1} \sin \mathbf{q}_{e,1}$$
$$\ddot{\mathbf{q}}_{e,2} = -g\Big(\frac{l_4}{2} + \frac{l_5}{2}\Big)^{-1} \sin \mathbf{q}_{e,2}.$$

As far as the reach component is concerned, exploiting Theorem 3.1 the corresponding mean-field units and the corresponding outputs are given by

$$\mathbf{x}_{s,\delta} = \frac{1}{2}(\mathbf{x}_{s,1} + \mathbf{x}_{s,2}), \quad \mathbf{y}_{s,\delta} = \frac{1}{2}(\mathbf{y}_{s,1} + \mathbf{y}_{s,2}).$$

One can verify, in general, that unless $m = m_i$ and $l = l_i$ ($i = 1, \ldots, 7$) the emergent dynamics is not attractive for all agents (i.e., asymptotic state multi-consensus cannot be achieved). Exploiting the result in (Qu (2009)) for single-consensus, by passivity of all agents with storage

$$S_i(x_i) = \frac{1}{2}\dot{q}_i^2 + \frac{g}{l_i}(1 - \cos q_i)$$

asymptotic output multi-consensus is guaranteed; namely, as $t \to \infty$ one gets

$$\dot{q}_i \to \mathbf{y}_{s,1} = \frac{1}{6}\dot{q}_1 + \frac{1}{3}\dot{q}_2 + \frac{1}{2}\dot{q}_3, \quad i = 1, 2, 3$$
$$\dot{q}_i \to \mathbf{y}_{s,2} = \frac{1}{2}(\dot{q}_4 + \dot{q}_6), \quad i = 4, 5$$
$$\dot{q}_i \to \mathbf{y}_{s,\delta} = \frac{1}{12}\dot{q}_1 + \frac{1}{6}\dot{q}_2 + \frac{1}{4}\dot{q}_3 + \frac{1}{4}(\dot{q}_4 + \dot{q}_5), \quad i = 6, 7.$$

For completeness, a simple simulation is reported in Figure 1 setting $\kappa = 10$, $m_i = i$ and $l_i = i + 1$ for $i = 1, \ldots, 7$. The result highlights that all agents belonging to the same reach of the AEP converge to a shared output trajectory provided by the weighted mean (through (3)) of the outputs of the corresponding agents; nodes in the common converge to a further output consensus dictated by the nodes in the exclusive reach only. As expended, despite output consensus is achieved, the connection fails in guaranteeing state consensus even when all agents have identical masses.

6 Conclusions

In this paper, the behaviors induced by the network interconnection of heterogeneous nonlinear systems under a fixed but general graph topology has been characterized. As suggested by the single-integrator case, it results that the network induces a specific partition of the nodes which also governs the cluster collective behaviors. Current work is addressing the case of time-varying topology and the possibility of embedding feedback design to change both the agents dynamics and the connecting topology to obey to certain specifications in a unifying framework for different problems such as randez-vous, gossiping, flocking or formation control. A preliminary work in this direction is Cacace et al. (2021) where the case of LTI homogeneous agents has been addressed.

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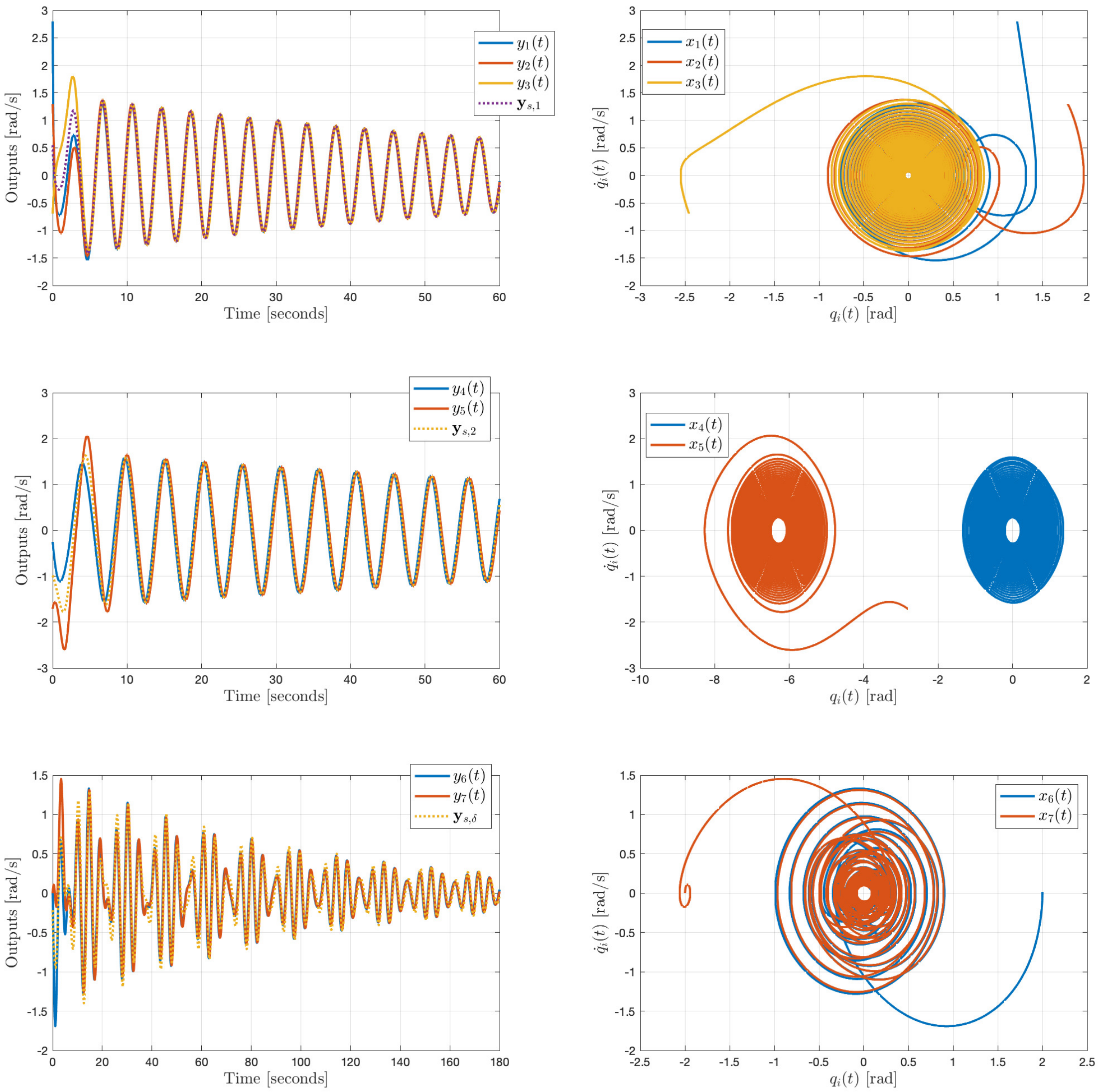

Fig. 1. Network of gravity pendula

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