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Normality of closure of nilpotent conjugacy classes

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Abstract

In this work the author studies the geometry of the conjugacy classes in the space of matrices and in its subspaces of symmetric and skew-symmetric matrices under the actions of the general linear group, the orthogonal group and the symplectic group. An extensive and self-contained review of the current state of the art concerning the description of nilpotent conjugacy classes and their closures is provided. Many geometrical properties of the closure of a conjugacy class are interesting from a representation theoretic viewpoint; in particular, their normality. A complete discussion of the normality of the closures of the conjugacy classes for the stated actions takes a prominent role in this work. The main new result completes the picture and it states that a nilpotent symmetric conjugacy class for the orthogonal group has normal closure if and only if the associated partition has consecutive parts of length differing by at most one.

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Contents

Introduction

The study of conjugacy classes of matrices connects several branches of mathematics such as linear algebra, classical and geometric invariant theory, Lie theory, algebraic geometry, combinatorics and quiver theory. In the present thesis we aim at showing at least some facets of that deep bond. In particular we deal with conjugacy classes of nilpotent matrices.

One the main theme of the thesis is that the theory of conjugacy classes may be developed for any linear reductive group. In particular, if one works with the following classical linear groups, the general linear group, the orthogonal group and the symplectic group, as we do here, one finds that most of the constructions may be carried on in the same way in each case and, generally, there is a close correspondence between the results in each case.

The nilpotent matrices studied will be taken in the space of all matrices, when we consider the general linear group, or in the space of the skew matrices or the space of the symmetric matrices, when we consider the other classical groups. First we describe combinatorially the conjugacy classes arising in each case using partitions, then we describe the Zariski closure of the conjugacy classes, finally we inspect some geometrical properties of the closures of the conjugacy classes, the main property being the normality.

Consider the following general problem. Let G be a reductive linear group acting on a space X. Suppose that there is a single open orbit $\mathcal{O} \subseteq X$ such that $X = \overline{\mathcal{O}}$, the closure in the Zariski topology. Many geometrical properties of X are interesting to study because of representation theory consequences. In particular, it is interesting to understand which type of singularities appear on the boundary $X \setminus \mathcal{O}$. In this thesis we are mainly concerned with the normality of X. Some of the reasons that explain the cruciality of this property are discussed in section 1.4. Many works of many mathematicians were dedicated to the study of this problem. The main paper on which the interest towards this problem has been raised is due to Kostant [Kos63].

Although this generality might be useful for some constructions, the work in this paper is focused on the adjoint action of a reductive group G on its Lie algebra $\mathfrak{g}$. The orbits in this case are called conjugacy classes for G. We can take X to be the Zariski closure in $\mathfrak{g}$ of any conjugacy classes; therefore the problem which we are going to inspect becomes the following: which closure of conjugacy classes are normal varieties? Actually, this is already the setting in Kostant [Kos63].

In subsequent years, Kostant and Rallis [KR71, KR69b, KR69a] developed the theory of symmetric pairs and symmetric spaces. In their settings there is still an adjoint action of a reductive group G, but they studied the restriction of the adjoint action to a subgroup K. Such an action leaves invariant two spaces $\mathfrak{k}$ and $\mathfrak{p}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{k}$ is the Lie algebra of K. See section 1.1.3 for major details in this setting. If one takes X to be the Zariski closure of a K-orbit in $\mathfrak{p}$, this analogue problem arises: which closure of K-conjugacy classes in $\mathfrak{p}$ are normal varieties?

We briefly summarize the main results obtained towards the solution of these problems. Kostant showed in [Kos63] that the nilpotent cone in $\mathfrak{g}$, which is the closure of the maximal nilpotent conjugacy class of $\mathfrak{g}$, also called the regular nilpotent conjugacy class, is always normal.

In [Bro94], Broer proved that a larger set of nilpotent conjugacy classes, such as the subregular conjugacy class, have normal closure.

Hesselink proved in [Hes76] that the normality of the closure of any conjugacy class can be reduced to the study of only nilpotent conjugacy classes (possibly more than one and possibly changing the acting group G).

In several papers ([KP79, KP80, PK78]) Kraft and Procesi showed that the conjugacy classes of usual matrices, which is the case $G = \mathrm{GL}_n$, always have normal closures. In [KP82], they worked also on the problem of conjugacy classes of the classical groups $G = O_n$ and $G = Sp_{2n}$, where they showed that the picture is much more complex in this case.

The exceptional cases has been solved in [Kra89, Bro98, Som03]. In [Som05], Sommers also concluded the analysis of the SO_n-conjugacy classes, started by Kraft and Procesi in [KP82].

Ohta adapted the method of Kraft and Procesi in [Oht86] to prove that the conjugacy classes of symmetric matrices under the action of Sp_{2n} have normal closure. He found only a partial result for the orthogonal conjugacy classes. The author completed that analysis for the orthogonal group in [Tre22].

The present thesis is structured in the following way. In chapter 1 we set the background of Lie theory, algebraic geometry and we state all the main theorems needed for the following chapters. We conclude chapter 1 with some motivations for the study of normality and the proof of Kostant, which serves as a model to the normality of other conjugacy classes.

Chapter 2 displays each action that we are interested in: the GL_n-conjugacy classes in $\mathfrak{gl}_n$; the conjugacy classes under O_n and under Sp_{2n} of both the skew and the symmetric matrices which are the spaces $\mathfrak{k}$ and $\mathfrak{p}$ in $\mathfrak{gl}_n$. We follow the approach of [KR71] for those actions. Then we describe the nilpotent conjugacy classes in each of those cases, which are classic, and we mainly follow [CM93].

In chapter 3 we study the closure $X = \overline{\mathcal{O}}$ of the conjugacy classes introduced in chapter 2, in particular we determine which nilpotent classes lie inside X, depending on $\mathcal{O}$. The theory of this chapter as well is classic; we follow mainly [CM93].

In chapter 4 we develop the theory of nilpotent pairs of linear maps, which are the representations of a particular quiver with relations. We will heavily rely on that theory in the last two chapters. The main references are [KP79, KP82].

We present an auxiliary variety Z to support the proof of the normality of X in chapter 5. Indeed, we construct a map $\Theta : Z \to X$ which is a quotient under a particular reductive group; therefore the normality of X is reduced to the normality of Z. We combine the methods first described in [KP79] and later adapted to the classical groups in [KP82, Oht86] in this and in the following chapters.

The study of the normality of Z leads to a stratification of Z which is the main topic of chapter 6.

Finally, chapter 7 is assigned to combine the results of chapter 5 and chapter 6 to prove the normality of the closure of many conjugacy classes. In fact, in section 7.2 we even prove which conjugacy class do not have normal closure. In the end we have simple combinatorial conditions to decide whether a nilpotent conjugacy class has normal closure or not.

Chapter 1

Background

In this thesis we will work exclusively with an algebraically closed field k of characteristic 0. For us a ring has 1.

By *variety* we mean a scheme such that it is reduced and of finite-type over $\mathrm{spec}(k)$. We denote the set of regular functions of a variety X by $k[X]$ or, sometimes, by $\mathcal{O}(X)$. We will work almost exclusively with affine varieties, so that $X = \mathrm{spec}(k[X])$.

1.1 Lie theory

Some standard references for this section are [Spr94, Hum12, Wey16].

A linear algebraic group G is a closed subgroup of the general linear group $\mathrm{GL}(V)$ of a finite dimensional k-vector space V. A G-representation is a k-vector space W equipped with a morphism $G \to \mathrm{GL}(W)$. A rational G-representation W is a G-representation such that for each $w \in W$ there exists a G-invariant finite-dimensional subspace $W_0 \subseteq W$ such that $w \in W_0$.

A G-space X is a variety equipped with a compatible action of the group: $G \times X \to X$, in particular the G-representation $k[X]$ is rational. We denote the action of an element $g \in G$ on a point $x \in X$ by $g.x \in X$, the orbit of x by $G.x \subseteq X$, the stabilizer of x by $G_x \subseteq G$. If one denotes the automorphisms of X by $\mathrm{Aut}(X)$, one can think about the G-space X as a group homomorphism $G \to \mathrm{Aut}(X)$.

For a positive integer n, the group U_n is the closed subrgroup of GL_n consisting of upper triangular matrices with 1s along the diagonal. A linear algebraic group G is called *unipotent* if it can be embedded as a closed subgroup of U_n for some $n \geq 1$.

A linear algebraic group G is *reductive* if the maximal connected closed unipotent subgroup of G is the trivial subgroup $\{1\}$. We will work almost exclusively with the classical groups GL_n, O_n, Sp_{2n}, all of which are reductive.

1.1.1 Adjoint action

Let G be a linear algebraic group. The Lie algebra of G is the tangent space $\mathfrak{g} := \mathrm{Lie}(G) := T_1 G$ to the identity element $1 \in G$.

Given a linear algebraic group G, one can consider the left action of G to itself by conjugation:

$$\begin{aligned} G \times G &\to G \\ (g, h) &\mapsto ghg^{-1}. \end{aligned}$$

For each $g \in G$, let $c_g : G \to G$ be the morphism given by the conjugation by g, i. e. $h \mapsto ghg^{-1}$. This way one can think about this action as the group homomorphism $G \to \mathrm{Aut}(G)$ such that $g \mapsto c_g$. One can consider the differential at 1 of any such automorphism

$$d(c_g) : \mathfrak{g} \to \mathfrak{g}.$$

Therefore one obtains a G-representation structure on the finite dimension space $\mathfrak{g}$ given in the following definition

Definition 1.1.1. Let G be a linear algebraic group. The adjoint action of G on its Lie algebra $\mathfrak{g}$ is given by

$$\begin{aligned} G \times \mathfrak{g} &\to \mathfrak{g} \\ g.v &= d(c_g)(v). \end{aligned}$$

We sometimes denote the action of a $g \in G$ by $\mathrm{Adj}(g)$.

In particular, the adjoint action may be seen as a homomorphism of linear algebraic groups $\mathrm{Adj} : G \to \mathrm{GL}(\mathfrak{g})$. If one considers the differential of Adj at 1, one may define a bilinear operation on $\mathfrak{g}$ by

$$\begin{aligned} \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g} \\ (v, w) &\mapsto [vw] := d(\mathrm{Adj})(v)(w). \end{aligned}$$

It is well known that $[-, -]$ equips the vector space $\mathfrak{g}$ with a Lie algebra structure.

Remark. The kernel of the group homomorphism $g \mapsto c_g$ is given by the center $Z(G)$ of G, by definition of center. Therefore one finds that the center $Z(G)$ is also the kernel of the adjoint action $G \to \mathrm{GL}(\mathfrak{g})$. In particular, one gets the faithful action

$$G/Z(G) \to \mathrm{GL}(\mathfrak{g}).$$

If $\dim Z(G) = 0$, this is exactly the adjoint action of the centerless group $G/Z(G)$ (G is reductive). If G is reductive, one has

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}],$$

the adjoint action stabilizes $[\mathfrak{g}, \mathfrak{g}]$ and fixes pointwise $\mathfrak{z}(\mathfrak{g})$. Therefore, in this case, all the interesting information of the adjoint action of the reductive group G lies in the adjoint action of the centerless group $G/Z(G)$. Moreover, its Lie algebra $\mathrm{Lie}(G/Z(G)) \simeq [\mathfrak{g}, \mathfrak{g}]$ is semisimple.

If G is connected, we define the *adjoint group* of G as

$$G^{\mathrm{ad}} := G/Z(G).$$

Although we will mostly work with adjoint actions of connected reductive groups G (so, basically, only with adjoint groups and their adjoint actions), there is a notable exception: the orthogonal group O_n. The orthogonal group has two connected components, distinguished by the determinant, and a finite center given by $\{\pm 1\} \simeq \mathbb{Z}/2$. The quotient $O_n/\{\pm 1\}$ is connected if and only if -1 does not belong to the same connected component of the identity, which happens if and only if $\det(-1) = -1$ if and only if n is odd.

The fact that $O_{2n}/\{\pm 1\}$ is not connected has profound consequences on the adjoint action of O_{2n}, among others, the fact that there exist not connected orbits. Quite surprisingly, this fact influences the geometry of the orbits in the adjoint actions of other classical groups, in particular the actions of O_{2n+1} and of Sp_{2n} (see proposition 4.4.9, as an example).

Remark. The adjoint action of GL_n on its Lie algebra $\mathfrak{gl}_n$ is given by the conjugation of matrices

$$g.A = gAg^{-1}$$

for every $g \in \mathrm{GL}_n$ and $A \in \mathfrak{gl}_n$.

Remark. For any closed subgroup $H \subseteq G$, the adjoint action of H is the restriction of the adjoint action of G to the subgroup H. Therefore, also for the other classical groups $G = O_n$ or $G = Sp_{2n}$ one gets that

$$g.A = gAg^{-1}$$

for every $g \in G$ and $G \in \mathfrak{g}$.

The orbits of the adjoint action of a group G are called the *conjugacy classes* of $\mathfrak{g}$. In particular, for $G = \mathrm{GL}_n$, one has that the orbit $\mathrm{GL}_n .A$ of the adjoint action are the usual matrices conjugated to A.

The orbit $G.x \subseteq \mathfrak{g}$ is a subvariety of the affine space $\mathfrak{g}$. One can take the Zariski closure $\overline{G.x} \subseteq \mathfrak{g}$. The main problem we deal with in this thesis is to decide whether the affine variety $\overline{G.x}$ is normal.

We recall some general facts about Jordan decomposition.

Proposition 1.1.2. *Let G be a linear algebraic group. Then, for every $g \in G$, there exist unique elements $g_s, g_n \in G$ such that:*

- *$g = g_s g_n = g_n g_s$ and*
- *for each closed embedding $f : G \to \mathrm{GL}_n$, $f(g_s)$ is a semisimple element in GL_n and $f(g_n)$ is a unipotent element in GL_n.*

One can also work only with a Lie algebra $\mathfrak{g}$. In that case one may write $x = x_s + x_n$ for any $x \in \mathfrak{g}$, where $[x_s, x_n] = 0$, x_s is semisimple, x_n is nilpotent. It is easy to construct such a Jordan decomposition if $\mathfrak{g}$ is semisimple, but one may obtain a useful Jordan decomposition even for $\mathfrak{g}$ reductive, by considering the center $\mathfrak{z}(\mathfrak{g})$ as consisting of semisimple elements.

The following are a couple of important facts involving nilpotent elements.

Theorem 1.1.3 (Jacobson-Morozov)**.** *Let $\mathfrak{g}$ be a reductive Lie algebra and let $X \in \mathfrak{g}$ be a non-zero nilpotent element. Then there is an injective morphism of Lie algebras $\rho : \mathfrak{sl}_2 \to \mathfrak{g}$ such that $\rho(e) = X$ where e is the nilpositive element of $\mathfrak{sl}_2$.*

Theorem 1.1.4 (Kostant)**.** *Let G be a reductive group with Lie algebra $\mathfrak{g}$ and let $\rho, \rho' : \mathfrak{sl}_2 \to \mathfrak{g}$ be two maps as in theorem 1.1.3. Then there is $g \in G$ such that $\rho' = \mathrm{Ad}(g) \circ \rho$.*

1.1.2 Quadratic spaces

In this section we recall some basic facts about orthogonal spaces and symplectic spaces.

Let ε be $+1$ or -1. A finite dimensional vector space V equipped with a non-degenerate bilinear form $(-,-)$ is called *quadratic space of type* ε if

$$(x, y) = \varepsilon(y, x) \quad \forall x, y \in V.$$

Or, in short, V is orthogonal if $\varepsilon = +1$ and V is symplectic if $\varepsilon = -1$. We denote by $G_\varepsilon(V)$ (or simply $G(V)$) the subgroup of $\mathrm{GL}(V)$ of elements leaving the form $(-,-)$ on V invariant. It is well known that, if $\dim V = n$, $G_{+1}(V) \simeq O_n$ (the orthogonal group) and $G_{-1}(V) \simeq Sp_n$ (the symplectic group).

As the form $(-,-)$ is non-degenerate, one has a linear involution $-^* : \mathrm{End}(V) \to \mathrm{End}(V)$, that we call adjoint, which maps D to the linear endomorphism D^* defined by

$$(Dx, y) = (x, D^* y) \quad \forall x, y \in V.$$

Remark. If one takes the standard bilinear form with $\varepsilon = +1$ on $V = k^n$, given by $(x, y) \mapsto x^t y$, the adjoint involution is $D^* = D^t$.

If $\varepsilon = +1$ the standard bilinear form on $V = k^n$ (in particular n must be even), is given by $(x, y) \mapsto x^t J y$, where $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and 1 is the identity matrix with size $n/2$. In this case, the adjoint involution is $D^* = -JD^t J$.

By definition of $G(V)$ we have

$$\begin{aligned} G(V) &= \{g \in GL(V) : g^* = g^{-1}\} \\ \mathfrak{g}(V) &:= \mathrm{Lie}\, G(V) = \{D \in \mathrm{End}(V) : D^* = -D\}. \end{aligned}$$

1.1.3 Symmetric pairs

We follow [KR71], but see also [KR69b, KR69a].

Let $\mathfrak{g}$ be a reductive Lie algebra and let $\theta : \mathfrak{g} \mapsto \mathfrak{g}$ be a Lie algebra-automorphism. Suppose also that $\theta^2 = 1_{\mathfrak{g}}$. Then one has a decomposition of $\mathfrak{g}$ into θ-eigenspaces

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k} = \{x \in \mathfrak{g} : \theta(x) = x\}$ and $\mathfrak{p} = \{x \in \mathfrak{g} : \theta(x) = -x\}$. One gets the following immediate facts:

- $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$,
- $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$,
- $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$.

In particular, $\mathfrak{k}$ is a Lie-subalgebra of $\mathfrak{g}$.

Definition 1.1.5. A pair of reductive Lie algebras $(\mathfrak{g}, \mathfrak{k})$ is called *symmetric* if $\mathfrak{k} \subseteq \mathfrak{g}$ and there exists an involutive Lie algebra-automorphism θ such that $\mathfrak{k}$ is the θ-fixed points of θ.

Let $(\mathfrak{g}, \mathfrak{k})$ be a symmetric pair. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, then $\mathfrak{p}$ is the *symmetric space* associated to the pair $(\mathfrak{g}, \mathfrak{k})$.

Example. Let $\varepsilon \in \{\pm 1\}$ and let V be a quadratic space of type ε. Let D^* be the adjoint endomorphism of D. It is possible to define an automorphism θ of $\mathfrak{gl}(V)$ by

$$\theta(D) = -D^*.$$

Indeed, θ is linear and

$$\theta([C, D]) = \theta(CD - DC) = -\theta(D)\theta(C) + \theta(C)\theta(D) = [\theta(C), \theta(D)].$$

Therefore, one gets a symmetric pair for each quadratic space V. We remark that

$$\mathfrak{k} = \{x \in \mathfrak{g} : \theta(x) = x\} = \{x \in \mathfrak{g} : x^* = -x\} = \mathfrak{g}(V),$$

that is, $\mathfrak{k}$ is the Lie algebra of skew matrices, while

$$\mathfrak{p} = \{x \in \mathfrak{g} : \theta(x) = -x\} = \{x \in \mathfrak{g} : x^* = x\}$$

is the subspace of symmetric matrices.

We will call $(\mathfrak{gl}_n, \mathfrak{o}_n)$ the orthogonal symmetric pair and $(\mathfrak{gl}_{2n}, \mathfrak{sp}_{2n})$ the symplectic symmetric pair. These symmetric pairs are one the main focus in this thesis.

Let G be the adjoint group of $\mathfrak{g}$, acting on $\mathfrak{g}$ with the adjoint action. One can define the subgroup of θ-commuting element of G

$$K_\theta := \{g \in G : \operatorname{Ad} g \circ \theta = \theta \circ \operatorname{Ad} g\}.$$

Clearly the action of K_θ stabilizes both $\mathfrak{k}$ and $\mathfrak{p}$, in particular, we are concerned with the latter action $K_\theta \to \operatorname{Aut}(\mathfrak{p})$.

In general, the group K_θ is not connected. Let $K := K_\theta^0$ be the connected component of the identity of K_θ. Then one has that $\operatorname{Lie} K = \mathfrak{k}$.

We can finally extend the problem of the normality of the orbits to the case of symmetric pairs. Let $x \in \mathfrak{p}$ be any symmetric element and $K_\theta.x$ its conjugacy class. When is the affine variety $\overline{K_\theta.x} \subseteq \mathfrak{p}$ normal? In fact we could also ask which differences appear if we take the orbit with respect to K instead of K_θ, but we will only tackle this problem tangentially: the main focus in this thesis lies on the problem with K_θ and only in the case of the orthogonal and symplectic symmetric pairs.

An element $x \in \mathfrak{g}$ is *semisimple* if $\operatorname{Ad}(x)$ is diagonalizable. An element $x \in \mathfrak{g}$ is *nilpotent* if $x \in [\mathfrak{g}, \mathfrak{g}]$ and $\operatorname{Ad}(x)$ is nilpotent. One has the following important result.

Proposition 1.1.6. *Let $(\mathfrak{g}, \mathfrak{k})$ be a symmetric pair with symmetric space $\mathfrak{p}$. Let $x \in \mathfrak{p}$ be any element and let $x = x_s + x_n$ be its Jordan decomposition in $\mathfrak{g}$. Then $x_s, x_n \in \mathfrak{p}$.*

Proof. By definition, $\theta(x) = -x$. Therefore, the Jordan decomposition of $\theta(x)$ is $(-x_s) + (-x_n)$.

On the other hand, $\theta(x) = \theta(x_s) + \theta(x_n)$. As θ is a Lie algebra automorphism, $\theta(x_s)$ and $\theta(x_n)$ commute, $\theta(x_s)$ is semisimple and $\theta(x_n)$ is nilpotent. The conclusion follows by the uniqueness of the Jordan decomposition in $\mathfrak{g}$, as $\theta(x_s) = -x_s$ and $\theta(x_n) = -x_n$. □

It is clear that both the sets of semisimple elements in $\mathfrak{p}$ and of nilpotent elements in $\mathfrak{p}$ are invariant by K_θ. In fact, one can show the analogue of theorem 1.4.9 for symmetric pairs and the problem of the normality of the orbits is reduced to the case of nilpotent orbits.

Proposition 1.1.6 is the starting point to the extension of most of the usual results of Lie theory to the symmetric pairs. We are not going to develop the full theory, but we are recall an important theorem.

Theorem 1.1.7 (Jacobson Morozov for symmetric pairs)**.** *Let $(\mathfrak{g}, \mathfrak{k})$ be a symmetric pair with symmetric space $\mathfrak{p}$. Let $x \in \mathfrak{p}$ be a nilpotent element such that $x \neq 0$. Then there is a Lie algebra morphism $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$ such that $\phi(e) = x$, $\phi(h) \in \mathfrak{k}$ and $\phi(f) \in \mathfrak{p}$.*

On the difference $\dim \mathfrak{p} - \dim \mathfrak{k}$

Let $x \in \mathfrak{g}$ be any element. We denote by $\mathfrak{g}^x$ its centralizer in $\mathfrak{g}$ and, similarly, we denote by $\mathfrak{k}^x$ (resp. $\mathfrak{p}^x$) the centralizer of x inside $\mathfrak{k}$ (resp. $\mathfrak{p}$).

Proposition 1.1.8. *Let $\mathfrak{p}$ be the symmetric space of the pair $(\mathfrak{g}, \mathfrak{k})$ and let $x \in \mathfrak{p}$. Then*

$$\dim \mathfrak{p}^x - \dim \mathfrak{k}^x = \dim \mathfrak{p} - \dim \mathfrak{k}.$$

Proof. As $[\mathfrak{g}, \mathfrak{g}]$ is semisimple, there exists a symmetric, non-singular, $\operatorname{Ad}\mathfrak{g}$-invariant bilinear form on $[\mathfrak{g}, \mathfrak{g}]$, namely the Killing form. This form is also θ-invariant, as θ is an automorphism of Lie algebras. We want to get a symmetric, non-singular, $\operatorname{Ad}\mathfrak{g}$-invariant, θ-invariant bilinear form B on $\mathfrak{g}$, by extending the Killing form on the center $\mathfrak{z}$ of $\mathfrak{g}$. We set $B([\mathfrak{g}, \mathfrak{g}], \mathfrak{z}) = 0$, so that it is enough to extend the Killing form with any symmetric, non-singular bilinear form on $\mathfrak{z}$. The form B obtained in this way is θ-invariant as $\mathfrak{z}$ is θ-invariant.

For any $x \in \mathfrak{p}$, let B_x be the antisymmetric bilinear form on $\mathfrak{g}$ defined by

$$B_x(y, z) = B(x, [z, y]).$$

We can see that the kernel of B_x is exactly $\mathfrak{g}^x$:

$$\begin{aligned} &B_x(y, z) = 0 \quad \forall y \Leftrightarrow \\ &B([x, z], y) = 0 \quad \forall y \Leftrightarrow \\ &[x, z] = 0. \end{aligned}$$

Therefore B_x defines a non-singular antisymmetric bilinear form $\tilde{B}_x$ on $\mathfrak{g}/\mathfrak{g}^x$. As $x \in \mathfrak{p}$, we have that $\mathfrak{k} \cap \mathfrak{g}^x = \mathfrak{k}^x$, $\mathfrak{p} \cap \mathfrak{g}^x = \mathfrak{p}^x$, and $\mathfrak{g}^x = \mathfrak{k}^x \oplus \mathfrak{p}^x$, therefore

$$\mathfrak{g}/\mathfrak{g}^x = \mathfrak{k}/\mathfrak{k}^x \oplus \mathfrak{p}/\mathfrak{p}^x.$$

Moreover,

$$B_x(\mathfrak{k},\mathfrak{k}) = B(x,[\mathfrak{k},\mathfrak{k}]) \subseteq B(\mathfrak{p},\mathfrak{k}) = 0$$

as B is θ-invariant, so $\mathfrak{p}$ and $\mathfrak{k}$ must be B-orthogonal. This implies that $\mathfrak{k}/\mathfrak{k}^x$ is an isotropic space with respect to $\tilde{B}_x$, and the same conclusion is obtained with $\mathfrak{p}/\mathfrak{p}^x$. However, $\tilde{B}_x$ is non-degenerate, therefore we cannot have isotropic subspaces with dimension greater than half of $\mathfrak{g}/\mathfrak{g}^x$. Therefore we conclude that

$$\dim\mathfrak{k} - \dim\mathfrak{k}^x = \frac{1}{2}(\dim\mathfrak{g} - \dim\mathfrak{g}^x) = \dim\mathfrak{p} - \dim\mathfrak{p}^x.$$

□

Corollary 1.1.9. *Let $x \in \mathfrak{p}$ and let X (resp. C) be the orbit of x in $\mathfrak{p}$ (resp. $\mathfrak{g}$). Then*

$$\dim C = 2\dim X.$$

Proof. In order to evaluate the dimension of an orbit through a point $x \in \mathfrak{g}$, we can just compute the dimension of its tangent space to x inside $\mathfrak{g}$. Therefore we need to compute the dimension of $[\mathfrak{g}, x]$ and the dimension of $[\mathfrak{k}, x]$. Therefore we must show that

$$\begin{aligned} \dim[\mathfrak{g},x] &= 2\dim[\mathfrak{k},x] \\ \dim\mathfrak{g} - \dim\mathfrak{g}^x &= 2(\dim\mathfrak{p} - \dim\mathfrak{p}^x) \\ \dim\mathfrak{k} + \dim\mathfrak{p} - \dim\mathfrak{k}^x - \dim\mathfrak{p}^x &= 2(\dim\mathfrak{p} - \dim\mathfrak{p}^x) \\ \dim\mathfrak{k} - \dim\mathfrak{k}^x &= \dim\mathfrak{p} - \dim\mathfrak{p}^x \end{aligned}$$

which is exactly the result of proposition 1.1.8. □

1.1.4 Graded Lie algebras or θ-groups

In this section we discuss a generalization of symmetric pairs, called θ-groups, due to Vinberg, see [Vin76].

Let $\mathfrak{g}$ be a reductive Lie algebra, m be a positive integer and $\theta : \mathfrak{g} \to \mathfrak{g}$ be a Lie algebra automorphism such that $\theta^m = 1_{\mathfrak{g}}$. As $\theta^m = 1_{\mathfrak{g}}$, θ is diagonalizable and its eigenspaces are

$$\mathfrak{g}_k := \{x \in \mathfrak{g} : \theta(x) = \exp(2\pi ik/m)x\},$$

for $k = 0, 1, \ldots, m-1$. Hence we get a decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{m-1}. \tag{1.1.1}$$

As θ is a Lie algebra automorphism, we get $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ (understanding that $i + j \in \mathbb{Z}/m$). Therefore, the decomposition (1.1.1) is a $\mathbb{Z}/m$-gradation.

One can recover the automorphism θ once a gradation like (1.1.1) is given. Indeed, one can define a linear endomorphism θ of $\mathfrak{g}$ by putting $\theta(x) = \exp(2\pi ik/m)x$ for $x \in \mathfrak{g}_k$. Then, by the $\mathbb{Z}/m$ gradation of (1.1.1), one gets that such a map θ preserves the bracket.

Definition 1.1.10. A *θ-group* or a *graded Lie algebra* is a triple $(\mathfrak{g}, \theta, m)$ where $\mathfrak{g}$ is a reductive Lie algebra, m is a positive integer, $\theta : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism such that $\theta^m = 1_{\mathfrak{g}}$.

Remark. We notice that, if $m = 2$, the θ-group $(\mathfrak{g}, \theta, 2)$ is exactly the symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$ in the sense of definition 1.1.5.

Let G be any connected algebraic group with Lie algebra $\operatorname{Lie} G = \mathfrak{g}$ and let G_0 be the connected subgroup of G which corresponds to the subalgebra $\mathfrak{g}_0$. From the property $[\mathfrak{g}_0, \mathfrak{g}_k] \subseteq \mathfrak{g}_k$, it follows that the adjoint representation of G induces, by restriction, a linear representation ρ_k of G_0 in $\mathfrak{g}_k$ for every k. As the adjoint representation annihilates the center of G, the linear group $\rho_k(G_0)$ does not depend on the choice of G among the algebraic groups with Lie algebra $\mathfrak{g}$. Moreover, the study of the linear representations ρ_k could be restricted to the case $k = 1$. Indeed, for any k, one may always change the Lie algebra $\mathfrak{g}$ to another $\bar{\mathfrak{g}}$, graded modulo $\bar{m} = m/(m, k)$, setting $\bar{\mathfrak{g}}_h = \mathfrak{g}_{hk}$ for $h \in \mathbb{Z}/\bar{m}$, so that $\bar{\mathfrak{g}}_0 = \mathfrak{g}_0$ and $\bar{\mathfrak{g}}_1 = \mathfrak{g}_k$.

The group $\rho_1(G_0)$ is said to be the group associated with the graded Lie algebra $(\mathfrak{g}, \theta, m)$. In the context of θ-groups, one is mainly concerned with the study of the representation of $\rho_1(G_0)$ in $\mathfrak{g}_1$.

Suppose that the automorphism θ of $\mathfrak{g}$ is induced by an automorphism of G, still denoted by θ. The group

$$G^\theta := \{g \in G : \theta(g) = g\} \subseteq G$$

acts on $\mathfrak{g}_1$ by restriction of the adjoint representation of G to G^θ. We denote this representation by ρ_1. Again, $\rho_1(G^\theta)$ does not depend on the choice of G (among the algebraic groups with Lie algebra $\mathfrak{g}$). Moreover, G_0 is the connected component of the identity of G^θ.

There are a few classical results in Lie theory which generalize easily in the context of θ-groups.

Theorem 1.1.11 (Jacobson-Morozov)**.** *Let $X \in \mathfrak{g}_1$ be a nilpotent element with $X \neq 0$. Then there are $H \in \mathfrak{g}_0$ and $Y \in \mathfrak{g}_{-1}$ such that (X, H, Y) is an $\mathfrak{sl}_2$-triple.*

Theorem 1.1.12 (Kostant)**.** *Let $X \in \mathfrak{g}_1$ be a nilpotent element and let (X, H, Y), (X, H', Y') be two $\mathfrak{sl}_2$-triples. Then they are conjugate under G_X.*

1.1.5 Quivers

Some standard references are [DW05, Bon91].

Let $Q = (Q_0, Q_1, s, t)$ be a quiver, where Q_0 is the set of vertices, Q_1 is the set of arrows, $s, t : Q_1 \to Q_0$ are the maps assigning to each arrow its source and target, respectively.

A dimension vector for Q is a function $\underline{n} : Q_0 \to \mathbb{Z}_{\geq 0}$, also denoted by $(n_i)_{i \in Q_0}$. A representation of Q with dimension vector $\underline{n}$ is a set of vector spaces V_i with dimension n_i for every vertex $i \in Q_0$ together with a set of linear maps $f_\alpha : V_{s(\alpha)} \to V_{t(\alpha)}$ for every arrow $\alpha \in Q_1$. If $f = (V_i, f_\alpha)$ and $g = (W_i, g_\alpha)$ are representations of Q a morphism of the representations of Q between f and g is a set of linear maps $u_i : V_i \to W_i$ such that, for each $\alpha \in Q_1$, $u_{t(\alpha)} f_\alpha = g_\alpha u_{s(\alpha)}$.

If one fixes $\underline{n}$ and the spaces V_i, the following definition is quite natural.

Definition 1.1.13. The representation space of the quiver Q for the dimension vector $\underline{n}$ is

$$\operatorname{Rep}(Q, \underline{n}) := \bigoplus_{\alpha \in Q_1} \operatorname{Hom}_k(k^{n_{s(\alpha)}}, k^{n_{t(\alpha)}}).$$

This is a vector space of dimension $\sum_{\alpha \in Q_1} n_{s(\alpha)} n_{t(\alpha)}$.

The group

$$\mathrm{GL}(\underline{n}) := \prod_{i \in Q_0},$$

called the structure group of $\mathrm{Rep}(Q, \underline{n})$, acts linearly on $\mathrm{Rep}(Q, \underline{n})$ on each space $\mathrm{Hom}_k(k^{n_{s(\alpha)}}, k^{n_{t(\alpha)}})$ by

$$(g_i)_{i \in Q_0}.x_\alpha := g_{t(\alpha)} x_\alpha g_{s(\alpha)}.$$

Hence $\mathrm{GL}(\underline{n})$ acts on the whole representation space $\mathrm{Rep}(Q, \underline{n})$.

Clearly, each point $x \in \mathrm{Rep}(Q, \underline{n})$ defines a representation M_x of Q. Moreover, two representations M_x, M_y are isomorphic if and only if x and y lie in the same $\mathrm{GL}(\underline{n})$-orbit. Therefore, the map $x \mapsto M_x$ gives a bijective correspondence from the set of $\mathrm{GL}(\underline{n})$-orbits in $\mathrm{Rep}(Q, \underline{n})$ and the set of isomorphism classes of representations of Q with dimension vector $\underline{n}$. Also, the stabilizer $\mathrm{GL}(\underline{n})_x$ of a point x is isomorphic to the automorphism group of the representation M_x.

The *path algebra* kQ of the quiver Q is a k-algebra, linearly spanned by the set of all paths in Q (including the trivial paths e_i without arrows, starting and ending in the vertex i). The product of two paths π_1, π_2 is given by the concatenation $\pi_2 \pi_1$ if π_1 ends where π_2 starts or zero otherwise.

A *relation* of Q is a subspace of kQ spanned by a linear combination of paths having a common source and a common target of length at least 2. A *quiver with relations* is a pair (Q, I) where Q is a quiver and I is a two sided ideal of kQ generated by relations. The *path algebra* of (Q, I) is the quotient algebra kQ/I.

1.2 Algebraic geometry

A standard reference is [GD60], but see also [Liu02, EH06].

Normal varieties

In this section we recall the definition of a normal variety and some important general facts about normality, since they play such an important role in this thesis.

Definition 1.2.1. Let R be a ring and $R \to S$ a R-algebra. An element $s \in S$ is said to be R-integral if there exists a monic polynomial $p \in R[x]$ such that s is a root of p.

A ring R is *integrally closed* if an element s in the fraction field $Frac(R)$ of R is R-integral if and only if $s \in R$.

Remark. One can show that if R is integrally closed and R contains a zero-divisor, then R is a product of integrally closed rings.

The key geometrical result is the fact that being integrally closed is a local property.

Proposition 1.2.2. *Let R be an integral domain. Then the following are equivalent:*

1. *R is integrally closed;*
2. *$R_{\mathfrak{p}}$ is integrally closed for every prime ideal $\mathfrak{p}$;*

3. *$R_{\mathfrak{m}}$ is integrally closed for every maximal ideal $\mathfrak{m}$.*

Definition 1.2.3. A ring R is *normal* if every localization $R_{\mathfrak{p}}$ at a prime $\mathfrak{p}$ is an integrally closed domain.

Remark. A normal ring R is reduced.

Remark. A normal ring R is integrally closed in its total ring of fractions $Q(R)$.

Definition 1.2.4. A variety X is *normal* if the local ring $\mathcal{O}_{X,x}$ is normal for each point $x \in X$.

Remark. By proposition 1.2.2, an affine variety $X = Spec(R)$ is normal if and only if R is a normal domain.

Serre's criterion

Definition 1.2.5. Let R be a ring. A *chain of prime ideals* is a sequence $\mathfrak{p}_0, \ldots, \mathfrak{p}_n$ of prime ideals $\mathfrak{p}_i \subset R$ such that $\mathfrak{p}_i \subset \mathfrak{p}_{i+1}$ and $\mathfrak{p}_i \neq \mathfrak{p}_{i+1}$. The *length* of the chain $\mathfrak{p}_0, \ldots, \mathfrak{p}_n$ is the natural number n.

The *Krull dimension* of R is the supremum of the lengths of the chains of prime ideals in R.

The *height* of a prime ideal $\mathfrak{p}$ of R, is the dimension of the local ring $R_{\mathfrak{p}}$.

Definition 1.2.6. Let R be a local Noetherian ring, let $\mathfrak{m}$ be its maximal ideal and $k = R/\mathfrak{m}$ be its residue field. The ring R is *regular* if

$$\dim(R) = \dim_k \mathfrak{m}/\mathfrak{m}^2.$$

Definition 1.2.7. Let R be a ring. A sequence of elements $f_1, \ldots, f_r \in R$ is a *regular sequence* if

1. f_i is a non zero divisor on $R/(f_1, \ldots, f_{i-1})$ for each $i = 1, \ldots, r$;
2. $(f_1, \ldots, f_r) \neq R$.

If each $f_i \in I$ for some ideal $I \subset R$, $(f_1, \ldots, f_r)$ is a *regular sequence in I*.

Definition 1.2.8. Let R be a local Noetherian ring and let $\mathfrak{m}$ be its maximal ideal. The *depth* of R, $\operatorname{depth}(R)$, is the supremum of the lengths of regular sequences in $\mathfrak{m}$.

Definition 1.2.9. A local Noetherian ring R is *Cohen-Macaulay* if

$$\operatorname{depth}(R) = \dim(R),$$

where $\dim(R)$ is the Krull dimension of R.

A Noetherian ring R is *Cohen-Macaulay* if $R_{\mathfrak{p}}$ is local Cohen-Macaulay for every prime $\mathfrak{p} \subset R$.

Remark. The global version of the definition 1.2.9 is consistent with the fact that every localization of a local Cohen-Macaulay ring is still a local Cohen-Macaulay ring.

Theorem 1.2.10. *A local Noetherian regular ring R is local Cohen-Macaulay.*

Definition 1.2.11. Let k be a field and A a finite type k-algebra. A is *complete intersection over k* if there exists a presentation

$$A \simeq k[x_1, \dots, x_n]/(f_1, \dots, f_c)$$

such that $\dim(A) = n - c$.

Theorem 1.2.12. *A complete intersection k-algebra A is Cohen-Macaulay.*

Definition 1.2.13. Let R be a Noetherian ring and $k \geq 0$ an integer.

- The ring R has property (R_k) (or is *regular in codimension* $\leq k$) if, for every prime $\mathfrak{p}$ of height $\leq k$, the local ring $R_\mathfrak{p}$ is regular.
- The ring R has property (S_k) if, for every prime $\mathfrak{p}$,

$$\text{depth}(R_\mathfrak{p}) \geq \min\{k, \dim(R_\mathfrak{p})\}.$$

Example. One can see some example of these properties.

- Every Noetherian ring has property (S_0).
- A local Noetherian ring is regular if and only if it has (R_k) for every $k \geq 0$.
- By definition, a Noetherian ring is Cohen-Macaulay if and only if it has (S_k) for every $k \geq 0$.

The main results concerning these properties are the following.

Theorem 1.2.14. *Let R be a Noetherian ring. Then R is reduced if and only if R has (R_0) and (S_1).*

Theorem 1.2.15 (Serre's criterion for normality)**.** *Let R be a Noetherian ring. Then R is normal if and only if R has (R_1) and (S_2).*

In this thesis we will primarily use the following corollary of Serre's criterion.

Corollary 1.2.16. *Let A be a complete intersection k-algebra. Then A is reduced if and only if A is regular in codimension* 0 *and A is normal if and only if A is regular in codimension* ≤ 1.

Proof. By theorem 1.2.12, the ring A is Cohen-Macaulay. By definition of Cohen-Macaulay, A has properties (S_k) for every $k \geq 0$. Therefore, Serre's criterion (theorem 1.2.14 and theorem 1.2.15) reduces to:

- A is reduced if and only if A has (R_0) and
- A is normal if and only if A has (R_1).

□

The following is another property involving (S_2) which we will use.

Proposition 1.2.17. *Let R be a Noetherian ring such that R has (S_2). Let $I \subset R$ be an ideal such that* $\dim(R/I) \leq \dim(R) - 2$. *Then every regular function on the complement of $V(I)$ in Spec(R) can be extended to a regular function on the whole Spec(R).*

1.3 Miscellaneous results

In this main section we recall some more well-known facts that will be important in some following chapters.

1.3.1 First Fundamental Theorem of invariant theory

We mainly follow [Pro06] and [DCP82].

Let G be a linear group acting on an affine variety X. There is a natural action of G on the ring of regular functions $k[X]$ on X. Let $k[X]^G \subseteq k[X]$ be the subring of the G-invariant regular functions. An affine variety Y such that $k[Y] = k[X]^G$ is a *quotient* of X under G.

Definition 1.3.1. Let X, Y be affine varieties, G be a linear group acting on X and $p : X \to Y$ a morphism such that the induced map on the regular functions $p^* : k[Y] \to k[X]$ is an isomorphism between $k[Y]$ and $k[X]^G$. We say that (Y, p) is a quotient of X under G and we write $Y = X/G$.

We are interested in these quotients for a few reasons. The first is that quotients preserve normality.

Proposition 1.3.2. *Let X be an affine G-space, with quotient $f : X \to X/G$. Suppose, moreover, that X is normal. Then X/G is normal.*

Proof. We work with the ring of regular functions. We know that $k[X]$ is normal, that is, the localization $k[X]_{\mathfrak{p}}$ is an integrally closed domain for every prime ideal $\mathfrak{p}$. We need to show that for every prime ideal $\mathfrak{q} \subseteq k[X]^G$, the localization $(k[X]^G)_{\mathfrak{q}}$ is an integrally closed domain.

Let $\mathfrak{p} \subseteq k[X]$ be a minimal prime containing $\mathfrak{q} \subseteq k[X]^G \subseteq k[X]$, so that $\mathfrak{p} \cap k[X]^G = \mathfrak{q}$. Then $(k[X]^G)_{\mathfrak{q}} \simeq (k[X]_{\mathfrak{p}})^G$, hence we can assume to be working with a local domain $A = k[X]_{\mathfrak{p}}$. As A is a domain and $A^G \subseteq A$ is a subring, A^G is a domain as well and we have an inclusion between the fields of fractions $Q(A^G) \subseteq Q(A)$. Let $a/b \in Q(A^G)$ be an integral element over A^G; in particular it is integral over A, hence $a/b = a'/1$ for some $a' \in A$, as A is integrally closed. Moreover, a/b is fixed by the action of G, as both $a, b \in A^G$, therefore, also a' must be fixed by G, hence the proposition is proven. □

Another pivotal fact for the construction we will describe in later chapters is the first fundamental theorem of invariant theory. There are many ways of stating this theorem. In this work we prefer to use the matricx version along with the definition of quotient map just introduced. We denote the space of $m \times n$ matrices by $M_{m,n}$.

As different classical linear groups have different versions of this theorem, we are introducing them separately. We start with the general linear group GL_n acting on k^n.

Theorem 1.3.3. *Let n, p, q be positive integers. The group $G = \mathrm{GL}_n$ acts on the space of pairs of matrices $M_{p,n} \times M_{n,q}$ by*

$$g.(A, B) = (Ag^{-1}, gB) \quad \forall g \in G, A \in M_{p,n}, B \in M_{n,q}.$$

Then the map

$$\begin{aligned}\pi : & M_{p,n} \times M_{n,q} \to M_{p,q} \\ & (A,B) \mapsto AB\end{aligned}$$

is the quotient of $M_{p,n} \times M_{n,q}$ under G (with image the matrices with rank $\leq n$).

In order to introduce the analogue theorem for the orthogonal group, we fix the standard symmetric non-degenerate bilinear form on k^n: $(x,y) \mapsto x^t y$. Hence $O_n = \{g \in \mathrm{GL}_n : g^t g = 1_n\}$.

Theorem 1.3.4. *Let n, p be positive integers. The group $G = O_n$ acts on the space of matrices $M_{p,n}$ by the left action*

$$g.A = Ag^t \quad \forall g \in G, A \in M_{p,n}.$$

Then the map

$$\begin{aligned}\pi : & M_{p,n} \to M_{p,p} \\ & A \mapsto AA^t\end{aligned}$$

is the quotient of $M_{p,n}$ under G (with image the matrices with rank $\leq n$).

Similarly, for the symplectic group we fix the standard symplectic non-degenerate bilinear form on k^{2n}: $(x,y) \mapsto x^t J_{2n} y$, where $J_{2n} = \left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right)$. Hence $Sp_{2n} = \{g \in \mathrm{GL}_{2n} : g^t J_{2n} g = J_{2n}\}$.

Theorem 1.3.5. *Let n, p be positive integers. The group $G = Sp_{2n}$ acts on the space of matrices $M_{p,2n}$ by the left action*

$$g.A = Ag^t \quad \forall g \in G, A \in M_{p,2n}.$$

Then the map

$$\begin{aligned}\pi : & M_{p,2n} \to M_{p,p} \\ & A \mapsto AJ_{2n}A^t\end{aligned}$$

is the quotient of $M_{p,2n}$ under G (with image the matrices with rank $\leq 2n$).

1.3.2 Classical and geometric invariant theory results

The following are well known results, the standard reference is [MFK94].

Proposition 1.3.6. *Let $f : X \to X/G$ be a quotient of affine varieties under a reductive group G. Let $Z \subseteq X$ a closed G-invariant subspace of X. Then $f(Z) \subseteq X/G$ is closed and $f|_Z : Z \to f(Z)$ is a quotient.*

Proposition 1.3.7. *Let $f : X \to X/G$ be a quotient of affine varieties under a reductive group G and let $g : Y \to X/G$ any morphism. Consider the following cartesian diagram:*

$$\begin{array}{ccc} Y \times_{X/G} X & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & X/G. \end{array}$$

Then $Y \times_{X/G} X$ is naturally a G-space and f' is a quotient under G.

1.3.3 A theorem on the orbits of the action of fixed points group

Let G be a group acting on a space X. Let $g \mapsto g^\sigma$ and $x \mapsto \sigma x$ be involutions of G and X respectively. We will denote by G^σ, X^σ the fixed points set. The following theorem is taken from [MWZ00] and describes a fortunate relation between a G^σ-orbit in X^σ and the intersection of a G-orbit in X with X^σ.

Theorem 1.3.8. *Let G, X be as above. Suppose moreover that*

1. $\sigma(g.\sigma(x)) = g^\sigma.x$,
2. *there exists a finite dimensional associative algebra M over $k = \bar{k}$ such that G is a subgroup of the invertible elements $M^\times$,*
3. *the anti-involution map $g \mapsto g^* := (g^\sigma)^{-1}$ extend to the whole M,*
4. *given $H = \mathrm{Stab}_G(x)$, $H = \langle H\rangle_k^\times$, that is the invertible elements of the subspace $\langle H\rangle$ in M.*

Then $G^\sigma.x = G.x \cap X^\sigma$ for every $x \in X^\sigma$.

Proof. First observe that X^σ is G^σ-invariant, as immediately follows from hypothesis 1. This implies the inclusion $\subseteq$.

Let us remark that the other inclusion is not a solely set theory result. It is equivalent to saying that, given $x \in X^\sigma$ and $g \in G$ such that $g.x \in X^\sigma$, there must be a $\tilde{g} \in G^\sigma$ such that $\tilde{g}.x = g.x$. Let us choose $\tilde{g} = gh^{-1}$ where $h \in H = \mathrm{Stab}_G(x)$. This way we ensure $\tilde{g}.x = g.x$. Now we look for a $h \in H$ such that $gh^{-1} \in G^\sigma$, i.e.

$$\begin{aligned}(gh^{-1})^\sigma &= gh^{-1}\\ g^\sigma h^{-1^\sigma} &= gh^{-1}\\ h^{-1^\sigma} h &= g^{\sigma-1} g\\ h^* h &= g^* g,\end{aligned}$$

by the third hypothesis.

Let us observe that $g^*g = (g^*g)^*$ (trivial) and $g^*g \in H$:

$$g^*g.x = (g^{-1})^\sigma.(g.x) = \sigma(g^{-1}.\sigma(g.x)) = \sigma(g^{-1}.g.x) = \sigma(x) = x$$

indeed G stabilizes X^σ, and $x \in X^\sigma$.

Any $l \in k[g^*g] \subseteq M$ satisfies $l^* = l$ and $l \in \langle H\rangle_k$. Therefore, using the hypothesis 4, the thesis follows as soon as we show the existence of an element $h \in k[g^*g]$ such that h is invertible and satisfies

$$h^2 = g^*g.$$

Let us name $A = g^*g$. We just need to solve the following

$$x^2 = A \tag{1.3.1}$$

inside $k[A] \simeq k[t]/\mu_A(t)$. Let us notice that $\mu_A(0) \neq 0$ as A is invertible. As k is algebraically closed, $\mu_A(t) = \prod(t - \lambda_i)^{m_i}$ for some $\lambda_i \in k^\times$. By the Chinese remainder theorem, $k[t]/\mu_A(t) \simeq \prod k[t]/(t-\lambda_i)^{m_i}$, so, in order to solve (1.3.1), it is enough to solve

$$x^2 = t \tag{1.3.2}$$

inside $k[t]/(t-\lambda_i)^{m_i}$ for each i and check that every solution found is invertible. Without loss of generality, by translating t, the equation (1.3.2) becomes

$$x^2 = p(t)$$

inside $k[t]/t^{m_i}$. Here $p(t)$ is some polynomial of t such that $p(0) \neq 0$. Now the Hensel lifting lemma grants us a solution for every positive integer m_i. Indeed the quadratic polynomial $Q(x) = x^2 - p$ has a root x_0 inside $k[t]/t^1 \simeq k$, since k is algebraically closed; moreover x_0 is simple since $Q'(x_0) = 2x_0 \neq 0$ (in fact we know that $p(0) \neq 0$).

Such an $x_0 \in k[t]/t^{m_i}$ must be invertible since $x_0(0) \neq 0$ and (t) is the unique maximal ideal of $k[t]/t^{m_i}$. □

1.4 Motivations

We follow [Kos63] and some general references like [Pro06].

Suppose that $X = \overline{\mathcal{O}}$ is a variety with a single open orbit $\mathcal{O}$ and suppose that the boundary $X \setminus \mathcal{O}$ has codimension at least 2 in X. One of the main reasons for which one is interested in the fact that a variety X is normal is given by the following well known theorem.

Theorem 1.4.1. *Let X be a normal variety and $Y \subset X$ a subvariety of codimension at least 2. Then any regular function on $X \setminus Y$ extends to a regular function on X, that is*

$$\mathcal{O}(X) = \mathcal{O}(X \setminus Y). \tag{1.4.1}$$

An important consequence of the equation (1.4.1) is given by the Frobenius reciprocity.

Theorem 1.4.2 (Frobenius reciprocity)**.** *Let G be an algebraic group and $H < G$ be a closed subgroup. Then, for every G-module M and for every H-module N,*

$$\mathrm{Hom}_G(M, \mathrm{Ind}_H^G(N)) \simeq \mathrm{Hom}_H(M|_H, N).$$

One can use this result to directly study regular functions on a G-orbit $\mathcal{O} = G.x$ in the following way. Let $G_x < G$ be the stabilizer of x, so that $G.x \simeq G/G_x$ and

$$k[G.x] \simeq k[G/G_x].$$

One can see that the latter term is an induced representation from the subgroup G_x.

Definition 1.4.3. Let G be an algebraic group and $H < G$ be a closed subgroup. Let M be a H-module. The induced representation of M is defined as

$$\mathrm{Ind}_H^G(M) := \{f : G \to M : f(gh^{-1}) = h.f(g), \forall h \in H, \forall g \in G\}.$$

If one calls k the trivial representation of G_x, i. e. the representation $\rho : G_x \to \mathrm{GL}(k)$ such that $\rho(h) = 1_k$ for every $h \in G_x$, then

$$\begin{aligned}\mathrm{Ind}_{G_x}^G(k) =& \{f : G \to k : f(gh^{-1}) = f(g), \forall h \in G_x, \forall g \in G\} \\ \simeq& \{f : G/G_x \to k\} = k[G/G_x].\end{aligned}$$

Therefore, the knowledge that X is normal lets us study the G-module structure of $k[X]$ by looking at $k[\mathcal{O}]$ or at the subgroup G_x, where $x \in \mathcal{O}$ is any point in the open orbit.

1.4.1 Reduction to nilpotent case

In this section we aim at reducing the problem of the normality of $\overline{G.x}$ to the case when x is a nilpotent element in $\mathfrak{g}$. We expand the argument of [Hes79].

We rely upon the following geometric facts.

Theorem 1.4.4. *Let X, Y be varieties. Then $X \times Y$ is normal if and only if both X and Y are normal.*

Theorem 1.4.5. *Let $f : X \to Y$ be a smooth morphism of varieties. Then X is normal if and only if Y is normal.*

Proposition 1.4.6. *Let G be any linear algebraic group. Then the constant map $G \to \{pt\}$ is smooth.*

Theorem 1.4.7. *Let $f : X \to S$ be a morphism and $g : Y \to S$ be a smooth morphism. Then the base change $g' : X \times_S Y \to X$ of g is smooth.*

Theorem 1.4.8. *Let G be a reductive group, $\mathfrak{g}$ its Lie algebra, $x \in \mathfrak{g}$ and $x = x_s + x_n$ its Jordan decomposition. Then the centralizer $G_{x_s} \leq G$ is reductive.*

Finally, we can prove the main theorem for this section.

Theorem 1.4.9. *Let G be a reductive group, $\mathfrak{g}$ its Lie algebra, $x \in \mathfrak{g}$ and $x = x_s + x_n$ its Jordan decomposition. Let $X = \overline{G.x} \subseteq \mathfrak{g}$ and $X' = \overline{G_{x_s}.x_n} \subseteq \mathrm{Lie}(G_{x_s})$. Then X' is normal if and only if X is normal.*

Proof. We consider the following map:

$$\begin{aligned}f : G \times X' &\to X \\ (g, y) &\mapsto g.(x_s + y).\end{aligned}$$

We want to prove that f is a well-defined, onto, smooth morphism.

One may check the following statements: the element $g.(x_s + y)$ actually belongs to X for every $(g, y) \in G \times X'$; f is onto; f is a morphism of varieties; f is smooth.

By theorem 1.4.5 on the morphism f, X is normal if and only if $G \times X'$ is normal. By theorem 1.4.7, the projection onto the second factor $\pi_2 : G \times X' \to X'$ is smooth, as $G \to \{pt\}$ is smooth by proposition 1.4.6. Still by theorem 1.4.5 (this time on the morphism π_2), X' is normal if and only if $G \times X'$ is normal, which concludes the proof. □

Theorem 1.4.9 implies that we can reduce the study of the normality of the closure of orbits $\overline{G.x} \subseteq \mathfrak{g}$ to the case when x is a nilpotent element (at the cost of changing the reductive group G to G_{x_s}).

We want to address the question in the case $G = \mathrm{GL}_n$ more in depth.

1.4.2 Normality of the nilpotent cone in reductive Lie algebras

In this section we will prove the normality of a particular orbit, the regular nilpotent orbit, for each reductive Lie algebra $\mathfrak{g}$. This proof is due to Kostant ([Kos63]). We chose to give it right now because, on one hand, it is pretty straightforward and, on the other hand, it uses the same general strategy to prove the normality of G-space that we will use in chapter 5. Therefore, one may immediately get a picture of what we are going to do later.

Theorem 1.4.10. *Let $\mathfrak{g}$ be a reductive Lie algebra and let $\mathcal{N}(\mathfrak{g})$ be the subvariety of nilpotent elements in $\mathfrak{g}$. Then $\mathcal{N}(\mathfrak{g})$ is normal.*

We will see in chapter 2 and in chapter 3 that this statement is a particular case of our problem. Indeed, one always has that there exists only finitely many nilpotent orbits in $\mathfrak{g}$ and that there is exactly one open orbit in $\mathcal{N}(\mathfrak{g})$, called the regular nilpotent orbit $\mathcal{O}$, and that $\overline{\mathcal{O}} = \mathcal{N}(\mathfrak{g})$.

The main step of the proof of theorem 1.4.10 is the Serre's criterion, corollary 1.2.16. In order to use it, we need to establish the following:

- $\mathcal{N}(\mathfrak{g})$ is a complete intersection variety;
- $\dim(\mathcal{N}(\mathfrak{g}) \setminus \mathcal{O}) \leq \dim \mathcal{N}(\mathfrak{g}) - 2$.

As $\mathcal{N}(\mathfrak{g})$ is G-invariant and a union of finitely many G-orbits, so is the boundary $\mathcal{N}(\mathfrak{g}) \setminus \mathcal{O}$. Therefore, the latter fact is a consequence of the following general theorem.

Theorem 1.4.11. *Let $\mathfrak{g}$ be a reductive Lie algebra, $x \in \mathfrak{g}$ be any element and G be the adjoint group of $\mathfrak{g}$. Then the conjugacy class $G.x$ has even dimension.*

In order to prove that $\mathcal{N}(\mathfrak{g})$ is a complete intersection variety, we want to find an affine space V such that there is a closed embedding $\mathcal{N}(\mathfrak{g}) \hookrightarrow V$ such that the ideal of $\mathcal{N}(\mathfrak{g})$ in V is generated by $(\dim V - \dim \mathcal{N}(\mathfrak{g}))$ polynomial equations. It turns out that one can take $V = \mathfrak{g}$. Let $J = k[\mathfrak{g}]^G$ be the set of G-invariant polynomials. Therefore, the ideal of the nilpotent cone $\mathcal{N}(\mathfrak{g})$ is the set $J^+ \subseteq J$ of invariant polynomials with zero constant term.

Let ℓ be the rank of $\mathfrak{g}$. A theorem by Chevalley states that J is a polynomial ring generated by ℓ homogeneous polynomials $u_1, \ldots, u_\ell$, with $\deg u_i = m_i + 1$, where m_i are the exponents of $\mathfrak{g}$. Let

$$u : \mathfrak{g} \to \mathbb{C}^\ell$$

the morphism given by putting $u(x) = (u_1(x), \ldots, u_\ell(x))$ for any $x \in \mathfrak{g}$. One can check if the differentials $(du_i)_x$ in a point $x \in \mathfrak{g}$ are linearly independent, that is, if u is smooth on a point $x \in \mathfrak{g}$. The general result is the following.

Theorem 1.4.12. *Let $x \in \mathfrak{g}$. The map u is smooth on x if and only if x is a regular element of $\mathfrak{g}$.*

In particular, u is smooth on all regular nilpotent elements. With these facts, one has the following theorem.

Theorem 1.4.13. *For every $\xi \in \mathbb{C}^\ell$, the fibre*

$$u^{-1}(\xi) \subseteq \mathfrak{g}$$

is the closure of the regular elements in $u^{-1}(\xi)$, has dimension $n-\ell$ and is a complete intersection.

As $\mathcal{N}(\mathfrak{g}) = u^{-1}(0)$, where 0 is the origin of $\mathbb{C}^\ell$, theorem 1.4.13 implies that $\mathcal{N}(\mathfrak{g})$ is a complete intersection, therefore concluding the proof of theorem 1.4.10.

Chapter 2

Nilpotent conjugacy classes

Let $\mathfrak{g}$ be a classic semisimple Lie algebra, so that there is a natural representation $\rho : \mathfrak{g} \to \mathfrak{gl}(k^n)$. For each $\mathfrak{sl}_2$-subalgebra $\mathfrak{s}$ of $\mathfrak{g}$, k^n is an $\mathfrak{s}$-module as well. By Weyl theorem, $\mathfrak{sl}_2$ is completely reducible and one may decompose k^n into irreducible representations of $\mathfrak{sl}_2$. It is well known that the isomorphism class of a finite dimensional irreducible representation of $\mathfrak{sl}_2$ is uniquely determined by its maximum weight $r \in \mathbb{Z}_{\geq 0}$. We denote it by $V(r)$. Hence we get a decomposition

$$k^n = \bigoplus_{i=1}^{h} V(r_i), \tag{2.0.1}$$

where h is the number of irreducible $\mathfrak{s}$-representations in k^n and r_i is the maximum weight of the i-th irreducible $\mathfrak{s}$-representation.

Since $\dim V(r) = r + 1$, we have $n = \sum_{i=1}^{h}(r_i + 1)$. If we sort the numbers r_i so that $i < j \Rightarrow r_i \geq r_j$, the sequence $\lambda = (r_1 + 1, r_2 + 1, \ldots, r_h + 1)$ becomes a partition of n. Clearly λ does not change if we take another $\mathfrak{sl}_2$-subalgebra $\mathfrak{s}'$, that is conjugated to $\mathfrak{s}$, to decompose k^n.

Let $X \in \mathfrak{g}$ be any nilpotent element. If $X \neq 0$, by Jacobson-Morozov theorem, we have an $\mathfrak{sl}_2$ triple $X, H, Y \in \mathfrak{g}$, that is, $\mathfrak{s} := \langle X, H, Y \rangle_k$ is an $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$. Therefore, we can attach to X the partition λ just obtained. By a theorem of Kostant, one obtains the same partition λ for each nilpotent element X' belonging to the same conjugacy class of X. Hence, the partition λ corresponding to a nilpotent element X is an invariant of the conjugacy class of X.

In this chapter we prove that λ is, in fact, the only invariant of a conjugacy class for each of the groups GL_n, O_n, Sp_{2n}, which are our only concern in this thesis. As a consequence, we often denote the corresponding conjugacy class by C_λ. We inspect which partition actually appears in each case. We study the orthogonal and symplectic pairs, as well, where we still found that λ is the only invariant under the orthogonal or the symplectic group. At the end of the chapter we compute the dimension of each conjugacy class, since those formulas will be needed later.

We mainly follow [CM93], since all the results in this chapter are classic, but we remark that a lot of matematicians worked on the problem of the classification of conjugacy classes in various contexts, see [Wal63, Sek87, Oht91a] as some examples.

2.1 Partitions

We start by setting up some notation for the partitions.

Definition 2.1.1. Let n be a positive integer. A *partition* of n is a sequence

$$\lambda = (\lambda_1, \dots, \lambda_h)$$

of integers λ_i such that $\lambda_i \geq \lambda_{i+1}$ for each $i = 1, \dots, h-1$.

Usually, we assume that $\lambda_h > 0$, so that each integer is positive, nevertheless, sometimes, it proves useful to consider partitions with $\lambda_h = 0$ as well. As usual, we denote by $\lambda \vdash n$ the fact that λ is a partition of n.

We define the set of all partitions of n as

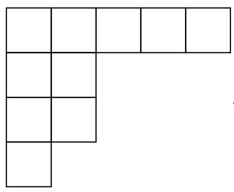
$$\mathcal{P}(n) := \{\lambda : \lambda \vdash n\};$$

the size of a partition λ by $|\lambda| := n = \lambda_1 + \cdots + \lambda_h$ and the dual partition $\widehat{\lambda}$ of a partition λ by

$$\widehat{\lambda}_i := |\{j : \lambda_j \geq i\}| \quad \forall i = 1, \dots, \lambda_1.$$

It is useful to think about a partition via its Young diagram. One may construct it in the following fashion. If $\lambda = (\lambda_1, \dots, \lambda_h)$ is a partition, one creates h rows of boxes, putting λ_i boxes on the i-th row. It is customary to draw them close together and push to the left side as in the following example.

Example. Let $\lambda = (5, 2, 2, 1)$ be a partition. Its Young diagram is



.

As λ_i can be thought as the length of the i-th row, $\widehat{\lambda}_j$ is the length of the j-th column of the Young diagram of λ. In particular, one obtains the Young diagram of $\widehat{\lambda}$ by transposing the Young diagram of λ. We identify a partition with its Young diagram, so that we will often speak of the rows and columns of a partition.

We introduce some combinatorial construction which will play important roles in the following chapters as well. Let $\lambda = (\lambda_1, \dots, \lambda_h)$ be a partition. We define the partition λ' as the partition obtained by deleting the first column of λ, or, more formally, as the partition such that

$$\lambda'_i := \lambda_i - 1.$$

In particular, we notice that $|\lambda'| = |\lambda| - h$. We will actually use often this construction performed multiple times in succession, that is, we delete several columns using this construction inductively:

$$\begin{aligned} \lambda^{(0)} &:= \lambda \\ \lambda^{(m+1)} &:= \left(\lambda^{(m)}\right)'. \end{aligned} \tag{2.1.1}$$

We will make use of the following construction as well. Let $\lambda = (\lambda_1, \dots, \lambda_h) \in \mathcal{P}(n)$ and $\mu = (\mu_1, \dots, \mu_k) \in \mathcal{P}(m)$. We define their *sum* as the partition $\lambda \oplus \mu \in P(n+m)$ obtained by sorting the sequence of integers

$$(\lambda_1, \dots, \lambda_h, \mu_1, \dots, \mu_k)$$

in order to ensure that $(\lambda \oplus \mu)_i \geq (\lambda \oplus \mu)_j$ for each $1 \leq i < j \leq h + k$. Notice that there is a simpler characterization for the the sum of partition by looking at the columns:

$$\left(\widehat{\lambda \oplus \mu}\right)_i = \widehat{\lambda}_i + \widehat{\mu}_i. \tag{2.1.2}$$

2.2 Nilpotent conjugacy classes under GL_n

Let $X \in \mathfrak{gl}_n$ be a nilpotent endomorphism of k^n. In classic Jordan theory, one proves that the conjugacy classes of $\mathcal{O}_X \subseteq \mathfrak{gl}_n$ of X is uniquely determined by the list of invariants $\operatorname{rk} X^1, \operatorname{rk} X^2, \dots, \operatorname{rk} X^n$. In order to identify the irreducible $\mathfrak{sl}_2$-modules of (2.0.1), one may proceed in several ways.

One may choose an $\mathfrak{sl}_2$-triple (X, H, Y) in $\mathfrak{gl}_n$, by the Jacobson-Morozov theorem (theorem 1.1.3), and look at the highest weight spaces $H(r_i) \subseteq V(r_i)$ for each i, which spans the kernel of X. Therefore, one identifies the isotypic component with the weight spaces decomposition of

$$\ker X = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} (\ker X)_i$$

given by the semisimple element H where $(\ker X)_i := \{v \in \ker X : Hv = iv\}$. Moreover the multiplicity of an irreducible module $V(r)$ in k^n

$$\operatorname{mul}_{k^n}(V(r)) = \dim (\ker X)_i \,.$$

By theorem 1.1.4, $\dim \ker X_i$ do not depend on the particular $\mathfrak{sl}_2$-triple (X, H, Y).

One may also proceed in a different way. It is useful to work with the lowest weight spaces $L(r_i) \subseteq V(r_i)$ for each i. Indeed, even if they are not uniquely determined by X, they have the following properties, which depend only on X: $L(r_i) \cap \operatorname{Im} X = 0$, $X^{r_i}(L(r_i)) = 0$ and $X^{r_i - 1}(L(r_i)) \neq 0$, for each i. Hence, one gets that the multiplicity of any irreducible module $V(r)$ in (2.0.1) is given by

$$\begin{aligned} \operatorname{mul}_{k^n}(V(r)) &= \left(\dim \ker X^{r+1} - \dim \ker X^r\right) - \left(\dim \ker X^{r+2} - \dim \ker X^{r+1}\right) \\ &= \left(\operatorname{rk} X^r - \operatorname{rk} X^{r+1}\right) - \left(\operatorname{rk} X^{r+1} - \operatorname{rk} X^{r+2}\right). \end{aligned}$$

Hence, the associated partition λ can be recovered more easily by its columns:

$$\widehat{\lambda}_i = \operatorname{rk} X^{i-1} - \operatorname{rk} X^i.$$

One may notice that each irreducible module $V(r_i)$ in (2.0.1) corresponds to a Jordan block of the Jordan normal form of the nilpotent element X, since they are generated (as a $k[X]$-module) by one lowest weight vector.

Finally we prove that, for any partition $\lambda = (\lambda_1, \dots, \lambda_h) \vdash n$, one may construct a nilpotent element X such that its partition is λ. Take the $\mathfrak{sl}_2$-representation

$\rho_\lambda : \mathfrak{sl}_2 \to \mathfrak{gl}_n$ given by the sum (2.0.1) where $r_i := \lambda_i - 1$. It is clear that the image $X = \rho_\lambda(e) \in \mathfrak{gl}_n$ of the nilpositive element $e \in \mathfrak{sl}_2$ is nilpotent,

$$\operatorname{rk} X^i = \sum_{j:r_j \geq i} (r_j + 1 - i) = \sum_{j:\lambda_j > i} (\lambda_j - i) = \sum_{k>i} \widehat{\lambda}_k, \tag{2.2.1}$$

and the partition associated with X is λ, as well. Therefore, each partition $\lambda \in \mathcal{P}(n)$ is attainable as the partition associated to a nilpotent conjugacy class in $\mathfrak{gl}_n$.

Remark. Notice that (2.2.1) may be expressed using the construction (2.1.1). Indeed,

$$\left|\lambda^{(i)}\right| = n - \left(\widehat{\lambda}_1 + \widehat{\lambda}_2 + \cdots + \widehat{\lambda}_i\right) = \sum_{k>i} \widehat{\lambda}_k = \operatorname{rk} X^i. \tag{2.2.2}$$

In the end, one has the following classification theorem.

Theorem 2.2.1. *There is a one-to-one correspondence between the nilpotent orbits in $\mathfrak{gl}_n$ and the set of partitions $P(n)$.*

In light of theorem 2.2.1, we denote by C_λ the nilpotent conjugacy class in $\mathfrak{gl}_n$ corresponding to the partition $\lambda \vdash n$.

We highlight some special conjugacy classes C_λ depending on λ. If $\lambda = (n)$, C_λ is called the *regular nilpotent conjugacy class.* If $\lambda = (n-1, 1)$, C_λ is called the *subregular nilpotent conjugacy class.* If $\lambda = (1^n) = \widehat{(n)}$, C_λ is the zero class. If $\lambda = (2, 1^{n-2}) = \widehat{(n-1, 1)}$, C_λ is called the *minimal nilpotent conjugacy class.*

2.3 Nilpotent conjugacy classes under the other classical groups

We turn to the other classical groups. In fact, we want to discuss simultaneously both the cases of quadratic spaces and the symmetric spaces of the symmetric orthogonal and symplectic pairs.

Let $\varepsilon \in \{\pm 1\}$ be a sign and V, equipped with a non-degenerate bilinear form $(-,-)$, be a quadratic space of type ε and of finite dimension $n := \dim V$. As discussed in section 1.1.3 and in section 1.1.4, if one defines $\theta(D) = -D^*$, the group of θ-fixed points $K_\theta \subseteq \mathrm{GL}(V)$, by restricting the adjoint action, acts on

$$\mathfrak{gl}(V) = \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

where $\mathfrak{g}_0 = \mathfrak{k} = \{X \in \mathfrak{gl}(V) : \theta(X) = X\}$ and $\mathfrak{g}_1 = \mathfrak{p} = \{X \in \mathfrak{gl}(V) : \theta(X) = -X\}$ and it acts on both $\mathfrak{g}_0$, $\mathfrak{g}_1$ as well.

For each choice of $\varepsilon \in \{\pm 1\}$ and $\eta \in \{0, 1\}$, we have the action of a group $K_\varepsilon := K_\theta \subseteq \mathrm{GL}(V)$ on a space $\mathfrak{g}_{\varepsilon,\eta} := \mathfrak{g}_\eta \subseteq \mathfrak{gl}(V)$. The cases $\eta = 0$ are the usual adjoint actions of K_θ in its Lie algebra, while the cases $\eta = 1$ are the action induced on the symmetric spaces. In the cases $\varepsilon = 1$, the group $K_\theta \simeq O_n$ is the orthogonal group; in the cases $\varepsilon = -1$, the group $K_\theta \simeq Sp_n$ is the symplectic group (and n is even).

In this section, we classify the nilpotent K_ε-conjugacy classes of $\mathfrak{g}_{\varepsilon,\eta}$ for each ε, η. The analysis is split in two parts. In the first part we prove that the invariants already described in theorem 2.2.1 uniquely identify a nilpotent K_ε-conjugacy class. That is equivalent to say that each nilpotent K_ε-conjugacy class corresponds to a partition in $\mathcal{P}(\dim V)$. In the second part we inspect, in each of the four cases, which partitions appear as the partition of a nilpotent K_ε-conjugacy class.

2.3.1 Partitions uniquely identify conjugacy classes

For the first part we use theorem 1.3.8 to prove the following.

Theorem 2.3.1. *Let $X \in \mathfrak{g}_{\varepsilon,\eta}$ be a nilpotent element. Then*

$$\mathrm{GL}(V).X \cap \mathfrak{g}_{\varepsilon,\eta} = K_\theta.X$$

is a unique K_θ-orbit.

Proof. We $G := \mathrm{GL}(V)$, $X := \mathfrak{gl}(V)$ and the involutions σ by

$$g^\sigma := (g^*)^{-1} \quad \forall g \in G,$$

and

$$\sigma x := (-1)^{\eta+1} x^* \quad \forall x \in X.$$

It is clear, therefore, that

$$\begin{aligned} G^\sigma &= \{g \in \mathrm{GL}(V) : g = (g^*)^{-1}\} = K_\theta, \\ X^\sigma &= \{x \in \mathfrak{gl}(V) : x = (-1)^{\eta+1} x^*\} = \mathfrak{g}_{\varepsilon,\eta}. \end{aligned}$$

We check each hypothesis of theorem 1.3.8. Since $\left(g^{-1}\right)^* = (g^*)^{-1}$,

$$\begin{aligned} \sigma(g.\sigma(x)) &= \sigma(g(-1)^{\eta+1} x^* g^{-1}) = (-1)^{\eta+1} \left(g(-1)^{\eta+1} x^* g^{-1}\right)^* = \left(g x^* g^{-1}\right)^* \\ &= \left(g^{-1}\right)^* x g^* = \left(g^{-1}\right)^* .x = g^\sigma.x. \end{aligned}$$

The group $\mathrm{GL}(V)$ is the group of the invertible elements of the finite dimensional algebra $M = \mathrm{End}(V) = \mathfrak{gl}(V)$. The map $g \mapsto (g^\sigma)^{-1}$ is the restriction to M of the linear anti-involution $g \mapsto g^*$. The stabilizer of an element x is

$$\mathrm{Stab}_G(x) = \{g \in \mathrm{GL}(V) : gxg^{-1} = x\}.$$

Consider the set

$$L := \{g \in M : gx - xg = 0\} \subseteq M.$$

Then, clearly L is a sub-vector space in M, $\mathrm{Stab}_G(x) \subseteq L$ and, for every $g \in L$ such that g is invertible, one gets $gxg^{-1} = x$, that is, $g \in \mathrm{Stab}_G(x)$. □

2.3.2 Classes of partitions

For each of the four cases, depending on the choices of $\varepsilon \in \{\pm 1\}$ and $\eta \in \{0, 1\}$, we define a class of partitions $\mathcal{P}_{\varepsilon,\eta}(n) \subseteq \mathcal{P}(n)$. We are going to prove that each partition in these classes $\mathcal{P}_{\varepsilon,\eta}(n)$ appears as the partition of a nilpotent element, according to the case (ε, η).

Definition 2.3.2. We define

$$\begin{aligned}\mathcal{P}_{1,0}(n) &:= \{\lambda \vdash n : \forall k \text{ even}, |\{i : \lambda_i = k\}| \text{ is even}\} \subseteq \mathcal{P}(n);\\ \mathcal{P}_{-1,0}(n) &:= \{\lambda \vdash n : \forall k \text{ odd}, |\{i : \lambda_i = k\}| \text{ is even}\} \subseteq \mathcal{P}(n);\\ \mathcal{P}_{1,1}(n) &:= \mathcal{P}(n);\\ \mathcal{P}_{-1,1}(n) &:= \{\lambda \vdash n : \forall k, |\{i : \lambda_i = k\}| \text{ is even}\} \subseteq \mathcal{P}(n).\end{aligned}$$

Remark. It is possible to characterize the partition classes of definition 2.3.2 by suitable conditions on the columns. In fact, it is easy to check that

$$\begin{aligned}\mathcal{P}_{1,0}(n) &= \{\lambda \vdash n : \forall k \text{ even}, \widehat{\lambda}_k + \widehat{\lambda}_{k+1} \text{ is even}\};\\ \mathcal{P}_{-1,0}(n) &= \{\lambda \vdash n : \forall k \text{ odd}, \widehat{\lambda}_k + \widehat{\lambda}_{k+1} \text{ is even}\};\\ \mathcal{P}_{-1,1}(n) &= \{\lambda \vdash n : \forall k, \widehat{\lambda}_k \text{ is even}\}.\end{aligned} \tag{2.3.1}$$

Remark. Since the sum of partitions behaves well on the columns by (2.1.2), by the previous remark it is clear that, for each $\varepsilon \in \{\pm 1\}$, $\eta \in \{0, 1\}$ we have that

$$\lambda \in \mathcal{P}_{\varepsilon,\eta}(n), \mu \in \mathcal{P}_{\varepsilon,\eta}(m) \;\Rightarrow\; \lambda \oplus \mu \in \mathcal{P}_{\varepsilon,\eta}(n+m).$$

Moreover, the sets $P_{\varepsilon,\eta}$ are closed *by differences* as well, in the sense that, if $\lambda \in \mathcal{P}_{\varepsilon,\eta}(n)$ and $\lambda \oplus \mu \in \mathcal{P}_{\varepsilon,\eta}(n+m)$ for some partition $\mu \in \mathcal{P}(m)$, then also $\mu \in \mathcal{P}_{\varepsilon,\eta}(m)$.

We may finally state the classification theorem.

Theorem 2.3.3. *For each $\varepsilon \in \{\pm 1\}$ and each $\eta \in \{0, 1\}$, there is a one-to-one correspondence between the nilpotent K_θ-conjugacy classes in $\mathfrak{g}_{\varepsilon,\eta} \subseteq \mathfrak{gl}_n$ and the partitions in $\mathcal{P}_{\varepsilon,\eta}(n)$.*

We denote the nilpotent K_ε-conjugacy class corresponding to a partition $\lambda \in \mathcal{P}_{\varepsilon,\eta}(n)$ by $C_{\varepsilon,\eta,\lambda}$.

In order to prove theorem 2.3.3, we will make use of some intermediate lemmas.

Lemma 2.3.4. *Let $X \in \mathfrak{g}_{\varepsilon,\eta}$ be a nilpotent endomorphism of the quadratic space $V = k^n$. Then $\operatorname{rk} X^i$ must be even for*

- *every odd $i \in \{0, \dots, n\}$ if $\varepsilon = 1$, $\eta = 0$;*
- *every even $i \in \{0, \dots, n\}$ if $\varepsilon = -1$, $\eta = 0$;*
- *no restrictions if $\varepsilon = 1$, $\eta = 1$;*
- *every $i \in \{0, \dots, n\}$ if $\varepsilon = -1$, $\eta = 1$.*

Proof. The key step lies in finding a suitable non-degenerate symplectic space. In the cases $\eta = 0$, by theorem 1.1.3, one may consider an $\mathfrak{sl}_2$-triple (X, H, Y) in $\mathfrak{g}_{\varepsilon,\eta}$. In the cases $\eta = 1$, one may use theorem 1.1.11 to still find an $\mathfrak{sl}_2$-triple (X, H, Y) having $X, Y \in \mathfrak{g}_{\varepsilon,\eta}$. For an index $i = 0, \dots, n$, let $W = X^i(V)$. One may define a bilinear form $\langle -, - \rangle$ in W using the bilinear form $(-, -)$ of V as

$$\langle u, v \rangle := (u, Y^i v)$$

for each $u, v \in W$.

We prove that $\langle -,- \rangle$ is non-degenerate in W. Let $X^i u \in W$ be a vector such that $\langle X^i u, X^i v\rangle = 0$ for all $v \in V$. We prove that $X^i u = 0$. As $(D^*)^* = D$ for every map D, one gets that $\big((X^*)^i (Y^*)^i X^i u, v\big) = 0$ for all $v \in V$. By the non-degeneracy of $(-,-)$, that implies that $(X^*)^i(Y^*)^i X^i u = 0$. As both $X, Y \in \mathfrak{g}_{\varepsilon,\eta}$, we have that $X^* = (-1)^{1+\eta} X$ and $Y^* = (-1)^{1+\eta} Y$, so that $X^i Y^i X^i u = 0$. As $\ker X$ is H-invariant, we may assume that u belongs to some r-weight space of V. We may also assume that u belongs to some $\mathfrak{sl}_2$-isotypic component, say that of maximum weight w. Therefore, $Y^i X^i u$ has weight r and $Y^i X^i u \in \ker X^i$. Thus $r + i > w$. However, $X^i u$ has weight $r+i$, hence $X^i u = 0$.

We finally prove that $\langle -,- \rangle$ is a symplectic form exactly in the cases we need. Since $Y \in \mathfrak{g}_{\varepsilon,\eta}$, $Y^* = (-1)^{\eta+1} Y$ and

$$\langle v, u\rangle = (v, Y^i u) = \varepsilon(Y^i u, v) = \varepsilon(u, (-1)^{(\eta+1)i} Y^i v) = \varepsilon(-1)^{(\eta+1)i}\langle u, v\rangle.$$

for each $v, u \in X^i(V)$. Since a non-degenerate symplectic space must have even dimension, the conclusion follows. □

Lemma 2.3.5. *The regular nilpotent conjugacy class of $\mathfrak{gl}_n$ meets $\mathfrak{g}_{\varepsilon,\eta}$ if one of the following holds:*

- $\varepsilon = 1$, $\eta = 0$ *and* n *odd;*
- $\varepsilon = -1$, $\eta = 0$ *and* n *even;*
- $\varepsilon = 1$ *and* $\eta = 1$.

Proof. In each case, we choose an ε-symmetric non-degenerate bilinear form $(-,-)$ and a matrix X such that each of its entry is 0 except the entries $[X]_i^{i+1}$ for every $i = 1, \dots, n-1$. We prove that $X^* = (-1)^{\eta+1} X$, where the adjoint is done under $(-,-)$, so that $X \in \mathfrak{g}_{\varepsilon,\eta}$. This is enough, any such endomorphism X is nilpotent, $\operatorname{rk} X = n-1$, so that $X \in C_{(n)}$.

We start by considering the case $\varepsilon = 1$. Let $J \in \mathrm{GL}_n$ be the symmetric anti-diagonal matrix given by $[J]_i^j = 1$ if $i + j = n+1$ or 0 otherwise. The bilinear form $(x, y) := x^t J y$ is symmetric. Under this form, the adjoint X^* of a matrix X is given by

$$[X^*]_i^j = [X]_{n+1-j}^{n+1-i}\,. \tag{2.3.2}$$

If $\eta = 1$, we pick the matrix X defined by $[X]_i^{i+1} = 1$ for each $i = 1, \dots, n-1$ (that is, the usual nilpotent Jordan block of size n). We immediately get $X^* = X$, therefore $X \in \mathfrak{g}_{1,1}$. If $\eta = 0$, we pick X such that $[X]_i^{i+1} = (-1)^i$. If n is odd, we get $X^* = -X$, therefore $X \in \mathfrak{g}_{1,0}$.

The case $\varepsilon = -1$ exists only if $n = 2m$ is even. In that case we may take the matrix $B = \left(\begin{smallmatrix} 1_m & 0 \\ 0 & -1_m \end{smallmatrix}\right)$, where 1_m is the identity of size m, and the bilinear form $(x, y) := x^t B J y$, that is symplectic. The adjoint X^* of a matrix X is given by taking the adjoint (2.3.2) and then conjugating by B. We may pick X such that $[X]_i^{i+1} = -1$ if $i \le m$ and $[X]_i^{i+1} = 1$ if $m < i < 2m$. One gets $X^* = -X$, therefore $X \in \mathfrak{g}_{-1,0}$. □

Lemma 2.3.6. *If $\lambda \in \mathcal{P}(n)$ is a partition such that each column $\widehat{\lambda}_i$ is even, then the nilpotent orbit in $\mathfrak{gl}_n$ associated to λ meets $\mathfrak{g}_{\varepsilon,\eta}$ for every ε, η.*

Proof. As every column of λ is even, one may define the partition $\mu \vdash m := n/2$ by $\widehat{\mu}_i := \widehat{\lambda}_i/2$. Therefore $\lambda = \mu \oplus \mu$ by (2.1.2).

Let $W = k^m$ and let $X \in \mathfrak{gl}_m$ be a nilpotent element with corresponding partition μ. Let $W^\times$ the linear dual space of W. On $V = W \oplus W^\times$, one may define the bilinear form

$$(u, \phi) \cdot (v, \psi) := \phi(v) + \varepsilon\psi(u),$$

which is clearly non-degenerate and ε-symmetric. Then, one may define consider the linear map $f : W \to V$ given by

$$f(X)(u, \phi) := (Xu, (-1)^{1+\eta} X^\times \phi),$$

where $X^\times$ is the dual of X. Therefore, one gets

$$\begin{aligned} f(X)(u, \phi) \cdot (v, \psi) &= (Xu, (-1)^{1+\eta} X^\times \phi) \cdot (v, \psi) \\ &= (-1)^{1+\eta} (X^\times \phi)(v) + \varepsilon\psi(Xu) \\ &= (-1)^{1+\eta} \phi(Xv) + \varepsilon(X^\times \psi)(u) \\ &= (-1)^{1+\eta} (u, \phi) \cdot (Xv, (-1)^{1+\eta} X^\times \psi) \\ &= (-1)^{1+\eta} (u, \phi) \cdot f(X)(v, \psi) \\ &= (u, \phi) \cdot \Big((-1)^{1+\eta} f(X)\Big) (v, \psi). \end{aligned}$$

Thus $f(X)^* = (-1)^{1+\eta} f(X)$. As $\operatorname{rk} f(X)^i = 2 \operatorname{rk} X^i$ for each i (as $X \simeq X^\times$), the partition of $f(X)$ is $\mu \oplus \mu = \lambda$ and we are done. □

We can finally prove theorem 2.3.3.

Proof of theorem 2.3.3. We use (eq. (2.3.1)). We first prove that if a partition $\lambda \vdash n$ is such that $\lambda \notin \mathcal{P}_{\varepsilon,\eta}(n)$ do not appear in nilpotent conjugacy classes of $\mathfrak{g}_{\varepsilon,\eta}$.

Let $X \in \mathfrak{g}_{\varepsilon,\eta}$ be a nilpotent element and let λ the corresponding partition. By lemma 2.3.4, $\operatorname{rk} X^k$ is even if

- k is even and $(\varepsilon, \eta) = (1, 0)$, or
- k is odd and $(\varepsilon, \eta) = (-1, 0)$, or
- k is any and $(\varepsilon, \eta) = (-1, 1)$.

By (2.2.1),

$$\widehat{\lambda}_{k+1} + \widehat{\lambda}_{k+2} = \operatorname{rk} X^k - \operatorname{rk} X^{k+2},$$

so it is even for every k even if $(\varepsilon, \eta) = (1, 0)$. The other cases are completely analogous.

It remains to prove that every partition $\lambda \in \mathcal{P}_{\varepsilon,\eta}(n)$ appears as the partition of some nilpotent K_ε-conjugacy class in $\mathfrak{g}_{\varepsilon,\eta}$. Let $X_1 \in \mathfrak{g}_{\varepsilon,\eta}(n_1)$ be a nilpotent element with partition λ^1 and $X_2 \in \mathfrak{g}_{\varepsilon,\eta}(n_2)$ be a nilpotent element with partition λ^2. One may construct a nilpotent element $X_1 \oplus X_2$ in the direct sum $k^{n_1} \oplus k^{n_2}$ by

$$(X_1 \oplus X_2)(u, v) := (X_1 u, X_2 v).$$

It is clear that $\mathrm{rk}(X_1 \oplus X_2) = \mathrm{rk}\, X_1 + \mathrm{rk}\, X_2$, therefore the partition of $X_1 \oplus X_2$, therefore $X_1 \oplus X_2$ is nilpotent, it has partition $\lambda^1 \oplus \lambda^2$ and $X_1 \oplus X_2 \in \mathfrak{g}_{\varepsilon,\eta}(n_1, n_2)$ (since, in $k^{n_1} \oplus k^{n_2}$, the subspaces k^{n_1}, k^{n_2} remain orthogonal). Therefore the sum of obtainable partitions is obtainable.

By lemma 2.3.5, depending on ε and η, some partition with a single row is obtainable. By lemma 2.3.6, each partition with two equal rows by lemma 2.3.6. Clearly every $\lambda \in \mathcal{P}_{\varepsilon,\eta}(n)$ may be expressed as the sum of those minimal partitions. □

2.4 Not connected classes

In this section we inspect the differences between K_θ-conjugacy classes and K_θ^0-conjugacy classes, where K_θ^0 is the connected component of the identity of K_θ. As a matter of fact, we operate this analysis when K_θ is not connected, which happens when $\varepsilon = 1$. In that case, $K_\theta \simeq O_n$, $K_\theta^0 \simeq SO_n$, and it may happen that some O_n-conjugacy class splits into multiple SO_n-conjugacy classes.

Let $X \in \mathfrak{gl}_n$ be a nilpotent element. As its conjugacy class $O_n.X$ is a homogeneous space, we can think of it as O_n/O_n^X and we have a canonical inclusion

$$SO_n/SO_n^X \hookrightarrow O_n/O_n^X.$$

Thus, there can be at most $|O_n/SO_n| = 2$ connected components of $O_n.X$ and there is exactly one component if and only if $SO_n^X \subsetneq O_n^X$. If n is odd, $\det(-1) = -1$ so, $-1 \in O_n \setminus SO_n$ commutes with each X, therefore we can always find $-1 \in O_n^X \setminus SO_n^X$. We are concerned only with the case n even. We need to decide whether there is a $Z \in O_n^X$ with $\det(Z) = -1$.

We need some preparation. Let $Z \in \mathrm{End}(U \oplus V)$. One might consider the elements $Z_U \in \mathrm{End}(U)$ defined by

$$Z_U = pr_U \circ Z|_U,$$

and similarly for Z_V.

Remark. If $Z \in \mathrm{GL}(U \oplus V)$ is invertible and $Z(U) \subseteq U$, then it is well-known that Z_U and Z_V are invertible and $\det(Z) = \det Z_U \det Z_V$.

We return to the nilpotent element $X \in \mathfrak{gl}_n$. We may take an $\mathfrak{sl}_2$-triple (X, H, Y) in $\mathfrak{gl}_n$ and, therefore, we obtain the isotypic components $M(d)$ of $\mathfrak{gl}_n$. We also denote by $L(d)$ the lowest weight space of $M(d)$. We recall that, if we denote by λ the partition of X, $\dim L(d) = |\{j : \lambda_j = d\}|$ and that $X|_{M(d)}$ has nilpotent order $d + 1$. We have the following.

Lemma 2.4.1. *Let $Z \in \mathrm{GL}_n^X$. Then $Z_{L(d)}$ are invertible and*

$$\det(Z) = \prod_d \det\left(Z_{L(d)}\right)^{d+1}.$$

Proof. We make use of the previous remark several times. As Z commutes with X, $Z(\ker X^m) \subseteq \ker X^m$. Let $W_m = \ker X^{m+1}/\ker X^m$. As X is nilpotent, $k^n \simeq \bigoplus_{m \geq 0} W_m$, hence

$$\det(Z) = \prod_{m \geq 0} \det\left(Z_{W_m}\right).$$

Moreover,

$$W_m \simeq L(m) \oplus X(W_{m+1}) \simeq \bigoplus_{d \geq m} X^{d-m}(L(d)).$$

As Z commutes with X, $Z(X^{d-m}(L(d))) \subseteq X^{d-m}(L(d))$, and, actually,

$$Z_{X^{d-m}(L(d))} = Z_{L(d)},$$

therefore we conclude by

$$\det(Z_{W_m}) = \prod_{d \geq m} \det\left(Z_{L(d)}\right).$$

□

We need, now, to consider a symmetric bilinear form $(-,-)$ on k^n, so that we can work with O_n and O_n^X. We notice that, if $X^* = \pm X$, then the restrictions $(-,-)|_{M(d)}$ of this form to $M(d)$ is non-degenerate for each d and $M(d) \perp M(e)$ for $d \neq e$. The form $(-,-)$ actually induces a non-degenerate bilinear form $\langle -,- \rangle_d$ of $L(d)$ by

$$\langle u,v \rangle_d := (u, X^d v) \quad \forall u,v \in L(d).$$

Lemma 2.4.2. *Let $Z \in O_n^X$. Then $Z_{L(d)}$ is skew with respect to the bilinear form $\langle -,- \rangle_d$ on $L(d)$.*

Proof. It follows from

$$(Zu, X^d v) = (u, -ZX^d v) = (u, X^d(-Z)(v)) \quad \forall u,v \in k^n$$

and from the fact that there is a perfect pairing between the highest weight space and lowest weight space of $M(d)$. □

Theorem 2.4.3. *Let $\eta \in \{0,1\}$, n a positive integer and let $X \in \mathfrak{g}_{1,\eta}$ be a nilpotent element. Let $\lambda = (\lambda_1, \ldots, \lambda_h) \vdash n$ be the partition corresponding to X. Then the conjugacy class $O_n.X \subseteq \mathfrak{g}_{1,\eta}$ is connected (and therefore it is a SO_n-conjugacy class) if and only if λ_i is odd for some i. Otherwise, $O_n.X$ is the union of two (connected) SO_n-orbit.*

Proof. From the previous discussion, it is enough to determine whether there is a $Z \in O_n^X$ such that $\det(Z) = -1$.

We start in the case λ_i is always an even integer. By lemma 2.4.2, $\det(Z_{L(d)}) = \pm 1$. As λ_i is even, every d is odd, therefore each exponent in the formula for $\det(Z)$ in lemma 2.4.1 is even. Hence $\det(Z) = 1$ and the conjugacy class $O_n.X$ is not connected.

In the other case, there is i such that $e + 1 := \lambda_i$ is odd. We show that there is a $Z \in O_n^X$ with $Z^2 = 1$ and $\det(Z) = -1$. We may take Z to leave invariant each isotypic component $M(d)$ and actually to be the identity in $M(d)$ for each $d \neq e$. We may choose a decomposition into irreducible and orthogonal modules $M(e) \simeq V(e)^m$, where $m = |\{i : \lambda_i = e+1\}|$. We may put Z to be -1 on the first module $V(e)$ and 1 on the remaining modules. This way $Z_{M(e)}$ is skew, $Z_{M(e)}$ commutes with X and $\det(Z_{L(e)}) = -1$ as $\dim\{v : Zv = -v\} \cap L(e) = 1$. By lemma 2.4.1, $\det(Z) = (-1)^{e+1} = -1$ and we are done. □

The result in theorem 2.4.3 suggests the following definition.

Definition 2.4.4. A partition $\lambda \vdash n$ is *very even* if every row and every column of λ is even.

Remark. It is clear that a very even partition $\lambda \vdash n$ belongs to $\mathcal{P}_{1,0}(n)$. Therefore there exists an O_n-conjugacy class $\mathcal{O}_X \subseteq \mathfrak{g}_{1,0}$ corresponding to this λ. Theorem 2.4.3 says that these conjugacy classes are exactly the one not connected in $\mathfrak{g}_{1,0}$.

2.5 Dimension of conjugacy classes

In this section we compute dimension formulas for each nilpotent conjugacy class in each of the cases examined, that is, for $\mathfrak{gl}_n$ and for $\mathfrak{g}_{\varepsilon,\eta}(n)$ in each of the cases $\varepsilon \in \{\pm 1\}$ and $\eta \in \{0,1\}$.

In each case we first compute the dimension of the centralizer $\mathfrak{g}^X$ of a suitable nilpotent element X in the Lie algebra $\mathfrak{g}$. Then, we deduce the dimension of the orbit by

$$\dim \mathcal{O}_X = \dim[X, \mathfrak{g}] = \dim \mathfrak{g} - \dim \mathfrak{g}^X. \tag{2.5.1}$$

Every formula depends on the partition λ corresponding to the nilpotent conjugacy class $C_\lambda = \mathcal{O}_X$.

2.5.1 For GL_n

Proposition 2.5.1. *Let n be a positive integer, $\lambda \vdash n$ be a partition and $\widehat{\lambda} = (\widehat{\lambda}_1, \dots, \widehat{\lambda}_t)$ its dual and C_λ be the corresponding nilpotent conjugacy class. Then*

$$\begin{aligned} \dim C_\lambda &= n^2 - \sum_{i=1}^{t} \left(\widehat{\lambda}_i\right)^2 \\ &= n^2 - \sum_{i,j} \min(\lambda_i, \lambda_j). \end{aligned}$$

By (2.5.1), the proof of proposition 2.5.1 follows immediately by the following lemma.

Lemma 2.5.2. *Let n be a positive integer, $\lambda \vdash n$ be a partition and $\widehat{\lambda} = (\widehat{\lambda}_1, \dots, \widehat{\lambda}_t)$ its dual. Let $X \in \mathfrak{gl}_n$ be a nilpotent element with partition λ. Let $\mathfrak{gl}_n^X \subseteq \mathfrak{gl}_n$ be the centralizer of X. Then*

$$\dim \mathfrak{gl}_n^X = \sum_{i=1}^{t} \left(\widehat{\lambda}_i\right)^2.$$

Proof. Let (X, H, Y) be an $\mathfrak{sl}_2$-triple in $\mathfrak{gl}_n$. One may restrict the action of $\mathfrak{gl}_n$ to the natural representation k^n to the $\mathfrak{sl}_2$-subalgebra spanned by X, H, Y. Therefore, one gets the isotypic components $M(d)$ for each non-negative weight d. Let $L(d)$ be the lowest weight space of $M(d)$ and let $L = \bigoplus_d L(d)$ the sum of all $L(d)$. As $L(d)$ generates the whole $M(d)$ as a $k[X]$-module, the subspace L generates k^n as a $k[X]$-module.

Let $Z \in \mathfrak{gl}_n^X$ be an element commuting with X. One may recover the matrix Z once it is known the restriction $Z|_L$; indeed, any vector $v \in k^n$ may be written as

$v = p(X)w$ for some polynomial p and some $w \in L$, hence $Zv = Zp(X)w = p(X)Zw$ is known as soon as Zw is known.

There is a limitation about the possible values for Zw. If $w \in L(d)$, $X^{d+1}w = 0$, hence $Zw \in \ker X^{d+1}$. Anyway, this is the only restriction on Z. Indeed, fix a basis $w_{d,i}$ for each space $L(d)$ and choose, for each d, i, any vector $v_{d,i} \in \ker X^{d+1}$. Put $ZX^m w_{d,i} := X^m v_{d,i}$ for every $d, i, m \leq d$. It is clear that $ZX = XZ$ for each element of the basis $X^m w_{d,i}$.

One can compute the dimension of $\mathfrak{gl}_n^X$ by projecting $\mathfrak{gl}_n^X$ into the spaces $\operatorname{Hom}_k(L(d), \ker X^{d+1})$. In fact, we already proved that

$$\dim \mathfrak{gl}_n^X = \sum_d \dim \operatorname{Hom}_k(L(d), \ker X^{d+1}) = \sum_d \dim L(d) \dim X^{d+1}.$$

By (2.2.1), $\dim \ker X^{d+1} = \sum_{i \leq d+1} \widehat{\lambda}_i$; while it is clear that $\dim L(d) = |\{i : \lambda_i = d+1\}| = \widehat{\lambda}_{d+1} - \widehat{\lambda}_{d+2}$. Hence

$$\begin{aligned} \dim \mathfrak{gl}_n^X &= \sum_d \sum_{i \leq d+1} \left(\widehat{\lambda}_i \left(\widehat{\lambda}_{d+1} - \widehat{\lambda}_{d+2}\right)\right) = \sum_i \left(\widehat{\lambda}_i \sum_{d \geq i-1} \left(\widehat{\lambda}_{d+1} - \widehat{\lambda}_{d+2}\right)\right) \\ &= \sum_i \left(\widehat{\lambda}_i \widehat{\lambda}_{i-1+1}\right) = \sum_i \left(\widehat{\lambda}_i\right)^2. \end{aligned}$$

□

2.5.2 For quadratic and symmetric spaces

Proposition 2.5.3. *Let $\varepsilon \in \{\pm 1\}$, $\eta = 0$, n be a positive integer. Let $\lambda \in \mathcal{P}_{\varepsilon,0}(n)$, $\widehat{\lambda} = (\widehat{\lambda}_1, \ldots, \widehat{\lambda}_t)$ its dual and $C_{\varepsilon,0,\lambda}$ be the corresponding nilpotent K_ε-conjugacy class in $\mathfrak{g}_{\varepsilon,0}$. Then*

$$\dim C_{\varepsilon,0,\lambda} = \frac{1}{2}\left(n^2 - \varepsilon n - \sum_{i=1}^t \left(\widehat{\lambda}_i\right)^2 + \varepsilon |\{i : \lambda_i \text{ is odd}\}|\right).$$

Similar to the case GL_n, by (2.5.1), the proof of proposition 2.5.3 follows immediately by the following lemma.

Lemma 2.5.4. *Let $\varepsilon \in \{\pm 1\}$, $\eta = 0$, n be a positive integer. Let $\lambda = (\lambda_1, \ldots, \lambda_h) \in \mathcal{P}_{\varepsilon,0}(n)$ and $\widehat{\lambda} = (\widehat{\lambda}_1, \ldots, \widehat{\lambda}_t)$ its dual. Let $X \in \mathfrak{g}_{\varepsilon,0}$ a nilpotent element with partition λ and let $\mathfrak{g}_{\varepsilon,0}^X \subseteq \mathfrak{g}_{\varepsilon,0}$ the centralizer of X. Then*

$$\dim \mathfrak{g}_{\varepsilon,0}^X = \frac{1}{2} \sum_{i=1}^t \left(\widehat{\lambda}_i\right)^2 - \frac{\varepsilon}{2} |\{i : \lambda_i \text{ is odd}\}|.$$

Proof. As the cases $\varepsilon = 1$ and $\varepsilon = -1$ are rather similar, we will only deal with the latter.

We proceed in a similar fashion to lemma 2.5.2, but we need to do a finer-grained analysis. We choose an $\mathfrak{sl}_2$-triple (X, H, Y) in $\mathfrak{g}_{\varepsilon,0}$, hence we may decompose the natural representation k^n of $\mathfrak{gl}_{\varepsilon,0}$ into irreducible $\mathfrak{sl}_2$-representations

$$k^n = \bigoplus_{i \in I} V(d_i),$$

where I is an index set parametrizing the irreducible representations in k^n (thus corresponds to the set of rows of λ) and d_i is the highest weight of the i-th module. As $X, H, Y \in \mathfrak{g}_{\varepsilon,0}$, they commute with their adjoint, thus we may assume that any two irreducible modules $V(d_i)$, $V(d_j)$ (possibly $i = j$) are orthogonal with each other or they are in perfect pairing with respect to non-degenerate bilinear form $(-,-)$ of k^n.

For each $i \in I$, let L_i be the lowest weight space of $V(d_i)$. As $V(d_i)$ is irreducible, $\dim L_i = 1$. By the same argument in the proof of lemma 2.5.2, a morphism $Z \in \mathfrak{gl}_n$ that commutes with X is completely determined by its value on each L_i and $Z(L_i) \subseteq \ker X^{i+1}$.

We project the subspace of commuting linear morphism $\mathfrak{g}^X_{\varepsilon,0} \subseteq \mathfrak{gl}_n$ into each space $L_{i,j} := \mathrm{Hom}_k(L_i, V(d_j))$, for each $i, j \in I$. If $Z \in \mathfrak{g}^X_{\varepsilon,0}$, we denote by $Z_{i,j} \in L_{i,j}$ the projection into this space. For a pair of indices (i, j), we distinguish two cases.

Suppose that $V(d_i) \perp V(d_j)$ and, without loss of generality, that $d_i \geq d_j$. Then, by $Z \in \mathfrak{g}_{\varepsilon,0}$, modulo composing $Z_{j,i}$ with the projection into the subspace of $V(d_i) \cap \ker X^{j+1}$, one gets that $Z_{j,i} = -Z_{i,j}$. Moreover, for each linear morphism $z \in L_{i,j}$ one may construct a $Z \in \mathfrak{g}^X_{\varepsilon,0}$ by $Z_{i,j} = z$, $Z_{j,i} = -z$ and each other projection null.

In the other case, suppose that $V(d_i)$ pairs non-degenerately with $V(d_j)$. We remark that this is possible only if $d_i = d_j$. Let $Z \in \mathfrak{g}_{\varepsilon,0}$ and let $v \in L_i$ not zero. As $V(d_i)$ is generated by v (by applying X several times) and the pairing between $V(d_i)$ and $V(d_j) \supseteq Z_{i,j}(L_i)$, if $Z_{i,j}(v) \neq 0$, there is some vector $X^m v \in V(d_i)$ such that $(Z_{i,j}(v), X^m v) \neq 0$. As Z and X commute, $X^* = -X$, $Z^*_{i,j} = -Z_{i,j}$, one gets

$$(Z_{i,j}(v), X^m v) = (-1)^{m+1}(X^m v, Z_{i,j}(v)) \;\Rightarrow\; (-1)^{m+1} = \varepsilon.$$

Thus m cannot be odd if $\varepsilon = 1$ and m cannot be even if $\varepsilon = -1$. Moreover, for each linear morphism $z \in L_{i,j}$ with image contained in the sum of the weight spaces in $V(d_j)$ of weight with the right parity, one may construct a $Z \in \mathfrak{g}^X_{\varepsilon,0}$ by $Z_{i,j} = z$ and each other projection null.

We can finally compute the dimension of $\mathfrak{g}^X_{\varepsilon,0}$. In fact, we compute $2\dim \mathfrak{g}^X_{\varepsilon,0} - \dim \mathfrak{gl}^X_n$. For each $i, j \in I$, if $V(d_i) \perp V(d_j)$, we proved

$$\dim\left(\mathfrak{g}^X_{\varepsilon,0} \cap (L_{i,j} \oplus L_{j,i})\right) = \frac{1}{2}\dim\left(\mathfrak{gl}^X_n \cap (L_{i,j} \oplus L_{j,i})\right).$$

If $V(d_i)$ pairs non-degenerately with $V(d_j)$, let $v \in L_i$ be a generator of $V(d_i)$; we proved

$$\begin{aligned}\dim\left(\mathfrak{g}^X_{\varepsilon,0} \cap L_{i,j}\right) &= |\{m : X^m v \neq 0 \text{ and } (-1)^{m+1} = \varepsilon\}| \\ &= \begin{cases} \frac{1}{2}\dim V(d_i) & \text{if } \dim V(d_i) \text{ is even;} \\ \frac{1}{2}\dim V(d_i) - \frac{\varepsilon}{2} & \text{if } \dim V(d_i) \text{ is odd.} \end{cases}\end{aligned}$$

As $\dim\left(\mathfrak{gl}^X_n \cap L_{i,j}\right) = \dim V(d_i)$, we get

$$2\dim \mathfrak{g}^X_{\varepsilon,0} - \dim \mathfrak{gl}^X_n = -\varepsilon|\{i \in I : \dim V(d_i) \text{ is odd}\}|.$$

Recalling that I corresponds to the set of rows of λ, we have $\dim V(d_i) = \lambda_i$, which concludes the proof. □

Proposition 2.5.5. *Let $\varepsilon \in \{\pm 1\}$, $\eta = 1$, n be a positive integer. Let $\lambda \in \mathcal{P}_{\varepsilon,1}(n)$, $\widehat{\lambda} = (\widehat{\lambda}_1, \ldots, \widehat{\lambda}_t)$ its dual and $C_{\varepsilon,1,\lambda}$ be the corresponding nilpotent K_ε-conjugacy class in $\mathfrak{g}_{\varepsilon,1}$. Then*

$$\dim C_{\varepsilon,1,\lambda} = \frac{1}{2}\left(n^2 - \sum_{i=1}^{t}\left(\widehat{\lambda}_i\right)^2\right).$$

Proof. As $\mathfrak{g}_{\varepsilon,1}(n)$ is the symmetric space of the symmetric pair $(\mathfrak{gl}_n, \mathfrak{g}_{\varepsilon,0})$, this case follows instantly by corollary 1.1.9 and by proposition 2.5.1. □

Chapter 3

Closure of conjugacy classes

Let $\mathfrak{V} = \mathfrak{g}$ be a Lie algebra or $\mathfrak{V} = \mathfrak{p}$ be the symmetric space of an orthogonal or symplectic pair $(\mathfrak{gl}_n, \mathfrak{g})$. In the last chapter we studied the nilpotent conjugacy classes with respect to some classical group G in $\mathfrak{V}$. In particular we obtained that they are finite and we parameterized them with some class of partition $\mathcal{Q} \subseteq \mathcal{P}(n)$. For a partition $\lambda \in \mathcal{Q}$, in this chapter we describe the closure $\overline{C_\lambda} \subseteq \mathfrak{V}$.

As $\overline{C_\lambda}$ is G-invariant in $\mathfrak{V}$, we get that $\overline{C_\lambda}$ is a union of G-orbit. It turns out that every G-orbit in $\overline{C_\lambda}$ is nilpotent as the whole set of nilpotent elements in $\mathfrak{V}$ is closed. Therefore, there are only finitely many G-orbits in $\overline{C_\lambda}$. By usual arguments, C_λ is open inside $\overline{C_\lambda}$ and the closed boundary $\overline{C_\lambda} \setminus C_\lambda$ has dimension strictly less than $\dim C_\lambda$. Hence, giving two partitions $\lambda, \mu \in \mathcal{Q}$, the relation $\overline{C_\lambda} \supseteq \overline{C_\mu}$ defines an order in $\mathcal{Q}$. The main result in this chapter is establishing that this order coincides with the dominace order, see definition 3.1.1, which is defined in a purely combinatorial way. As a warm-up, we prove that that is true when $\mu = (1^n)$, the lowest partition, which corresponds to the null orbit $\{0\}$.

Remark. For every nilpotent $X \in \mathfrak{V}$, $0 \in \overline{\mathcal{O}_X}$. The key step is that $tX \in \mathcal{O}_X$ for every non-zero scalar t, so that $0 = 0X$ would belong to the line kX.

One may consider an $\mathfrak{sl}_2$-triple X, H, Y in a suitable Lie algebra. In $\mathfrak{sl}_2$, every multiple te of the nilpositive element e is nilpotent. Moreover, there are only two nilpotent conjugacy classes in $\mathfrak{sl}_2$: 0 and the regular one, hence te is regular for every $t \neq 0$, so te is SL_2-conjugated to e. The element $tX \neq 0$, being the image of te, is still SL_2-conjugated to X, therefore $tX \in \mathcal{O}_X$.

Remark. The previous remark actually proved that for every nilpotent class C_λ, the closure $\overline{C_\lambda}$ is a cone in $\mathfrak{V}$ centered at 0. In particular it is connected.

3.1 Dominance order

There is an important partial order for the set $\mathcal{P}(n)$ called the dominance order. A standard reference here is [BB06].

Definition 3.1.1. Given $\lambda, \mu \in \mathcal{P}(n)$, we say that $\lambda \geq \mu$ when

$$\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i \quad \forall j.$$

The order $\leq$ of $\mathcal{P}(n)$ is called *dominance order.*

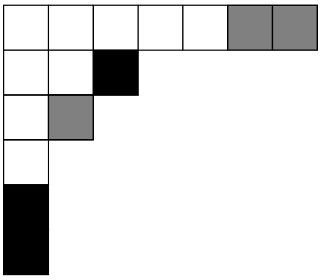
In order to better understand the dominance order, we introduce a function on the comparable pairs $\lambda \geq \mu$ of partitions $\lambda, \mu \in \mathcal{P}(n)$.

Definition 3.1.2. If $\lambda = (\lambda_1, \dots, \lambda_k) \in P(n)$ and $\mu = (\mu_1, \dots, \mu_k) \in P(n)$, possibly allowing $\lambda_i = 0$ for some i, we define

$$q(\lambda, \mu) = \frac{1}{2} \sum_{i=1}^{k} |\lambda_i - \mu_i|.$$

Remark. By an easy computation modulo 2, it can be seen that $q(\lambda, \mu)$ is an integer.

Example. Let $\lambda = (7, 2, 2, 1)$ and $\mu = (5, 3, 1, 1, 1, 1)$. We can draw the Young diagram of both partitions and overlay one over the other, so that λ is given by white and gray boxes while μ is given by white and black boxes in the following diagram:



We can compute the q difference:

$$q(\lambda, \mu) = \frac{1}{2} ((7 - 5) + (3 - 2) + (2 - 1) + (1 - 1) + 1 + 1) = 3.$$

We can check that in this case $q(\lambda, \mu)$ is the number of gray boxes or the number of black boxes. This is an easy fact (it is not even needed that $\lambda > \mu$) that can be derived from the following lemma.

Lemma 3.1.3. *Let $\lambda > \mu$ be two partitions of n. Then there exist partitions*

$$\lambda = \lambda^0 > \lambda^1 > \dots > \lambda^{q(\lambda,\mu)} = \mu$$

such that $q(\lambda^i, \lambda^{i+1}) = 1$ for each $i = 0, \dots, q(\lambda, \mu) - 1$.

Moreover, we can choose λ^i such that for each row r and for each column c the sequences of integers $\lambda^0_r, \dots, \lambda^{q(\lambda,\mu)}_r$ and $\widehat{\lambda}^0_c, \dots, \widehat{\lambda}^{q(\lambda,\mu)}_c$ are monotone.

Proof. We proceed by induction on $q = q(\lambda, \mu)$. If $q = 1$ we trivially have $\lambda^1 = \mu$. If $q > 1$, we are going to build λ^1 such that $q(\lambda, \lambda^1) = 1$ and $q(\lambda^1, \mu) = q(\lambda, \mu) - 1$.

We build λ^1 starting from λ in this way. Let $\tilde{\imath}$ be the first index such that $\lambda_{\tilde{\imath}} > \mu_{\tilde{\imath}}$ and let j be the first index such that $\lambda_j < \mu_j$. Clearly there must exists such indices because $\lambda \neq \mu$ and $|\lambda| = |\mu|$. Let $i \geq \tilde{\imath}$ be the last index such that $\lambda_i = \lambda_{\tilde{\imath}}$ (possibly $i = \tilde{\imath}$). As $\lambda > \mu$, we must have $\tilde{\imath} < j$ and therefore $i < j$. We define

$$\lambda^1 = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \dots, \lambda_{j-1}, \lambda_j + 1, \lambda_{j+1}, \dots, \lambda_k).$$

The sequence λ^1 is actually a partition of n as $|\lambda^1| = |\lambda| = n$, $\lambda_i - 1 \geq \lambda_{i+1}$ by the maximality of i and $\lambda_{j-1} \geq \lambda_j + 1$ by the minimality of j.

Moreover, both the following hold: $q(\lambda, \lambda^1) = 1$ and $q(\lambda^1, \mu) = q - 1$. We apply induction on the pair (λ^1, μ) and we conclude the proof of the first part of the lemma.

To conclude the proof, we notice that if, for a row r, $\lambda_r > \mu_r$ (resp. $\lambda_r < \mu_r$), then we have $\lambda^i_r \geq \lambda^{i+1}_r$ (resp. $\lambda^i_r \leq \lambda^{i+1}_r$) by construction. The same holds true for the column lengths. □

Example. Taking the partitions λ, μ defined in the previous example, the construction explained in the lemma produces the following sequence of partitions:

$$(7, 2, 2, 1) > (6, 3, 2, 1) > (5, 3, 2, 1, 1) > (5, 3, 1, 1, 1).$$

Remark. In view of lemma 3.1.3 we can interpret $q(\lambda, \mu)$ as the minimum number of boxes needed to lower in order to obtain μ from λ.

Remark. We can state an equivalent definition of the dominance order. Indeed,

$$\lambda \geq \mu \quad \Leftrightarrow \quad \sum_{k>j} \widehat{\lambda}_k \geq \sum_{k>j} \widehat{\mu}_k \quad \forall j. \tag{3.1.1}$$

This may be proved by looking at the $q(\lambda, \mu)$ boxes moved in lemma 3.1.3. Each of those boxes is moved at least one row down and at least one column left. Therefore, the number of boxes in the columns after the j-th is non-increasing along the succession $\lambda^1, \lambda^2, \ldots, \lambda^{q(\lambda,\mu)}$.

Remark. The duality of partitions $\lambda \leftrightarrow \widehat{\lambda}$ reverses the dominance order. By lemma 3.1.3, it is enough to prove that this is the case when $q(\lambda, \mu) = 1$. In that case, there is a single box B which is moved at least one row down and at least one column left from λ to μ. Therefore, B is moved at least one row up and at least one column right from $\widehat{\lambda}$ to $\widehat{\mu}$.

Remark. The one-column partition $\mu = (1^n)$ is the minimum of $\mathcal{P}(n)$, with respect to the dominance order. Indeed, for every partition $\lambda \in \mathcal{P}(n)$, for every $j \geq 1$,

$$\sum_{k>j} \widehat{\lambda}_k \geq 0 = \sum_{k>j} \widehat{\mu}_k.$$

By the previous remark, one gets that the one-row partition $\lambda = (n)$ is the maximum of $\mathcal{P}(n)$.

3.1.1 Combinatorics of minimal degenerations

In this section we inspect some properties of the dominance order (definition 3.1.1). The main result is the description of the cover relation $\lambda \gtrdot \mu$ induced by the dominance order $\geq$. We show some of the results of [Hes76] and of [Oht86].

As we will not always deal with the whole partition set $\mathcal{P}(n)$, but also with some subset $\mathcal{Q} \subseteq \mathcal{P}(n)$, we notice that $\gtrdot$ depends on $\mathcal{Q}$. Indeed some partition $\nu \in \mathcal{P}(n)$, intermediate between $\mu \leq \nu \leq \lambda$, may be missing $\mathcal{Q}$ so that $\mu \lessdot \lambda$ in $\mathcal{Q}$, even if $\mu \not\lessdot \lambda$ in $\mathcal{P}(n)$.

Definition 3.1.4. Let $\mathcal{Q} \subseteq \mathcal{P}(n)$ be a subset of the partitions and let $\lambda, \mu \in Q$. We say that μ is a *minimal degeneration* of λ if $\mu \lessdot \lambda$ with respect to the dominance order $\leq$ restricted to $\mathcal{Q}$.

Definition 3.1.5. We say that a pair (λ, μ) of partitions in $P(n)$ can be *reduced by erasing rows* (resp. *columns*) to a pair (λ', μ') if there is a positive integer k such that $\lambda_i = \mu_i$ for $i = 1, \dots, k$ (resp. $\widehat{\lambda}_i = \widehat{\mu}_i$ for $i = 1, \dots, k$) and λ' may be obtained from λ by dropping the first k rows $\lambda_1, \dots, \lambda_k$ (resp. columns $\widehat{\lambda}_1, \dots, \widehat{\lambda}_k$) and μ' may be obtained from μ by dropping the first k rows $\lambda_1, \dots, \lambda_k$ (resp. columns $\widehat{\mu}_1, \dots, \widehat{\mu}_k$).

A pair of partitions (λ, μ) is *irreducible* if it cannot be reduced by erasing rows or columns to any other pair.

Remark. If $\lambda > \mu$ is a minimal degeneration and $\lambda' > \mu'$ is a degeneration obtained by erasing rows or columns from $\lambda > \mu$, then also $\lambda' > \mu'$ is minimal.

Remark. A degeneration $\lambda > \mu$ may be obtained uniquely from an irreducible degeneration by adding rows and columns.

There is an interesting result that justifies the introduction of this relation. It involves a straightforward computation on the dimension of the conjugacy classes as shown in the following propositions.

Proposition 3.1.6. *Let $\lambda, \mu \in \mathcal{P}(n)$ and let (λ', μ') be a pair of partitions obtained by erasing the first r rows and the first s columns from (λ, μ). Then*

$$\dim C_\lambda - \dim C_\mu = \dim C_{\lambda'} - \dim C_{\mu'}.$$

Proof. By induction on r, s, it is enough to prove the cases $r = 0$, $s = 1$ and $r = 1$, $s = 0$.

Let $\widehat{\lambda} = (\widehat{\lambda}_1, \dots, \widehat{\lambda}_t)$ and $\widehat{\mu} = (\widehat{\mu}_1, \dots, \widehat{\mu}_t)$, possibly with some $\widehat{\lambda}_i = 0$ or $\widehat{\mu}_i = 0$. We define

$$D(\lambda, \mu) := \sum_{i=1}^{t} \left((\widehat{\mu}_i)^2 - (\widehat{\lambda}_i)^2 \right). \tag{3.1.2}$$

By proposition 2.5.1,

$$\begin{aligned} \dim C_\lambda - \dim C_\mu &= \left(|\lambda|^2 - \sum_{i=1}^{t} (\widehat{\lambda}_i)^2 \right) - \left(|\mu|^2 - \sum_{i=1}^{t} (\widehat{\mu}_i)^2 \right) \\ &= D(\lambda, \mu). \end{aligned}$$

Hence, one has to show that $D(\lambda, \mu) = D(\lambda', \mu')$.

If $r = 0$, $s = 1$, the conclusion follows immediately, as the first term in the sum (3.1.2) cancel out.

If $r = 1$, $s = 0$, one has $\widehat{\lambda}_t \neq 0$, $\widehat{\mu}_t \neq 0$, $\widehat{\lambda'}_i = \widehat{\lambda}_i - 1$ and $\widehat{\mu'}_i = \widehat{\mu}_i - 1$ for each $i = 1, \dots, t$. Therefore,

$$
\begin{aligned}
D(\lambda,\mu) &= \sum_{i=1}^{t}\Big((\widehat{\mu}_i)^2-(\widehat{\lambda}_i)^2\Big)\\
&= \sum_{i=1}^{t}\Big((\widehat{\mu}'_i+1)^2-(\widehat{\lambda}'_i+1)^2\Big)\\
&= \sum_{i=1}^{t}\Big((\widehat{\mu}'_i)^2+2\widehat{\mu}'_i+1-(\widehat{\lambda}'_i)^2-2\widehat{\lambda}'_i-1\Big)\\
&= D(\lambda',\mu')+2\sum_{i=1}^{t}\Big(\widehat{\mu}'_i-\widehat{\lambda}'_i\Big)\\
&= D(\lambda',\mu')+|\mu'|-|\lambda'| = D(\lambda',\mu').
\end{aligned} \tag{3.1.3}
$$

□

We turn to the study the minimal irreducible degenerations in $\mathcal{P}(n)$.

Lemma 3.1.7. *Let $\lambda,\mu\in\mathcal{P}(n)$. If (λ,μ) is a minimal irreducible degeneration, then*

$$\lambda=(n),\quad \mu=(n-1,1)$$

or

$$\lambda=(2,1^{n-1}),\quad \mu=(1^n).$$

Proof. By lemma 3.1.3, $q(\lambda,\mu)=1$, otherwise the degeneration is not minimal. In fact the single box B moving down from λ to μ cannot move by more than one row and more than one column. Indeed, B can be moved down by one row if the difference between its row and the next row is at least two. But, if the difference between two consecutive rows is always at most one, then B can be moved by only one column to the left.

In the first case there are no more than two rows involved in the degeneration (λ,μ) therefore, the irreducible degeneration obtained by erasing rows and columns is $((n),(n-1,1))$. The second case is symmetric by the duality of partitions. □

We want to perform a similar analysis also for the minimal irreducible degenerations in $\mathcal{Q}=\mathcal{P}_{1,0}(n),\mathcal{P}_{-1,0}(n),\mathcal{P}_{-1,1}(n)$.

Remark. If $\lambda>\mu$ is a degeneration in $\mathcal{P}_{1,0}$ (resp. $\mathcal{P}_{-1,0}$) and $\lambda'>\mu'$ is obtained from $(\lambda>\mu)$ by erasing one column, then all parity of the rows in λ' and μ' are changed, hence $\lambda'>\mu'$ is a degeneration in $\mathcal{P}_{-1,0}$ (resp. $\mathcal{P}_{1,0}$).

Proposition 3.1.8. *Let $\varepsilon\in\{\pm1\}$ and $\eta\in\{0,1\}$. Let $\lambda,\mu\in\mathcal{P}_{\varepsilon,\eta}(n)$ and let (λ',μ') be a pair of partitions in $P_{\varepsilon',\eta}(m)$ obtained by erasing the first r rows and the first s columns from (λ,μ) ($\varepsilon':=\varepsilon(-1)^{s(1+\eta)}$). Then*

$$\dim C_{\varepsilon,\eta,\lambda}-\dim C_{\varepsilon,\eta,\mu}=\dim C_{\varepsilon',\eta,\lambda'}-\dim C_{\varepsilon',\eta,\mu'}.$$

Proof. We distinguish $\eta=0$ and $\eta=1$.

If $\eta=1$, $\varepsilon=\varepsilon'$ and, by proposition 2.5.5, $2\dim C_{\varepsilon,1,\lambda}=\dim C_\lambda$. Hence the conclusion follows by proposition 3.1.6.

If $\eta = 0$, by induction, we reduce to the cases $r = 0$, $s = 1$ and $r = 1$, $s = 0$ and define $D(\lambda, \mu)$ as (3.1.2).

Suppose $r = 0$, $s = 1$. Then $\varepsilon' = -\varepsilon$, $\widehat{\lambda}_1 = \widehat{\mu}_1$, $\lambda'_i = \lambda_i - 1$, $\mu'_i = \mu_i - 1$ for each i. Moreover, $|\{i : \lambda_i \text{ is odd}\}| = |\{i : \lambda'_i \text{ is even}\}| = \widehat{\lambda}_1 - |\{i : \lambda'_i \text{ is odd}\}|$ and it is clear that $D(\lambda, \mu) = D(\lambda', \mu')$. Hence, by proposition 2.5.3, one gets

$$\begin{aligned}\dim C_{\varepsilon,0,\lambda} - \dim C_{\varepsilon,0,\mu} = &\left(\frac{1}{2}\left(n^2 - \varepsilon n - \sum_{i=1}^{t}\left(\widehat{\lambda}_i\right)^2 + \varepsilon|\{i : \lambda_i \text{ is odd}\}|\right)\right) + \\ &- \left(\frac{1}{2}\left(n^2 - \varepsilon n - \sum_{i=1}^{t}\left(\widehat{\mu}_i\right)^2 + \varepsilon|\{i : \mu_i \text{ is odd}\}|\right)\right) \\ =&\frac{1}{2}D(\lambda, \mu) + \frac{\varepsilon}{2}\left(|\{i : \lambda_i \text{ is odd}\}| - |\{i : \mu_i \text{ is odd}\}|\right) \\ =&\frac{1}{2}D(\lambda', \mu') - \frac{\varepsilon}{2}\left(|\{i : \lambda'_i \text{ is odd}\}| - |\{i : \mu'_i \text{ is odd}\}|\right) \\ =&\dim C_{\varepsilon',0,\lambda'} - \dim C_{\varepsilon',0,\mu'}.\end{aligned}$$

Suppose $r = 1$, $s = 0$. Then $\varepsilon' = \varepsilon$,

$$\left(|\{i : \lambda_i \text{ is odd}\}| - |\{i : \mu_i \text{ is odd}\}|\right) = \left(|\{i : \lambda'_i \text{ is odd}\}| - |\{i : \mu'_i \text{ is odd}\}|\right)$$

since $\lambda_1 = \mu_1$ has the same parity and $D(\lambda, \mu) = D(\lambda', \mu')$ by (3.1.3). Hence the conclusion follows as in the previous case. □

Lemma 3.1.9. *Let $\varepsilon \in \{\pm 1\}$ and $\eta \in \{0, 1\}$. Let $\lambda > \mu$ be a minimal degeneration in $\mathcal{P}_{\varepsilon,\eta}(n)$. Then the irreducible degeneration (λ', μ') obtained by erasing rows and columns from (λ, μ) is one of the cases reported in table 3.1.*

Proof. In the case $\varepsilon = 1$, $\eta = 1$, $\mathcal{P}_{1,1}(n) = \mathcal{P}(n)$ and the result follows by lemma 3.1.7.

If $\varepsilon = -1$, $\eta = 1$, there is a one-to-one correspondence between $\mathcal{P}_{-1,1}(n)$ and $\mathcal{P}(n/2)$ given by halving each column (which always has an even length). The result follows by lemma 3.1.7 applied on $\mathcal{P}(n/2)$.

We deal with the remaining cases ($\eta = 0$) together, since ε change when we erase rows.

The condition $\lambda \in \mathcal{P}_{\varepsilon,0}(n)$ may be stated as $\widehat{\lambda}_i + \widehat{\lambda}_{i+1}$ *is even* for each i even (if $\varepsilon = 1$) or i odd (if $\varepsilon = -1$). In particular, for such i, $0 \leq \widehat{\lambda}_i - \widehat{\lambda}_{i+1}$ is even. We group the indices of columns in pairs $(i, i+1)$ according to the parity of i, in such a way that two columns paired have even boxes in total (if $\varepsilon = 1$, the first column is left unpaired).

Let $(i, i+1)$ be the pair with maximum i such that there is a column $j \in \{i, i+1\}$ such that $\widehat{\lambda}_j \neq \widehat{\mu}_j$. Clearly $\widehat{\lambda}_j > \widehat{\mu}_j$ since $\lambda > \mu$. If $\widehat{\lambda}_i < \widehat{\mu}_i$ then $j = i + 1$, therefore $\widehat{\lambda}_{i+1} - \widehat{\mu}_{i+1} = 1 = \widehat{\mu}_i - \widehat{\lambda}_i$ and each other column is unchanged by the minimality of $\lambda > \mu$. Those are the cases of types a and g.

From now on, suppose $\widehat{\lambda}_i \geq \widehat{\mu}_i$. Let $D := \widehat{\lambda}_i + \widehat{\lambda}_{i+1} - \widehat{\mu}_i - \widehat{\mu}_{i+1}$ be the number of boxes moved from the pair $(i, i+1)$. One has $D \geq 2$, since $\widehat{\lambda}_i - \widehat{\mu}_i \geq 0$, $\widehat{\lambda}_{i+1} - \widehat{\mu}_{i+1} \geq 0$, $\widehat{\lambda}_j - \widehat{\mu}_j > 0$, so $D > 0$; and D is even. In particular, $i > 1$.

If $\widehat{\lambda}_i - \widehat{\lambda}_{i+1} \geq 2$ (resp. $\widehat{\lambda}_i - \widehat{\lambda}_{i+1} = 0$), we claim that $j = i$ (resp. $\widehat{\lambda}_i - \widehat{\mu}_i > 0$). Otherwise, in both cases take the partition $\nu \vdash n$ such that $\widehat{\nu}_k := \widehat{\mu}_k$ for each

Table 3.1. Irreducible minimal degenerations

(a) Symplectic ($\varepsilon = -1$, $\eta = 0$)

type	a	b	d	g
		$n > 1$	$n > 0$	$n > 1$
λ	(2)	$(2n)$	$(2n+1, 2n+1)$	$(2, 1^{2n-2})$
μ	$(1,1)$	$(2n-2, 2)$	$(2n, 2n, 2)$	(1^{2n})
$\operatorname{codim}_{C_\lambda} C_\mu$	2	2	2	$2n$

(b) Orthogonal ($\varepsilon = 1$, $\eta = 0$)

type	c	e	f	h
	$n > 0$	$n > 0$	$n > 1$	$n > 2$
λ	$(2n+1)$	$(2n, 2n)$	$(2, 2, 1^{2n-3})$	$(2, 2, 1^{2n-4})$
μ	$(2n-1, 1, 1)$	$(2n-1, 2n-1, 1, 1)$	(1^{2n+1})	(1^{2n})
$\operatorname{codim}_{C_\lambda} C_\mu$	2	2	$4n-4$	$4n-6$

(c) Symmetric orthogonal ($\varepsilon = 1$, $\eta = 1$)

type	x	x^*
	$n > 1$	$n > 1$
λ	(n)	$(2, 1^{n-2})$
μ	$(n-1, 1)$	(1^n)
$\operatorname{codim}_{C_\lambda} C_\mu$	1	$n-1$

(d) Symmetric symplectic ($\varepsilon = -1$, $\eta = 1$)

type	y	y^*
	$n > 1$	$n > 1$
λ	(n, n)	$(2, 2, 1^{n-2}, 1^{n-2})$
μ	$(n-1, n-1, 1, 1)$	$(1^n, 1^n)$
$\operatorname{codim}_{C_\lambda} C_\mu$	4	$4n-4$

$k \notin \{i, i+1\}$, $\widehat{\nu}_i := \widehat{\mu}_i - 1$, $\widehat{\nu}_{i+1} := \widehat{\mu}_{i+1} + 1$. Clearly, $\nu > \mu$ is a degeneration of type a or g and $\nu \leq \lambda$ by a straightforward check in both cases. By the minimality of $\lambda > \mu$, we must have $\lambda = \nu$, but that contradicts $\widehat{\nu}_i = \widehat{\lambda}_i \geq \widehat{\mu}_i$.

Suppose that $\widehat{\lambda}_{i-1} = \widehat{\lambda}_i$ and let $r = \widehat{\lambda}_{i-1} + 1$ be the first row index such that λ_r has less than $\widehat{\lambda}_{i-1}$ boxes (possibly $\lambda_r = 0$). Case *(i)*: λ_r is odd and $\varepsilon = 1$ or λ_r is even and $\varepsilon = -1$; case *(ii)*: otherwise. Notice that in case *(ii)*, $\lambda_{r+1} = \lambda_r$. We build a partition $\nu \vdash n$ starting from λ, by moving down 2 boxes out of the D boxes in the columns pair $(i, i+1)$: if we are in case *(i)*, we put those 2 boxes on $\nu_r := \lambda_r + 2$, otherwise we put $\nu_r := \lambda_r + 1$ and $\nu_{r+1} := \lambda_{r+1} + 1$. In each case we have $\mu \leq \nu < \lambda$. Therefore $\mu = \nu$ by the minimality of $\mu < \lambda$. By the claim of the previous paragraph, in case *(i)*, that degeneration has type b (resp. d) if $\widehat{\lambda}_i = \widehat{\lambda}_{i+1}$ (resp. $\widehat{\lambda}_i > \widehat{\lambda}_{i+1}$); similarly, in case *(ii)* that degeneration has type c or e.

Suppose, instead, that $\widehat{\lambda}_{i-1} > \widehat{\lambda}_i$. We claim that $\widehat{\lambda}_i > \widehat{\lambda}_{i+1}$. Otherwise, take the partition $\nu \vdash n$ such that $\widehat{\nu}_k = \widehat{\lambda}_k$ for $k \notin \{i, i+1\}$, $\widehat{\nu}_i = \widehat{\lambda}_i + 1$, $\widehat{\nu}_{i+1} = \widehat{\lambda}_{i+1} - 1$. Clearly $\mu \leq \nu < \lambda$, hence $\nu = \mu$, but that contradicts $\widehat{\mu}_i = \widehat{\nu}_i \leq \widehat{\lambda}_i$. In particular, we have $j = i$.

If $i = 2$ or $\widehat{\lambda}_{i-2} > \widehat{\lambda}_{i-1}$, we take a partition $\nu \vdash n$ such that $\widehat{\nu}_k = \widehat{\lambda}_k$ for $k \notin \{i-1, i\}$, $\widehat{\nu}_{i-1} = \widehat{\lambda}_{i-1} + 2$, $\widehat{\nu}_i = \widehat{\lambda}_i - 2$. Clearly $\mu \leq \nu < \lambda$. Therefore $\mu = \nu$ and the degeneration $\mu < \lambda$ has type f or h (depending on the parity of $\widehat{\lambda}_{i-1}$).

Finally, suppose that $i - 1 > 1$ and $\widehat{\lambda}_{i-2} = \widehat{\lambda}_{i-1}$. We look for a contradiction. If $\widehat{\lambda}_{i-2} > \widehat{\mu}_{i-2}$ take $\nu \vdash n$ such that $\widehat{\nu}_k = \widehat{\lambda}_k$ for $k \notin \{i-2, i-1\}$, $\widehat{\nu}_{i-2} = \widehat{\lambda}_{i-2} - 1$, $\widehat{\nu}_{i-1} = \widehat{\lambda}_{i-1} + 1$. Then clearly $\mu \leq \nu < \lambda$, so $\nu = \mu$ hence $j = i - 1 \notin \{i, i+1\}$. Otherwise, take $\nu \vdash n$ such that $\widehat{\nu}_k = \widehat{\mu}_k$ for $k \notin \{i-2, i-1, i\}$, $\widehat{\nu}_{i-2} = \widehat{\mu}_{i-2} - 1$, $\widehat{\nu}_{i-1} = \widehat{\mu}_{i-1} - 1$, $\widehat{\nu}_i = \widehat{\mu}_i + 2$. Then $\mu < \nu \leq \lambda$, so $\nu = \lambda$. However, we get $\widehat{\lambda}_{i-2} > \widehat{\mu}_{i-2}$, which we supposed was not the case. □

3.2 The closure order

In this section we prove the main result of the chapter: the order between nilpotent conjugacy classes induced by the inclusion of the closure is equivalent to the dominance order of partitions, via the usual correspondence between nilpotent classes and partitions.

Deciding which nilpotent orbits is contained in the closure of another nilpotent orbit is a classical problem studied by several authors (see [Ger61b, Ger61a, Hes76, DF74] for example) and it has been extended to some other cases as well ([Djo81, Djo82, Oht91b, BS98, CBS17] are only some of many interesting examples). As usual one implication is pretty straightforward: we prove it in proposition 3.2.1. To prove the converse implication, on the other hand, one usually explicitly provides a closed curve touching two conjugacy classes C_λ, C_μ for each $\lambda > \mu$, see proposition 3.2.3 and proposition 3.2.4.

Proposition 3.2.1. *Let $\mathfrak{V} \subseteq \mathfrak{gl}_n$ be a Lie algebra or a symmetric space of a symmetric pair and let G be the group acting on $\mathfrak{V}$. Let λ (resp. μ) be a partition such that there is $X \in \mathfrak{V}$ (resp. $Y \in \mathfrak{V}$) such that the partition of X is λ (resp. of Y is μ). Then*

$$\overline{G.X} \supseteq \overline{G.Y} \Rightarrow \lambda \geq \mu.$$

Proof. It is well-known that the determinantal varieties

$$d_r := \{Z \in \mathfrak{gl}_n : \operatorname{rk} Z \leq r\}$$

are closed subvarieties of $\mathfrak{gl}_n$. For every integer $m \geq 0$, the m-power $-^m : \mathfrak{gl}_n \to \mathfrak{gl}_n$ given by $Z \mapsto Z^m$ is a morphism of affine varieties. Therefore, for every $r \geq 0$, $m \geq 0$, the spaces

$$s_{r,m} := \{Z \in \mathfrak{gl}_n : \operatorname{rk} Z^m \leq r\}$$

are closed in $\mathfrak{gl}_n$. As the rank of a matrix is invariant by conjugation, one gets that $X \in s_{r,m}$ if and only if $G.X \subseteq s_{r,m}$ if and only if $\overline{G.X} \subseteq s_{r,m}$. Hence, by hypothesis, $Y \in s_{r,m}$, that is

$$\overline{G.X} \supseteq \overline{G.Y} \Rightarrow \operatorname{rk} X^m \geq \operatorname{rk} Y^m \quad \forall m \geq 0.$$

By (2.2.1), $\operatorname{rk} X^m = \sum_{i>m} \widehat{\lambda}_i$ and $\operatorname{rk} Y^m = \sum_{i>m} \widehat{\mu}_i$. The conclusion follows by the characterization of the dominance order given by (3.1.1). □

The strategy to prove the converse is simple, yet quite tedious. For every pair of partitions $\lambda, \mu \in \mathcal{Q}$ such that $\lambda > \mu$, one may show that there is an affine line $P : \mathbb{A}^1 \to \mathfrak{V}$ such that only $P(0) \in C_\mu$, while every other point $t \neq 0$ is mapped to $P(t) \in C_\lambda$. We call such a map P a *path connecting* λ *and* μ.

We remark that Bongartz, in [Bon96], uses an interesting method consisting in the following idea. If a module B is an extension of the modules A and C, that is

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact, then one may prove that B degenerates to the direct sum $A \oplus C$, by a general construction of a path connecting the orbit of B to the orbit of $A \oplus C$. Therefore, one may prove the existence of such an exact sequence for each pair $\lambda > \mu$. See also [Zwa00]. However, one may always translate that method to our paths $P(t)$ and conversely a path to $P(t)$ basically because our paths are affine lines. We prefer the more direct way of the paths $P(t)$ in the remaining part of the current section.

Before starting the construction of $P(t)$ for each pair $\lambda > \mu$, one might consider some reductions. First, one may prove the existence of lines connecting only minimal degeneration $\lambda \gtrdot \mu$ in $\mathcal{Q}$. A second reduction is justified by the following.

Lemma 3.2.2. *Let $\lambda, \mu \in \mathcal{Q}(n)$ and let P be a path connecting $\lambda > \mu$. Let $\nu \in \mathcal{Q}(m)$. Then $\lambda \oplus \nu > \mu \oplus \nu$ is a degeneration of $Q(n+m)$ and one may construct a path connecting $\lambda \oplus \nu$ and $\mu \oplus \nu$.*

Proof. Let $\mathfrak{V}(n)$ be the space containing the classes C_λ and C_μ. Let $\mathfrak{V}(m)$ be the space containing the class C_ν. As $k^{n+m} \simeq k^n \oplus k^m$, there are canonical immersions $\mathfrak{gl}_n, \mathfrak{gl}_m \hookrightarrow \mathfrak{gl}_n \oplus \mathfrak{gl}_m \hookrightarrow \mathfrak{gl}_{n+m}$ and there are canonical immersions $\mathfrak{V}(n), \mathfrak{V}(m) \hookrightarrow \mathfrak{V}(n) \oplus \mathfrak{V}(m) \hookrightarrow \mathfrak{V}(n+m)$. Let $Z \in C_\nu$ and define the map $i_Z : \mathfrak{V}(n) \to \mathfrak{V}(n) \oplus \mathfrak{V}(m)$ by $i_Z(X) := (X, Z)$. We claim that $i_Z \circ P$ is a path in $\mathfrak{V}(n+m)$ connecting $\lambda \oplus \nu$ and $\mu \oplus \nu$.

We check that $i_Z(P(t)) \in C_{\lambda \oplus \nu}$ if $t \neq 0$ and $i_Z(P(0)) \in C_{\mu \oplus \nu}$. In fact a more general statement is true: for any nilpotent $(X, Z) \in \mathfrak{V}(n+m)$ with $X \in C_{\lambda'}$, $Z \in C_{\nu'}$, where $\lambda' \vdash n$ and $\nu' \vdash m$, one has $(X, Z) \in C_{\lambda' \oplus \nu'}$. That is an immediate consequence of comparing the ranks

$$\operatorname{rk}(X, Z)^k = \operatorname{rk}(X^k, Z^k) = \operatorname{rk} X^k + \operatorname{rk} Z^k,$$

with the columns lengths

$$\left(\widehat{\lambda' \oplus \nu'}\right)_k = \widehat{\lambda'}_k + \widehat{\nu'}_k.$$

□

We take on the case $\mathfrak{gl}_n$.

Proposition 3.2.3. *Let n be a positive integer and let $\lambda, \mu \in \mathcal{P}(n)$. Let $C_\lambda, C_\mu \subseteq \mathfrak{gl}_n$ be the respective nilpotent GL_n-conjugacy classes. Then*

$$\lambda \geq \mu \quad \Rightarrow \quad \overline{C_\lambda} \supseteq \overline{C_\mu}.$$

Proof. We may assume that $\lambda \gtrdot \mu$ is a minimal degeneration. By lemma 3.1.7, there are ato most two rows i_1, i_2 such that for all rows $i \neq i_1, i_2$, one has $\lambda_i = \mu_i$. Therefore, one may construct a partition μ with all these rows $i \neq i_1, i_2$ and obtain partitions λ', μ' such that $\lambda = \lambda' \oplus \nu$ and $\mu = \mu' \oplus \nu$. Therefore, we may assume that λ and μ have at most two rows.

We are constructing an explicit path $P : \mathbb{A}^1 \to \mathfrak{gl}_n$ connecting $\lambda = (\lambda_1, \lambda_2)$, $\mu = (\mu_1, \mu_2)$. For each entry $[P]_i^j$ we choose a polynomial in t of degree ≤ 1 in the following way: $[P]_i^{i+1} = 1$ for each $i = 1, \dots, n-2$, with $i \neq \lambda_2$; $[P]_{n-1}^n = t$; $[P]_{\lambda_2}^n = 1$ (only if $\lambda_2 \geq 1$); zero everywhere else. We draw a picture of the action of $P(t)$ on the canonical basis $e_1, \dots, e_n$.

$$\begin{array}{ccccccccc}
1 & \longleftarrow & 2 & \longleftarrow & \dots & \longleftarrow & \lambda_2 - 1 & \longleftarrow & \lambda_2 \\
 & & & & & & & & \big\uparrow \\
\lambda_2 + 1 & \longleftarrow & \lambda_2 + 2 & \longleftarrow & \dots & \longleftarrow & \lambda_2 + \lambda_1 - 1 & \overset{t}{\longleftarrow} & \lambda_2 + \lambda_1
\end{array}$$

As $P(t)$ is strictly upper triangular, $P(t)$ is nilpotent. The first $n-2$ rows of $P(t)$ are linearly independent, as they contain a single 1 in different columns, so $\operatorname{rk} P(t) \geq n-2$. In that case $\dim\ker P(t) \leq 2$ so the partition $\lambda(t)$ of $P(t)$ has at most 2 rows and it may be recovered knowing only the nilpotency order of $P(t)$, that is, the minimum $m > 0$ such that $P(t)^m = 0$. In that case $\lambda(t)_1 = m$ and $\lambda(t)_2 = n - m$ necessarily.

If $t \neq 0$, $P(t)^{\lambda_1 - 1}(e_n) = te_{\lambda_2+1} \neq 0$, as $\lambda_2 \leq \lambda_1 - 2$ (so the upper path is shorter than the lower path), while $P(t)^{\lambda_1} = 0$. Therefore $\lambda(t) = (\lambda_1, \lambda_2) = \lambda$ (if $t \neq 0$).

If $t = 0$, the upper path is still not longer than the lower path, therefore the nilpotency order of $P(0)$ is $\lambda_1 - 1$, so $\lambda(0) = (\lambda_1 - 1, \lambda_2 + 1) = \mu$ as $\mu \lessdot \lambda$. $\square$

In order to prove the analogous proposition for the other cases, we definitely prefer to fix a bilinear form of k^n of type ε. For $\varepsilon = 1$ we choose $(x, y) \mapsto x^t F_1 y$ where F_1 is the anti-identity matrix $[F]_{n-i}^i = 1$ and zero everywhere else. If $\varepsilon = -1$ we choose $(x, y) \mapsto x^t F_{-1} y$ where $F_{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) F_1$.

Proposition 3.2.4. *Let n be a positive integer, $\varepsilon \in \{\pm 1\}$, $\eta \in \{0, 1\}$ and let $\lambda, \mu \in \mathcal{P}_{\varepsilon,\eta}(n)$. Let $C_\lambda, C_\mu \subseteq \mathfrak{g}_{\varepsilon,\eta}$ be the respective nilpotent K_ε-conjugacy classes. Then*

$$\lambda \geq \mu \quad \Rightarrow \quad \overline{C_\lambda} \supseteq \overline{C_\mu}.$$

We start with a useful construction.

Lemma 3.2.5. *Let $\lambda, \mu \in \mathcal{P}_{\varepsilon,\eta}(n)$ such that there are $\lambda', \mu' \in \mathcal{P}(n/2)$ such that $\lambda = \lambda' \oplus \lambda'$ and $\mu = \mu' \oplus \mu'$. Then*

$$\lambda \geq \mu \quad \Rightarrow \quad \overline{C_{\varepsilon,\eta,\lambda}} \supseteq \overline{C_{\varepsilon,\eta,\mu}}.$$

Proof. Let $m := n/2$ and let V be a vector space of dimension m and $V^\times$ its dual. One may equip $V \oplus V^\times$ with a non-degenerate bilinear form $\langle -, - \rangle$ of type ε by putting

$$\langle (u, \phi), (v, \psi) \rangle := \psi(u) + \varepsilon \phi(v). \tag{3.2.1}$$

Let $\tilde{\varepsilon} := -1$ if $\eta = 0$ and $\tilde{\varepsilon} := 1$ if $\eta = 1$. One may define a map $f : \mathfrak{gl}(V) \to \mathfrak{gl}(V) \oplus \mathfrak{gl}(V^\times) \subseteq \mathfrak{gl}(V \oplus V^\times)$ by

$$D \mapsto (D, \tilde{\varepsilon} D^\times)$$

where $D^\times$ is the linear dual map of D. One may see that the image of f is contained in $\mathfrak{g}_{\varepsilon,\eta}(V \oplus V^\times)$ since $(D, \tilde{\varepsilon} D^\times)^* = \tilde{\varepsilon}(D, \tilde{\varepsilon} D^\times)$ as

$$\begin{aligned} \langle (D, \tilde{\varepsilon} D^\times)(u, \phi), (v, \psi) \rangle &= \langle (Du, \tilde{\varepsilon} D^\times \phi), (v, \psi) \rangle \\ &= \psi(Du) + \varepsilon \tilde{\varepsilon} D^\times \phi(v) \\ &= D^\times \psi(u) + \varepsilon \tilde{\varepsilon} \phi(Dv) \\ &= \langle (u, \phi), (\tilde{\varepsilon} Dv, D^\times \psi) \rangle \\ &= \langle (u, \phi), (\tilde{\varepsilon} D, D^\times)(v, \psi) \rangle. \end{aligned} \tag{3.2.2}$$

Thus we have a linear map $\mathfrak{gl}(V) \to \mathfrak{g}_{\varepsilon,\eta}(V \oplus V^\times)$. One may also notice that $\mathrm{rk}(fD) = 2 \,\mathrm{rk}\, D$ for any linear map D; in particular, if $D \in C_{\nu'}$ for some partition $\nu' \vdash m$, then $fD \in C_{\nu' \oplus \nu'}$.

By (3.1.1) and by (2.1.2), $\lambda \geq \mu$ if and only if $\lambda' \geq \mu'$. Thus one may take a path $P : \mathbb{A}^1 \to \overline{C_{\lambda'}}$ such that $P(0) \in C_{\mu'}$ and $P(t) \in C_{\lambda'}$ for $t \neq 0$, as we did in the proof of proposition 3.2.3. Therefore, the composition $f \circ P : \mathbb{A}^1 \to \mathfrak{g}_{\varepsilon,\eta}(V \oplus V^\times)$ is a curve such that

$$\begin{aligned} &f \circ P(0) \in C_\mu \cap \mathfrak{g}_{\varepsilon,\eta}(V \oplus V^\times) = C_{\varepsilon,\eta,\mu}, \\ &f \circ P(t) \in C_\lambda \cap \mathfrak{g}_{\varepsilon,\eta}(V \oplus V^\times) = C_{\varepsilon,\eta,\lambda}, \quad \forall t \neq 0. \end{aligned}$$

Hence, we built a path connecting $C_{\varepsilon,\eta,\mu}$ and $C_{\varepsilon,\eta,\lambda}$. □

That last result proves the closure relation for many of the degenerations of table 3.1. Now we deal with the remaining cases.

Proof of proposition 3.2.4. It is enough to prove the thesis in the case $\mu \lessdot \lambda$ is a minimal degeneration. By lemma 3.1.9, the reduced degeneration $\mu' < \lambda'$ is one of the cases in table 3.1. In particular there are at most four different rows between μ and λ, that is, there are partitions $\nu, \tilde{\mu}, \tilde{\lambda}$ such that $\mu = \nu \oplus \tilde{\mu}$, $\lambda = \nu \oplus \tilde{\lambda}$, $\widehat{\tilde{\mu}}_1 \leq 4$, $\tilde{\mu}_i \neq \tilde{\lambda}_i$ for each $i = 1, \dots, 4$ such that $\tilde{\mu}_i > 0$. Depending on the type of $\mu' < \lambda'$, we prove that there is a path connecting $\tilde{\mu} < \tilde{\lambda}$; then one may use lemma 3.2.2.

In order to simplify the notation, we denote $\tilde{\mu}, \tilde{\lambda}, |\tilde{\lambda}|$ by μ, λ, n, respectively, for the rest of the proof.

Types e, f, h, y, y^*. In each of these cases, we have that each column of λ and of μ are even. Therefore, by halving the columns, we get partitions $\bar{\lambda}, \bar{\mu} \in \mathcal{P}(n/2)$ such that $\lambda = \bar{\lambda} \oplus \bar{\lambda}$ and $\mu = \bar{\mu} \oplus \bar{\mu}$. Therefore, the conclusion follows by lemma 3.2.5.

From now on, we fix the vector space $V = k^n$. We also fix the bilinear form $(-,-)_\varepsilon$ given by $(u, v) \mapsto u^t F_\varepsilon v$ where F_1 is the anti-diagonal symmetric matrix with $[F_1]_i^{n+1-i} = 1$ for each $i = 1, \dots, n$ and, if n is even, F_{-1} is the anti-diagonal skew matrix with $[F_{-1}]_i^{n+1-i} = 1$ for $i = 1, \dots, n/2$ and $[F_{-1}]_i^{n+1-i} = -1$ for $i = n/2+1, \dots, n$. One may notice that the adjoint D^* of a linear map $D \in \mathfrak{gl}_n$ with respect to the form $(-,-)_\varepsilon$ is given by

$$[D^*]_i^j := \tilde{\varepsilon}[D]_{n+1-j}^{n+1-i}, \tag{3.2.3}$$

where $\tilde{\varepsilon} = -1$ if $\varepsilon = 1$ and $\left(i - \frac{n+1}{2}\right)\left(j - \frac{n+1}{2}\right) < 0$ or $\tilde{\varepsilon} = 1$ in each other case. For each case we build a path $P(t)$ connecting C_λ and C_μ by specifying each entry $[P(t)]_i^j$. We also draw the action of $P(t)$ on the canonical basis $e_1, \dots, e_n$.

Even if each $P(t)$ constructed stabilizes the standard complete flag $0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \dots \subseteq V$, we also prove that there is a (possibly shorter) flag of subspaces $0 = V_0 \subseteq \dots \subseteq V_k = V$ strictly stabilized by $P(t)$ ($P(t)(V_{i+1}) \subseteq V_i$). That flag is useful in order to obtain not only that $P(t)$ is nilpotent but also to compute the nilpotency order of $P(t)$. We will define such a flag by declaring a complement of V_i inside V_{i+1} for each $i = 0, \dots, k-1$.

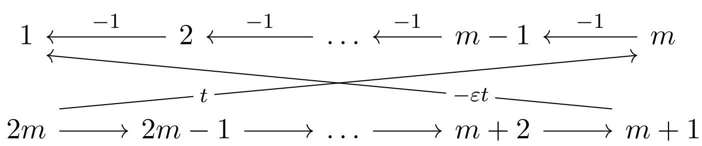
For each $t \in \mathbb{A}^1$, we denote by $\lambda(t)$ the partition corresponding to $P(t)$. Therefore one needs $\lambda(0) = \mu$ and $\lambda(t) = \lambda$ for $t \neq 0$. By (3.2.3), one checks immediately that $P(t)^* = -P(t)$ for types a, b, c, d, g (for both orthogonal and symplectic spaces); while for types x, x^* one checks that $P(t)^* = P(t)$.

In the following cases λ and μ has at most two rows. That condition may be checked for $\lambda(t)$ by showing that $\operatorname{rk} P(t) \geq n-2$. Thus, in order to recognize $\lambda(t)$ it is enough to compute the nilpotency order of $P(t)$.

Types a**,** g**.** $n = 2m$ is even, $\mu = (m,m)$, $\lambda = (m+1, m-1)$.

The case $m = 1$ is clear as $\mu = (1,1)$ corresponds to the conjugacy class of 0.

$[P(t)]_i^{i+1} = -1$ for $i = 1, \dots, m-1$; $[P(t)]_i^{i+1} = 1$ for $i = m+1, \dots, 2m$; $[P(t)]_1^{m+1} = -\varepsilon t$; $[P(t)]_m^{2m} = t$; 0 otherwise.



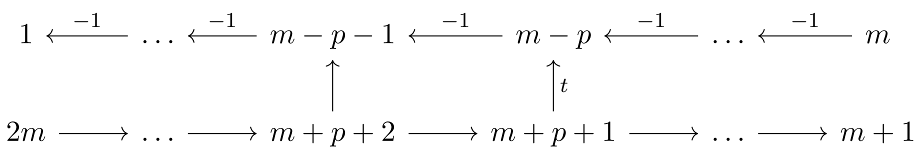
The images of e_i, for $i \notin \{1, m+1\}$, are independent.

The flag strictly stabilized is given by $\langle e_1 \rangle$, $\langle e_i, e_{m+i-1} \rangle$ for $i = 2, \dots, m$, $\langle e_{2m} \rangle$.

$P(t)^{m-1}(e_m) = (-1)^{m-1}e_1 \neq 0$ and $P(t)^m(e_{2m}) = ((-1)^{m-1}t - \varepsilon t)e_1 \neq 0$ if $t \neq 0$, m even and $\varepsilon = 1$ or m odd and $\varepsilon = -1$.

Type b**,** $\varepsilon = -1$**.** $n = 2m$ is even, $\mu = (2m-2p-2, 2p+2)$, $\lambda = (2m-2p, 2p)$, m even and $0 \leq p < m/2$.

$[P(t)]_i^{i+1} = -1$ for $i = 1, \dots, m-1$; $[P(t)]_i^{i+1} = 1$ for $i = m+1, \dots, 2m$; $[P(t)]_{m-p-1}^{m+p+2} = 1$; $[P(t)]_{m-p}^{m+p+1} = t$; 0 otherwise.



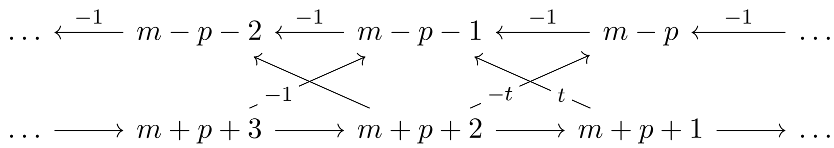
The images of e_i, for $i \notin \{1, m+1\}$, are independent.

The flag strictly stabilized is given by $\langle e_i \rangle$ for $i = 1, \dots, m-2p$, $\langle e_i, e_{i+2p} \rangle$ for $i = m-2p+1, \dots, m$, $\langle e_{i+2p} \rangle$ for $i = m+1, \dots, 2m-2p$.

$P(t)^{2m-2p-3}(e_{2m}) = (-1)^{m-p} e_1 + (-1)^{m-p} t e_3 \neq 0$ ($e_1 + e_3 \neq 0$ if $m = 2$), $P(t)^{2m-2p-1}(e_{2m}) = (-1)^{m-p} t e_1 \neq 0$ for $t \neq 0$.

Type b, $\varepsilon = 1$. $n = 2m$ is even, $\mu = (2m-2p-3, 2p+3)$, $\lambda = (2m-2p-1, 2p+1)$, $m \geq 3$ odd and $0 \leq p < m/2 - 1$.

$[P(t)]_i^{i+1} = -1$ for $i = 1, \dots, m-1$; $[P(t)]_i^{i+1} = 1$ for $i = m+1, \dots, 2m$; $[P(t)]_{m-p-1}^{m+p+3} = -1$; $[P(t)]_{m-p-2}^{m+p+2} = 1$; $[P(t)]_{m-p}^{m+p+2} = -t$; $[P(t)]_{m-p-1}^{m+p+1} = t$; 0 otherwise.



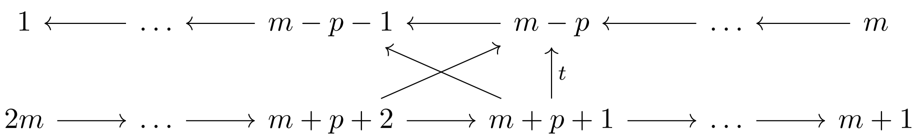
The images of e_i, for $i \notin \{1, m+1\}$, are independent.

The flag strictly stabilized is given by $\langle e_i \rangle$ for $i = 1, \dots, m-2p-1$, $\langle e_i, e_{i+2p+1} \rangle$ for $i = m-2p, \dots, m$, $\langle e_{i+2p+1} \rangle$ for $i = m+1, \dots, 2m-2p-1$.

$P(t)^{2m-2p-4}(e_{2m}) = (-1)^{m-p-1} 2e_1 + (-1)^{m-p} 2te_3 \neq 0$ ($e_1 + e_3 \neq 0$ if $m = 2$), $P(t)^{2m-2p-2}(e_{2m}) = (-1)^{m-p} 2te_1 \neq 0$ if $t \neq 0$.

Types x, x^*, λ_1 and λ_2 even. $n = 2m$, $\mu = (2m-2p-1, 2p+1)$, $\lambda = (2m-2p, 2p)$, $0 \leq p \leq (m-1)/2$.

$[P(t)]_i^{i+1} = 1$ for $i \neq m$; $[P(t)]_{m-p}^{m+p+2} = 1$; $[P(t)]_{m-p-1}^{m+p+1} = 1$; $[P(t)]_{m-p}^{m+p+1} = t$; 0 otherwise.



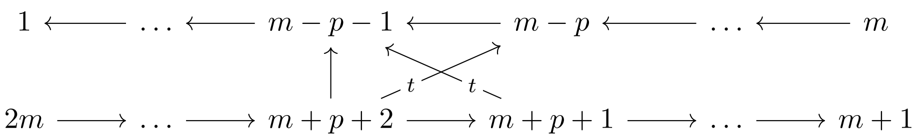
The images of e_i, for $i \notin \{1, m+1\}$, are independent.

The flag strictly stabilized is given by $\langle e_i \rangle$ for $i = 1, \dots, m-2p$, $\langle e_i, e_{i+2p} \rangle$ for $i = m-2p+1, \dots, m$, $\langle e_{i+2p} \rangle$ for $i = m+1, \dots, 2m-2p$.

$P(t)^{2m-2p-2}(e_{2m}) = 2e_1 + te_2 \neq 0$; $P(t)^{2m-2p-1}(e_{2m}) = te_1 \neq 0$ if $t \neq 0$.

Types x, x^*, λ_1 and λ_2 odd. $n = 2m$, $\mu = (2m-2p-2, 2p+2)$, $\lambda = (2m-2p-1, 2p+1)$, $0 \leq p \leq m/2 - 1$.

$[P(t)]_i^{i+1} = 1$ for $i \neq m$; $[P(t)]_{m-p-1}^{m+p+2} = 1$; $[P(t)]_{m-p}^{m+p+2} = t$; $[P(t)]_{m-p-1}^{m+p+1} = t$; 0 otherwise.



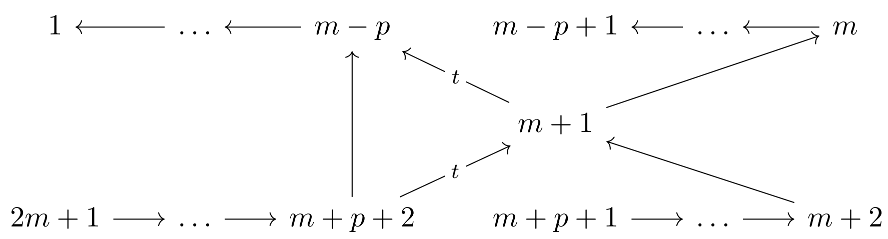
The images of e_i, for $i \notin \{1, m+1\}$, are independent.

The flag strictly stabilized is given by $\langle e_i \rangle$ for $i = 1, \ldots, m-2p-1$, $\langle e_i, e_{i+2p+1} \rangle$ for $i = m-2p, \ldots, m$, $\langle e_{i+2p+1} \rangle$ for $i = m+1, \ldots, 2m-2p-1$.

$P(t)^{2m-2p-3}(e_{2m}) = e_1 + 2te_2 \neq 0$; $P(t)^{2m-2p-2}(e_{2m}) = 2te_1 \neq 0$ if $t \neq 0$.

Types x, x^*, λ_1 **odd,** λ_2 **even.** $n = 2m+1$, $\mu = (2m-2p, 2p+1)$, $\lambda = (2m-2p+1, 2p)$, $0 \leq p < m/2$.

$[P(t)]_i^{i+1} = 1$ for $i \neq m-p, m+p+1$; $[P(t)]_{m-p}^{m+p+2} = 1$; $[P(t)]_{m-p}^{m+1} = t$; $[P(t)]_{m+1}^{m+p+2} = t$; 0 otherwise.



The images of e_i, for $i \notin \{1, m-p+1\}$, are independent.

The flag strictly stabilized is given by $\langle e_i \rangle$ for $i = 1, \ldots, m-2p$, $\langle e_i, e_{i+p} \rangle$ for $i = m-2p+1, \ldots, m-p$, $\langle e_m \rangle$, $\langle e_{i+p}, e_{i+2p} \rangle$ for $i = m-p+2, \ldots, m+2$, $\langle e_{i+2p} \rangle$ for $i = m+3 \ldots, 2m-2p+1$.

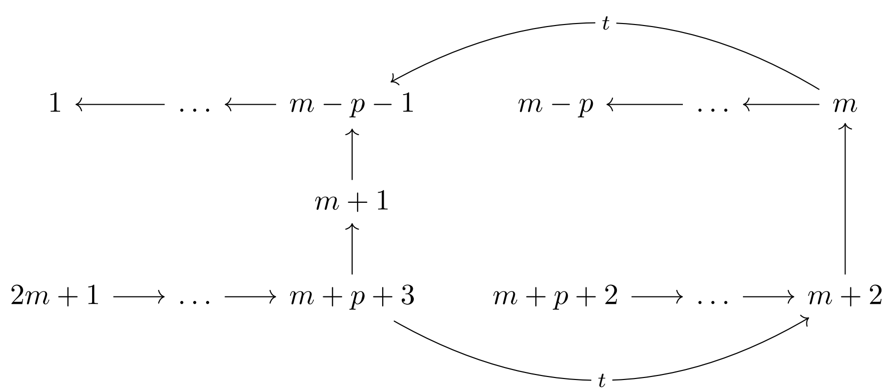
$P(t)^{2m-2p-1}(e_{2m+1}) = e_1 + t^2e_2 \neq 0$ ($e_1 + t^2e_2 + te_{m-p+1} \neq 0$ if $2p = m-1$); $P(t)^{2m-2p}(e_{2m+1}) = t^2e_1 \neq 0$ if $t \neq 0$.

Types x, x^*, λ_1 **even,** λ_2 **odd.** $n = 2m+1$, $\mu = (2m-2p-1, 2p+2)$, $\lambda = (2m-2p, 2p+1)$, $0 \leq p < (m-1)/2$.

$[P(t)]_i^{i+1} = 1$ for $i \neq m-p-1, m+p+2$; $[P(t)]_m^{m+2} = 1$; $[P(t)]_{m-p}^{m+1} = 1$; $[P(t)]_{m+1}^{m+p+2} = 1$; $[P(t)]_{m-p}^{m} = t$; $[P(t)]_{m+2}^{m+p+2} = t$; 0 otherwise.

The images of e_i, for $i \notin \{1, m-p\}$, are independent.

The flag strictly stabilized is given by $\langle e_i \rangle$ for $i = 1, \ldots, m-2p-1$, $\langle e_i, e_{i+p} \rangle$ for $i = m-2p, \ldots, m-p-1$, $\langle e_m, e_{m+1} \rangle$, $\langle e_{m+2} \rangle$, $\langle e_{i+p+1}, e_{i+2p+1} \rangle$ for $i = m-p+2, \ldots, m+1$, $\langle e_{i+2p+1} \rangle$ for $i = m+2 \ldots, 2m-2p$.

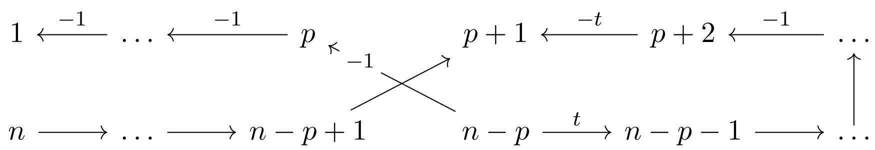


$P(t)^{2m-2p-2}(e_{2m+1}) = e_1 + t^2 e_2 \neq 0$ ($e_1 + t^2 e_2 + t e_{m-p} \neq 0$ if $2p = m-2$);
$P(t)^{2m-2p}(e_{2m+1}) = t^2 e_1 \neq 0$ if $t \neq 0$.

In the last couple of cases, λ and μ have at most three rows, therefore we show that $\operatorname{rk} P(t) \geq n-3$.

Type c. $\mu = (n-2p-2, p+1, p+1)$, $\lambda = (n-2p, p, p)$, $0 \leq p \leq n/3 - 1$.

$[P(t)]_i^{i+1} = -1$ for $i \leq (n-1)/2$, $i \neq p$; $[P(t)]_i^{i+1} = 1$ for $i > (n-1)/2$, $i \neq n-p$; $[P(t)]_p^{n-p} = -1$ (if $p > 0$); $[P(t)]_{p+1}^{n-p+1} = 1$ (if $p > 0$); $[P(t)]_{p+1}^{p+2} = -t$; $[P(t)]_{n-p-1}^{n-p} = t$; 0 otherwise.



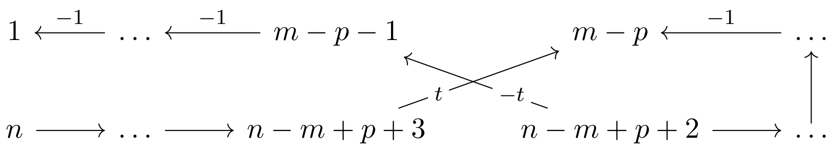
The images of e_i, for $i \notin \{1, p+1, p+2\}$, are independent.

The flag strictly stabilized is given as follows. For $i = 1, \ldots, n-2p$, we place $e_{i+p} \in V_i$; for $i = 2, \ldots, p$, we place $e_{i+n-p-1} \in V_i$; for $i = n-3p, \ldots, n-2p-1$, we place $e_{i+n+3p+1} \in V_i$.

If $t = 0$, clearly $\lambda(0) = \mu$, since $P(t)$ has three Jordan blocks of size $p+1, n-2p-2, p+1$ respectively. Moreover, $P(t)^{n-2p-1}(e_{n-p}) = \pm t^2 e_{p+1} \neq 0$ if $t \neq 0$ and $\ker P(t)^p$ clearly contains $e_1, \ldots, e_p$, $e_{p+1}, \ldots, e_{2p}$, $e_{n-p+1}, \ldots, e_{n-1}$ and $\pm t e_n - e_{2p}$, hence e_n if $t \neq 0$. Hence the sum of the first p columns of $\lambda(t)$ is at least $3p$ if $t \neq 0$.

Type d. $\mu = (m-p-1, m-p-1, 2p+n-2m+2)$, $\lambda = (m-p, m-p, 2p+n-2m)$, $m := \lfloor n/2 \rfloor$, $0 \leq p \leq m-1-n/3$.

$[P(t)]_i^{i+1} = -1$ for $i \leq (n-1)/2$, $i \neq m-p-1$; $[P(t)]_i^{i+1} = 1$ for $i > (n-1)/2$, $i \neq n-m+p+2$; $[P(t)]_{m-p-1}^{n-m+p+2} = -t$; $[P(t)]_{m-p}^{n-m+p+3} = t$; 0 otherwise.



The images of e_i, for $i \notin \{1, m-p, n-m+p+3\}$, are independent.

The flag strictly stabilized is given as follows. For $i = 1, \ldots, m-p-1$, we place $e_i \in V_i$; for $i = 2, \ldots, m-p$, we place $e_{i+n-m+p} \in V_i$; we place $e_{n-m+p+2} \in V_{m-p}$, $e_{m-p} \in V_1$; for $i = 2, \ldots, n-2m+2p+2$ ($< m-p$) we place $e_{i+m-p-1} \in V_i$.

$P(t)^{m-p-2}(e_{m-p-1}) = \pm e_1 \neq 0$, $P(t)^{m-p-2}(e_n) = e_{n-m+p+3} \neq 0$, hence $\operatorname{rk} P(t)^{m-p-2} \geq 2$ and the column $\widehat{\lambda(t)}_{m-p-1} \geq 2$. $P(t)^{m-p-1}(e_{n-m+p+2}) = \pm t e_1 \neq 0$, if $t \neq 0$, $P(t)^{m-p-1}(e_n) = t e_{m-p} \neq 0$, if $t \neq 0$, hence the column $\widehat{\lambda(t)}_{m-p} \geq 2$ if $t \neq 0$.

□

Remark. Proposition 3.2.4 and proposition 3.2.1 may be summed up in the following fashion, which resembles theorem 2.3.1. Let $X \in \mathfrak{g}_{\varepsilon,\eta}$ be a nilpotent element. Then

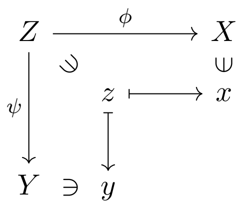
$$\overline{\mathrm{GL}(V).X} \cap \mathfrak{g}_{\varepsilon,\eta} = \overline{K_\theta.X}. \tag{3.2.4}$$

However an equation like this is not a general fact like theorem 1.3.8.

3.3 Smoothly equivalent singularities

In this section we present a tool to study the singularities of a closure of a conjugacy class. The main reference is [Slo80].

Definition 3.3.1. Let X, Y be varieties and let $x \in X$, $y \in Y$ be two points. We say that *the singularity of X at x is smoothly equivalent to the singularity of Y at y* if there is a variety Z and a point $z \in Z$ and two maps $\phi : Z \to X$, $\psi : Z \to Y$ such that $\phi(z) = x$ and $\phi(z) = y$ and ϕ and ψ are smooth in z.



Remark. It is clear that the relation between pointed varieties of definition 3.3.1 is an equivalence. We will denote the equivalence class of (X, x) by $\mathrm{Sing}(X, x)$.

Remark. Suppose that an algebraic group G acts on a variety X and let $O \subseteq X$ be a G-orbit. Then $\mathrm{Sing}(X, x) = \mathrm{Sing}(X, x')$ for all $x, x' \in O$. In this case we denote the equivalence class also by $\mathrm{Sing}(X, O)$.

The reason one introduces this equivalence relation for a pointed variety (X, x) is that it preserves several geometrical properties of a neighbour of the point $x \in X$. The main concern for us is the normality of X in x which is an immediate consequence of theorem 1.4.5. However, it may be proved that smoothness, seminormality, unibranchness, Cohen-Macaulay and rational singularity is preserved as well.

The main theorem involving the smoothly equivalence relation we will prove is the following.

Proposition 3.3.2. *Let $\lambda, \mu \in \mathcal{P}(n)$ be partitions such that $\lambda > \mu$ and let (λ', μ') be a pair of partitions obtained by erasing r rows and s columns from (λ, μ). Then*

$$\mathrm{Sing}(\overline{C_\lambda}, C_\mu) = \mathrm{Sing}(\overline{C_{\lambda'}}, C_{\mu'}).$$

There is also a corresponding version for conjugacy classes of quadratic spaces and symmetric spaces.

Proposition 3.3.3. *Let $\varepsilon \in \{\pm 1\}$ and $\eta \in \{0, 1\}$. Let $\lambda, \mu \in \mathcal{P}_{\varepsilon,\eta}(n)$ be partitions such that $\lambda > \mu$ and let (λ', μ') be a pair of partitions obtained by erasing r rows and s columns from (λ, μ). Then*

$$\mathrm{Sing}(\overline{C_{\varepsilon,\eta,\lambda}}, C_{\varepsilon,\eta,\mu}) = \mathrm{Sing}(\overline{C_{\varepsilon',\eta,\lambda'}}, C_{\varepsilon',\eta,\mu'}),$$

where $\varepsilon' = \varepsilon(-1)^{c(1+\eta)}$ as in proposition 3.1.8.

Since the study of the normality of a conjugacy class C_λ may be carried on by looking at each degeneration $C_\mu \subseteq \overline{C_\lambda}$, one might just focus on the smoothly equivalent class $\mathrm{Sing}(\overline{C_\lambda}, C_\mu)$ for each such μ. With the help of these propositions, one is able to reduce this study to just the cases of irreducible degenerations $\mu < \lambda$. This is particularly helpful if one needs information on just the minimal degeneration, as those belong to a small list of known cases. See the various tables at section 3.1.1.

The proofs of proposition 3.3.2 and proposition 3.3.3 are split into two parts. We will deal with the *erasing rows* reduction in this section. However, the proof of that case depends on the normality of the closure of conjugacy classes of $\mathfrak{gl}_n$, see corollary 7.1.5. Instead, the *erasing columns* reduction will be carried on in section 4.2 and in section 4.4.

3.3.1 Cross sections and erasing rows

Definition 3.3.4. Let G be an algebraic group, X a G-space and $x \in X$. A locally closed subvariety $S \subseteq X$ is a *cross section at x* if $x \in S$ and the restriction of the G-action $G \times S \to X$ is smooth at $(e, x) \in G \times S$.

Remark. If $S \subseteq X$ is a cross section at $x \in X$, then $\mathrm{Sing}(X, x) = \mathrm{Sing}(S, x)$, since one could take the variety $Z = G \times S$ with the point $z = (e, x)$ (and the obvious maps).

There is a natural way to construct cross sections for affine G-varieties $X \subseteq V$, where V is a G-module. Let $x \in X$ be a point, $T_x(G.x) \subseteq V$ the tangent at x to the orbit $G.x \subseteq X$ and choose one of its complementary subspace $N \subseteq V$. One may define the schematic intersection $S := (x + N) \cap X$. Let $a : G \times (x + N) \to V$ be the restriction of the action of G to $(x + N)$, so that there is a fibre product

$$\begin{array}{ccc} G \times (x+N) & \xrightarrow{a} & V \\ \uparrow & & \uparrow \\ G \times S & \longrightarrow & X. \end{array}$$

Lemma 3.3.5. *With the notations above, modulo intersecting with a neighbour of x, one has*

1. *S is reduced and a cross section at x;*
2. *x is an isolated point in $S \cap G.x$;*
3. *if X is equidimensional (e.g. irreducible),* $\dim_x S = \mathrm{codim}_X(G.x)$.

Proof. Since N is a complementary subspace to the tangent $T_x(G.x) \subseteq V$, the differential $da_{(e,x)} : \mathfrak{g} \times N \to V$ is surjective. Therefore a is smooth at (e, x). Hence $G \times S \to X$ is smooth at (e, x) by theorem 1.4.7, so $G \times S$ is reduced at (e, x) and S is a variety. Clearly, the intersection $(x + N) \cap X$ is closed in X, therefore S is a cross section at x and *1.* follows.

Since a is smooth at (e, x), it is flat on neighbour of (e, x) and one has

$$\dim \mathfrak{g} + \dim N - \dim V = \dim \mathfrak{g} + \dim_x S - \dim_x X.$$

Hence $\dim_x S = \dim_x X - \dim G.x$, or, if X is equidimensional ($\dim_x X$ does not depend on x), $\dim_x S = \operatorname{codim}_X(G.x)$, which is *3.*

If one takes the fibre product

$$\begin{array}{ccc} G \times (x+N) & \xrightarrow{a} & V \\ \uparrow & & \uparrow \\ G \times (S \cap G.x) & \longrightarrow & G.x, \end{array}$$

one still has that $G \times (S \cap G.x) \to G.x$ is smooth at (e, x), therefore $\dim_x(S \cap G.x) = 0$. Hence *2.* follows as well. □

Remark. One may define the concept of *transversal slice* as in [Slo80] as a cross section of minimal dimension. The existence of transversal slices is always guaranteed with a construction similar to the one we gave for $S = (x + N) \cap X$. Also, the dimension or a cross section at x must always be at least the codimension of $G.x$ in X, and a transversal slice always has this dimension in characteristic 0. In particular we notice that the cross section $S = (x + N) \cap X$ is a transversal slice.

Proposition 3.3.6. *Let G be an algebraic group, $G', H \subseteq G$ closed groups such that $H' := G' \cap H$ is reductive. Let V be a G-module, $V' \subseteq V$ a G'-module, $W \subseteq V$ an H-module so that $W' := V' \cap W$ is an H'-module. Let $x \in W'$ and let $y \in \overline{H'.x}$. Assume*

1. $\operatorname{codim}_{G'.x}(G'.y) = \operatorname{codim}_{G.x}(G.y)$
2. $\overline{G'.x} \cap W' = \overline{H'.x}$
3. $\overline{G.x}$ *is normal in y.*

Then $\operatorname{Sing}(\overline{H.x}, y) = \operatorname{Sing}(\overline{H'.x}, y)$.

Proof. Since H' is reductive, one may find an H'-stable decomposition $V = W' \oplus M' \oplus M_0 \oplus D$ such that $V' = W' \oplus M'$, $W = W' \oplus M_0$. Since the tangent space $T := T_y(G.y) \subseteq V$ is H'-invariant as $H' \subseteq G$, one may decompose

$$T = (T \cap W') \oplus (T \cap M') \oplus (T \cap M_0) \oplus (T \cap D)$$

and one obtains $T_y(G'.y) = (T \cap W') \oplus (T \cap M')$, $T_y(H.y) = (T \cap W') \oplus (T \cap M_0)$, $T_y(H'.y) = T \cap W'$. By the reductivity of H', one may find H'-stable decompositions

$$\begin{aligned} W' &= (T \cap W') \oplus N_1, \quad & M' &= (T \cap M') \oplus N_2, \\ M_0 &= (T \cap M_0) \oplus N_3, \quad & D &= (T \cap D) \oplus N_4. \end{aligned}$$

If one defines

$$N := N_1 \oplus N_2 \oplus N_3 \oplus N_4, \quad N' := N_1 \oplus N_2, \quad N_0 := N_1 \oplus N_3, \quad N_0' := N_1,$$

one gets that $N' = N \cap V'$, $N_0 = N \cap W$, $N_0' = N \cap W'$ and

$$\begin{aligned} V &= T_y(G.y) \oplus N, \quad & V' &= T_y(G'.y) \oplus N', \\ W &= T_y(H.y) \oplus N_0, \quad & W' &= T_y(H'.y) \oplus N_0'. \end{aligned}$$

Hence, one may construct cross sections of the closure of the orbits of x under the action of the groups G, G', H, H' using lemma 3.3.5 by putting

$$\begin{aligned} S &:= (y+N)\cap\overline{G.x}, & S' &:= (y+N')\cap\overline{G'.x},\\ S_0 &:= (y+N_0)\cap\overline{H.x}, & S'_0 &:= (y+N'_0)\cap\overline{H'.x}. \end{aligned}$$

We claim that S and S' coincide in a suitable neighbour of y. Indeed $S'\subseteq S$ is closed by construction and $\dim_y S' = \dim_y S$ by assumption *2.* and by lemma 3.3.5 (homogeneous spaces are equidimensional hence so are they closures). By assumption *3.* and by lemma 3.3.5, S is normal in y, hence the claim follows.

We also claim that S_0 and S'_0 coincide in a suitable neighbour of y. Indeed, $S\cap W$ and $S'\cap W$ coincide in a suitable neighbour of y by previous claim. Moreover $S\cap W\supseteq S_0\supseteq S'_0$ by construction and

$$\begin{aligned} S'\cap W &= ((y+N')\cap W)\cap\left(W\cap\overline{G'.x}\right)\\ &= (y+(N'\cap W))\cap W\cap V'\cap\overline{G'.x}\\ &= (y+N'_0)\cap W'\cap\overline{G'.x}\\ &= (y+N'_0)\cap\overline{H'.x} = S'_0; \end{aligned}$$

therefore also the second claim follows.

Finally, $\operatorname{Sing}(\overline{H.x},y)=\operatorname{Sing}(S_0,y)=\operatorname{Sing}(S'_0,y)=\operatorname{Sing}(\overline{H'.x},y)$. □

We now prove one case of proposition 3.3.2 and proposition 3.3.3, although, we have to defer an hypothesis about the normality of GL_n-conjugacy classes until corollary 7.1.5 (which does not use any result about smoothly equivalent singularities).

Proposition 3.3.7. *Let $\lambda\geq\mu$ be partitions and let (λ',μ') be a degeneration obtained by erasing rows from (λ,μ). Assume that $\overline{C_\lambda}$ is normal. Then* $\operatorname{Sing}(\overline{C_\lambda},C_\mu) = \operatorname{Sing}(\overline{C_{\lambda'}},C_{\mu'})$.

Proof. Since the hypothesis on (λ',μ'), one may form a partition ν by taking the first common rows of λ and μ in such a way that $\lambda=\nu\oplus\lambda'$ and $\mu=\nu\oplus\mu'$. Let $U_1, U_2, U := U_1\oplus U_2$ be vector spaces of dimension $|\nu|$, $|\lambda'|$ and $|\lambda|$ respectively. One may find nilpotent endomorphisms $D, E\in L(U)$, $F\in L(U_1)$ and $D', E'\in L(U_2)$ belonging to the nilpotent conjugacy classes C_λ, C_μ, C_ν and $C_{\lambda'}, C_{\mu'}$, respectively, such that $D|_{U_1}=E|_{U_1}=F$ and $D|_{U_2}=D'$ and $E|_{U_2}=E'$.

We apply proposition 3.3.6 to the following. $G=H$, $V=W$ (so, in particular, $G'=H'$ and $V'=W'$); $G=\mathrm{GL}(U)$, $G'=\mathrm{GL}(U_1)\times\mathrm{GL}(U_2)$ and $V=\mathfrak{gl}(U)$, $V'=\mathfrak{gl}(U_1)\oplus\mathfrak{gl}(U_2)$ with the adjoint action. Condition *1.* is proposition 3.1.6. Condition *2.* is empty as $G=H$. Condition *3.* is assumed in the hypothesis. Hence $\operatorname{Sing}(\overline{G.D},E)=\operatorname{Sing}(\overline{G'.D},E)$. Since $\overline{G'.D}=\overline{\mathrm{GL}(U_1).F}\times\overline{\mathrm{GL}(U_2).D'}$, the immersion $\mathrm{GL}(U_1).F\times\overline{\mathrm{GL}(U_2).D'}\hookrightarrow\overline{G'.D}$ is open (and therefore smooth) and E lies in this open subset, one gets $\operatorname{Sing}(\overline{G'.D},E)=\operatorname{Sing}(\overline{\mathrm{GL}(U_2).D'},E')$ as well and we are done. □

We have a pretty similar proof in the quadratic cases.

Proposition 3.3.8. *Let $\lambda\geq\mu$ be partitions in $\mathcal{P}_{\varepsilon,\eta}(n)$ for some $\varepsilon\in\{\pm1\}$, $\eta\in\{0,1\}$ and n. Let (λ',μ') be a degeneration obtained by erasing rows from (λ,μ). Assume that $\overline{C_\lambda}$ is normal. Then* $\operatorname{Sing}(\overline{C_{\varepsilon,\eta,\lambda}},C_{\varepsilon,\eta,\mu})=\operatorname{Sing}(\overline{C_{\varepsilon,\eta,\lambda'}},C_{\varepsilon,\eta,\mu'})$.

Proof. In a completely analogous manner as in the proof of proposition 3.3.7, one may find a partition ν such that $\lambda = \nu \oplus \lambda'$ and $\mu = \nu \oplus \mu'$; vector spaces $U_1, U_2, U := U_1 \oplus U_2$ of dimension $|\nu|$, $|\lambda'|$ and $|\lambda|$ respectively; nilpotent endomorphisms $D, E \in \mathfrak{g}_{\varepsilon,\eta}(U)$, $F \in \mathfrak{g}_{\varepsilon,\eta}(U_1)$ and $D', E' \in \mathfrak{g}_{\varepsilon,\eta}(U_2)$ belonging to the nilpotent conjugacy classes $C_{\varepsilon,\eta,\lambda}, C_{\varepsilon,\eta,\mu}$, $C_{\varepsilon,\eta,\nu}$ and $C_{\varepsilon,\eta,\lambda'}, C_{\varepsilon,\eta,\mu'}$, respectively, such that $D|_{U_1} = E|_{U_1} = F$ and $D|_{U_2} = D'$ and $E|_{U_2} = E'$.

We apply proposition 3.3.6 to the following. $G = \mathrm{GL}(U)$, $G' = \mathrm{GL}(U_1) \times \mathrm{GL}(U_2)$, $H = K_\varepsilon(U)$, $H' = K_\varepsilon(U_1) \times K_\varepsilon(U_2)$ and $V = \mathfrak{gl}(U)$, $V' = \mathfrak{gl}(U_1) \oplus \mathfrak{gl}(U_2)$, $W = \mathfrak{g}_{\varepsilon,\eta}(U)$, $W' = \mathfrak{g}_{\varepsilon,\eta}(U_1) \oplus \mathfrak{g}_{\varepsilon,\eta}(U_2)$. Condition *1.* is proposition 3.1.8. Condition *2.* is (3.2.4). Condition *3.* is assumed in the hypothesis. Hence the conclusion coincides with that of proposition 3.3.7. □

3.3.2 Subregular singularity in the nilpotent cone

In this section we explicitly compute a cross section of the singular locus of the nilpotent cone in order to get the class of the singularity of the subregular nilpotent class inside the closure of the regular nilpotent class. Although one may develop the very same idea to identify the class of such a singularity for every case we have been dealing with, we limit ourselves to the symmetric orthogonal case, that is $\varepsilon = 1$, $\eta = 1$, since we will use this case in chapter 7 to completely recognize the classes with normal closure. The method that we are following here is taken from [Sek84], which elaborates the methods of [Bri70, Slo80].

Theorem 3.3.9. *Let $n \geq 2$ be an integer, $\mathcal{C}_n = \{x^n - y^2 = 0\} \subseteq k^2$ be an affine plane curve, $\lambda = (n)$, $\mu = (n-1, 1)$ be the partitions corresponding to the regular and subregular conjugacy class in $\mathfrak{g}_{1,1}$. Then,*

$$\mathrm{Sing}(\overline{C_{1,1,\lambda}}, C_{1,1,\mu}) = \mathrm{Sing}(\mathcal{C}, 0).$$

Proof. Let $V = k^n$ be a vector space equipped with the symmetric bilinear form $(-,-)$ which pairs e_i to e_{n-i} and let

$$\mathfrak{p} := \{X \in \mathfrak{gl}(V) : \mathrm{Tr}(X) = 0, X^* = X\} = \mathfrak{sl}(V) \cap \mathfrak{g}_{1,1}(V).$$

Let $\mathcal{N} \subseteq \mathfrak{p}$ be the cone of the nilpotent elements in $\mathfrak{p}$.

We construct a cross section S of C_μ through a particular point X_0 inside the affine space $\mathfrak{p}$ using a complement N of the tangent space $T_{X_0}(C_\mu)$ in $\mathfrak{p}$, as seen in the previous section. Therefore, the intersection

$$S := \overline{C_\lambda} \cap (X_0 + N)$$

yields a cross section of C_μ. Then we will further study the affine subspace $X_0 + N$ and its subvariety S in order to recognize the curve $\mathcal{C}$.

First we distinguish $n = 2$. In that case $\dim \mathfrak{p} = 2$, $\dim C_{(n)} = 1$ and

$$\dim C_{(n-1,1)} = \dim C_{(1,1)} = 0.$$

Therefore $N := \mathfrak{p}$ is already a complement to the tangent $T_0(C_{1,1}) = 0$, therefore $S = \overline{C_{(2)}}$ is already a cross section through $X_0 = 0$. Moreover, a point $\begin{pmatrix} 0 & x_1 \\ x_2 & 0 \end{pmatrix} \in \mathfrak{p}$ is nilpotent if and only if $x_1 x_2 = 0$, therefore, by a linear change of the coordinates,

$$S = \overline{C_{(2)}} = \{(x_1, x_2) : x_1 x_2 = 0\} \simeq \{(x, y) : x^2 - y^2 = 0\} = \mathcal{C}_2.$$

Assume now that $n > 2$. If C is the conjugacy class of any element $X \in \mathfrak{p}$, then tangent space $T_X C = [X, \mathfrak{gl}(V)] = [X, \mathfrak{g}_{1,0}(V) \cap \mathfrak{sl}(V)]$. We distinguish two cases depending on the parity of n.

If $n = 2m+1$ is odd, let X_0 be the matrix such that $[X_0]_i^{i+1} = 1$ for $1 \le i \le 2m$ and $i \neq m, m+1$, $[X_0]_m^{m+2} = 1$ and 0 otherwise. Clearly X_0 is nilpotent and its partition is $(2m, 1) = \mu$.

Let $N \subseteq \mathfrak{gl}_n$ be the subspace of the matrices Y such that, for some value of the variables $t, x_1, x_2, \dots, x_{n-1}$, $[Y]_i^i = t$ for $i \neq m+1$, $[Y]_{m+1}^{m+1} = -2mt$, $[Y]_{i+1}^1 = [Y]_{2m+1}^{2m+1-i} = x_i$ and 0 otherwise. Clearly $\operatorname{Tr} Y = 0$ and $Y^* = Y$, so $N \subseteq \mathfrak{p}$ and $\dim N = n$. A computation yields $[X_0, \mathfrak{g}_{1,0}(V) \cap \mathfrak{sl}(V)] \cap N = 0$, hence N is a complement of $T_{X_0} C_\mu$ in $\mathfrak{p}$.

If $n = 2m$ is even, let X_0 be the matrix such that $[X_0]_i^{i+1} = 1$ for $1 \le i < 2m$ and $i \neq m$, $[X_0]_{m-1}^{m+1} = [X_0]_m^{m+2} = 1$ and 0 otherwise. Clearly X_0 is nilpotent and its partition is $(2m-1, 1) = \mu$.

Let $N \subseteq \mathfrak{gl}_n$ be the subspace of the matrices Y such that, for some value of the variables $t, x_1, x_2, \dots, x_{n-1}$, $[Y]_i^i = t$ for $i \neq m, m+1$, $[Y]_m^m = [Y]_{m+1}^{m+1} = (1-m)t$, $[Y]_{i+1}^1 = [Y]_{2m}^{2m-i} = x_i$ and 0 otherwise. Clearly $\operatorname{Tr} Y = 0$ and $Y^* = Y$, so $N \subseteq \mathfrak{p}$ and $\dim N = n$. A similar computation as before yields $[X_0, \mathfrak{g}_{1,0}(V) \cap \mathfrak{sl}(V)] \cap N = 0$, hence N is a complement of $T_{X_0} C_\mu$ in $\mathfrak{p}$ in this case as well. For each case, we have a cross section $S = \overline{C_{(n)}} \cap (X_0 + N)$.

For $X \in \mathfrak{p}$, we define the polynomial functions $P_2(X), \dots, P_n(X)$ by

$$\det(\lambda 1 + X) = \lambda^n + P_2(X)\lambda^{n-2} + \cdots + P_n(X),$$

so that $X \mapsto (P_2(X), \dots, P_n(X))$ is a morphism $\phi : \mathfrak{p} \to k^{n-1}$. We want to study the fibres $\phi^{-1}(\xi) \cap (X_0 + N)$ of the restriction $\phi|_{(X_0+N)}$ for each $\xi = (\xi_2, \dots, \xi_n) \in k^{n-1}$. Notice that $\phi^{-1}(0) = \overline{C_{(n)}}$ is the nilpotent cone, in particular $S = \phi^{-1}(0) \cap (X_0 + N)$.

We claim that, for each $\xi \in k^{n-1}$, there is a morphism $\psi_\xi : \mathbb{A}^2 \to k^{n-2}$ such that $\phi^{-1}(\xi) \cap (X_0 + N)$ is a closed subvariety of the graph $\Gamma(\psi_\xi) \subseteq \mathbb{A}^2 \times k^{n-2} \simeq (X_0 + N)$. We then prove that the image of $\phi^{-1}(\xi) \cap (X_0 + N)$ through the projection $X_0 + N \to \mathbb{A}^2$ is a curve $\mathcal{C}_{n,\xi}$ such that $\mathcal{C}_{n,0} = \mathcal{C}_n$ and the projection of X_0 is $(0,0) \in \mathbb{A}^2$. That would yield

$$\operatorname{Sing}(\overline{C_\lambda}, C_\mu) = \operatorname{Sing}(S, X_0) = \operatorname{Sing}(\mathcal{C}_n, (0,0)),$$

since the projection $\Gamma(\psi_\xi) \to \mathbb{A}^2$ is smooth.

If $X \in X_0 + N$, $\det(\lambda 1 + X)$ is a polynomial in $\lambda, t, x_1, \dots, x_{n-1}$. We claim that there is a grading of $\lambda, t, x_1, \dots, x_{n-1}$ such that $\det(\lambda 1 + X)$ is homogeneous of degree n. If $n = 2m+1$ is odd, we put $\deg \lambda = \deg t = 1$, $\deg x_i = i+1$ for $i = 1, \dots, m$, $\deg x_{m+1} = m + 1/2$, $\deg x_i = i$ for $i = m+2, \dots, 2m$. If $n = 2m$ is even, we put $\deg \lambda = \deg t = 1$, $\deg x_i = i+1$ for $i = 1, \dots, m-1$, $\deg x_i = i$ for $i = m, \dots, 2m-1$. One may prove the homogeneity of $\det(\lambda 1 + X)$ by performing the Laplace expansion on both the first column and the last column:

$$\det(\lambda 1 + X) = [\lambda 1 + X]_n^1 \det A_{\{n\},\{1\}} + \sum_{(i,j) \neq (n,1)} [\lambda 1 + X]_i^1 [\lambda 1 + X]_n^j \det A_{\{i,n\},\{1,j\}} \tag{3.3.1}$$

where $A_{I,J}$ is the cofactor of $\lambda 1 + X$ corresponding to the rows I and the columns J. The cofactors $A_{\{i,n\},\{1,j\}}$ are lower triangular matrices, while $A_{\{n\},\{1\}}$ is lower triangular in blocks with a single block of size greater than one in the middle (it has size 2 if n is odd or size 3 if n is even). In every case, those cofactors depend only on λ, t. We claim that each summand of (3.3.1) is homogeneous of degree n. Since $[\lambda 1 + X]_i^1 = x_{i-1} = [\lambda 1 + X]_n^{n-i+1}$ for $i > 1$ and $[\lambda 1 + X]_1^1 = [\lambda 1 + X]_n^n = \lambda + t$ the homogeneity of each summand of (3.3.1) follows by counting how many λ appears on the diagonal of the cofactor (and by the fact that determinant of $A_{\{n\},\{1\}}$ is $(n-1)t - \lambda$ if n odd and $(n-2)t - \lambda$ if n even, hence it has degree 1).

In fact, one may carry on the computation of (3.3.1) in order to show that the coefficient of $\lambda^{\deg x_i} x_i$ is ± 2, in particular not zero, if $i \neq m$ and $n = 2m+1$ is odd or if $i \neq m-1, m$ if $n = 2m$ is even. In the latter case, one may see that the coefficients of $\lambda^m x_{m-1}$ and $\lambda^m x_m$ are the same (by the symmetry of the columns and the rows $m, m+1$) and they are not zero. Therefore, the equations defining the fibre $\phi^{-1}(\xi) \cap (X_0 + N)$, $P_i(X) = \xi_i$ become $\pm 2x_{i+1} + R_{i+1}(X) = \xi_i$ for $i < n/2 - 1$ and $\pm 2x_i + R_i(X) = \xi_i$ for $i > n/2$, where $R_i(X)$ are homogeneous polynomials of degree $\deg x_i$ not depending on x_i. If $n = 2m$ is even, one also gets a similar equation $\pm 2(x_{m-1} + x_m) + R_{m-1}(X) = \xi_m$.

Hence, one may construct the morphism ψ_ξ as follows. If $n = 2m+1$, fixing the values of ξ and t (or that of $2mt$) gives unique values for x_i for each $i \neq m$ by solving those equations recursively. Hence there is a morphism $\psi_\xi : \mathbb{A}^2 \to k^{2m-1}$

$$(2mt, x_m) \mapsto (x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_{2m}).$$

Similarly, if $n = 2m$, fixing the values of ξ, t (or that of $(m-1)t$), and $x_{m-1} - x_m$ gives unique values for x_i for each $i \neq m$ and we still get a morphism $\psi_\xi : \mathbb{A}^2 \to k^{2m-2}$

$$((m-1)t, x_{m-1} - x_m) \mapsto (x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_{2m-1}).$$

Let

$$\mathcal{C}_{n,\xi} := \left\{ (x, y) \in \mathcal{A}^2 : x^n + \xi_2 x^{n-2} + \cdots + \xi_n = y^2 \right\}.$$

Clearly, $\mathcal{C}_{n,0} = \mathcal{C}_n$. It remains to prove that the projection $pr : \Gamma(\psi_\xi) \to \mathbb{A}^2$ sends $\phi^{-1}(\xi) \cap (X_0 + N)$ to $\mathcal{C}_{n,\xi}$. If $n = 2m+1$, the central entry of $(2mt)1 + X$ is 0 for each $X \in X_0 + N$, therefore the column (resp. row) $m+1$ has only one not null entry, the one in the last row (resp. first column). Hence, one may expand with Laplace along column $m+1$ and row $m+1$ to find

$$\det((2mt)1 + X) = (x_m)^2 \det A_{\{m+1,2m\},\{1,m+1\}} = (x_m)^2.$$

Therefore, if $X \in \phi^{-1}(\xi) \cap (X_0 + N)$,

$$(2mt)^n + \xi_2 (2mt)^{n-2} + \cdots + \xi_n = (x_m)^2,$$

that is $pr(X) = (2mt, x_m) \in \mathcal{C}_{n,\xi}$. If $n = 2m$, the columns m, $m+1$ of $(m-1)t1 + X$ are linearly dependent (actually equal) if $x_{m-1} = x_m$. Therefore, one may subtract the m-th column to the $m+1$-th column so that only the last entry of the $m+1$-th column is possibly not null. One may operate similarly on the rows m and $m+1$, then expanding along column $m+1$ and row $m+1$ to get

$$\det(((m-1)t)1 + X) = (x_m)^2 \det A_{\{m+1,2m+1\},\{1,m+1\}} = (x_{m-1} - x_m)^2.$$

Therefore, if $X \in \phi^{-1}(\xi) \cap (X_0 + N)$,

$$((m-1)t)^n + \xi_2((m-1)t)^{n-2} + \cdots + \xi_n = (x_{m-1} - x_m)^2,$$

that is $pr(X) = ((m-1)t, x_{m-1} - x_m) \in \mathcal{C}_{n,\xi}$.

□

Chapter 4

Pairs of linear maps

In this chapter we define nilpotent pairs of linear maps between two vector spaces. We develop a theory similar to the Jordan normal form, in particular we will classify the nilpotent pairs into finitely many conjugacy classes. At the end of the chapter, we will also compute a dimension formula for each of these conjugacy classes.

4.1 Nilpotent pairs

Let V and U be complex finite dimensional vector spaces. We denote $L(V,U) := \mathrm{Hom}_k(V,U)$ the vector space of linear homomorphisms. We also introduce the notation $L(V) := L(V,V) = \mathrm{End}(V,U)$. We are interested in the study of pairs of linear maps between the spaces V and U.

Definition 4.1.1. We define the vector space of pairs of maps as

$$L_{V,U} := L(V,U) \times L(U,V).$$

In the following paragraphs we are going to give some different interpretations to the space $L_{V,U}$ of pairs of maps.

First of all, one can consider the following quiver:

$$Q = \; 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \; . \tag{4.1.1}$$

If one fixes the dimension vector to be $\underline{d} = (\dim U, \dim V)$, one gets that the space of pairs of maps is the variety of representations of Q with dimension vector $\underline{d}$:

$$L_{V,U} = Rep_{\underline{d}}(Q).$$

As usual in the theory of quivers, we have the structural group $\mathrm{GL}(\underline{d}) := \mathrm{GL}(V) \times \mathrm{GL}(U)$. The group $\mathrm{GL}(\underline{d})$ naturally acts on the left on the affine space $L_{V,U}$ by change of basis. This action is given by

$$(g,h).(A,B) = (hAg^{-1}, gBh^{-1}),$$

where $(g,h) \in \mathrm{GL}(V) \times \mathrm{GL}(U) = \mathrm{GL}(\underline{d})$ and $(A,B) \in L_{V,U}$.

Another interpretation of the space $L_{V,U}$ is as a symmetric space as in definition 1.1.5. We use the language of the θ-groups, described in section 1.1.4.

Let $W := V \oplus U$ and consider the automorphisms θ of $L(W)$ and of $\mathrm{GL}(W)$ (both denoted by θ) given by the conjugation with

$$J = \begin{pmatrix} 1_V & 0 \\ 0 & -1_U \end{pmatrix}.$$

As $J^2 = 1_W$, we get that $\theta^2 = 1_{L(W)}$. Therefore we can decompose $L(W)$ into the θ-eigenspaces:

$$L(W) = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \tag{4.1.2}$$

where $\mathfrak{g}_0$ is the 1-eigenspace and $\mathfrak{g}_1$ is the -1-eigenspace, that is

$$\mathfrak{g}_0 = \{w \in W : \theta(w) = w\} \text{ and } \mathfrak{g}_1 = \{w \in W : \theta(w) = -w\}.$$

Moreover, we have the identifications

$$\mathfrak{g}_0 \simeq L(V) \times L(U) \text{ and } \mathfrak{g}_1 \simeq L_{V,U}.$$

Finally, θ is also a $\mathrm{GL}(W)$ automorphism and $\mathrm{GL}(V) \times \mathrm{GL}(U)$ is the subgroup of the θ-fixed points, which acts on both $\mathfrak{g}_0$ and $\mathfrak{g}_1$.

In the language symmetric pairs, $(L(W), L(V) \times L(U))$ is a symmetric pair and $\mathfrak{g}_1 \simeq L_{V,U}$ is its symmetric space. In the language of graded Lie algebras, $(L(W), \theta, 2)$ is a graded Lie algebra, $L := \mathrm{GL}(V) \times \mathrm{GL}(U)$ is the associated linear group and $V = \mathfrak{g}_1$ is the space in which L acts.

Later we will consider the action of groups different from $\mathrm{GL}(V) \times \mathrm{GL}(U)$. Whenever we choose subgroups $G(V) \subseteq \mathrm{GL}(V)$ and $G(U) \subseteq \mathrm{GL}(U)$, the restriction of the action of $\mathrm{GL}(V) \times \mathrm{GL}(U)$ on $L_{V,U}$ gives rise to an action of $G(U) \times G(V)$ on $L_{V,U}$. This study will be carried on starting from section 4.4.

The main definition of this chapter is the following.

Definition 4.1.2. A pair of maps $(A, B) \in L_{V,U}$ is said to be *nilpotent* if AB is a nilpotent endomorphism. We define

$$N_{V,U} := \{(A, B) \in L_{V,U} : AB \text{ is nilpotent}\}.$$

Remark. As AB is nilpotent if and only if BA is nilpotent, one can check the nilpotency of the latter product.

Remark. As the product $L_{V,U} \to L(U)$ given by $(A, B) \mapsto AB$ is a morphism of varieties and the nilpotent cone N_U is a closed subvariety of $L(U)$, one gets that $N_{V,U}$ is a closed subvariety of $L_{V,U}$.

The action of $\mathrm{GL}(V) \times \mathrm{GL}(U)$, or any of its subgroups $H \subseteq \mathrm{GL}(V) \times \mathrm{GL}(U)$, stabilizes $N_{V,U}$. Therefore we say that an H-orbit is nilpotent if it contains a nilpotent pair.

In the next sections we will give a classification of the finitely many nilpotent orbits of the $\mathrm{GL}(V) \times \mathrm{GL}(U)$-action in $N_{V,U}$ and we will study their dimension. We will also conduct a similar study for the orbits of subgroups $K_\theta(V) \times K_\theta(U)$ where $K_\theta(V)$ (and $K_\theta(U)$) is the orthogonal or the symplectic group.

4.2 The maps π and ρ and erasing columns

Definition 4.2.1. Let V and U be two vector spaces with $\dim U \leq \dim V$. We define the maps π, ρ

$$\begin{array}{ccc} L_{V,U} & \xrightarrow{\ \pi\ } & L(U) \\ \Big\downarrow{\scriptstyle \rho} & & \\ L(V) & & \end{array}$$

by $\pi(A,B) = AB$, $\rho(A,B) = BA$, for each $(A,B) \in L_{V,U}$.

Theorem 4.2.2 (First fundamental theorem of invariant theory)**.** *The maps π, ρ are quotient maps under* $\mathrm{GL}(V)$*,* $\mathrm{GL}(U)$ *respectively. The image of ρ is the determinantal variety of matrices of rank $\leq \dim U$.*

One may observe that, by definition of $N_{V,U}$, we have $\pi(N_{V,U}) \subseteq N_U$ and $\rho(N_{V,U}) \subseteq N_V$. In fact we have a more precise result as shown by the following lemma.

Lemma 4.2.3. *Let $\lambda \vdash \dim V$ be a partition such that $|\lambda'| = \dim U$. Let $C_\lambda \subseteq L(V)$, $C_{\lambda'} \subseteq L(U)$ be the corresponding nilpotent conjugacy classes. Let $N_\lambda := \pi^{-1}(\overline{C_{\lambda'}})$. Then $\rho(N_\lambda) = \overline{C_\lambda}$.*

$$\begin{array}{ccc} N_\lambda & \xrightarrow{\ \pi\ } & \overline{C_{\lambda'}} \\ \Big\downarrow{\scriptstyle \rho} & & \\ \overline{C_\lambda} & & \end{array}$$

Proof. First we prove that $\rho(N_\lambda) \subseteq \overline{C_\lambda}$. For an endomorphism D we have that $D \in \overline{C_\lambda}$ if and only if $\mathrm{rk}\, D^i \leq |\lambda^{(i)}|$ for all $i = 0, \ldots, t$.

Let $(A,B) \in N_\lambda$. By definition, we have $AB \in \overline{C_{\lambda'}}$. Moreover, as $(BA)^i = B(AB)^{i-1}A$, we have that

$$\mathrm{rk}(BA)^i = \mathrm{rk}\, B(AB)^{i-1}A \leq \mathrm{rk}(AB)^{i-1} \leq |\lambda'^{(i-1)}| = |\lambda^i|,$$

thus proving the first inclusion.

In order to prove $\rho(N_\lambda) \supseteq \overline{C_\lambda}$, we first prove that $C_\lambda \subseteq \rho(N_\lambda)$. Let $D \in C_\lambda$. As $\dim U = |\lambda^{(1)}| = \mathrm{rk}\, D^1$, we identify $U \simeq D(V)$. Therefore, the map D factor through its image

$$\begin{array}{ccc} V & \xrightarrow{\ D\ } & V \\ & {\scriptstyle A}\searrow \quad \nearrow{\scriptstyle B} & \\ & U & \end{array}$$

where B is the canonical inclusion and A coincides with D. The composition AB is the restriction $D|_{D(V)}$, therefore $AB \in C_{\lambda'}$ and $(A,B) \in N_\lambda$, while $BA = D \in C_\lambda$ implies $\rho(N_\lambda) \supseteq C_\lambda$.

Finally, ρ is a quotient map by theorem 4.2.2. By proposition 1.3.6 a closed invariant subspace is mapped to a closed space. Since N_λ is closed by construction, the image $\rho(N_\lambda) \subseteq L(V)$ is a closed space containing C_λ. $\square$

We investigate further the geometry of the maps π, ρ. Let

$$L'_{V,U} := \{(A,B) \in L_{V,U} : \operatorname{rk} A = \operatorname{rk} B = \dim U\}.$$

(We recall that we are assuming $\dim U \leq \dim V$.)

Lemma 4.2.4.

1. *$\rho(L'_{V,U}) = \{X \in L(V) : \operatorname{rk} X = \dim U\}$ and $\rho^{-1}(\rho(L'_{V,U})) = L'_{V,U}$;*
2. *for each $(A,B) \in L'_{V,U}$, the stabilizer of (A,B) in $\mathrm{GL}(U)$ is trivial;*
3. *for each $(A,B) \in L'_{V,U}$, $\rho^{-1}(\rho(A,B))$ is a single orbit under $\mathrm{GL}(U)$;*
4. *the restriction $\pi|_{L'_{V,U}} : L'_{V,U} \to \pi(L'_{V,U})$ is smooth;*
5. *the restriction $\rho|_{L'_{V,U}} : L'_{V,U} \to \rho(L'_{V,U})$ is smooth.*

Proof.

1. Since $(A,B) \in L'_{V,U}$ if and only if A is surjective and B injective, clearly $\operatorname{rk} BA = \dim U$. On the other hand, for $X \in L(V)$ such that $\operatorname{rk} X = \dim U$, as in the proof of lemma 4.2.3, one may identify $U = \operatorname{Im}(X)$ and take $A = X$, $B : \operatorname{Im}(X) \hookrightarrow V$ so that A, B have maximal rank and so $(A,B) \in L'_{V,U}$. Finally, $\dim U = \operatorname{rk} BA \leq \operatorname{rk} A$, implies that A is surjective and, similarly, that B is injective.

2. The action of $\mathrm{GL}(U)$ on $L_{V,U}$ is given by

   $$h.(A,B) = (hA, Bh^{-1}) \quad \forall h \in \mathrm{GL}(U), (A,B) \in L_{V,U}.$$

   If h stabilizes $(A,B) \in L'_{V,U}$, $hA = A \Rightarrow h = 1$ as A is surjective.

3. Let $(C,D) \in \rho^{-1}(\rho(A,B))$, that is $DC = BA$. By *1.*, C must be surjective and D must be injective and then $\ker C = \ker DC = \ker BA = \ker A$. Therefore there exists a $h \in \mathrm{GL}(U)$ such that $hC = A$ and we have $Dh^{-1} = B$, since $DC = BhA$.

4. The tangent map $d\pi_{(A,B)} : L_{V,U} \to L(U)$ is given by $(X,Y) \mapsto AY + XB$. If A is surjective, there is a right inverse $\tilde{A}$ ($A\tilde{A} = 1$), therefore for every $W \in L(U)$ we may take $(X,Y) = (0, \tilde{A}W)$, thus showing that $d\pi_{(A,B)}$ is surjective.

5. Let $G := \mathrm{GL}(V) \times \mathrm{GL}(U) \times \mathrm{GL}(V)$ acting on $L_{V,U}$ by

   $$(g', h, g).(A,B) := (hAg^{-1}, g'Bh^{-1}) \quad \forall (g', h, g) \in G, (A,B) \in L_{V,U}.$$

   Under this action $L'_{V,U}$ is a G-orbit and ρ is G-equivariant (with the obvious G-action on $L(V)$), in particular $\rho(L'_{V,U})$ is an orbit. Therefore there are closed groups $H \subseteq H' \subseteq G$ such that the restriction $\rho|_{L'_{V,U}}$ is the projection $G/H \to G/H'$. In particular, $\rho|_{L'_{V,U}}$ is smooth.

□

Lemma 4.2.5. *In the settings of lemma 4.2.3, we have*

1. $\rho^{-1}(C_\lambda)$ *is a single* $\mathrm{GL}(V) \times \mathrm{GL}(U)$*-orbit contained in* $N_\lambda \cap L'_{V,U}$*;*

2. $\pi(\rho^{-1}(C_\lambda)) = C_{\lambda'}$.

Proof. By lemma 4.2.4, *3.*, $\rho^{-1}(X)$ is a single $\mathrm{GL}(U)$-orbit for each X such that $\operatorname{rk} X = \dim U$; in particular for $X \in C_\lambda$. Therefore $\rho^{-1}(C_\lambda)$ is a single $\mathrm{GL}(V) \times \mathrm{GL}(U)$-orbit. By lemma 4.2.4, *1.*, $\rho^{-1}(C_\lambda) \subseteq L'_{V,U}$. For every $X \in C_\lambda$ and for every $(A, B) \in \rho^{-1}(X)$, one has $\pi(A, B) = AB = X|_{\operatorname{Im} X} \in C_{\lambda'}$. Therefore $\rho^{-1}(C_\lambda) \subseteq N_\lambda$ and $\pi(\rho^{-1}(C_\lambda)) = C_{\lambda'}$ follow. □

With these results, we are now able to prove the *erasing columns* part of proposition 3.3.2.

Proposition 4.2.6. *Let* $\lambda > \mu$ *a degeneration of partitions in* $\mathcal{P}(n)$ *such that* $\widehat{\lambda}_1 = \widehat{\mu}_1$ *and let* (λ', μ') *be the degeneration obtained from* (λ, μ) *by erasing the first column. Then* $\mathrm{Sing}(\overline{C_\lambda}, C_\mu) = \mathrm{Sing}(\overline{C_{\lambda'}}, C_{\mu'})$.

Proof. With the notation of lemma 4.2.3, we have the following diagram:

$$\begin{array}{ccc} N_\lambda & \xrightarrow{\pi} & \overline{C_{\lambda'}} \\ \Big\downarrow{\scriptstyle\rho} & & \\ \overline{C_\lambda} & & \end{array}$$

Let $Z = N_\lambda$ and $z \in \rho^{-1}(C_\mu) \subseteq N_\lambda$. We claim that

$$\rho(z) \in C_\mu, \pi(z) \in C_{\mu'}, \rho \text{ and } \pi \text{ are smooth in } z.$$

The first claim follows by construction. The second follows by lemma 4.2.5, *2.*, with $\lambda := \mu$ (notice that $|\mu^{(1)}| = |\lambda^{(1)}|$ since the first columns of λ and μ coincide).

The subvariety $N'_\lambda := N_\lambda \cap L'_{V,U}$ is locally closed in $L'_{V,U}$, hence both π and ρ are smooth in N'_λ by lemma 4.2.4, *4.* and *5.*. Moreover, $\rho^{-1}(C_\mu) \subseteq N_\mu \cap L'_{V,U} \subseteq N'_\lambda$ by lemma 4.2.5, *1.* and the last claim follows.

The conclusion follows as we found a pointed variety (Z, z) with the requirements met. □

4.3 *ab*-diagrams

In this section we want to develop a theory similar to the Jordan normal form of nilpotent matrices in the case of nilpotent pairs. We start with a suitable definition analogue to the Young diagrams.

Definition 4.3.1. An *ab*-string is a string with one or more letters which alternates between a and b. An *ab*-diagram is a sequence of *ab*-strings.

Example. The following is an *ab*-diagram with 5 as and 6 bs:

$$\delta = \begin{array}{l} ababa \\ baba \\ b \\ b \end{array} . \tag{4.3.1}$$

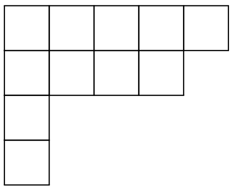
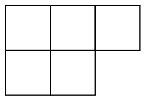
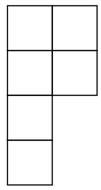
As we did for partitions, we define the sum $\delta \oplus \delta'$ of ab-diagrams δ, δ' by taking the disjoint union of the rows of δ and δ'.

From each ab-diagram δ we can build three Young diagrams.

Definition 4.3.2. Let δ be an ab-diagram. We define the Young diagrams

- $\bar{\delta}$ by replacing each letter a and b in δ with a box;
- $\pi(\delta)$ by deleting the letters b (and replacing each a with a box);
- $\rho(\delta)$ by deleting the letters a (and replacing each b with a box).

Example. If we take the ab-diagram δ from (4.3.1), we get the following Young diagrams:

$\bar{\delta} =$  ; $\pi(\delta) =$  ; $\rho(\delta) =$  .

For any ab-diagram δ, we can construct a nilpotent pair in a similar fashion with the Jordan normal form and the Young tableaux. Let V (resp. U) be a vector space with basis given by the letters a (resp. b) in δ. We give both maps (A, B) simultaneously by indicating where to map each element in the basis. The letters in the last positions in a row are mapped to 0. Each other letter is mapped to its right adjacent letter. We call this pair (A, B) the normal form with ab-diagram δ and we denote by O_δ its $\mathrm{GL}(V) \times \mathrm{GL}(U)$-orbit.

We observe that the orbit $\mathrm{GL}(V \oplus U).O_\delta \subseteq \mathrm{End}(U \oplus V)$ is a nilpotent conjugacy class of the space $U \oplus V$ with Young diagram equal to $\bar{\delta}$. Similarly, $\pi(O_\delta) \subseteq \mathrm{End}(U)$ (resp. $\rho(O_\delta) \subseteq \mathrm{End}(V)$) is a nilpotent conjugacy class of the space U (resp. V) with Young diagram equal to $\pi(\delta)$ (resp. $\rho(\delta)$).

We want to prove the following lemma.

Theorem 4.3.3. *Each nilpotent pair (A, B) in $L(V, U)$ is $\mathrm{GL}(V) \times \mathrm{GL}(U)$-conjugate to a normal form of some ab-diagram δ with $\dim V$ letters a and $\dim U$ letters b.*

A consequence of theorem 4.3.3 is that the nilpotent $\mathrm{GL}(V) \times \mathrm{GL}(U)$-orbits in $L(V, U)$ are in one to one correspondence to the ab-diagrams. Therefore, any orbit has the form O_δ for some ab-diagram δ.

Proof. of theorem 4.3.3. Let $\tilde{U} = V \oplus U$ and let $\tilde{A} = \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix}$, so that $\tilde{A} \in \mathrm{End}(\tilde{U})$. As (A, B) is a nilpotent pair, also $\tilde{A}$ is nilpotent, in particular, there exists a Young diagram η corresponding to $\tilde{A}$. We want to show that we can choose a Jordan basis $\mathcal{B}$ of $\tilde{U}$ such that each vector of $\mathcal{B}$ belongs to U or V. This would give us an ab-diagram δ such that $\bar{\delta} = \eta$.

Let $K_i := \ker \tilde{A}^i$ for each natural number i. As $\tilde{A}$ lies in the -1-eigenspace of the automorphism θ, and θ is the conjugation with J, we get that K_i is J-stable. This yields that $K_i = (K_i \cap V) \oplus (K_i \cap U)$. Moreover, for any i we can find a supplementary subspace of K_i inside K_{i+1} such that it is J-stable (it is enough to take supplementary subspaces in both U and V). This shows that we can fix a basis of each of these supplementary subspaces picking only elements of V or U.

As in the usual Jordan normal form, a choice of a basis of each supplementary of K_i inside K_{i+1} uniquely determines a Jordan basis $\mathcal{B}$ for the whole space $\tilde{U}$. We further observe that if $v \in V$ (resp. $u \in U$), then $\tilde{A}v \in U$ (resp. $\tilde{A}u \in V$). Therefore each element of $\mathcal{B}$ either belongs to V or to U. □

Lemma 4.3.4. *Let $(A, B) \in N_{V,U}$ be a nilpotent pair with ab-diagram δ. Then*

1. *A is injective if and only if each ab-string of δ ends with b;*
2. *A is surjective if and only if each ab-string of δ starts with a;*
3. *B is injective if and only if each ab-string of δ ends with a;*
4. *AB is surjective if and only if each ab-string of δ starts with b.*

Proof. Each conclusion follows immediately by using the basis of V and U corresponding to the diagram δ, as described in the proof of theorem 4.3.3. □

Let $(A, B) \in L_{V,U}$ be a nilpotent pair with ab-diagram δ. For every $h \geq 1$, we want to infer the rank of the linear map $(AB)^{h-1}A : V \to V$ (resp. $(BA)^{h-1}B : U \to U$) from δ. If $\gamma = c_1 \cdots c_\ell$ is a string, for any $1 \leq i \leq j \leq \ell$, we call $c_i \cdots c_j$ a *substring* of γ. A *substring* of δ is a substring of one of its ab-strings. For example, for $h = 1$, the rank of A equals the number of occurrences of the letter a which are not in the last position of a string; this is hence the number of occurrences of the sub-string $(ab) = (ab)^h$ in δ. With this in mind we prove the following lemma.

Lemma 4.3.5. *Let (A, B) be a nilpotent pair with ab-diagram δ. For every $h \geq 1$, the* $\mathrm{rk}((AB)^{h-1}A)$ *(resp.* $\mathrm{rk}((BA)^{h-1}B)$*) equals the number of occurrences of the substring $(ab)^h$ (resp. $(ba)^h$) in δ.*

Proof. The ab-diagram δ suggests the choice of basis of V and U for which the behaviour of A and B is straightforward. For each ab-string $\delta_i = c_{i1}c_{i2} \dots c_{i\ell_i}$ of δ (where $c_{ij} \in \{a, b\}$), for each $j = 1, \dots, \ell_i$, we pick vectors $v_{ij} \in V$ (resp. $u_{ij} \in U$) if $c_{ij} = a$ (resp. $c_{ij} = b$), with the following properties: $\{v_{ij}\}_{i,j}$ (resp. $\{u_{ij}\}_{i,j}$) is a basis of V (resp. U) and $Av_{ij} = u_{i,j+1}$ (resp. $Bu_{ij} = v_{i,j+1}$) if $j+1 \leq \ell_i$ or $Av_{i\ell_i} = 0$ (resp. $Bu_{i\ell_i} = 0$).

Notice that the map $(AB)^{h-1}A$ carries v_{ij} to $u_{i(j+2h-1)}$ if $j + 2h - 1 \leq \ell_i$ or to 0 otherwise. Therefore, $\mathrm{rk}((AB)^{h-1}A)$ equals the number of pairs (i, j) such that $c_{ij} = a$ and $j + 2h - 1 \leq \ell_i$. Moreover, every such pair (i, j) corresponds one to one to the substring $c_{ij} \dots c_{i(j+2h-1)}$ of δ, which is the substring $(ab)^h$. A similar argument computes $\mathrm{rk}((BA)^{h-1}B)$. □

Lemma 4.3.6. *Let (A, B) be a nilpotent pair with ab-diagram δ. For every $h \geq 1$, let $U_1 := \mathrm{Im}((BA)^{h-1}B)$, $U_2 := \ker((AB)^{h-1}A)$. Then $\dim U_1/(U_1 \cap U_2)$ is the number of occurrences of the substring $(ba)^{2h-1}b$ in δ.*

Proof. As in the proof of lemma 4.3.5, we choose the basis v_{ij} of V and u_{ij} of U. Then $U_2 = \ker((AB)^{h-1}A)$ is spanned by the basis elements v_{ij} such that $j + 2h - 1 > \ell_i$; U_2 is spanned by the basis elements v_{ij} such that $j - 2h + 1 > 0$; $U_1 \cap U_2$ is spanned by the basis elements v_{ij} such that $j + 2h - 1 > \ell_i$ and $j - 2h + 1 > 0$; hence the projection $U_1 \to U_1/U_1 \cap U_2$ maps the vectors v_{ij} such that $2h - 1 < j \leq \ell_i - 2h + 1$ to a basis of $U_1/U_1 \cap U_2$. Each such pair (i, j) may be identified as the middle letter of a substring $(ba)^{2h-1}b$ in δ. □

Remark. Let $G = \mathrm{GL}(V) \times \mathrm{GL}(U)$ and δ be an ab-diagram. Since O_δ is a G-orbit, there is a closed subgroup $H \subseteq G$ such that $O_\delta \simeq G/H$. Moreover, the restrictions

$$\begin{aligned} \pi|_{O_\delta} &: O_\delta \to C_{\pi(\delta)} \\ \rho|_{O_\delta} &: O_\delta \to C_{\rho(\delta)} \end{aligned} \tag{4.3.2}$$

are still G-equivariant, so $C_{\pi(\delta)}$ (resp. $C_{\rho(\delta)}$) is a single G-orbit and there is another closed subgroup $H \subseteq H' \subseteq G$ such that $C_{\pi(\delta)} \simeq G/H'$ (resp. $C_{\rho(\delta)} \simeq G/H'$). Therefore, the maps $\pi|_{O_\delta}$, $\rho|_{O_\delta}$ are smooth. (See also the proof of proposition 4.2.6, since $O_\delta \subseteq N_{\pi(\delta)}$.)

4.4 Quadratic spaces

In this section we introduce bilinear forms on the spaces V, U.

Definition 4.4.1. Let V (resp. U) be a vector space endowed with a non-degenerate ε-symmetric (resp. ε'-symmetric) bilinear form $(-,-)_U$ (resp. $(-,-)_V$). Let $X \in L(V, U)$. The adjoint map of X is defined as the unique linear map $X^* \in L(U, V)$ such that

$$(Xv, u)_U = (v, X^*u)_V \qquad \forall v \in V, u \in U.$$

Lemma 4.4.2. *In the settings of definition 4.4.1, we have*

$$(X^*)^* = \varepsilon\varepsilon' X.$$

Proof. For each $v \in V$, $u \in U$, we have

$$(Xv, u)_U = (v, X^*u)_V = \varepsilon(X^*u, v)_V = \varepsilon(u, (X^*)^*v)_U = \varepsilon\varepsilon'((X^*)^*v, u)_U,$$

therefore $X = \varepsilon\varepsilon'(X^*)^*$. □

Lemma 4.4.3. *Let $A \in L(V, U)$ be a linear map. Then*

$$\operatorname{rk} A = \operatorname{rk} A^*.$$

Proof. By definition 4.4.1, we get that $\ker A \perp \operatorname{Im} A^*$. Therefore $\operatorname{Im} A^* \subseteq \ker A^\perp$ so, as $(-,-)_V$ is non-degenerate, we have $\operatorname{rk} A^* \leq \operatorname{codim} \ker A = \operatorname{rk} A$. Using the symmetry of $(-,-)_V$ and $(-,-)_U$ one also obtains $\operatorname{rk} A \leq \operatorname{rk} A^*$. □

In this context, we are primarily interested in working with pairs (A, B) such that $B = A^*$. We introduce involutions on $L_{V,U}$, $L(V)$, $L(U)$, $\mathrm{GL}(V)\times\mathrm{GL}(U)$, each called σ by

$$\begin{aligned}
&\sigma(A,B) := (\varepsilon\varepsilon' B^*, A^*), \quad \forall (A,B) \in L_{V,U};\\
&\sigma(X) := \varepsilon\varepsilon' X^*, \quad \forall X \in L(V);\\
&\sigma(Y) := \varepsilon\varepsilon' Y^*, \quad \forall Y \in L(U);\\
&\sigma(g,h) := ((g^*)^{-1}, (h^*)^{-1}), \quad \forall (g,h) \in \mathrm{GL}(V)\times\mathrm{GL}(U).
\end{aligned} \tag{4.4.1}$$

Notice that σ on $L_{V,U}$ is an involution by lemma 4.4.2, since

$$\sigma(\sigma(A,B)) = \sigma(\varepsilon\varepsilon' B^*, A^*) = (\varepsilon\varepsilon'\varepsilon\varepsilon' A, \varepsilon\varepsilon'\varepsilon\varepsilon' B) = (A,B).$$

Giving an involution is equivalent to giving a $\mathbb{Z}/2$-action. Therefore, one might interpret $L_{V,U}$, $L(V)$, $L(U)$ and $\mathrm{GL}(V)\times\mathrm{GL}(U)$ as $\mathbb{Z}/2$-spaces and one may consider the fixed points sets

$$\begin{aligned}
&L^\sigma_{V,U} = \{(A,B) \in L_{V,U} : B = A^*\} \subseteq L_{V,U},\\
&L^\sigma(V) \subseteq L(V),\\
&L^\sigma(U) \subseteq L(U),\\
&(\mathrm{GL}(V)\times\mathrm{GL}(U))^\sigma = K_\varepsilon(V)\times K_{\varepsilon'}(U) \subseteq \mathrm{GL}(V)\times\mathrm{GL}(U).
\end{aligned}$$

Notice that, if $\varepsilon' = -\varepsilon$, $L^\sigma(V) = \mathfrak{g}_{\varepsilon,0}$, $L^\sigma(U) = \mathfrak{g}_{\varepsilon',0}$ are the skew linear morphisms, while, if $\varepsilon' = \varepsilon$, $L^\sigma(V) = \mathfrak{g}_{\varepsilon,1}$, $L^\sigma(U) = \mathfrak{g}_{\varepsilon',1}$ are the symmetric linear morphisms.

Lemma 4.4.4. *The maps π, ρ and the action of $\mathrm{GL}(V)\times\mathrm{GL}(U)$ on $L_{V,U}$ are $\mathbb{Z}/2$-equivariant.*

Proof.

$$\sigma\rho(A,B) = \varepsilon\varepsilon'(BA)^* = \varepsilon\varepsilon' A^*B^* = \rho(\varepsilon\varepsilon' B^*, A^*) = \rho(\sigma(A,B))$$

and

$$\begin{aligned}
\sigma\left((g,h).(A,B)\right) &= \sigma(hAg^{-1}, gBh^{-1})\\
&= \left(\varepsilon\varepsilon'(gBh^{-1})^*, (hAg^{-1})^*\right)\\
&= \left(\varepsilon\varepsilon'(h^{-1})^*B^*g^*, (g^{-1})^*A^*h^*\right)\\
&= \left((g^*)^{-1}, (h^*)^{-1}\right).(\varepsilon\varepsilon' B^*, A^*)\\
&= (\sigma(g,h)).\sigma(A,B).
\end{aligned}$$

□

Therefore, one immediately gets the following.

Corollary 4.4.5.

1. *$\rho(L^\sigma_{V,U}) \subseteq L^\sigma(V)$;*

2. $\pi(L^\sigma_{V,U}) \subseteq L^\sigma(U)$*;*
3. $K_\varepsilon(V) \times K_{\varepsilon'}(U)$ *stabilizes* $L^\sigma_{V,U}$*.*

We proceed to study the geometry of the restrictions of the maps π and ρ to $L^\sigma_{V,U}$. We will get results completely analogous to the results of section 4.2.

By corollary 4.4.5, the restrictions of ρ, π are maps $\rho|_{L^\sigma_{V,U}} L^\sigma_{V,U} \to L^\sigma(V)$ and $\pi|_{L^\sigma_{V,U}} L^\sigma_{V,U} \to L^\sigma(U)$. For the rest of the current section, we will denote these restrictions still by ρ and π (respectively). We fix $\eta = 0$ (resp. $\eta = 1$) if $\varepsilon' = \varepsilon$ (resp. $\varepsilon' = \varepsilon$).

Lemma 4.4.6. *Let* $\lambda \in \mathcal{P}_{\varepsilon,\eta}(\dim V)$ *be a partition such that* $\lambda' \in \mathcal{P}_{\varepsilon',\eta}(\dim U)$*. Let* $C_{\varepsilon,\eta,\lambda} \subseteq L^\sigma(V)$ *(resp.* $C_{\varepsilon',\eta,\lambda'} \subseteq L^\sigma(U)$*) be the corresponding* K_ε*(resp.* $K_{\varepsilon'}$*)-conjugacy class. Let* $N^\sigma_\lambda := \pi^{-1}(\overline{C_{\varepsilon',\eta,\lambda'}})$*. Then* $\rho(N^\sigma_\lambda) = \overline{C_{\varepsilon,\eta,\lambda}}$*.*

$$\begin{array}{ccc} N^\sigma_\lambda & \xrightarrow{\pi} & \overline{C_{\varepsilon',\eta,\lambda'}} \\ \downarrow{\scriptstyle\rho} & & \\ \overline{C_{\varepsilon,\eta,\lambda}} & & \end{array}$$

Proof. The spaces $C_{\varepsilon,\eta,\lambda}$, $\overline{C_{\varepsilon,\eta,\lambda}}$, $C_{\varepsilon',\eta,\lambda'}$, $\overline{C_{\varepsilon',\eta,\lambda'}}$ are the fixed points loci of C_λ, $\overline{C_\lambda}$, $C_{\lambda'}$, $\overline{C_{\lambda'}}$, respectively, under the involution σ of $L(V)$, $L(U)$ by theorem 2.3.1 and by (3.2.4). Therefore N^σ_λ is the fixed points locus of the space N_λ of lemma 4.2.3 under the involution σ of $L_{V,U}$, and one gets $\rho(N^\sigma_\lambda) \subseteq \overline{C_{\varepsilon,\eta,\lambda}}$.

In order to prove $\rho(N^\sigma_\lambda) \supseteq \overline{C_{\varepsilon,\eta,\lambda}}$, we first prove that $C_{\varepsilon,\eta,\lambda} \subseteq \rho(N^\sigma_\lambda)$. Let $D \in C_{\varepsilon,\eta,\lambda}$. As $\dim U = |\lambda^{(1)}| = \operatorname{rk} D^1$, we identify $U \simeq D(V)$. We define a bilinear form $\langle -,- \rangle$ on V by

$$\langle v, u \rangle := (v, Du) \quad \forall u, v \in V.$$

As $D^* = \tilde{\varepsilon} D$, the kernel of $\langle -,- \rangle_i$ is clearly $\ker D$. Moreover, $\langle -,- \rangle$ is ε'-symmetric. Hence, the form $\langle -,- \rangle$ induces a non-degenerate bilinear form of type ε' on the quotient space $V/\ker D \simeq D(V) \simeq U$ that, consequently, we may identify with $(-,-)_U$. The map D factor through its image

$$\begin{array}{ccc} V & \xrightarrow{D} & V \\ {\scriptstyle A}\searrow & & \nearrow{\scriptstyle B} \\ & U & \end{array}$$

where B is the canonical inclusion and A coincides with D. The composition AB is the restriction $D|_{D(V)}$, therefore $AB \in C_{\lambda'}$. We check that $B = A^*$.

$$\begin{array}{rll} (Av, u)_U = (v, Bu)_V & \forall v \in V, u \in D(V) & \Leftrightarrow \\ (Av, A\bar{u})_U = (v, BA\bar{u})_V & \forall v, \bar{u} \in V & \Leftrightarrow \\ \langle v, \bar{u} \rangle = (v, D\bar{u})_V & \forall v, \bar{u} \in V & \Leftrightarrow \\ (v, D\bar{u})_V = (v, D\bar{u})_V & \forall v, \bar{u} \in V. & \end{array}$$

Hence $(A, A^*) \in L^\sigma_{V,U}$. Moreover, since $\pi(A, A^*) = AA^* = D|_{\operatorname{Im} D} \in C_{\varepsilon',\eta,\lambda'}$, $(A, A^*) \in N^\sigma_\lambda$ and $\rho(A, A^*) = A^*A = D \in C_{\varepsilon,\eta,\lambda}$ implies that $\rho(N^\sigma_\lambda) \supseteq C_{\varepsilon,\eta,\lambda}$.

Finally, ρ is a quotient map by theorem 1.3.4 or by theorem 1.3.5. By proposition 1.3.6 a closed invariant subspace is mapped to a closed space. Since N^σ_λ is closed by construction, the image $\rho(N^\sigma_\lambda) \subseteq L^\sigma(V)$ is a closed space containing $C_{\varepsilon,\eta,\lambda}$. □

Let

$$L'^{\sigma} := \{(A, A^*) \in L^{\sigma}_{V,U} : \operatorname{rk} A = \dim U\}.$$

Lemma 4.4.7.

1. *$\rho(L'^{\sigma}) = \{X \in L^{\sigma}(V) : \operatorname{rk} X = \dim U\}$ and $\rho^{-1}(\rho(L'^{\sigma})) = L'^{\sigma}$;*

2. *for each $(A, B) \in L'^{\sigma}$, the stabilizer of (A, B) in $K_{\varepsilon'}(U)$ is trivial;*

3. *for each $(A, B) \in L'^{\sigma}$, $\rho^{-1}(\rho(A, B))$ is a single orbit under $K_{\varepsilon'}(U)$;*

4. *the restriction $\pi|_{L'^{\sigma}} : L'^{\sigma} \to \pi(L'^{\sigma})$ is smooth;*

5. *the restriction $\rho|_{L'^{\sigma}} : L'^{\sigma} \to \rho(L'^{\sigma})$ is smooth.*

Proof. 1. Clearly $\operatorname{rk} AA^* = \dim U$ is $\operatorname{rk} A = \dim U$. As in the proof of lemma 4.4.6, for every $X \in L^{\sigma}(V)$ one may find $(A, A^*) \in L^{\sigma}(V, U)$ such that $X = A^*A$ and A surjective. The last equality is the same as in lemma 4.2.4, *1.*.

2. Immediate from lemma 4.2.4, *2.*.

3. Let $(C, C^*) \in \rho^{-1}(\rho(A, A^*))$, by lemma 4.2.4, *3.*, we get $h \in \mathrm{GL}(U)$ such that $hC = A$, so $C^*C = A^*A = C^*h^*hC$. Since C is surjective and C^* injective, we get $h^*h = 1$, therefore $h \in K_{\varepsilon'}(U)$.

4. The tangent map $d\pi_{(A,A^*)} : L^{\sigma}_{V,U} \to L^{\sigma}(U)$ is the image of the linear map $d\pi_(A, A^*) : L_{V,U} \to L(U)$ through the functor of the $\mathbb{Z}/2$-invariants $V \mapsto V^{\sigma}$. This functor is $\mathrm{Hom}_{\mathbb{Z}/2}(k, -)$ (where k is the trivial $k[\mathbb{Z}/2]$-module), therefore one gets its exactness as soon as k is a projective $k[\mathbb{Z}/2]$-module. This is obviously true as $\mathbb{Z}/2$ is finite (and $\operatorname{char} k = 0$). Therefore $d\pi_{(A,A^*)}$ is surjective as well.

5. Let $\mathrm{GL}(V)$ act on $L^{\sigma}_{V,U}$ by

$$g.(A, A^*) := (Ag^*, gA^*) \quad \forall g \in \mathrm{GL}(V), (A, A^*) \in L^{\sigma}_{V,U}.$$

Under this action L'^{σ} is a $\mathrm{GL}(V)$-orbit and ρ is $\mathrm{GL}(V)$-equivariant (with the $\mathrm{GL}(V)$-action on $L(V)$ given by $X \mapsto gXg^*$), in particular $\rho(L'^{\sigma})$ is an orbit. Therefore there are closed groups $H \subseteq H' \subseteq \mathrm{GL}(V)$ such that the restriction $\rho|_{L'^{\sigma}}$ is the projection $\mathrm{GL}(V)/H \to \mathrm{GL}(V)/H'$. In particular, $\rho|_{L'^{\sigma}}$ is smooth. □

Lemma 4.4.8. *In the settings of lemma 4.4.6, we have*

1. *$\rho^{-1}(C_{\varepsilon,\eta,\lambda})$ is a single $K_{\varepsilon}(V) \times K_{\varepsilon'}(U)$-orbit contained in $N^{\sigma}_{\lambda} \cap L'^{\sigma}$;*

2. *$\pi(\rho^{-1}(C_{\varepsilon,\eta,\lambda})) = C_{\varepsilon,\eta,\lambda'}$.*

Proof. By lemma 4.4.7, *3.*, $\rho^{-1}(X)$ is a single $K_{\varepsilon'}(U)$-orbit for each X such that $\operatorname{rk} X = \dim U$; in particular for $X \in C_{\varepsilon,\eta,\lambda}$. Therefore $\rho^{-1}(C_{\varepsilon,\eta,\lambda})$ is a single $K_{\varepsilon}(V) \times K_{\varepsilon'}(U)$-orbit. By lemma 4.4.7, *1.*, $\rho^{-1}(C_{\varepsilon,\eta,\lambda}) \subseteq L'^{\sigma}$. For every $X \in C_{\varepsilon,\eta,\lambda}$ and for every $(A, A^*) \in \rho^{-1}(X)$, one has $\pi(A, A^*) = AA^* = X|_{\operatorname{Im} X} \in C_{\varepsilon,\eta,\lambda'}$. Therefore $\rho^{-1}(C_{\varepsilon,\eta,\lambda}) \subseteq N_{\lambda}$ and $\pi(\rho^{-1}(C_{\varepsilon,\eta,\lambda})) = C_{\varepsilon,\eta,\lambda'}$ follow. □

We prove the *erasing columns* part of proposition 3.3.3.

Proposition 4.4.9. *Let $\varepsilon \in \{\pm 1\}$ and $\eta \in \{0,1\}$. Let $\lambda, \mu \in \mathcal{P}_{\varepsilon,\eta}(n)$ be partitions such that $\lambda > \mu$ and $\widehat{\lambda}_1 = \widehat{\mu}_1$ and let (λ', μ') be a pair of partitions obtained by erasing the first column from (λ, μ). Then*

$$\mathrm{Sing}(\overline{C_{\varepsilon,\eta,\lambda}}, C_{\varepsilon,\eta,\mu}) = \mathrm{Sing}(\overline{C_{\varepsilon',\eta,\lambda'}}, C_{\varepsilon',\eta,\mu'}).$$

Proof. With the notation of lemma 4.4.6, we have the following diagram:

$$\begin{array}{ccc} N_\lambda^\sigma & \xrightarrow{\pi} & \overline{C_{\varepsilon',\eta,\lambda'}} \\ \Big\downarrow{\scriptstyle \rho} & & \\ \overline{C_{\varepsilon,\eta,\lambda}} & & \end{array}$$

Let $Z = N_\lambda^\sigma$ and $z \in \rho^{-1}(C_{\varepsilon,\eta,\mu}) \subseteq N_\lambda$. We claim that

$$\rho(z) \in C_{\varepsilon,\eta,\mu}, \pi(z) \in C_{\varepsilon,\eta,\mu'}, \rho \text{ and } \pi \text{ are smooth in } z.$$

The first claim follows by construction. The second follows by lemma 4.4.8, *2.*, with $\lambda := \mu$ (notice that $|\mu^{(1)}| = |\lambda^{(1)}|$ since the first columns of λ and μ coincide).

The subvariety $N'^\sigma_\lambda := N_\lambda^\sigma \cap L'^\sigma$ is locally closed in L'^σ, hence both π and ρ are smooth in N'^σ_λ by lemma 4.4.7, *4.* and *5.*. Moreover, $\rho^{-1}(C_{\varepsilon,\eta,\mu}) \subseteq N_\mu^\sigma \cap L'^\sigma \subseteq N'^\sigma_\lambda$ by lemma 4.4.8, *1.* and the last claim follows and the proof as well. □

4.5 *ab*-diagrams for quadratic spaces

Since the *ab*-diagram of any nilpotent pair in $L_{V,U}$ is invariant under $\mathrm{GL}(V) \times \mathrm{GL}(U)$, it is also invariant under $K_\varepsilon(V) \times K_{\varepsilon'}(U)$. In fact, for a symmetric pair $(A, B) \in L^\sigma_{V,U}$ this is the only invariant as stated by the following theorem.

Proposition 4.5.1. *Let $(A, A^*) \in L^\sigma_{V,U}$ be a nilpotent pair. Then*

$$K_\varepsilon(V) \times K_{\varepsilon'}(U).(A, A^*) = \mathrm{GL}(V) \times \mathrm{GL}(U).(A, A^*) \cap L^\sigma_{V,U}.$$

Proof. Let $G := \mathrm{GL}(V) \times \mathrm{GL}(U)$, $X := L_{V,U}$ and let σ be the involutions defined by (4.4.1). We already notice that $G^\sigma = K_\varepsilon(V) \times K_{\varepsilon'}(U)$ and $X^\sigma = L^\sigma(V, U)$. Therefore, the conclusion is a matter of checking the hypothesis of theorem 1.3.8.

1. The compatibility has already been proved in lemma 4.4.4.

2. G is obviously the group of invertible elements of the finite dimensional algebra $M := L(V) \times L(U)$.

3. The linear map $M \to M$ given by $(g, h) \mapsto (g^*, h^*)$ is well-defined for each $(g, h) \in M$ and an anti-involution. When restricted to G, it is obviously the inverse of σ.

4. Let us fix an element $(A, A^*) \in X^\sigma$. Then,

$$H := \mathrm{Stab}_G(A, A^*) = \{(g, h) \in G : hAg^{-1} = A \text{ and } gA^*h^{-1} = A^*\}.$$

Clearly the linear subspace

$$\{(g, h) \in M : hA = Ag \text{ and } gA^* = A^*h\} \subseteq M$$

contains H. Moreover, if (g, h) lies in this subspace and it is invertible, then it satisfies the equations defining H as well, that is $\langle H\rangle_k^\times \subseteq H$.

□

However, not each *ab*-diagram is the *ab*-diagram of a nilpotent pair in $L^\sigma_{V,U}$. We say that an *ab*-diagram δ has type $(\varepsilon, \varepsilon')$ if there is a nilpotent pair $(A, A^*) \in L^\sigma_{V,U}$ with *ab*-diagram δ.

In each of the following tables, for each *ab*-diagram, we state the number of occurrences of the letters a and b. We remind that $\dim V = \#a$ and $\dim U = \#b$. Also we state any further condition on the non-negative integer k.

Table 4.1. Indecomposable *ab*-diagrams

(a) Ortho-symplectic (type $(\varepsilon, \varepsilon') = (1, -1)$)

type	α_k	β_k	γ_k	δ_k	ε_k
	$aba\dots a$	$bab\dots b$	$aba\dots a$ $aba\dots a$	$bab\dots b$ $bab\dots b$	$aba\dots b$ $bab\dots a$
$\#a$	$k+1$	$k-1$	$2k+2$	$2k-2$	$2k$
$\#b$	k	k	$2k$	$2k$	$2k$
	$k \geq 0$ even	$k \geq 2$ even	$k \geq 1$ odd	$k \geq 1$ odd	$k \geq 1$

(b) Ortho-symmetric (type $(\varepsilon, \varepsilon') = (1, 1)$)

type	α_k	β_k	ε_k
	$abab\dots a$	$baba\dots b$	$abab\dots b$ $baba\dots a$
$\#a$	$k+1$	k	$2k$
$\#b$	k	$k+1$	$2k$
	$k \geq 0$	$k \geq 0$	$k \geq 1$

(c) Symplectic-symmetric (type $(\varepsilon, \varepsilon') = (-1, -1)$)

type	γ_k	δ_k	ε_k
	$abab\dots a$ $abab\dots a$	$baba\dots b$ $baba\dots b$	$abab\dots b$ $baba\dots a$
$\#a$	$2k+2$	$2k$	$2k$
$\#b$	$2k$	$2k+2$	$2k$
	$k \geq 0$	$k \geq 0$	$k \geq 1$

Proposition 4.5.2. *Let $(A, A^*) \in L^\sigma_{V,U}$ be a nilpotent pair and let δ be its ab-diagram. Then δ is a sum of finitely many ab-diagrams picked from table 4.1,*

according to the signs signs ε, ε' of the quadratic spaces V, U, such that δ has $\dim V$ *letters a and* $\dim U$ *letters b. Moreover, for any such ab-diagrams δ, there exists a nilpotent pair (A, A^*) with ab-diagram δ.*

The proof of proposition 4.5.2 is split into lemma 4.5.4 and lemma 4.5.5, which appear later in this section.

We denote by $O_{\varepsilon,\varepsilon',\delta}$ the $K_\varepsilon(V) \times K_{\varepsilon'}(U)$-orbit of the nilpotent pairs in $L^\sigma_{V,U}$ with ab-diagram δ.

Definition 4.5.3. Let $\varepsilon, \varepsilon' \in \{\pm 1\}$ and let δ be an ab-diagram of type $(\varepsilon, \varepsilon')$. We call such a δ

- *ortho-symplectic* if $\varepsilon = -\varepsilon'$;
- *ortho-symmetric* if $\varepsilon = \varepsilon' = 1$;
- *symplectic-symmetric* if $\varepsilon = \varepsilon' = -1$.

Example. Both δ_1, δ_2 are ab-diagrams, but only δ_1 is ortho-symmetric:

$$\delta_1 = \begin{matrix} abab \\ abab \\ baba \\ baba \\ a \end{matrix}, \qquad \delta_2 = \begin{matrix} abab \\ abab \\ baba \\ a \end{matrix}.$$

Remark. Let $G = K_\varepsilon(V) \times K_{\varepsilon'}(U)$ and δ be an ab-diagram of type $\varepsilon, \varepsilon'$. Since $O_{\varepsilon,\varepsilon',\delta}$ is G-homogeneous and the maps π, ρ are G-equivariant, the restrictions

$$\begin{aligned} \pi|_{O_{\varepsilon,\varepsilon',\delta}} &: O_{\varepsilon,\varepsilon',\delta} \to C_{\varepsilon,\eta,\pi(\delta)} \\ \rho|_{O_{\varepsilon,\varepsilon',\delta}} &: O_{\varepsilon,\varepsilon',\delta} \to C_{\varepsilon,\eta,\rho(\delta)} \end{aligned} \tag{4.5.1}$$

are smooth. (See also the proof of proposition 4.4.9, since $O_{\varepsilon,\varepsilon',\delta} \subseteq N^\sigma_{\pi(\delta)}$.)

Lemma 4.5.4 (Necessity condition of proposition 4.5.2)**.** *Let $(A, A^*) \in N^\sigma_{V,U}$ be a symmetric pair and δ its ab-diagram. Then δ is a sum of ab-diagrams from table 4.1.*

Proof. We say that an ab-diagram η has *property* $\mathcal{P}$ if, for every $h \geq 1$, the number of occurrences of $(ab)^h$ in η equals the number of occurrences of $(ba)^h$ in η.

Every ab-diagrams $\alpha_k, \beta_k, \gamma_k, \delta_k, \varepsilon_k$ in table 4.1 has property $\mathcal{P}$. Indeed in the diagrams α_k and β_k, for every $h \geq 1$, there are exactly $k-h+1$ occurrences of $(ab)^h$ and of $(ba)^h$; therefore the diagrams γ_k, δ_k have property $\mathcal{P}$ as well. The remaining diagrams ε_k has property $\mathcal{P}$ by symmetry of its two ab-strings.

Let $(A, A^*) \in N^\sigma_{V,U}$ be a nilpotent symmetric pair with ab-diagram δ, an ortho-symmetric ab-diagram. For every positive integer h, we notice that the pair $((AA^*)^{h-1}A, (A^*A)^{h-1}A^*)$ is nilpotent symmetric. By lemma 4.4.3, we have:

$$\mathrm{rk}(AA^*)^{h-1}A = \mathrm{rk}(A^*A)^{h-1}A^*,$$

which, by lemma 4.3.5, implies that δ has property $\mathcal{P}$.

We claim that an *ab*-diagram δ with property $\mathcal{P}$ is a sum of finitely many *ab*-diagrams of types α_k, β_k, ε_k. We proceed by contradiction: suppose that there exist *ab*-diagrams with property $\mathcal{P}$ which are not the sum of finitely many *ab*-diagrams of types α_k, β_k, ε_k and take δ one with the least number of *ab*-strings. Let s be one of the *ab*-strings in δ with maximal length and let l be its length.

If $l = 2k+1$ is odd, $s = \alpha_k$ or $s = \beta_k$. Let δ' be the *ab*-diagram such that $\delta = s \oplus \delta'$ (we cannot have $\delta = s$, otherwise δ is already of type α_k or β_k.

If $l = 2k$ is even, without loss of generality, we assume that s starts with a. Then, in s there exists exactly one occurrence of $(ab)^k$ and no occurrence of $(ba)^k$. As δ satisfies property $\mathcal{P}$, by maximality of l, there must be an *ab*-string s' in δ of length $l = 2k$ starting with b. Thus $s \oplus s' = \varepsilon_k$, so let δ' be the *ab*-diagram such that $\delta = s \oplus s' \oplus \delta'$ (again, it cannot be empty).

In both cases, δ and $\delta \setminus \delta'$ have property $\mathcal{P}$, therefore also δ' has property $\mathcal{P}$. By minimality of δ, δ' is sum of finitely many *ab*-diagrams of types α_k, β_k, ε_k, thus so is δ. This contradicts the existence of such a δ and concludes the proposition if neither V nor U is symplectic.

For the rest of the proof we assume that V is symplectic. We say that an *ab*-diagram η has *property $\mathcal{Q}_a$ (resp. $\mathcal{Q}_b$)* if, for every $h \geq 1$, the number of occurrences of the substring $(ba)^{2h-1}b$ (resp. $(ab)^{2h-1}a$) in η is even.

The *ab*-diagrams $\gamma_k, \delta_k, \varepsilon_k$ in table 4.1 have properties $\mathcal{Q}_a$ and $\mathcal{Q}_b$ by symmetry of their two rows. The *ab*-diagrams α_k have property $\mathcal{Q}_a$ only for k even. A similar conclusion may be drawn about property $\mathcal{Q}_b$ and the *ab*-diagrams β_k.

Let $U_1 := \operatorname{Im}(A^*A)^{h-1}A^* \subseteq V$ and $U_2 := \ker(AA^*)^{h-1}A \subseteq V$. Since the pair $((AA^*)^{h-1}A, (A^*A)^{h-1}A^*)$ is nilpotent symmetric, $U_1^\perp = U_2$. Therefore, the restriction of the form $(-,-)_V$ on $U_1 + U_2 \subseteq V$ has kernel $U_1 \cap U_2$, hence it induces a symplectic form on $W := (U_1+U_2)/(U_1 \cap U_2)$. Since $U_1/(U_1 \cap U_2) \subseteq W$ trivially intersects its orthogonal, the restriction of the form to $U_1/(U_1 \cap U_2)$ is non-degenerate. Therefore $\dim(U_1/(U_1 \cap U_2))$ must be even. By lemma 4.3.6, we get that the number of occurrences of $(ba)^{2h-1}b$ in δ must be even. Since that must be true for each $h \geq 1$, δ has property $\mathcal{Q}_a$.

Similarly to what we did with property $\mathcal{P}$, we may show that δ uses the *ab*-strings of type α_k with an odd number of as (resp. β_k with an odd number of as) only inside *ab*-diagrams of type γ_k (resp. δ_k). Indeed one may choose the longest *ab*-string s of δ and find another *ab*-string $s' = s$ in δ if $s = \alpha_k$ with an odd number of as, by property $\mathcal{Q}_a$, and proceed with an *ab*-diagram δ' with fewer *ab*-strings. This concludes the proof if V is symplectic. The same reasoning concludes in the case U is symplectic as well, using property $\mathcal{Q}_b$. $\square$

Lemma 4.5.5 (Sufficiency condition of proposition 4.5.2)**.** *Let δ be an ab-diagram of table 4.1. Then there is a symmetric pair $(A, A^*) \in N^\sigma_{V,U}$ that has ab-diagram δ.*

Proof. We just need to construct a symmetric pair (A, A^*) associated to each *ab*-diagram α_k, β_k, γ_k, δ_k, ε_k.

α_k**:** Let V be a quadratic vector space of dimension $k+1$ and of type ε, which is possible only if $\varepsilon = 1$ or $k+1$ is even. Let $D \in \mathfrak{gl}(V)$ be a regular nilpotent element such that $D \in \mathfrak{g}_{\varepsilon,\eta}(V)$, which is possible only for some choices of ε and η, by lemma 2.3.5. Let $D = I \circ X$ be the canonical decomposition of the map

D through its image $D(V)$, so that $X : V \to D(V)$ and $I : D(V) \hookrightarrow V$. Then, as shown in the proof of lemma 4.4.6, D induces a non-degenerate bilinear form of type $\varepsilon(-1)^{1+\eta}$ in $D(V)$, and $X = I^*$. As $\dim D(V) = k$ and both X, I have maximal rank, we immediately get that the ab-diagram of the symmetric nilpotent pair (X, I) is α_k.

β_k: Let (A, A^*) be any pair with ab-diagram of type α_k. Then (A^*, A) is a pair with ab-diagram of type β_k.

γ_k, δ_k, ε_k: Let (V, U) be a pair of spaces respectively of dimensions $(k+1, k)$, $(k, k+1)$, (k, k), depending on whether δ is γ_k, δ_k or ε_k. Let (A, B) be any nilpotent (not symmetric) pair between the spaces (V, U) of type α_k, β_k, $ba \cdots a$, respectively. Then we can equip $V \oplus V^*$ (resp. $U \oplus U^*$) with the non-degenerate bilinear form of type ε (resp. ε'), given by (3.2.1). Therefore $\tilde{A} := (A, \varepsilon\varepsilon' B^*), \tilde{B} := (B, \varepsilon\varepsilon' A^*)$ is a nilpotent pair such that $\tilde{B}^* = \varepsilon\varepsilon'\tilde{A}$ (by the same computation of (3.2.2)) and its ab-diagram has type γ_k, δ_k, ε_k.

□

4.6 Dimension of orbits of nilpotent pairs

In this section we examine the dimensions of the orbits in each of the cases analysed previously in this chapter. We want to give formulas for the orbit O_δ depending solely on the ab-diagram δ.

We introduce some notation on the ab-diagrams.

Definition 4.6.1. Let δ be an ab-diagram. We define a_i (resp. b_i) to be the number of rows in δ starting by a (resp. b) with length i. We also define:

$$\Delta(\delta) := \sum_{i \text{ odd}} a_i b_i$$

and

$$o(\delta) := \sum_{i \text{ odd}} a_i + b_i.$$

Proposition 4.6.2. *Let δ be an ab-diagram and let $O_\delta \subseteq L_{V,U}$ be the nilpotent orbit with ab-diagram δ. Then*

$$\dim O_\delta = \frac{1}{2}(\dim C_{\pi(\delta)} + \dim C_{\rho(\delta)}) + \dim V \cdot \dim U - \Delta(\delta),$$

where $\dim V = |\rho(\delta)|$ *and* $\dim U = |\pi(\delta)|$.

Proof. As $L_{V,U}$ is the symmetric space of the symmetric pair $(\mathrm{End}(V \oplus U), \mathrm{End}(V) \oplus \mathrm{End}(U))$, we can put $C = \mathrm{GL}(V \oplus U).O_\delta \subseteq \mathrm{End}(V \oplus U)$; by corollary 1.1.9 one gets

$$\dim C = 2 \dim O_\delta.$$

Let $m := \dim V$, $n := \dim U$, $\lambda = \pi(\delta)$ and $\mu = \rho(\delta)$. As C is the nilpotent conjugacy class with partition $\bar{\delta}$, by proposition 2.5.1, we get

$$\dim C = (n+m)^2 - \sum_{i,j} \min(\bar{\delta}_i, \bar{\delta}_j).$$

Moreover, $\bar{\delta}_i = \lambda_i + \mu_i$ and $|\lambda_i - \mu_i| \leq 1$ by definition of ab-diagram, therefore we get that

$$\min(\bar{\delta}_i, \bar{\delta}_j) = \begin{cases} \min(\lambda_i, \lambda_j) + \min(\mu_i, \mu_j) + 1 \text{ if } \bar{\delta}_i = \bar{\delta}_j \text{ is odd and } \lambda_i = \mu_j, \lambda_j = \mu_i \\ \min(\lambda_i, \lambda_j) + \min(\mu_i, \mu_j) \text{ otherwise.} \end{cases}$$

Thus,

$$\begin{aligned} \dim C &= (n+m)^2 - \left(\sum_{i,j} \min(\lambda_i, \lambda_j) + \sum_{i,j} \min(\mu_i, \mu_j) + 2 \sum_{i \text{ odd}} a_i b_i \right) \\ &= \dim \pi(O_\delta) + \dim \rho(O_\delta) + 2mn - 2\Delta(\delta), \end{aligned}$$

hence the required dimension formula. □

We want to give an interpretation of $L^\sigma_{V,U}$ as a θ-group, similar to the previous case. Let θ be the automorphism of $\tilde{\mathfrak{g}} = \operatorname{End}(V \oplus U)$ defined by:

$$\theta \begin{pmatrix} A & B \\ C & D \end{pmatrix} = - \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}.$$

As θ has order 4, we get a Lie algebra $\mathbb{Z}/4$-gradation $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}^{(0)} \oplus \tilde{\mathfrak{g}}^{(1)} \oplus \tilde{\mathfrak{g}}^{(2)} \oplus \tilde{\mathfrak{g}}^{(3)}$, where $\tilde{\mathfrak{g}}^{(1)} \simeq L^\sigma_{(V,U)}$ as G-modules. We observe that the θ-fixed points locus of $\tilde{G} := \operatorname{GL}(V \oplus U)$ is $G := K_\varepsilon(V) \times K_{\varepsilon'}(U)$ and the θ^2-fixed points locus is $G' := \operatorname{GL}(V) \times \operatorname{GL}(U)$. We also have $\operatorname{Lie} G' = \tilde{\mathfrak{g}}^{(0)} \oplus \tilde{\mathfrak{g}}^{(2)} =: \mathfrak{g}'$ and $\operatorname{Lie} G = \tilde{\mathfrak{g}}^{(0)} =: \mathfrak{g}$.

Proposition 4.6.3. *Let $\varepsilon = 1$, $\varepsilon' = -1$, let δ be an orthosymplectic ab-diagram and let $O_\delta \subseteq L^\sigma_{V,U}$ be the nilpotent orthosymplectic orbit with ab-diagram δ. Then*

$$\dim O_\delta = \frac{1}{2}(\dim C_{\pi(\delta)} + \dim C_{\rho(\delta)} + \dim V \cdot \dim U - \Delta(\delta)).$$

Proof. Let $X \in O_\delta$ be any nilpotent representation with associated ab-diagram δ. Let $\tilde{O}_X := \tilde{G}.X$, $O'_X := G'.X$, $O_X := G.X$ be the orbits of X generated by the groups $\tilde{G}$, G' and G respectively. We want to compute a dimension formula for the dimension of $O_X = O_\delta$. Also, proposition 4.6.2 gives us a dimension formula for O'_X. This proof will proceed as follows: first we gather some facts about an $\mathfrak{sl}_2$ structure of $\tilde{\mathfrak{g}}$, $\mathfrak{g}'$ and $\mathfrak{g}$; then we evaluate the difference $2 \dim O_X - \dim O'_X$; finally we conclude the computation with some combinatorics.

The $\mathfrak{sl}_2$ representation structure we are looking for is granted by theorem 1.1.11. In fact we can choose an $\mathfrak{sl}_2$-triple (X, H, Y) with $H \in \tilde{\mathfrak{g}}^{(0)}$ and $Y \in \tilde{\mathfrak{g}}^{(3)}$. Therefore, as in standard $\mathfrak{sl}_2$ representation theory we obtain the following results.

The semisimple element H defines a $\mathbb{Z}$-gradation of $\tilde{\mathfrak{g}}$ given by

$$\tilde{\mathfrak{g}} = \bigoplus_{i \in \mathbb{Z}} \tilde{\mathfrak{g}}_i, \quad \text{where } \tilde{\mathfrak{g}}_i := \{Z \in \tilde{\mathfrak{g}} : [H, Z] = iZ\}. \tag{4.6.1}$$

We define

$$\tilde{\mathfrak{p}} := \bigoplus_{i \geq 0} \tilde{\mathfrak{g}}_i, \quad \tilde{\mathfrak{n}} := \bigoplus_{i > 0} \tilde{\mathfrak{g}}_i, \quad \tilde{\mathfrak{n}}_2 := [X, \tilde{\mathfrak{p}}] = \bigoplus_{i \geq 2} \tilde{\mathfrak{g}}_i,$$

therefore $\tilde{\mathfrak{p}}$ is a parabolic subalgebra of $\tilde{\mathfrak{g}}$, $\tilde{\mathfrak{n}}$ is its nilradical, $\tilde{\mathfrak{p}} = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{n}}$ is its Levi decomposition; moreover $X \in \tilde{\mathfrak{n}}_2$ and the centralizer $\tilde{\mathfrak{g}}^X \subseteq \tilde{\mathfrak{p}}$. We also have that $\dim \tilde{\mathfrak{g}}^X = \dim \tilde{\mathfrak{g}}_0 + \dim \tilde{\mathfrak{g}}_1$ or

$$\dim \tilde{O}_X = \dim[\tilde{\mathfrak{g}}, X] = \dim \tilde{\mathfrak{g}} - \dim \tilde{\mathfrak{g}}^X = \dim \tilde{\mathfrak{n}} + \dim \tilde{\mathfrak{n}}_2.$$

As $H \in \tilde{\mathfrak{g}}^{(0)}$, the gradation in (4.6.1) is also θ-invariant. This implies the gradation in (4.6.1) induces a gradation of $\mathfrak{g}'$ and a gradation of $\mathfrak{g}$. In particular we get the following subalgebras:

$$\mathfrak{p}' := \tilde{\mathfrak{p}} \cap \mathfrak{g}' \quad \mathfrak{n}' := \tilde{\mathfrak{n}} \cap \mathfrak{g}' \quad \mathfrak{n}_2' := [X, \mathfrak{p}']$$

and

$$\mathfrak{p} := \tilde{\mathfrak{p}} \cap \mathfrak{g} \quad \mathfrak{n} := \tilde{\mathfrak{n}} \cap \mathfrak{g} \quad \mathfrak{n}_2 := [X, \mathfrak{p}].$$

Similarly, we have the Levi decompositions $\mathfrak{p}' = \mathfrak{g}_0' \oplus \mathfrak{n}'$, $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{n}$. Also note that $\mathfrak{n}_2' \subseteq \tilde{\mathfrak{g}}^{(1)} \oplus \tilde{\mathfrak{g}}^{(3)}$ and $\mathfrak{n}' \subseteq \tilde{\mathfrak{g}}^{(1)}$ because $X \in \tilde{\mathfrak{g}}^{(1)}$ and $\mathfrak{p}' \subseteq \tilde{\mathfrak{g}}^{(0)} \oplus \tilde{\mathfrak{g}}^{(2)}$, $\mathfrak{p} \subseteq \tilde{\mathfrak{g}}^{(0)}$. We also have analogous dimension formulas for the orbits:

$$\begin{aligned}\dim O_X' &= \dim[\mathfrak{g}', X] = \dim \mathfrak{g}' - \dim \mathfrak{g}'^X = \dim \mathfrak{g}' - \dim \mathfrak{p}'^X \\ &= \dim \mathfrak{g}' - (\dim \mathfrak{p}' - [\mathfrak{p}', X]) = (\dim \mathfrak{g}' - \dim \mathfrak{p}') + \dim \mathfrak{n}_2' \\ &= \dim \mathfrak{n}' + \dim \mathfrak{n}_2';\end{aligned}$$

and, similarly,

$$\dim O_X = \dim \mathfrak{n} + \dim \mathfrak{n}_2.$$

We can now deal with the main computation:

$$4 \dim O_X - 2 \dim O_X' = (4 \dim \mathfrak{n} - 2 \dim \mathfrak{n}') + 2(2 \dim \mathfrak{n}_2 - \dim \mathfrak{n}_2').$$

We observe that we have the following.

Lemma 4.6.4. *Let O_X', O_X be the orbits of X under G', G respectively. Let $\mathfrak{n}_2'$, $\mathfrak{n}_2$ be the corresponding nilpotent subalgebras. Then*

$$\dim \mathfrak{n}_2' = 2 \dim \mathfrak{n}_2.$$

Proof of lemma 4.6.4. The key fact is the isomorphism of the following weight spaces:

$$\tilde{\mathfrak{g}}^{(1)} \cap \tilde{\mathfrak{g}}_i \simeq \tilde{\mathfrak{g}}^{(3)} \cap \tilde{\mathfrak{g}}_i, \tag{4.6.2}$$

for each integer i. The isomorphism is given by θ^2, which is the conjugation by $J = \begin{pmatrix} 1_V & 0 \\ 0 & -1_U \end{pmatrix}$. In fact $\theta(J) = -J$, so $\theta(J\tilde{\mathfrak{g}}^{(1)}) = -J\theta(\tilde{\mathfrak{g}}^{(1)}) = J\theta(\tilde{\mathfrak{g}}^{(3)})$, therefore $J\tilde{\mathfrak{g}}^{(1)} = \mathfrak{g}^{(3)}$. Moreover, $H \in \mathfrak{g}^{(0)}$, that is $\theta(H) = H$, therefore $JH = HJ$, in particular J preserves each of the weight spaces $\tilde{\mathfrak{g}}_i$. Therefore J is the isomorphism we were looking for in (4.6.2).

The conclusion follows as $\mathfrak{n}_2' = (\tilde{\mathfrak{g}}^{(1)} \cap \tilde{\mathfrak{n}}_2) \oplus (\tilde{\mathfrak{g}}^{(3)} \cap \tilde{\mathfrak{n}}_2)$ and $\mathfrak{n}_2 = (\tilde{\mathfrak{g}}^{(1)} \cap \tilde{\mathfrak{n}}_2)$, by taking the dimensions of both spaces. □

Therefore, as $\dim \mathfrak{g} = 2\dim \mathfrak{n} + \dim \mathfrak{g}_0$ and $\dim \mathfrak{g}' = 2\dim \mathfrak{n}' + \dim \mathfrak{g}_0'$, we are left with

$$\begin{aligned} 4\dim O_X - 2\dim O_X' =&(4\dim \mathfrak{n} - 2\dim \mathfrak{n}') \\ =&(2\dim \mathfrak{g} - \dim \mathfrak{g}') - (2\dim \mathfrak{g}_0 - \dim \mathfrak{g}_0') \\ =&(2\dim \mathfrak{o}(V) - \dim \mathfrak{gl}(V)) + (2\dim \mathfrak{sp}(U) - \dim \mathfrak{gl}(U)) \\ &- (2\dim \mathfrak{g}_0 - \dim \mathfrak{g}_0') \\ =&(-\dim V) + (\dim U) - (2\dim \mathfrak{g}_0 - \dim \mathfrak{g}_0'). \end{aligned}$$

Before we can proceed, we introduce some combinatorial quantity. Let d_a (resp. d_b) the number of rows of the ab-diagram δ with an odd number of a (resp. b) and an even number of b (resp. a). Then we have the following.

Lemma 4.6.5. *The following holds:*

$$2\dim \mathfrak{g}_0 - \dim \mathfrak{g}_0' = d_b - d_a.$$

Proof of lemma 4.6.5. The $\mathfrak{sl}_2$ representation of (X, H, Y) induces a $\mathbb{Z}$-gradation on $V \oplus U$; moreover, this gradation is θ stable, therefore both V and U as a $\mathbb{Z}$-gradation which is given by

$$\begin{aligned} V = \bigoplus_{i\in\mathbb{Z}} V_i, &\quad \text{where } V_i = \{v \in V : Hv = iv\}; \\ U = \bigoplus_{j\in\mathbb{Z}} U_j, &\quad \text{where } V_j = \{u \in U : Hu = ju\}. \end{aligned}$$

By definition, $\mathfrak{g}_0'$ and $\mathfrak{g}_0$ are the stabilizers of H in $\mathfrak{g}'$ and $\mathfrak{g}$, respectively. Therefore we find

$$\mathfrak{g}_0' = \left(\bigoplus_{i\in\mathbb{Z}} \mathfrak{gl}(V_i)\right) \oplus \left(\bigoplus_{j\in\mathbb{Z}} \mathfrak{gl}(U_j)\right).$$

In order to get a similar decomposition for $\mathfrak{g}_0$, we notice that V_i and V_{-i} are in perfect pairing with respect to the orthogonal form of V and, similarly, the same holds true for U_j and U_{-j}. As $\mathfrak{g}_0$ is fixed by θ, we obtain:

$$\mathfrak{g}_0 \simeq \left(\mathfrak{o}(V_0) \oplus \bigoplus_{i>0} \mathfrak{gl}(V_i)\right) \oplus \left(\mathfrak{sp}(U_0) \oplus \bigoplus_{j>0} \mathfrak{gl}(U_j)\right).$$

We can finally compute:

$$\begin{aligned} 2\dim \mathfrak{g}_0 - \dim \mathfrak{g}_0' =&2(\dim \mathfrak{o}(V_0) + \dim \mathfrak{sp}(U_0)) - (\dim \mathfrak{gl}(V_0) + \dim \mathfrak{gl}(U_0)) \\ =&(2\dim \mathfrak{o}(V_0) - \dim \mathfrak{gl}(V_0)) + (2\dim \mathfrak{sp}(U_0) - \mathfrak{gl}(U_0)) \\ =&(\dim V_0(\dim V_0 - 1) - (\dim V_0)^2)+ \\ &+ (\dim U_0(\dim U_0 + 1) - (\dim U_0)^2) \\ =&\dim U_0 - \dim V_0. \end{aligned}$$

We only need to prove that $\dim U_0 = d_b$ and $\dim V_0 = d_a$. We recall that, for each row δ_i of the ab-diagram δ the vectors corresponding to the letters of δ_i span an irreducible representation of (X, H, Y). Thus, the zero-weight space $V_0 \oplus U_0$ is spanned by the vectors corresponding to the middle letter of δ_i, for each row which has an odd length. The distinction between V_0 and U_0 is the same as the distinction between as and bs, respectively, among these letters in the centres. This concludes the lemma. □

We can continue the main computation

$$4 \dim O_X - 2 \dim O'_X = \dim U - \dim V + d_a - d_b.$$

Let $C_{\pi(\delta)} = \pi(O_X) \subseteq \mathfrak{o}(V)$ and $C_{\rho(\delta)} = \rho(O_X) \subseteq \mathfrak{sp}(U)$ be the conjugacy classes we stated on the thesis; similarly, let $C'_{\pi(\delta)} = \pi(O'_X) \subseteq \mathfrak{gl}(V)$ and $C'_{\rho(\delta)} = \rho(O'_X) \subseteq \mathfrak{gl}(U)$ be the corresponding conjugacy classes generated by $C_{\pi(\delta)}$ and $C_{\rho(\delta)}$, respectively. By proposition 4.6.2, we have

$$\begin{aligned}
4 \dim O_X =& 2 \dim O'_X + \dim U - \dim V + d_a - d_b \\
=& \left(\dim C'_{\pi(\delta)} + \dim C'_{\rho(\delta)} + 2 \dim V \cdot \dim U - 2\Delta(\delta)\right) + \\
&+ \dim U - \dim V + d_a - d_b.
\end{aligned}$$

By proposition 2.5.3 we also have

$$\begin{aligned}
\dim C'_{\pi(\delta)} &= 2 \dim C_{\pi(\delta)} + \dim V - r_a \\
\dim C'_{\rho(\delta)} &= 2 \dim C_{\rho(\delta)} - \dim U + r_b,
\end{aligned}$$

where r_a (resp. r_b) is the number of odd rows in the Young diagram $\pi(\delta)$ (resp. $\rho(\delta)$). We still denote by δ_i the i-th row of δ; we finally have that

$$\begin{aligned}
r_a - r_b =& |\{i : \pi(\delta_i) \text{ is odd}\}| - |\{i : \rho(\delta_i) \text{ is odd}\}| \\
=& |\{i : \delta_i \text{ has odd } a\text{s}\}| - |\{i : \delta_i \text{ has odd } b\text{s}\}| \\
=& |\{i : \delta_i \text{ has odd } a\text{s and even } b\text{s}\}| - |\{i : \delta_i \text{ has odd } b\text{s and even } a\text{s}\}| \\
=& d_a - d_b.
\end{aligned}$$

We can finally conclude with

$$\begin{aligned}
4 \dim O_X =& \left(2 \dim C_{\pi(\delta)} + \dim V - r_a\right) + \left(2 \dim C_{\rho(\delta)} - \dim U + r_b\right) + \\
&+ \dim U - \dim V + d_a - d_b + 2 \dim V \cdot \dim U - 2\Delta(\delta) \\
=& 2 \left(\dim C_{\pi(\delta)} + \dim C_{\rho(\delta)} + \dim V \cdot \dim U - \Delta(\delta)\right).
\end{aligned}$$

□

Finally, we turn to the dimension formula in the ε-symmetric case. As in the previous case, we want to give an interpretation of $L^{\sigma}_{V,U}$ as a θ-group, but, differently from that case, now we have two different θ-group structures.

Let V, U be two quadratic spaces of type ε. Let θ be the automorphism of $\mathrm{End}(V \oplus U)$ defined by the conjugation with $J = \begin{pmatrix} 1_V & 0 \\ 0 & -1_U \end{pmatrix}$ and let σ be the automorphism of $\mathrm{End}(V \oplus U)$ defined by

$$\sigma \begin{pmatrix} A & B \\ C & D \end{pmatrix} = - \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix},$$

where $A \in \mathrm{End}(V)$, $B \in L(U,V)$, $C \in L(V,U)$, $D \in \mathrm{End}(U)$.

Both θ and σ are involutions of $\mathrm{End}(V \oplus U)$, as both V and U have the same type ε. Moreover, $\theta \circ \sigma = \sigma \circ \theta$. Let $\tilde{\mathfrak{g}} = \mathfrak{gl}(V \oplus U)$; therefore θ and σ become Lie algebra automorphism of order two, so we get a direct sum decomposition

$$\tilde{\mathfrak{g}} = (\tilde{\mathfrak{g}}^{\theta} \cap \tilde{\mathfrak{g}}^{\sigma}) \oplus (\tilde{\mathfrak{g}}^{\theta} \cap \tilde{\mathfrak{g}}^{-\sigma}) \oplus (\tilde{\mathfrak{g}}^{-\theta} \cap \tilde{\mathfrak{g}}^{\sigma}) \oplus (\tilde{\mathfrak{g}}^{-\theta} \cap \tilde{\mathfrak{g}}^{-\sigma}),$$

where $\tilde{\mathfrak{g}}^{\tau} = \{X \in \tilde{\mathfrak{g}} : \tau(X) = X\}$ is the fixed point subset for any linear map τ. We highlight that

$$\tilde{\mathfrak{g}}^{-\theta} \cap \tilde{\mathfrak{g}}^{\sigma} = \left\{ \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix} : B \in L(V,U) \right\}$$

and

$$\tilde{\mathfrak{g}}^{-\theta} \cap \tilde{\mathfrak{g}}^{-\sigma} = \left\{ \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} : B \in L(V,U) \right\}.$$

We also observe that the θ-fixed points of $\tilde{G} := \mathrm{GL}(V \oplus U)$ is $G' := \mathrm{GL}(V) \oplus \mathrm{GL}(U)$ and the σ-fixed points of G' is $G := K_\varepsilon(V) \times K_\varepsilon(U)$ and $G = \tilde{G}^{\theta} \cap \tilde{G}^{\sigma}$. In particular $L^{\sigma}_{V,U} \simeq \tilde{\mathfrak{g}}^{-\theta} \cap \tilde{\mathfrak{g}}^{-\sigma}$ as G-modules.

Proposition 4.6.6. *Let δ be an ε-symmetric ab-diagram and let $O_\delta \subseteq L^{\sigma}_{V,U}$ the nilpotent ε-symmetric orbit with ab-diagram δ. Then*

$$\dim O_\delta = \frac{1}{2}(\dim C_{\pi(\delta)} + \dim C_{\rho(\delta)} + \dim V \cdot \dim U - \Delta(\delta)) - \frac{\varepsilon}{4}(\dim V + \dim U - o(\delta)),$$

where $o(\delta)$ is the number of rows of δ having odd length.

Proof. The proof will proceed in a rather analogous way to the one given to proposition 4.6.3. Let $X \in O_\delta$ be any nilpotent representation with corresponding *ab*-diagram δ. Let $\tilde{O}_X := \tilde{G}.X$, $O'_X := G'.X$, $O_X := G.X$ be the orbits of X generated by the groups $\tilde{G}$, G' and G respectively. We know a dimension formula for O'_X thanks to proposition 4.6.2. We will still put an $\mathfrak{sl}_2$ structure in $\tilde{\mathfrak{g}}$, $\mathfrak{g}'$ and $\mathfrak{g}$; then we evaluate the difference $2\dim O_X - \dim O'_X$; finally we will conclude the computation with some combinatorics.

The Lie algebra $\tilde{\mathfrak{g}}^{\sigma}$ together with the involution θ is a θ-group of order 2. By theorem 1.1.11, we can choose an $\mathfrak{sl}_2$ triple (X, H, Y) with $H \in \mathfrak{g}^{\theta} \cap \tilde{\mathfrak{g}}^{\sigma}$ and $Y \in \mathfrak{g}^{-\theta} \cap \tilde{\mathfrak{g}}^{\sigma}$.

The semisimple element H defines a $\mathbb{Z}$-gradation of $\tilde{\mathfrak{g}}$ given by

$$\tilde{\mathfrak{g}} = \bigoplus_{i \in \mathbb{Z}} \tilde{\mathfrak{g}}_i, \quad \text{where } \tilde{\mathfrak{g}}_i := \{Z \in \tilde{\mathfrak{g}} : [H, Z] = iZ\}; \tag{4.6.3}$$

we define

$$\tilde{\mathfrak{p}} := \bigoplus_{i \geq 0} \tilde{\mathfrak{g}}_i, \quad \tilde{\mathfrak{n}} := \bigoplus_{i > 0} \tilde{\mathfrak{g}}_i, \quad \tilde{\mathfrak{n}}_2 := [X, \tilde{\mathfrak{p}}] = \bigoplus_{i \geq 2} \tilde{\mathfrak{g}}_i,$$

therefore $\tilde{\mathfrak{p}}$ is a parabolic subalgebra of $\tilde{\mathfrak{g}}$, $\tilde{\mathfrak{n}}$ is its nilradical, $\tilde{\mathfrak{p}} = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{n}}$ is its Levi decomposition; $X \in \tilde{\mathfrak{n}}_2$ and the centralizer $\tilde{\mathfrak{g}}^X \subseteq \tilde{\mathfrak{p}}$. We also have that

$$\dim \tilde{O}_X = \dim[\tilde{\mathfrak{g}}, X] = \dim \tilde{\mathfrak{g}} - \dim \tilde{\mathfrak{g}}^X = \dim \tilde{\mathfrak{n}} + \dim \tilde{\mathfrak{n}}_2.$$

As $H \in \tilde{\mathfrak{g}}^\theta \cap \mathfrak{g}^\sigma$, the gradation in (4.6.3) is also θ-invariant and σ-invariant, therefore the gradation in (4.6.1) induces a gradation of $\mathfrak{g}'$ and a gradation of $\mathfrak{g}$. In particular we get the following subalgebras:

$$\mathfrak{p}' := \tilde{\mathfrak{p}} \cap \mathfrak{g}' \quad \mathfrak{n}' := \tilde{\mathfrak{n}} \cap \mathfrak{g}' \quad \mathfrak{n}_2' := [X, \mathfrak{p}']$$
$$\mathfrak{p} := \tilde{\mathfrak{p}} \cap \mathfrak{g} \quad \mathfrak{n} := \tilde{\mathfrak{n}} \cap \mathfrak{g} \quad \mathfrak{n}_2 := [X, \mathfrak{p}].$$

and we have the Levi decompositions $\mathfrak{p}' = \mathfrak{g}_0' \oplus \mathfrak{n}'$, $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{n}$. We have analogous dimension formulas for the orbits:

$$\begin{aligned} \dim O_X' &= \dim[\mathfrak{g}', X] = \dim \mathfrak{g}' - \dim \mathfrak{g}'^X = \dim \mathfrak{g}' - \dim \mathfrak{p}'^X \\ &= \dim \mathfrak{g}' - (\dim \mathfrak{p}' - [\mathfrak{p}', X]) = (\dim \mathfrak{g}' - \dim \mathfrak{p}') + \dim \mathfrak{n}_2' \\ &= \dim \mathfrak{n}' + \dim \mathfrak{n}_2'; \end{aligned}$$

and

$$\dim O_X = \dim \mathfrak{n} + \dim \mathfrak{n}_2.$$

We have, similarly, the same result as in lemma 4.6.4, but that requires a different proof.

Lemma 4.6.7. *Let O_X', O_X be the orbits of X under G', G respectively. Let $\mathfrak{n}_2'$, $\mathfrak{n}_2$ be the corresponding nilpotent subalgebras. Then*

$$\dim \mathfrak{n}_2' = 2 \dim \mathfrak{n}_2.$$

Proof of lemma 4.6.7. The key fact now is that J induces an isomorphism of the following weight spaces:

$$\tilde{\mathfrak{g}}^{-\theta} \cap \tilde{\mathfrak{g}}^{\sigma} \cap \tilde{\mathfrak{g}}_i \simeq \tilde{\mathfrak{g}}^{-\theta} \cap \tilde{\mathfrak{g}}^{-\sigma} \cap \tilde{\mathfrak{g}}_i,$$

for each integer i.

The conclusion follows as

$$\mathfrak{n}_2' = (\tilde{\mathfrak{g}}^{-\theta} \cap \tilde{\mathfrak{g}}^{\sigma} \cap \tilde{\mathfrak{n}}_2) \oplus (\tilde{\mathfrak{g}}^{-\theta} \cap \tilde{\mathfrak{g}}^{-\sigma} \cap \tilde{\mathfrak{n}}_2) = \mathfrak{n}_2 \oplus J\mathfrak{n}_2.$$

□

Before tackling the main computation, we also state a new version of lemma 4.6.5. We still denote as d_a (resp. d_b) the number of rows of the ab-diagram δ with an odd number of a (resp. b) and an even number of b (resp. a).

Lemma 4.6.8. *The following holds:*

$$2 \dim \mathfrak{g}_0 - \dim \mathfrak{g}_0' = -\varepsilon(d_a + d_b).$$

Proof of lemma 4.6.8. We perform the same argument as in lemma 4.6.5, starting with the $\mathbb{Z}$-gradation gradation induced on U and V by H given by

$$V = \bigoplus_{i \in \mathbb{Z}} V_i, \quad \text{where } V_i = \{v \in V : Hv = iv\};$$
$$U = \bigoplus_{j \in \mathbb{Z}} U_j, \quad \text{where } U_j = \{u \in U : Hu = ju\}.$$

We still have

$$\mathfrak{g}_0' = \left(\bigoplus_{i\in\mathbb{Z}} \mathfrak{gl}(V_i)\right) \oplus \left(\bigoplus_{j\in\mathbb{Z}} \mathfrak{gl}(U_j)\right);$$
$$\mathfrak{g}_0 \simeq \left(\mathfrak{g}(V_0) \oplus \bigoplus_{i>0} \mathfrak{gl}(V_i)\right) \oplus \left(\mathfrak{g}(U_0) \oplus \bigoplus_{j>0} \mathfrak{gl}(U_j)\right).$$

We can finally compute:

$$\begin{aligned} 2\dim\mathfrak{g}_0 - \dim\mathfrak{g}_0' =& 2(\dim\mathfrak{g}(V_0) + \dim\mathfrak{g}(U_0)) - (\dim\mathfrak{gl}(V_0) + \dim\mathfrak{gl}(U_0)) \\ =& (2\dim\mathfrak{g}(V_0) - \dim\mathfrak{gl}(V_0)) + (2\dim\mathfrak{g}(U_0) - \mathfrak{gl}(U_0)) \\ =& (\dim V_0(\dim V_0 - \varepsilon) - (\dim V_0)^2) + \\ & + (\dim U_0(\dim U_0 - \varepsilon) - (\dim U_0)^2) \\ =& -\varepsilon(\dim U_0 + \dim V_0). \end{aligned}$$

As in the proof of lemma 4.6.5, we obtain that $\dim U_0 = d_b$ and $\dim V_0 = d_a$. □

We can deal with the main computation.

$$\begin{aligned} 4\dim O_X - 2\dim O_X' =& (4\dim\mathfrak{n} - 2\dim\mathfrak{n}') + 2(2\dim\mathfrak{n}_2 - \dim\mathfrak{n}_2') \\ =& (2\dim\mathfrak{g} - \dim\mathfrak{g}') - (2\dim\mathfrak{g}_0 - \dim\mathfrak{g}_0') + 2(0) \\ =& -\varepsilon(\dim V + \dim U) + \varepsilon(d_a + d_b). \end{aligned}$$

Let $C_{\pi(\delta)} = \pi(O_X) \subseteq \mathfrak{g}(V)$ and $C_{\rho(\delta)} = \rho(O_X) \subseteq \mathfrak{g}(U)$ be the conjugacy classes with respect to the groups $K_\varepsilon(V)$ and $K_\varepsilon(U)$ and, similarly, let $C'_{\pi(\delta)} = \pi(O_X') \subseteq \mathfrak{gl}(V)$ and $C'_{\rho(\delta)} = \rho(O_X') \subseteq \mathfrak{gl}(U)$ be the corresponding conjugacy classes generated by $C_{\pi(\delta)}$ and $C_{\rho(\delta)}$, respectively. By proposition 4.6.2, we have

$$\begin{aligned} 4\dim O_X =& 2\dim O_X' - \varepsilon(\dim V + \dim U) + \varepsilon(d_a + d_b) \\ =& \Big(\dim C'_{\pi(\delta)} + \dim C'_{\rho(\delta)} + 2\dim V \cdot \dim U - 2\Delta(\delta)\Big) + \\ & - \varepsilon(\dim V + \dim U) + \varepsilon(d_a + d_b) \end{aligned}$$

By proposition 2.5.5 we have

$$\begin{aligned} \dim C'_{\pi(\delta)} &= 2\dim C_{\pi(\delta)} \\ \dim C'_{\rho(\delta)} &= 2\dim C_{\rho(\delta)}, \end{aligned}$$

and $d_a + d_b = o(\delta)$ as each odd row in δ either has an odd number of as and an even number of bs or vice versa. Therefore

$$\begin{aligned} 4\dim O_X =& 2(\dim C_{\pi(\delta)} + \dim C_{\rho(\delta)} + \dim V \cdot \dim U - \Delta(\delta)) + \\ & - \varepsilon(\dim V + \dim U - d_a - d_b) \\ =& 2(\dim C_{\pi(\delta)} + \dim C_{\rho(\delta)} + \dim V \cdot \dim U - \Delta(\delta)) + \\ & - \varepsilon(\dim V + \dim U - o(\delta)). \end{aligned}$$

□

Chapter 5

The Variety Z

In this chapter we will construct an auxiliary variety Z. This construction is performed for every nilpotent orbit C_λ in each of the debated cases, namely orbits with respect to the groups GL, Sp, O. We will study the geometry of the variety Z, the action of a reductive group G on Z, the smooth locus of Z and, finally, a map $\Theta : Z \to \overline{C_\lambda}$ which will be the key to prove the normality of the latter space.

We will proceed as follows. First we describe the construction in the case useful for the GL-orbits. Then we modify some important details in order to build a Z useful to the cases where some bilinear maps are involved. Finally we prove that a certain hypothesis related to the dimension of a subvariety of Z is enough to prove the normality of Z and of $\overline{C_\lambda}$ as a consequence.

In the current chapter and in the following chapters we are combining mainly the following references: [KP79, KP82, Oht86, Tre22].

5.1 For GL_n

Let $\lambda \vdash n$ be a partition. Recall the construction performed in section 2.1: one may construct a sequence of partitions $\lambda = \lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \ldots$ where $\lambda^{(i+1)} := \lambda^{(i)\prime}$. We define a sequence of nonnegative integers

$$n_i := |\lambda^{(i)}| = \sum_{j>i}^{t} \widehat{\lambda}_j;$$

in particular $n_0 = n$ and $n_t = 0$. By (2.2.2), one gets that $n_i = \operatorname{rk} X^i$ for each $X \in C_\lambda$. The sequence $(n_0, \ldots, n_t)$ will play an important role in this and in the next chapters.

5.1.1 Z as a schematic fibre

For every $i = 0, \ldots, t$, we fix a vector space V_i of dimension n_i. We define the following vector spaces:

$$M := L_{V_0,V_1} \times L_{V_1,V_2} \times \cdots \times L_{V_{t-1},V_t}$$
$$N := L(V_1) \times L(V_2) \times \cdots \times L(V_{t-1}).$$

We denote an element of M as

$$x = (A_1, B_1, \dots, A_t, B_t) \in M$$

where $A_i \in L(V_{i-1}, V_i)$ and $B_i \in L(V_i, V_{i-1})$. Notice that, as $n_t = 0$, V_t is the null space and, therefore the maps A_t, B_t are always the null maps.

We also define a map $\Phi : M \to N$ by

$$\Phi : (A_1, B_1, \dots, A_t, B_t) \mapsto (A_1B_1 - B_2A_2, \dots, A_{t-1}B_{t-1} - B_tA_t).$$

Definition 5.1.1. Let $\lambda \in \mathcal{P}(n)$. We define the variety

$$Z := Z^{(\lambda)} := \Phi^{-1}(0)$$

as the schematic fibre over the point $0 = (0, 0, \dots, 0) \in N$.

Remark. We remark that this definition does not ensure that Z is a reduced scheme. In fact we will do quite some work in order to prove this. In the meantime we denote by Z_{red} the canonical reduced subscheme of M with topological space the closed set $\Phi^{-1}(0)$.

As V_t is the null space, we could have introduced Z in a slightly different way. We could have define only the spaces $V_0, \dots, V_{t-1}$ and we could have changed the definition of the last component of Φ to be $A_{t-1}B_{t-1}$ instead of $A_{t-1}B_{t-1} - B_tA_t$. While this would be perfectly equivalent, we prefer the other way with the dummy space V_t in order to give a more uniform definition of Φ.

5.1.2 Z as a quiver representation

We can give another interpretation of the reduced scheme Z_{red} as a quiver representation.

With the notation as above, let us define the quiver

$$Q_t = \quad 0 \underset{y}{\overset{x}{\rightleftarrows}} 1 \underset{y}{\overset{x}{\rightleftarrows}} \cdots \underset{y}{\overset{x}{\rightleftarrows}} t.$$

Let us consider also the relations $\mathcal{I}$:

$$xy = yx,$$

and the dimension vector $\underline{n} = (n_0, \dots, n_t)$. We have that

$$Z_{\text{red}} = \operatorname{Rep}(Q_t, \mathcal{I}, \underline{n}),$$

that is, Z_{red} is the variety of the representations of the quiver Q_t with relations $\mathcal{I}$ with dimension vector $\underline{n}$. Thus, each point $z \in \operatorname{Rep}(Q_t, \mathcal{I}, \underline{n})$ is a sequence of maps

$$(A_1, B_1, \dots, A_t, B_t)$$

such that $A_i \in L(k^{n_{i-1}}, k^{n_i})$, $B_i \in L(k^{n_i}, k^{n_{i-1}})$, and, by the relations $\mathcal{I}$, we have that $A_iB_i = B_{i+1}A_{i+1}$. This equations just require that the two compositions

$$V_{i-1} \rightleftarrows V_i \rightleftarrows V_{i+1}$$

yield the same endomorphism of V_i.

5.1.3 The map Θ

The interpretation of Z as a representation of the quiver Q_t suggests considering the action of the structural group of Q_t. Let

$$G := \mathrm{GL}(V_0) \times \cdots \times \mathrm{GL}(V_t).$$

Then G acts on the affine variety M in the usual way by change of basis as follows. If $g = (g_0, \ldots, g_t) \in G$ and $x = (A_1, B_1, \ldots, A_t, B_t) \in M$, we set

$$g.x := (g_i A_i g_{i-1}^{-1}, g_{i-1} B_i g_i^{-1})_{i=1,\ldots,t} \in M.$$

We also have that G acts on N, factor by factor, by change of basis. The map Φ between M and N is equivariant with respect to these G-actions as

$$\left(g_{i-1} A_{i-1} g_{i-2}^{-1}\right)\left(g_{i-2} B_{i-1} g_{i-1}^{-1}\right) - \left(g_{i-1} B_i g_i^{-1}\right)\left(g_i A_i g_{i-1}^{-1}\right) = \\ = g_{i-1}(A_{i-1}B_{i-1} - B_i A_i) g_{i-1}^{-1}.$$

As $0 \in N$ is a fixed point of the action of G, Z is invariant with respect to the action of G in M.

We can introduce an important morphism.

Definition 5.1.2. We define a morphism Θ by

$$\begin{aligned} \Theta : & Z_{\mathrm{red}} \to L(V_0) \\ & (A_1, B_1, \ldots, A_t, B_t) \mapsto B_1 A_1. \end{aligned}$$

Notice that G acts on $L(V_0)$ by change of basis using only the first factor $\mathrm{GL}(V_0)$, while leaving the other factors $\mathrm{GL}(V_i)$ to act trivially. We define

$$H := \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_t) < G$$

as the subgroup of G with first entry the identity of $\mathrm{GL}(V_0)$.

Remark. The map Θ is equivariant with respect to the actions of G on Z and on $L(V_0)$. This is equivalent to say that Θ is H-invariant and $\mathrm{GL}(V_0) \simeq G/H$-equivariant.

Proposition 5.1.3. $\Theta(Z_{\mathrm{red}}) = \overline{C_\lambda}$.

Proof. In fact we prove by induction on $i = t, t-1, \ldots, 1$ that the maps $\Theta_i : Z_{\mathrm{red}} \to L(V_{i-1})$ given by

$$\Theta_i(A_1, B_1, \ldots, A_t, B_t) = B_i A_i$$

satisfy $\Theta_i(Z_{\mathrm{red}}) = \overline{C_{\lambda^{(i-1)}}}$. For the base step, we have that for any A_t, B_t, $B_t A_t = 0$. Moreover $\lambda^{(t-1)}$ is a partition with a single column, so $C_{\lambda^{(t-1)}} = \{0\}$ and $\overline{C_{\lambda^{(t-1)}}} = \{0\}$, therefore we are done.

For the induction step, we use lemma 4.2.3. We may assume that $B_{i+1}A_{i+1} \in \overline{C_{\lambda^{(i+1)}}}$. As $A_i B_i = B_{i+1} A_{i+1}$, we have that $(A_i, B_i) \in N_{\lambda^{(i)}}$. By lemma 4.2.3, $B_i A_i \in \overline{C_{\lambda^{(i)}}}$. Moreover, still by lemma 4.2.3, Θ_i maps Z_{red} onto $\overline{C_{\lambda^{(i)}}}$, so we are done. ∎

Remark. As Θ is G-equivariant, we get that Z_{red} is a disjoint union of the G-invariant subspaces $\Theta^{-1}(C_\mu) \subseteq Z_{\mathrm{red}}$, where μ is any partition such that $\mu \leq \lambda$.

We may now state the following consequence of theorem 4.2.2.

Proposition 5.1.4. *The map $\Theta : Z_{\mathrm{red}} \to \overline{C_\lambda}$ is a quotient under H.*

Proof. In order to see that Θ is a quotient under H, we perform the quotients of each $\mathrm{GL}(V_i)$ in succession.

First, we need to define the intermediate spaces and maps. For each $i = 1, \ldots, t$ we define the vector spaces

$$\begin{aligned} M_i &:= L(V_0, V_1) \times L(V_1, V_0) \times \cdots \times L(V_{i-1}, V_i) \times L(V_i, V_{i-1}) \\ N_i &:= L(V_1) \times \cdots \times L(V_i); \end{aligned}$$

the maps

$$\begin{aligned} \Phi_i :& M_i \to N_i \\ & (A_1, B_1, \ldots, A_i, B_i) \mapsto (A_1 B_1 - B_2 A_2, \ldots, A_{i-1} B_{i-1} - B_i A_i, A_i B_i); \end{aligned}$$

the varieties

$$Z_i := \left(\Phi_i^{-1}(0 \times \cdots \times 0 \times \overline{C_{\lambda^{(i)}}})\right)_{\mathrm{red}};$$

the maps

$$\begin{aligned} \vartheta_i :& M_i \to M_{i-1} \times L(V_{i-1}) \\ & (A_1, B_1, \ldots, A_{i-1}, B_{i-1}, A_i, B_i) \mapsto (A_1, B_1, \ldots, A_{i-1}, B_{i-1}, B_i A_i); \end{aligned}$$

the maps

$$\begin{aligned} \gamma_i :& M_i \to L(V_i) \\ & (A_1, B_1, \ldots, A_i, B_i) \mapsto A_i B_i \end{aligned}$$

and the varieties $\Gamma_i \subseteq M_i \times L(V_i)$ given by the graphs of γ_i. Notice that $M = M_t$, $N = N_t$, $\Phi = \Phi_t$, $Z_{\mathrm{red}} = Z_t$.

We remark on some property of this construction. The group G acts on each M_i, N_i, Z_i as usual by change of basis. The map ϑ_i is a product of the identity on M_{i-1} and, by theorem 4.2.2, of a quotient under the group $\mathrm{GL}(V_i)$; therefore, by proposition 1.3.7, ϑ_i is a quotient under the group $\mathrm{GL}(V_i)$. Moreover, by proposition 1.3.6, as Z_i is closed and G-invariant, also the restrictions $\vartheta_i|_{Z_i}$ are quotients under $\mathrm{GL}(V_i)$.

As $B_i A_i = A_{i-1} B_{i-1}$ for the points in Z_i,

$$\vartheta_i(Z_i) = \Gamma_{i-1} \cap (Z_{i-1} \times L(V_{i-1})). \tag{5.1.1}$$

Moreover, $\Gamma_i \simeq M_i$, as Γ_i is a graph of a function; similarly, as $\Gamma_{i-1} \cap (Z_{i-1} \times L(V_{i-1}))$ is the graph of the restriction $\gamma_i|_{Z_{i-1}}$, we also have that

$$\Gamma_{i-1} \cap (Z_{i-1} \times L(V_{i-1})) \simeq Z_{i-1}. \tag{5.1.2}$$

We prove by induction on $i = t, t-1, \ldots, 1$ that there is a quotient map

$$Z_{\mathrm{red}} \to Z_i$$

under the group $\mathrm{GL}(V_{i+1}) \times \cdots \times \mathrm{GL}(V_t)$. The base step $i = t$ is tautological as $Z_t = Z_{\mathrm{red}}$.

For the induction step, we may assume that we have a quotient map $Z_{\mathrm{red}} \to Z_{i+1}$. By (5.1.1) and (5.1.2), we get a map $Z_{i+1} \to Z_i$ which is a quotient under $GL(V_{i+1})$, as ϑ_{i+1} is a quotient under $GL(V_{i+1})$, while (5.1.2) is an isomorphism. We conclude the induction by composing the two maps.

In the end we have a quotient $Z_{\mathrm{red}} \to Z_1$ under $\mathrm{GL}(V_2) \times \cdots \times \mathrm{GL}(V_t)$. We remark that

$$Z_1 = \{(A_1, B_1) \in L(V_0, V_1) \times L(V_1, V_0) : A_1 B_1 \in \overline{C_{\lambda^{(1)}}}\} = N_{\lambda^{(0)}} = N_\lambda.$$

The conclusion of the proof follows as $\rho : N_\lambda \to \overline{C_\lambda}$ is a quotient under $\mathrm{GL}(V_1)$ by theorem 4.2.2. □

Remark. In order to display the reasoning in a more suggestive way, we propose the construction of the varieties Z_{red} and Z_i for each $i = 1, \ldots, t$ seen in the proof of proposition 5.1.4 with a different picture.

We inductively build varieties $Z(i, j)$ as the reduced fibre product, starting from the varieties $N_{\lambda^{(k)}}$ as shown in the following diagram (here we show an example with $t = 4$):

$$\begin{array}{ccccccccc}
Z(0,4) & \longrightarrow & Z(0,3) & \longrightarrow & Z(0,2) & \longrightarrow & N_{\lambda^{(0)}} & \longrightarrow & \overline{C_{\lambda^{(0)}}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
Z(1,4) & \longrightarrow & Z(1,3) & \longrightarrow & N_{\lambda^{(1)}} & \longrightarrow & \overline{C_{\lambda^{(1)}}} & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
Z(2,4) & \longrightarrow & N_{\lambda^{(2)}} & \longrightarrow & \overline{C_{\lambda^{(2)}}} & & & & \\
\downarrow & & \downarrow & & & & & & \\
N_{\lambda^{(3)}} & \longrightarrow & \overline{C_{\lambda^{(3)}}} & & & & & & \\
\downarrow & & & & & & & & \\
\overline{C_{\lambda^{(4)}}} & & & & & & & &
\end{array} \tag{5.1.3}$$

Indeed, we have that $Z_i \simeq Z(0, i)$, $Z_{\mathrm{red}} \simeq Z(0, t)$ and, by proposition 1.3.7, each map in the diagram is a quotient under some group $\mathrm{GL}(V_i)$ (the group is constant on each column or on each row). In particular we get that the compositions along the rows $Z(i, j) \to \overline{C_{\lambda^{(i)}}}$ are quotients under the products $\mathrm{GL}(V_j) \times \cdots \times \mathrm{GL}(V_{i+1})$.

5.1.4 Smooth points in Z

We need to get some information on the smooth points of Z. Let us consider the following subset of M:

$$M^0 := \{(A_1, B_1, \ldots, A_t, B_t) \in M : \forall i = 1, \ldots, t, A_i \text{ or } B_i \text{ has maximal rank}\}.$$

As the condition $\operatorname{rk} D \geq k$ is open, M^0 clearly is a (non-empty) open subvariety of M.

As M and N are affine varieties, for each point $x \in M$, $y \in N$, we may identify the tangent spaces $T_M(x)$ (resp. $T_N(y)$) as M (resp. N) itself. We have the following.

Proposition 5.1.5. *The differential $d\Phi : T_M \to T_N$ of the map $\Phi : M \to N$ is onto at every point $x \in M^0$.*

Proof. Let $x = (A_1, B_1, \dots, A_t, B_t) \in M^0$ be a point, $v = (X_1, Y_1, \dots, X_t, Y_t) \in M = T_M(x)$ be a vector. We have that the differential $d\Phi$ at the point x is:

$$d\Phi_x(v) = (A_1Y_1 + X_1B_1 - B_2X_2 - Y_2A_2, \dots, A_{t-1}Y_{t-1} + X_{t-1}B_{t-1} - B_tX_t - Y_tA_t).$$

Let $w = (W_1, W_2, \dots, W_t) \in N = T_N(\Phi(x))$ be any vector. We claim that there exists at least a vector $v \in T_M(x)$ such that $d\Phi_x(v) = w$. In order to prove this, it is enough to solve the following linear system of matrices in X_i, Y_i:

$$\begin{aligned}
&A_1Y_1 + X_1B_1 - B_2X_2 - Y_2A_2 = W_1\\
&\quad\vdots\\
&A_iY_i + X_iB_i - B_{i+1}X_{i+1} - Y_{i+1}A_{i+1} = W_i\\
&\quad\vdots\\
&A_{t-1}Y_{t-1} + X_{t-1}B_{t-1} - B_tX_t - Y_tA_t = W_{t-1}.
\end{aligned}$$

We build a solution recursively on $i = t, t-1, \dots, 1$, starting from $X_t = Y_t = 0$.

We may assume that we have already built maps $X_{i+1}, Y_{i+1}, \dots, X_t, Y_t$. As $x \in M^0$, at least one of the maps A_i, B_i has maximal rank. In order to find maps X_i, Y_i, as the two conditions are completely analogous, we suppose that A_i has maximal rank. This implies that A_i is onto, as $\dim V_{i-1} > \dim V_i$. In particular, there exists a right inverse map $\tilde{A}_i : V_i \to V_{i-1}$ of A_i, so that $A_i\tilde{A}_i = 1_{V_{i-1}}$. We set $X_i = 0$, so the i-th equation becomes:

$$A_iY_i = B_{i+1}X_{i+1} + Y_{i+1}A_{i+1} + W_i.$$

A solution for Y_i is given by

$$Y_i = \tilde{A}_i\left(B_{i+1}X_{i+1} + Y_{i+1}A_{i+1} + W_i\right).$$

(We remark that in the other case, B_i is injective, so there exists a left inverse map $\tilde{B}_i : V_{i-1} \to V_i$, so we solve the equation $X_iB_i = R_i$ by setting $X_i = R_i\tilde{B}_i$.) $\square$

Let $Z^0 := Z \cap M^0$. The main consequence that we obtain is the following corollary.

Corollary 5.1.6. *The subvariety $Z^0 \subseteq Z$ is open, non-empty,*

$$\operatorname{codim}_M(Z^0) = \dim N$$

and it is contained in the smooth locus of Z.

Proof. It is clear that Z^0 is open. As soon as we prove that it is not empty, the statement about the codimension follows at once as a consequence of the previous proposition, while the fact that Z^0 is smooth follows directly by proposition 5.1.5.

In order to prove that Z^0 is non-empty, we show there exists a point $z \in Z^0$. Let $D \in C_\lambda$. For each $i = 1, \ldots, t$, as $\dim V_i = |\lambda^{(i)}| = \operatorname{rk} D^i$, we may identify $V_i \simeq D^i(V_0)$. Set

$$A_i = D|_{D^{i-1}(V_0)} = D|_{V_{i-1}}$$

(changing the codomain from V_0 to $D^i(V_0) = V_i$) and

$$B_i : D^i(V_0) \hookrightarrow D^{i-1}(V_0)$$

the canonical inclusion. We collect all these maps into $z = (A_1, B_1, \ldots, A_t, B_t) \in M$.

By construction, both A_i and B_i have maximal rank, therefore $z \in M^0$. Still by construction,

$$A_i B_i = D|_{D^i} = B_{i+1} A_{i+1}.$$

Therefore $z \in Z$ and we are done. □

Remark. One can ask to have both A_i and B_i with maximal rank. This way, one still finds an open subvariety $Z' \subseteq Z^0 \subseteq Z$, which is not empty as the point $z \in Z$ constructed in the proof of corollary 5.1.6 lies in Z'. It can be seen that this Z' is an orbit under the group G and, in fact, that it is the unique open G-orbit in Z.

5.2 For quadratic and symmetric spaces

In this section we want to mimic the construction of the variety Z to the case in which we equip each space V_i with a non-degenerate bilinear form. In this way we will construct an auxiliary variety Z useful for the cases of quadratic spaces and symmetric spaces, which were not covered with the construction done in section 5.1.

Although some details in the construction of the variety Z are different, the main properties remain true. Namely, we will prove the existence of a quotient map $Z \to \overline{C_\lambda}$ and we will find an open subvariety $Z^0 \subseteq Z$ contained in the smooth locus of Z of an adequate dimension.

5.2.1 $Z_{(\varepsilon_0, \varepsilon_1)}$ as a schematic fibre

As in section 5.1.1, for each $i = 0, \ldots, t$, we fix a vector space V_i of dimension n_i. Moreover, for each $i = 0, \ldots, t$, we fix signs $\varepsilon_i \in \{+1, -1\}$ and non-degenerate quadratic forms $(-, -)_i : V_i \times V_i \to k$ of type ε_i (clearly n_i must be even if $\varepsilon_i = -1$). We define the vector space

$$M^\sigma := L^\sigma_{V_0,V_1} \times L^\sigma_{V_1,V_2} \times \cdots \times L^\sigma_{V_{t-1},V_t}.$$

We denote an element $x \in M^\sigma$ as

$$x = (A_1, A_1^*, A_2, A_2^*, \ldots, A_t, A_t^*).$$

Remark. We remark that we need not carry the adjoint maps A_i^*, as the map

$$\begin{aligned} L(V_0, V_1) \times L(V_1, V_2) \times \cdots \times L(V_{t-1}, V_t) &\to M^\sigma \\ (A_1, A_2, \ldots, A_t) &\mapsto x \end{aligned}$$

is onto. In spite of this redundancy, we will use the former notation, as we believe that the groups actions will be more clear in this way.

Before we can introduce the vector space N^σ and the map Φ, we need to take care of the choice of the signs ε_i. By corollary 4.4.5, the maps π, ρ restricted to the spaces $L^\sigma_{V_i, V_{i+1}}$ are not onto, in fact their images are only symmetric or skew-symmetric depending on the sign of the product $\varepsilon_i \varepsilon_{i+1}$. One may notice that if $\varepsilon_i = \varepsilon_{i+2}$, each product $A_i A_i^*$ and $A_i^* A_i$ is consistently symmetric or skew-symmetric for each $i = 0, \ldots, t$, depending only on $\varepsilon_0 \varepsilon_1$. From now on, we assume that

$$\varepsilon_i = \varepsilon_{i+2} \quad \forall i = 0, 1, \ldots, t-2$$

so that each sign is fixed by the choice of ε_0, ε_1 and we denote $\tilde{\varepsilon} := \varepsilon_0 \varepsilon_1$.

Remark. As $n_i = \dim V_i$ must be even if $\varepsilon_i = -1$, one gets the following restriction on λ, depending on the choice of $(\varepsilon_0, \varepsilon_1) \in \{\pm 1\}^2$. (Recall definition 2.3.2.)

$(\varepsilon_0, \varepsilon_1) = (1, 1)$ In this case there are no restrictions, so $\lambda \in \mathcal{P}(n) = \mathcal{P}_{1,1}(n)$.

$(\varepsilon_0, \varepsilon_1) = (1, -1)$ In this case, n_i must be even for i odd. Therefore $\lambda \in \mathcal{P}_{1,0}(n)$.

$(\varepsilon_0, \varepsilon_1) = (-1, 1)$ In this case, n_i must be even for i even. Therefore $\lambda \in \mathcal{P}_{-1,0}(n)$.

$(\varepsilon_0, \varepsilon_1) = (-1, -1)$ In the last case n_i must always be even, therefore $\lambda \in \mathcal{P}_{-1,1}(n)$.

The proof of this fact is the same used in theorem 2.3.3.

Finally, we can define the vector space

$$N^\sigma := L^{\tilde{\varepsilon}}(V_1) \times L^{\tilde{\varepsilon}}(V_2) \times \cdots \times L^{\tilde{\varepsilon}}(V_{t-1})$$

and the map $\Phi^\sigma : M^\sigma \to N^\sigma$ by

$$\Phi^\sigma : (A_1, A_1^*, \ldots, A_t, A_t^*) \mapsto (A_1 A_1^* - A_2^* A_2, \ldots, A_{t-1} A_{t-1}^* - A_t^* A_t).$$

Definition 5.2.1. Let $\varepsilon_0, \varepsilon_1 \in \{\pm 1\}$ and $\lambda \in \mathcal{P}_{\varepsilon,\eta}(n)$ (where (ε, η) agrees with $(\varepsilon_0, \varepsilon_1)$). We define the variety

$$Z_{(\varepsilon_0, \varepsilon_1)} := Z^{(\lambda)}_{(\varepsilon_0, \varepsilon_1)} := \Phi^{-1}(0)$$

as the schematic fibre over the point $0 = (0, 0, \ldots, 0) \in N^\sigma$.

5.2.2 The map Θ^σ

One may interpret $Z_{(\varepsilon_0,\varepsilon_1)}$ as the fixed point set in the variety Z of definition 5.1.1 with respect to an involution. In fact we define involutions on the spaces M and N, too.

Let $\sigma : M \to M$ be a map defined by

$$(A_1, B_1, \dots, A_t, B_t) \mapsto (B_1^*, A_1^*, \dots, B_t^*, A_t^*);$$

let (still) $\sigma : N \to N$ be a map defined by

$$(X_1, \dots, X_{t-1}) \mapsto (\tilde{\varepsilon}X_1^*, \dots, \tilde{\varepsilon}X_{t-1}^*).$$

Then σ is an involution on both M and N, it can be thought as a $\mathbb{Z}/2$-action and the map $\Phi : M \to N$ is σ-equivariant. Moreover $M^\sigma \subseteq M$ and $N^\sigma \subseteq N$ are the fixed point loci of this action, so Φ may be restricted to the map $\Phi^\sigma : M^\sigma \to N^\sigma$, whose fibres $(\Phi^\sigma)^{-1}(n)$ are the fixed point set of the fibres $\left(\Phi^{-1}(n)\right)^\sigma$. In particular

$$Z_{(\varepsilon_0,\varepsilon_1)} = Z^\sigma.$$

One may define an $\mathbb{Z}/2$-action on the structural group G of the quiver Q_t by

$$g^\sigma = (g_0, \dots, g_t)^\sigma := \left((g_0^{-1})^*, \dots, (g_t^{-1})^*\right) \quad \forall g \in G.$$

It is clear that this action is compatible with the action on M, as $\sigma(g.\sigma(x)) = g^\sigma.x$ for every $x \in M$, $g \in G$. The fixed point group is

$$G^\sigma \simeq K_{\varepsilon_0}(V_0) \times \dots \times K_{\varepsilon_t}(V_t).$$

Definition 5.2.2. The morphism $\Theta : Z_{\mathrm{red}} \to L(V_0)$ of definition 5.1.2 restricts to a morphism

$$\begin{aligned} \Theta^\sigma : &Z_{(\varepsilon_0,\varepsilon_1)\,\mathrm{red}} \to L^{\tilde{\varepsilon}}(V_0) \\ &(A_1, A_1^*, \dots, A_t, A_t^*) \mapsto A_1^* A_1. \end{aligned}$$

Remark. The map Θ^σ is H^σ-invariant and $K_{\varepsilon_0}(V_0) \simeq G^\sigma/H^\sigma$-equivariant.

Proposition 5.2.3. $\Theta^\sigma(Z_{(\varepsilon_0,\varepsilon_1)\,\mathrm{red}}) = \overline{C_{\varepsilon,\eta,\lambda}}$.

Proof. The involution σ acts on $L(V_0)$ as a quotient of N. Therefore, by (3.2.4),

$$\Theta^\sigma(Z_{(\varepsilon_0,\varepsilon_1)\,\mathrm{red}}) \subseteq (\Theta(Z_{\mathrm{red}}))^\sigma = \overline{C_\lambda}^\sigma = \overline{C_{\varepsilon,\eta,\lambda}}.$$

The reverse inclusion may be proved in the same way as the proof of proposition 5.1.3, except we use lemma 4.4.6 instead of lemma 4.2.3. □

Proposition 5.2.4. *The map $\Theta^\sigma : Z_{(\varepsilon_0,\varepsilon_1)\,\mathrm{red}} \to \overline{C_{\varepsilon,\eta,\lambda}}$ is a quotient under H^σ.*

Proof. The proof of this proposition is the same as the proof of proposition 5.1.4. Just use theorem 1.3.4 and theorem 1.3.5 instead of theorem 4.2.2. □

Remark. One may still describe $Z_{(\varepsilon_0,\varepsilon_1)\,\mathrm{red}}$ as a fibre product.

We inductively build varieties $Z_{(\varepsilon_0,\varepsilon_1)}(i,j)$ as the reduced fibre product, starting from the varieties $N^\sigma_{\lambda(k)}$ as shown in the following diagram (here we show an example with $t = 4$, $\varepsilon_0 = 1$, $\varepsilon_1 = 1$):

$$\begin{array}{ccccccccc}
Z_{(1,1)}(0,4) & \longrightarrow & Z_{(1,1)}(0,3) & \longrightarrow & Z_{(1,1)}(0,2) & \longrightarrow & N^\sigma_{\lambda(0)} & \longrightarrow & \overline{C_{1,1,\lambda(0)}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
Z_{(1,1)}(1,4) & \longrightarrow & Z_{(1,1)}(1,3) & \longrightarrow & N^\sigma_{\lambda(1)} & \longrightarrow & \overline{C_{1,1,\lambda(1)}} & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
Z_{(1,1)}(2,4) & \longrightarrow & N^\sigma_{\lambda(2)} & \longrightarrow & \overline{C_{1,1,\lambda(2)}} & & & & \\
\downarrow & & \downarrow & & & & & & \\
N^\sigma_{\lambda(3)} & \longrightarrow & \overline{C_{1,1,\lambda(3)}} & & & & & & \\
\downarrow & & & & & & & & \\
\overline{C_{1,1,\lambda(4)}} & & & & & & & &
\end{array}$$

Again, $Z_{(\varepsilon_0,\varepsilon_1)\,\mathrm{red}} \simeq Z_{(\varepsilon_0,\varepsilon_1)}(0,t)$, each map in the diagram is a quotient under some group $K_{\varepsilon_i}(V_i)$ (the group is constant on each column or on each row), the compositions along the rows $Z_{(\varepsilon_0,\varepsilon_1)}(i,j) \to \overline{C_{\lambda(i)}}$ are quotients under the products $K_{\varepsilon_j}(V_j) \times \cdots \times K_{\varepsilon_{i+1}}(V_{i+1})$.

5.2.3 Smooth points in $Z_{(\varepsilon_0,\varepsilon_1)}$

Let
$$M^{\sigma,0} := M^\sigma \cap M^0$$
be the open set of the points $(A_1, A_1^*, \dots, A_t, A_t^*) \in M^\sigma$ with maximal rank (both A_i and A_i^* must have maximal rank, by lemma 4.4.3).

Proposition 5.2.5. *The differential $d\Phi^\sigma : T_{M^\sigma} \to T_{N^\sigma}$ of the map $\Phi^\sigma : M^\sigma \to N^\sigma$ is onto at every point $x \in M^{\sigma,0}$.*

Proof. If $x \in M^\sigma$, the linear map $d\Phi^\sigma_x : T_{x,M^\sigma} \to T_{\Phi(x),N^\sigma}$ is the image of the linear map $d\Phi_x : T_{x,M} \to T_{\Phi(x),N}$ through the functor of the $\mathbb{Z}/2$-invariants
$$V \mapsto V^\sigma.$$
This functor may be expressed as $\mathrm{Hom}_{\mathbb{Z}/2}(k, -)$ (where k is the trivial $k[\mathbb{Z}/2]$-module), therefore one gets its exactness as soon as k is a projective $k[\mathbb{Z}/2]$-module. This is obviously true as $\mathbb{Z}/2$ is finite (and $\operatorname{char} k = 0$). Therefore $d\Phi^\sigma_x$ is surjective on each point x such that $d\Phi_x$ is surjective. The latter certainly happens on each point $x \in M^{\sigma,0} \subseteq M^0$, by proposition 5.1.5. □

Let $Z^0_{(\varepsilon_0,\varepsilon_1)} := Z_{(\varepsilon_0,\varepsilon_1)} \cap M^{\sigma,0}$.

Corollary 5.2.6. *The subvariety* $Z^0_{(\varepsilon_0,\varepsilon_1)} \subseteq Z_{(\varepsilon_0,\varepsilon_1)}$ *is open, non-empty,*

$$\operatorname{codim}_{M^\sigma}(Z^0_{(\varepsilon_0,\varepsilon_1)}) = \dim N^\sigma$$

and it is contained in the smooth locus of $Z_{(\varepsilon_0,\varepsilon_1)}$.

Proof. As in corollary 5.1.6, one needs only prove that $Z^0_{(\varepsilon_0,\varepsilon_1)}$ is non-empty.

Let $D \in C_{\varepsilon,\eta,\lambda} \subseteq \mathfrak{g}_{\varepsilon,\eta}$. For each $i = 1, \dots, t$, as $\dim V_i = |\lambda^{(i)}| = \operatorname{rk} D^i$, we may identify $V_i \simeq D^i(V_0)$. The space V_0 is already equipped with a non-degenerate bilinear form $(-,-)_0$ of type $\varepsilon = \varepsilon_0$. For each $i = 1, \dots, t$, we introduce bilinear forms in V_i by first defining forms $\langle -,- \rangle_i$ on V_0 by

$$\langle u, v\rangle_i := (u, D^i v)_0 \quad \forall u, v \in V_0.$$

As $D^* = \tilde{\varepsilon} D$, the kernel of $\langle -,- \rangle_i$ is clearly $\ker D^i$. Hence, the form $\langle -,- \rangle_i$ induces a non-degenerate bilinear form $(-,-)_i$ on the quotient space $V_0/\ker D^i \simeq D^i(V_0) \simeq V_i$, for each $i = 1, \dots, t$. As $D^* = \tilde{\varepsilon} D$, $\langle -,- \rangle_i$ has type $\varepsilon_0 \tilde{\varepsilon}^i = \varepsilon_i$, therefore $(-,-)_i$ has type ε_i.

Set

$$A_i = D|_{D^{i-1}(V_0)} = D|_{V_{i-1}}$$

(changing the codomain from V_0 to $D^i(V_0) = V_i$) and

$$B_i : D^i(V_0) \hookrightarrow D^{i-1}(V_0)$$

the canonical inclusion. We check that $A_i^* = B_i$:

$$\begin{array}{rll} (A_i u, v)_i = (u, B_i v)_{i-1} & \forall u \in D^{i-1}(V_0), v \in D^i(V_0) & \Leftrightarrow \\ (D^i \bar{u}, D^i \bar{v})_i = (D^{i-1}\bar{u}, D^i \bar{v})_{i-1} & \forall \bar{u}, \bar{v} \in V_0 & \Leftrightarrow \\ \langle \bar{u}, \bar{v}\rangle_i = \langle \bar{u}, D\bar{v}\rangle_{i-1} & \forall \bar{u}, \bar{v} \in V_0 & \Leftrightarrow \\ (\bar{u}, D^i \bar{v})_0 = (\bar{u}, D^i \bar{v})_0 & \forall \bar{u}, \bar{v} \in V_0. & \end{array}$$

Hence $z = (A_1, A_1^*, \dots, A_t, A_t^*) \in M^\sigma$. By construction, A_i has maximal rank, therefore $z \in M^{\sigma,0}$. Still by construction,

$$A_i A_i^* = D|_{D^i} = A_{i+1}^* A_{i+1}.$$

Therefore $z \in Z^0_{(\varepsilon_0,\varepsilon_1)}$ and we are done. □

5.3 Geometry of Z

In this section we drop the differences in notation between the case of the general linear group $(M, N, Z, \dots)$ and the case of quadratic spaces and symmetric spaces $(M^\sigma, N^\sigma, Z_{(\varepsilon_0,\varepsilon_1)}, \dots)$. We give all the proofs involving the geometry of the varieties Z. In particular we reduce the study of the normality of Z to the combinatorics of the stratification of the complement $Z \setminus Z^0$ of Z^0 in Z which will be carried on in the next chapter.

In each case we dealt with, we took a partition $\lambda \vdash n$, we built a scheme $Z \subseteq M$ and a map $Z \to \overline{C_\lambda} \subseteq \mathfrak{V}$ which is a quotient under a reductive group G. We also considered an open subscheme $Z^0 \subseteq Z$ such that $\operatorname{codim}_M Z^0 = \dim N$.

The results of this section depend on the following fact:

$$\dim(Z \setminus Z^0) \leq \dim Z^0 - 2. \tag{5.3.1}$$

In some cases, depending on λ, ε_0, ε_1, we might find that (5.3.1) is not true. In most of those cases, we are at least able to obtain

$$\dim(Z \setminus Z^0) < \dim Z^0. \tag{5.3.2}$$

Proposition 5.3.1. *Let n be a positive integer, let $\lambda \vdash n$ be a partition and let Z be the scheme introduced in section 5.1.1 or, fixed $\varepsilon_0, \varepsilon_1 \in \{\pm 1\}$, in section 5.2.1. Suppose that (5.3.2) holds. Then Z is reduced and complete intersection.*

Proof. By (5.3.2), one gets that $\dim Z = \dim Z^0$. In particular, by corollary 5.1.6 or by corollary 5.2.6, $\operatorname{codim}_M(Z) = \dim N$ which means that Z is complete intersection, as N is an affine space.

Since each irreducible component Z_i of Z must have $\dim Z_i \geq \dim M - \dim N = \dim Z$, Z_i must meet Z^0, therefore Z^0 is dense in Z. As Z^0 is contained in the smooth locus of Z (by corollary 5.1.6 or by corollary 5.2.6) the singular locus of Z is contained in $Z \setminus Z^0$ that has codimension at least 1 by (5.3.2). Since Z is complete intersection, Z is reduced by corollary 1.2.16. □

In particular we have that $Z_{\text{red}} = Z$.

Theorem 5.3.2. *In the same settings of proposition 5.3.1, additionally suppose that (5.3.1) holds. Then Z and $\overline{C_\lambda}$ are normal.*

Proof. By proposition 5.3.1, Z is complete intersection. By corollary 5.1.6 or by corollary 5.2.6, the singular locus of Z is contained in $Z \setminus Z^0$. By (5.3.1), it has codimension in Z at least 2. By corollary 1.2.16, Z is normal.

By proposition 5.1.4 or by proposition 5.2.4, $\overline{C_\lambda}$ is a quotient of Z under some reductive group G. By proposition 1.3.2, $\overline{C_\lambda}$ is normal as well. □

Remark. Since Z is defined with homogeneous equations via Φ, Z is always a connected variety. In the settings of theorem 5.3.2, i. e. if (5.3.1) holds, one gets that Z is normal, hence irreducible. Even without that hypothesis, one could obtain the same conclusion provided that the open subset Z^0 is irreducible, since Z^0 is dense, at least in the settings of proposition 5.3.1. We will see that this is often the case.

Chapter 6

Stratification of Z

In this chapter we inspect the complement of the open subvariety Z^0 in Z.

6.1 Sequences of ab-diagrams

Let $\lambda \in \mathcal{P}(n)$ be a partition. In section 5.1.1 we defined a scheme $Z \subseteq M$.

Lemma 6.1.1. *Let*

$$z = (A_1, B_1, \dots, A_t, B_t) \in Z.$$

Then, for every $i = 1, \dots, t$

$$(B_i A_i)^{t-i+1} = 0 \tag{6.1.1}$$

and

$$(A_i B_i)^{t-i} = 0. \tag{6.1.2}$$

Proof. As $z \in Z$, we have that $A_{i-1}B_{i-1} = B_i A_i$ for every i, so (6.1.1) and (6.1.2) are equivalent. We just need to prove (6.1.1).

We proceed by reverse induction on i. The base step, $i = t$ is trivial as $A_t = B_t = 0$. For the induction step, let us suppose that $(B_{i+1}A_{i+1})^{t-i} = 0$. Therefore

$$(B_i A_i)^{t-i+1} = B_i (A_i B_i)^{t-i} A_i = B_i (B_{i+1} A_{i+1})^{t-i} A_i = B_i 0 A_i = 0.$$

□

In section 5.1.3, we showed the existence of an action of the structural group G of the quiver Q_t on M and on Z. For every $i = 1, \dots, t$, G acts on L_{V_{i-1},V_i} as well and the projection map p_i such that

$$\begin{aligned} M &\to L_{V_{i-1},V_i} \\ (A_1, B_1, \dots, A_t, B_t) &\mapsto (A_i, B_i), \end{aligned}$$

is G-equivariant. As a consequence of lemma 6.1.1, one gets $p_i(Z) \subseteq N_{V_{i-1},V_i}$.

The action of G on L_{V_{i-1},V_i} reduces to the action of $\mathrm{GL}(V_{i-1}) \times \mathrm{GL}(V_i)$. Since an ab-diagram is an invariant of a nilpotent pair in N_{V_{i-1},V_i} under the action $\mathrm{GL}(V_{i-1}) \times \mathrm{GL}(V_i)$, we may associate to a point $(A_1, B_1, \dots, A_t, B_t) \in Z$ an ab-diagram τ_i. Thus we have defined a sequence of G-invariants

$$(A_1, B_1, \dots, A_t, B_t) \mapsto (\tau_1, \dots, \tau_t) =: \tau.$$

There are some constraints in the sequence of ab-diagrams $\tau_1, \ldots, \tau_t$. Recall the Young diagrams of definition 4.3.2.

Lemma 6.1.2. *Let $\tau = (\tau_1, \ldots \tau_t)$ be a sequence of ab-diagrams corresponding to a point $(A_1, B_1, \ldots, A_t, B_t) \in Z$. Then*

1. *τ_i has n_{i-1} as and n_i bs for each $i = 1, \ldots, t$;*
2. *$\pi(\tau_{i+1}) = \rho(\tau_i)$ for each $i = 1, \ldots, t-1$.*

Proof. The first claim follows as τ_i is the ab-diagram of a pair $(A_i, B_i) \in N_{V_{i-1},V_i}$, therefore the number of as (resp. bs) in τ_i is $\dim V_{i-1} = n_{i-1}$ (resp. $\dim V_i = n_i$).

As $(A_1, B_1, \ldots, A_t, B_t) \in Z$, $A_iB_i = B_{i+1}A_{i+1}$ for each $i = 1, \ldots, t-1$. Since $\rho(\tau_{i-1})$ (resp. $\pi(\tau_i)$) is the Young diagram of the nilpotent endomorphism $A_{i-1}B_{i-1}$ (resp. B_iA_i), the second claim follows as well. □

Let

$$\Lambda := \{(\tau_1, \ldots, \tau_t) : |\pi(\tau_i)| = n_{i-1}, |\rho(\tau_i)| = n_i; \pi(\tau_{i+1}) = \rho(\tau_i)\ \forall i\}$$

be the set of sequences of ab-diagrams satisfying the conditions of lemma 6.1.2. It is clear that the sequences in Λ are exactly the sequences of G-invariants that appear to be associated to at least one point $z \in Z$.

Definition 6.1.3. Let $\tau = (\tau_1, \ldots, \tau_t) \in \Lambda$. We define the subvariety

$$Z_\tau := \{(A_1, B_1, \ldots, A_t, B_t) \in Z : \text{ the } ab\text{-diagram of } (A_i, B_i) \text{ is } \tau_i\ \forall i = 1, \ldots, t\}.$$

We want to prove that $(Z_\tau)_{\tau \in \Lambda}$ is a *stratification* of Z, i. e. a decomposition of Z into finitely many pairwise disjoint, locally closed, smooth subsets called *strata* such that the closure of a stratum is a union of strata.

Proposition 6.1.4. *For each $\tau \in \Lambda$, Z_τ is a locally closed, G-stable, smooth and irreducible subvariety of Z. Moreover the varieties Z_τ form a partition of Z and the closure of each Z_τ is a union of $Z_{\tau'}$ for some $\tau' \in \Lambda$.*

Proof. It is clear from definition 6.1.3 that the sets Z_τ, $Z_{\tau'}$ are disjoint for $\tau \neq \tau'$ and that they partition Z.

Let $\tau = (\tau_1, \ldots, \tau_t) \in \Lambda$ and put $\mu_i = \pi(\tau_{i+1}) = \rho(\tau_i)$ for $i = 1, \ldots, t-1$ and $\mu = \mu_0 = \pi(\tau_1)$. One may see Z_τ as an iterated fibre product subjected to the diagram (5.1.3). We display a diagram (6.1.3) as an example with $t = 4$.

For each $i = 1, \ldots, t$, let $p_i : Z \to N_{V_{i-1},V_i}$ be the canonical projections. Clearly the closure $\overline{Z_\tau}$ is contained in the intersection of the closed subsets $p_i^{-1}(\overline{O_{\tau_i}})$. In fact Z_τ is open and dense inside this intersection, as O_{τ_i} is open and dense inside $\overline{O_{\tau_i}}$. Therefore Z_τ is locally closed.

Moreover, since $\overline{O_{\tau_i}}$ is a union of $\mathrm{GL}(V_{i-1}) \times \mathrm{GL}(V_i)$-orbits O_δ for some ab-diagrams δ, closure $\overline{Z_\tau}$ is the union of $Z_{\tau'}$ for $\tau' = (\tau_1', \ldots \tau_t') \in \Lambda$ such that $O_{\tau_i'} \subseteq \overline{O_{\tau_i}}$.

Since O_{τ_i} is G-stable for each $i = 1, \ldots, t$ and the projections π_i are G-equivariant, each Z_τ is G-stable.

$$\begin{array}{ccccccccc}
Z_\tau & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & O_{\tau_1} & \longrightarrow & C_{\mu_0}\\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
Z_{\tau_2,\tau_3,\tau_4} & \longrightarrow & \cdot & \longrightarrow & O_{\tau_2} & \longrightarrow & C_{\mu_1} & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
Z_{\tau_3,\tau_4} & \longrightarrow & O_{\tau_3} & \longrightarrow & C_{\mu_2} & & & & \\
\downarrow & & \downarrow & & & & & & \\
O_{\tau_4} & \longrightarrow & C_{\mu_3} & & & & & & \\
\downarrow & & & & & & & & \\
C_{\mu_4} & & & & & & & &
\end{array} \tag{6.1.3}$$

As shown in (4.3.2), the maps

$$\begin{array}{ccc}
O_{\tau_i} & \xrightarrow{\ \pi\ } & C_{\mu_{i-1}}\\
\downarrow{\scriptstyle \rho} & & \\
C_{\mu_i} & &
\end{array}$$

are smooth, so each map in the diagram (6.1.3) is smooth. In particular Z_τ is a smooth subvariety of Z since fibre products preserve smooth maps by theorem 1.4.7.

Since O_{τ_i} is a $\mathrm{GL}(V_{i-1}) \times \mathrm{GL}(V_i)$-homogeneous space and $\mathrm{GL}(V_{i-1}) \times \mathrm{GL}(V_i)$ is connected and fibre products of connected spaces is connected, Z_τ is connected as well. The irreducibility of Z_τ follows from the connectedness and the smoothness. □

We will refer to Z_τ as the *strata* of Z.

Remark. In spite of O_{τ_i} being G-homogeneous, Z_τ is not G-homogeneous. Let $z = (A_1, B_1, \ldots, A_t, B_t)$. Clearly, for every path in the quiver Q_t, the rank of the corresponding composition of the maps A_i, B_i is G-invariant. On the other hand, the list of invariants $(\tau_1, \ldots, \tau_t)$ considers only the ranks of the paths between two consecutive vertices of Q_t.

As an example, take $\lambda = (3)$, so that $n_0 = 3$, $n_1 = 2$, $n_2 = 1$, $n_3 = 0$ and take the points $z, z' \in Z$ given by

$$\begin{array}{llll}
A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & B_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} & A_2 = \begin{pmatrix} 1 & 0 \end{pmatrix} & B_2 = 0,\\
A_1' = A_1 & B_1' = B_1 & A_2' = \begin{pmatrix} 0 & 1 \end{pmatrix} & B_2' = 0.
\end{array}$$

Clearly both $z, z' \in Z_\tau$ where

$$\tau = \left(\begin{array}{l|l|l} aba & ab & a \\ a & a & \\ b & & \end{array} \right)$$

but $\operatorname{rk} A_2 A_1 = 1 \neq 0 = \operatorname{rk} A_2' A_1'$.

Remark. One may actually find infinitely many G-orbits inside Z as soon as Q_t has more than two vertices. Take $t = 2$ so that Q_2 is

$$0 \underset{B_1}{\overset{A_1}{\rightleftarrows}} 1 \underset{B_2}{\overset{A_2}{\rightleftarrows}} 2.$$

We choose the dimension vector $\underline{n} = (2, 2, 2)$. For every $x \in k$ we define the representation $M(x) \in \operatorname{Rep}(Q_2, \underline{n})$ by

$$A_1(x) = \begin{pmatrix} 0 & 1 \\ 0 & x \end{pmatrix} \quad B_1(x) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad A_2(x) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad B_2(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The representations $M(x)$, $M(y)$ lie in the same G-orbit if and only if there are $g_1, g_2, g_3 \in \mathrm{GL}_2$ such that $(g_1, g_2, g_3).M(x) = M(y)$. By the definition of the action of G in $\operatorname{Rep}(Q_2, \underline{n})$ one gets the the following linear equations on the entries of g_1, g_2, g_3:

$$g_2 \begin{pmatrix} 0 & 1 \\ 0 & x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & y \end{pmatrix} g_1$$
$$g_1 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} g_2$$
$$g_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} g_2$$
$$g_2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} g_3$$

From the last two equations, one gets that g_2 is diagonal. From the second equation, one gets that g_2 is scalar and g_1 is upper triangular. Let $a = [g_2]_1^1 = [g_2]_2^2$ and $b = [g_1]_2^2$. From the first equation, checking only the second column, one gets that $a = b$ and $ax = yb$. As $a \neq 0$, one gets that $M(x)$ and $M(y)$ are equivalent if and only if $x = y$.

Therefore, there are infinitely many G-orbit in $\operatorname{Rep}(Q_2, \underline{n})$. One may notice that the representations $M(x)$ satisfy the relations needed for the variety Z, as $A_1(x)B_1(x) = 0 = B_2(x)A_2(x)$ and that both pairs $(A_1(x), B_1(x))$, $(A_2(x), B_2(x))$ are nilpotent. Therefore we actually find the representations $M(x)$ in the variety $Z^{(\lambda)}$ for some partition $\lambda \vdash n$, modulo adding a null summand with suitable dimensions. Taking $\lambda = (4)$ is such an example.

The fibre product in diagram (6.1.3) implies other facts like the following propositions. We recall the map $\Theta : Z \to \overline{C_\lambda}$.

Proposition 6.1.5. *For each $\tau = (\tau_1, \ldots, \tau_t) \in \Lambda$, $\Theta(Z_\tau) = C_{\pi(\tau_1)}$.*

Proof. By (6.1.3), the map Θ is the composition of the canonical projection $Z_\tau \to O_{\tau_0}$ and the quotient map $O_{\tau_0} \to C_{\pi(\tau_0)}$, therefore is onto as it is the composition of surjective maps. $\square$

We recall definition 4.6.1.

Proposition 6.1.6. *Let* $\tau = (\tau_1, \ldots, \tau_t) \in \Lambda$. *Then*

$$\dim Z_\tau = \frac{1}{2} \dim C_{\pi(\tau_1)} + \sum_{i=1}^{t} \left(n_{i-1} n_i - \Delta(\tau_i) \right).$$

Proof. In a cartesian fibre product

$$\begin{array}{ccc} X \times_S Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & S, \end{array} \tag{6.1.4}$$

if both maps $X \to S$, $Y \to S$ are smooth, it is well known that one has

$$\dim X \times_S Y - \dim X = \dim Y - \dim S. \tag{6.1.5}$$

Since Z_τ is an iterated fibre product, one may compute its dimension by summing the dimension of the factor O_{τ_i} for $i = 1, \ldots, t$ and subtract the dimension of the base spaces C_{μ_i} for $i = 1, \ldots, t-1$. By proposition 4.6.2 one gets

$$\begin{aligned}
\dim Z_\tau =& \sum_{i=1}^{t} \dim O_{\tau_i} - \sum_{i=1}^{t-1} \dim C_{\mu_i} \\
=& \sum_{i=1}^{t} \left(\frac{1}{2} (\dim C_{\pi(\tau_i)} + \dim C_{\rho(\tau_i)}) + n_{i-1} \cdot n_i - \Delta(\tau_i) \right) - \sum_{i=1}^{t-1} \dim C_{\mu_i} \\
=& \sum_{i=1}^{t} \left(\frac{1}{2} (\dim C_{\mu_i} + \dim C_{\mu_{i-1}}) - \dim C_{\mu_i} \right) + \dim C_{\mu_t} + \\
& + \sum_{i=1}^{t} (n_{i-1} n_i - \Delta(\tau_i)) \\
=& \frac{1}{2} \sum_{i=1}^{t} (\dim C_{\mu_{i-1}} - \dim C_{\mu_i}) + \sum_{i=1}^{t} (n_{i-1} n_i - \Delta(\tau_i)) \\
=& \frac{1}{2} (\dim C_{\mu_0} - \dim C_{\mu_t}) + \sum_{i=1}^{t} (n_{i-1} n_i - \Delta(\tau_i)) \\
=& \frac{1}{2} \dim C_{\pi(\tau_1)} + \sum_{i=1}^{t} (n_{i-1} n_i - \Delta(\tau_i))
\end{aligned}$$

as $\dim C_{\mu_t} = 0$ since $C_{\mu_t} \subseteq V_t = 0$. □

6.1.1 Special stratum τ^0

There is a special stratum Z_{τ^0} which plays an important role in the following chapter.

Definition 6.1.7. For each $i = 1, \ldots, t$ we define an ab-diagram τ_i^0 by taking the Young diagram of the partition $\lambda^{(i-1)} \vdash n_{i-1}$, replacing the boxes with as and inserting a b for each two consecutive a.

We define the sequence of ab-diagrams $\tau^0 := (\tau_1^0, \ldots, \tau_t^0)$.

Remark. It is clear that $\pi(\tau_i^0) = \lambda^{(i-1)}$ and $\rho(\tau_i^0) = \lambda^{(i)}$. Hence $\tau^0 \in \Lambda$.

Lemma 6.1.8. *One has*

$$\dim Z_{\tau^0} = \frac{1}{2}\dim C_\lambda + \sum_{i=1}^{t} n_{i-1}n_i.$$

Proof. Immediate by proposition 6.1.6. Notice that $\pi(\tau_1^0) = \lambda^{(0)} = \lambda$ by definition and that, for each $i = 1, \dots, t$, τ_i^0 has no rows starting with b, therefore $\Delta(\tau_i^0) = 0$. □

The stratum Z_{τ^0} has a very special property.

Proposition 6.1.9. $\Theta^{-1}(C_\lambda) = Z_{\tau^0}$

Proof. By proposition 6.1.5, for each $\tau \in \Lambda$, $\Theta(Z_\tau) = C_{\pi(\tau_1)}$. Therefore we only need to decide for which $\tau \in \Lambda$ we have $\pi(\tau_1) = \lambda$.

By definition 6.1.7, $\pi(\tau_1^0) = \lambda$.

Let $\tau \in \Lambda$. We prove that, if $\pi(\tau_i) = \lambda^{(i-1)}$ then $\rho(\tau_i) = \lambda^{(i)}$. Indeed, take such an ab-diagram τ_i. The condition $\pi(\tau_i) = \lambda^{(i-1)}$ says that some as of τ_i must lie in a same row (according to the row of $\lambda^{(i-1)}$). For each row of τ_i, there must be a b between each pair of consecutive as. Clearly the j-th row of $\lambda^{(i-1)}$ has $\lambda_j^{(i-1)}$ as, hence $\lambda_j^{(i-1)} - 1 = \lambda_j^{(i)}$ bs. Therefore there are already $|\lambda^{(i)}| = n_i$ bs fixed in τ_i. As τ_i contains exactly n_i bs, there is only one possible ab-diagram τ_i, which is τ_i^0. Therefore $\rho(\tau_i^0) = \lambda^{(i)}$ and we are done. □

Proposition 6.1.10. *For each $\tau \in \Lambda$, $Z_\tau \subseteq Z^0$ or $Z_\tau \subseteq Z \setminus Z^0$. For $\tau = \tau^0$, $Z_{\tau^0} \subseteq Z^0$. In particular, Z is regular in Z_{τ^0}.*

Proof. An element $z \in Z$ belongs to Z^0 if and only if for each $i = 1, \dots, t$, A_i or B_i has maximal rank. Let $\tau \in \Lambda$ the sequence of ab-diagrams such that $z \in Z_\tau$. By lemma 4.3.4, that may be checked by looking at the ab-diagram τ_i. Hence $z \in Z^0$ if and only if $z' \in Z^0$ for each $z' \in Z_\tau$.

By definition of τ^0, for each $i = 1, \dots, t$, each row of τ_i^0 starts with a and does not end with b. Therefore, in every pair $(A_i, B_i) \in O_{\tau_i^0}$, A_i is surjective and B_i is injective. In particular both A_i, B_i have maximal rank. □

Remark. In spite of the fact that the strata Z_τ are not G-orbits in general, Z_{τ^0} is G-homogeneous.

We would like to carry on an analysis of the sequences $\tau \in \Lambda$ with the property that $\pi(\tau_1)$ is a fixed partition $\mu \leq \lambda$. By proposition 6.1.5, such sequences τ appear in $\Theta^{-1}(C_\mu)$. Proposition 6.1.9 explains the particularly simple situation in case $\mu = \lambda$. The general situation is very intricate, but one may still get useful information.

Definition 6.1.11. Let δ be an ab-diagram and let d be a list of some a and some b (possibly empty). We define $\mathrm{aug}_\delta(d)$ to be the set of ab-diagrams obtainable from δ by adding all the letters in d.

We denote by d^a (resp. d^b) the number of as (resp. b) in the list d.

Example. Let $d = (a, a)$ (that is $d^a = 2$, $d^b = 0$) and

$$\delta = \begin{matrix} aba \\ aba \\ b \end{matrix} .$$

Then $\mathrm{aug}_\delta(d)$ has four elements:

$$\mathrm{aug}_\delta(d) = \left\{ \begin{matrix} aba \\ aba \\ aba \end{matrix} , \quad \begin{matrix} aba \\ aba \\ ab \\ a \end{matrix} , \quad \begin{matrix} aba \\ aba \\ ba \\ a \end{matrix} , \quad \begin{matrix} aba \\ aba \\ b \\ a \\ a \end{matrix} \right\} .$$

The construction of definition 6.1.7 of the sequence $\tau^0 =: \tau^0(\lambda)$ depends on a partition λ. One might construct that sequence starting with a different partition μ: let $\sigma^0 := \tau^0(\mu)$ the sequence of ab-diagrams thus constructed. We remark that, for $\mu \neq \lambda$, $\sigma^0 \notin \Lambda$, because the number of as and bs obtained with μ ($|\mu^{(i)}|$) are different from those obtained with λ ($|\lambda^{(i)}|$).

Since $\mu \leq \lambda$ if and only if $|\mu^{(i)}| \leq |\lambda^{(i)}|$ for each $i = 1, \ldots, t$, one may define a list of letters d_i for each $i = 1, \ldots, t$ (allowing, possibly, $\widehat{\mu}_j = 0$) by setting

$$d_i^a := \sum_{j=i}^{t} \left(\widehat{\lambda}_j - \widehat{\mu}_j \right), \quad d_i^b := d_{i+1}^a. \tag{6.1.6}$$

Notice that $d_i^a \geq 0$ for each $i = 1, \ldots, t$ is precisely the condition $\mu \leq \lambda$.

Lemma 6.1.12. *Let $\mu \leq \lambda$ be partitions in $\mathcal{P}(n)$ and let σ^0 be the sequence of ab-diagrams of definition 6.1.7 defined on μ as just introduced. Also, for each $i = 1, \ldots, t$, let d_i be the list of d_i^a as and d_i^b bs as defined by (6.1.6). Then, for each $\tau \in \Lambda$ such that $\pi(\tau_1) = \mu$, one has*

$$\tau_i \in \mathrm{aug}_{\sigma_i^0}(d_i) \ \forall i = 1, \ldots, t.$$

Proof. The condition $\tau_i \in \mathrm{aug}_{\sigma_i^0}(d_i)$ is equivalent to say that τ_i has a certain number of letters as and bs and that we can obtain σ_i^0 as a sub-ab-diagram of τ_i.

The number of a in an ab-diagram in $\mathrm{aug}_{\sigma_i^0}(d_i)$ is the number of a in σ_i^0, which is $|\mu^{(i)}|$, plus d_i^a. By (2.2.2) and by (6.1.6), this number is $|\lambda^{(i)}| = n_i$. A similar argument proves the correct number for the bs.

The latter condition may be obtained from the second condition of lemma 6.1.2. The key step is the following: if the j-th ab-string of τ_i contains at least $k \geq 1$ as, then the corresponding ab-string of τ_{i+1} contains at least $k-1$ as. Indeed, such an ab-string must contain at least $k-1$ bs, since the letters alternate. Therefore, the j-th row of $\rho(\tau_i)$ has length $k-1$ at least. The claim follows as $\rho(\tau_i) = \pi(\tau_{i+1})$.

For each row j of μ, τ_1 has a row with μ_j as since $\pi(\tau_1) = \mu$. By our remark, this implies that the corresponding row of τ_i contains at least $\mu_j - i + 1 = \mu_j^{(i-1)}$ as. Therefore $\pi(\tau_i)$ is a partition extending $\mu^{(i-1)}$, which is equivalent to say that τ_i extends σ_i^0. □

6.2 Quadratic and symmetric spaces

In this section we construct the stratification of the varieties $Z_{(\varepsilon_0,\varepsilon_1)}$ analogous to the stratification $(Z_\tau)_{\tau\in\Lambda}$ built earlier in this chapter.

As $Z_{(\varepsilon_0,\varepsilon_1)} \subseteq Z$, by lemma 6.1.1, for every point

$$z = (A_1, A_1^*, \ldots, A_t, A_t^*) \in Z_{(\varepsilon_0,\varepsilon_1)}$$

the pair (A_i, A_i^*) is nilpotent for each $i = 1, \ldots, t$. Therefore we may still associate a sequence of ab-diagrams $\tau = (\tau_1, \ldots, \tau_t)$ to each $z \in Z_{(\varepsilon_0,\varepsilon_1)}$.

Lemma 6.2.1. *Let $\tau = (\tau_1, \ldots \tau_t)$ be a sequence of ab-diagrams corresponding to a point $(A_1, B_1, \ldots, A_t, B_t) \in Z_{(\varepsilon_0,\varepsilon_1)}$. Then*

1. *τ_i has n_{i-1} as and n_i bs for each $i = 1, \ldots, t$;*
2. *$\pi(\tau_{i+1}) = \rho(\tau_i)$ for each $i = 1, \ldots, t-1$;*
3. *depending on the signs $\varepsilon_{i-1}, \varepsilon_i$, τ_i is ortho-symplectic or ortho-symmetric or symplectic-symmetric, according to definition 4.5.3.*

Proof. By lemma 6.1.2, we already have the first two conditions. The latter follows as τ_i is the ab-diagram of a nilpotent pair in $L^\sigma_{V_{i-1},V_i}$ and V_{i-1} (resp. V_i) has an ε_{i-1}-bilinear (resp. ε_i-bilinear) form. □

We define the $\Lambda_{(\varepsilon_0,\varepsilon_1)} \subseteq \Lambda$ as the set of sequences of ab-diagrams satisfying all the conditions of lemma 6.2.1. It is clear that the sequences in $\Lambda_{(\varepsilon_0,\varepsilon_1)}$ are exactly the sequences of G^σ-invariants that appear to be associated to at least one point $z \in Z_{(\varepsilon_0,\varepsilon_1)}$.

Definition 6.2.2. Let $\tau \in \Lambda_{(\varepsilon_0,\varepsilon_1)}$. We define the subvariety $Z_{(\varepsilon_0,\varepsilon_1),\tau} := Z_\tau \cap Z_{(\varepsilon_0,\varepsilon_1)}$.

Proposition 6.2.3. *For each $\tau \in \Lambda_{(\varepsilon_0,\varepsilon_1)}$, $Z_{(\varepsilon_0,\varepsilon_1),\tau}$ is a locally closed, G^σ-stable and smooth subvariety of $Z_{(\varepsilon_0,\varepsilon_1)}$. Moreover the varieties $Z_{(\varepsilon_0,\varepsilon_1),\tau}$ form a partition of $Z_{(\varepsilon_0,\varepsilon_1)}$ and the closure of each $Z_{(\varepsilon_0,\varepsilon_1),\tau}$ is a union of $Z_{(\varepsilon_0,\varepsilon_1),\tau'}$ for some $\tau' \in \Lambda_{(\varepsilon_0,\varepsilon_1)}$.*

Proof. By proposition 6.1.4, $Z_{(\varepsilon_0,\varepsilon_1),\tau}$ is a locally closed, G-stable subvariety and the collections of $Z_{(\varepsilon_0,\varepsilon_1),\tau}$ for $\tau \in \Lambda_{(\varepsilon_0,\varepsilon_1)}$ forms a stratification of $Z_{(\varepsilon_0,\varepsilon_1)}$. The G^σ-invariance is immediate as $Z_{(\varepsilon_0,\varepsilon_1)}$ is G^σ-invariant.

As in the proof of proposition 6.1.4, we have the diagram of fibre products in (6.2.1).

As shown in (4.5.1), the maps

$$\begin{array}{ccc} O_{\varepsilon_{i-1},\varepsilon_i,\tau_i} & \xrightarrow{\pi} & C_{\varepsilon_{i-1},\eta,\mu_{i-1}} \\ \downarrow \rho & & \\ C_{\varepsilon_i,\eta,\mu_i} & & \end{array}$$

are smooth, so each map in the diagram (6.2.1) is smooth, in particular $Z_{(\varepsilon_0,\varepsilon_1),\tau}$ is a smooth subvariety of $Z_{(\varepsilon_0,\varepsilon_1)}$. □

$$
\begin{array}{ccccccccc}
Z_{(\varepsilon_0,\varepsilon_1),\tau} & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & O_{\varepsilon_0,\varepsilon_1,\tau_1} & \longrightarrow & C_{\varepsilon_0,\eta,\mu_0} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdot & \longrightarrow & \cdot & \longrightarrow & O_{\varepsilon_1,\varepsilon_2,\tau_2} & \longrightarrow & C_{\varepsilon_1,\eta,\mu_1} & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
\cdot & \longrightarrow & O_{\varepsilon_2,\varepsilon_3,\tau_3} & \longrightarrow & C_{\varepsilon_2,\eta,\mu_2} & & & & \\
\downarrow & & \downarrow & & & & & & \\
O_{\varepsilon_3,\varepsilon_4,\tau_4} & \longrightarrow & C_{\varepsilon_3,\eta,\mu_3} & & & & & & \\
\downarrow & & & & & & & & \\
C_{\varepsilon_4,\eta,\mu_4} & & & & & & & &
\end{array}
\qquad (6.2.1)
$$

We prove the dimension formulas analogous to proposition 6.1.6. We choose to distinguish the cases $\varepsilon_0\varepsilon_1 = -1$ and $\varepsilon_0\varepsilon_1 = 1$. We start with the case of quadratic spaces.

Proposition 6.2.4. *Let* $\varepsilon_0 = -\varepsilon_1 \in \{\pm 1\}$ *and let* $\tau = (\tau_1, \ldots, \tau_t) \in \Lambda_{(\varepsilon_0,\varepsilon_1)}$. *Denote* $Z_\tau = \dim Z_{(\varepsilon_0,\varepsilon_1),\tau}$ *and* $C_{\pi(\tau_1)} = C_{\varepsilon_0,0,\pi(\tau_1)}$ *as well. Then*

$$\dim Z_\tau = \frac{1}{2}\dim C_{\pi(\tau_1)} + \frac{1}{2}\sum_{i=1}^{t}\left(n_{i-1}n_i - \Delta(\tau_i)\right).$$

Proof. We use that in a fibre product (6.1.4) we have (6.1.5). Since Z_τ is an iterated fibre product by (6.2.1), one may proceed as in the proof of proposition 6.1.6. By proposition 4.6.3 one gets

$$
\begin{aligned}
\dim Z_\tau =& \sum_{i=1}^{t}\dim O_{\varepsilon_{i-1},\varepsilon_i,\tau_i} - \sum_{i=1}^{t-1}\dim C_{\varepsilon_i,0,\mu_i} \\
=& \frac{1}{2}\sum_{i=1}^{t}\Big(\dim C_{\varepsilon_{i-1},0,\pi(\tau_i)} + \dim C_{\varepsilon_i,0,\rho(\tau_i)} + n_{i-1}n_i - \Delta(\tau_i)\Big) + \\
& - \sum_{i=1}^{t-1}\dim C_{\varepsilon_i,0,\mu_i} \\
=& \frac{1}{2}\sum_{i=1}^{t}\left(\dim C_{\varepsilon_{i-1},0,\mu_{i-1}} + \dim C_{\varepsilon_i,0,\mu_i} - 2\dim C_{\varepsilon_i,0,\mu_i}\right) + \\
& + \dim C_{\varepsilon_t,0,\mu_t} + \frac{1}{2}\sum_{i=1}^{t}\left(n_{i-1}n_i - \Delta(\tau_i)\right) \\
=& \frac{1}{2}\left(\dim C_{\varepsilon_0,0,\mu_0} - \dim C_{\varepsilon_t,0,\mu_t}\right) + \frac{1}{2}\sum_{i=1}^{t}\left(n_{i-1}n_i - \Delta(\tau_i)\right)
\end{aligned}
$$

$$=\frac{1}{2}\dim C_{\varepsilon_0,0,\pi(\tau_1)}+\frac{1}{2}\sum_{i=1}^{t}\left(n_{i-1}n_i-\Delta(\tau_i)\right)$$

as $\dim C_{\varepsilon_t,0,\mu_t}=0$ since $C_{\varepsilon_t,0,\mu_t}\subseteq V_t=0$. □

We turn to the case of symmetric spaces.

Proposition 6.2.5. *Let $\varepsilon=\varepsilon_0=\varepsilon_1\in\{\pm1\}$ and let $\tau=(\tau_1,\dots,\tau_t)\in\Lambda_{(\varepsilon_0,\varepsilon_1)}$. Denote $Z_\tau=Z_{(\varepsilon_0,\varepsilon_1),\tau}$ and $C_{\pi(\tau_1)}=C_{\varepsilon,1,\pi(\tau_1)}$. Then*

$$\dim Z_\tau=\frac{1}{2}\dim C_{\pi(\tau_1)}+\sum_{i=1}^{t}\left(\frac{1}{2}n_{i-1}n_i-\frac{\varepsilon}{4}(n_{i-1}+n_i)\right)+\sum_{i=1}^{t}\left(\frac{\varepsilon}{4}o(\tau_i)-\frac{1}{2}\Delta(\tau_i)\right).$$

Proof. We still use (6.1.5) and (6.2.1). By proposition 4.6.6 one gets

$$\begin{aligned}
\dim Z_\tau=&\sum_{i=1}^{t}\dim O_{\varepsilon,\varepsilon,\tau_i}-\sum_{i=1}^{t-1}\dim C_{\varepsilon,1,\mu_i}\\
=&\frac{1}{2}\sum_{i=1}^{t}\Big(\dim C_{\varepsilon,1,\pi(\tau_i)}+\dim C_{\varepsilon,1,\rho(\tau_i)}+n_{i-1}n_i-\Delta(\tau_i)\Big)+\\
&-\frac{\varepsilon}{4}\sum_{i=1}^{t}\left(n_{i-1}+n_i-o(\tau_i)\right)-\sum_{i=1}^{t-1}\dim C_{\varepsilon,1,\mu_i}\\
=&\frac{1}{2}\sum_{i=1}^{t}\left(\dim C_{\varepsilon,1,\mu_{i-1}}+\dim C_{\varepsilon,1,\mu_i}-2\dim C_{\varepsilon,1,\mu_i}\right)+\dim C_{\varepsilon_t,0,\mu_t}+\\
&+\sum_{i=1}^{t}\left(\frac{1}{2}n_{i-1}n_i-\frac{\varepsilon}{4}(n_{i-1}+n_i)\right)+\sum_{i=1}^{t}\left(-\frac{1}{2}\Delta(\tau_i)+\frac{\varepsilon}{4}o(\tau_i)\right)\\
=&\frac{1}{2}\left(\dim C_{\varepsilon,1,\mu_0}-\dim C_{\varepsilon,1,\mu_t}\right)+\\
&+\sum_{i=1}^{t}\left(\frac{1}{2}n_{i-1}n_i-\frac{\varepsilon}{4}(n_{i-1}+n_i)\right)+\sum_{i=1}^{t}\left(\frac{\varepsilon}{4}o(\tau_i)-\frac{1}{2}\Delta(\tau_i)\right)\\
=&\frac{1}{2}\dim C_{\varepsilon,1,\pi(\tau_1)}+\\
&+\sum_{i=1}^{t}\left(\frac{1}{2}n_{i-1}n_i-\frac{\varepsilon}{4}(n_{i-1}+n_i)\right)+\sum_{i=1}^{t}\left(\frac{\varepsilon}{4}o(\tau_i)-\frac{1}{2}\Delta(\tau_i)\right)
\end{aligned}$$

as $\dim C_{\varepsilon,1,\mu_t}=0$ since $C_{\varepsilon,1,\mu_t}\subseteq V_t=0$. □

6.2.1 Special stratum τ^0

The special stratum Z_{τ^0} of definition 6.1.7 meets $Z_{(\varepsilon_0,\varepsilon_1)}$ as shown in the following lemma.

Lemma 6.2.6. *Let $\varepsilon_0,\varepsilon_1\in\{\pm1\}$ and let $\varepsilon\in\{\pm1\}$ and $\eta\in\{0,1\}$ be the corresponding indices. Let $\lambda\in\mathcal{P}_{\varepsilon,\eta}(n)$ and let $\tau^0=(\tau^0_1,\dots,\tau^0_t)\in\Lambda^{(\lambda)}$ be the sequence of ab-diagrams of definition 6.1.7. Then τ^0_i has type $(\varepsilon_{i-1},\varepsilon_i)$.*

Proof. Each row of τ_i^0 starts and ends with a. Therefore one may express τ_i^0 as a sum of ab-diagrams of only types α_k and γ_k. We distinguish the four cases.

If $\varepsilon_0 = \varepsilon_1 = 1$, we may use α_k for each k, therefore the conclusion is clear.

If $\varepsilon_0 = \varepsilon_1 = -1$, each column of λ is even, therefore each column of $\lambda^{(i)}$ is even, so the rows appear with even multiplicity. The conclusion follows since diagrams of type γ_k contain pairs of rows of the same length.

In the other cases, we distinguish whether $\varepsilon_{i-1} = 1$, when we can use the ab-diagram α_k only for k even, or $\varepsilon_{i-1} = -1$, when we can use the ab-diagram α_k only for k odd. We analyse the former as the latter is clear by symmetry. In that case, we have $\pi(\tau_i^0) = \lambda^{(i-1)} \in \mathcal{P}_{1,0}(n_{i-1})$. Therefore the rows of τ_i^0 with an even number of as occur with even multiplicities and we may use the diagrams γ_k, with even k, for those lengths. □

Of course we have similar results for $Z_{(\varepsilon_0,\varepsilon_1),\tau^0}$ as we had for Z_{τ^0}.

Proposition 6.2.7. *We have* $(\Theta^\sigma)^{-1}(C_{\varepsilon,\eta,\lambda}) = Z_{(\varepsilon_0,\varepsilon_1),\tau^0}$ *and* $Z_{(\varepsilon_0,\varepsilon_1),\tau^0} \subseteq Z^0_{(\varepsilon_0,\varepsilon_1)}$. *In particular* $Z_{(\varepsilon_0,\varepsilon_1)}$ *is regular in* $Z_{(\varepsilon_0,\varepsilon_1),\tau^0}$.

We also extend definition 6.1.11 to the case of quadratic and symmetric spaces.

Definition 6.2.8. Let $\varepsilon, \varepsilon' \in \{\pm 1\}$, let δ be an ab-diagram of type $(\varepsilon, \varepsilon')$ and let d be a list of some a and some b (possibly empty). We define $\mathrm{aug}_{\varepsilon,\varepsilon',\delta}(d)$ to be the set of ab-diagrams of type $(\varepsilon, \varepsilon')$ obtainable from δ by adding all the letters in d.

By lemma 6.2.1, it is immediate to check the extension of lemma 6.1.12.

Lemma 6.2.9. *Let* $\varepsilon_0, \varepsilon_1 \in \{\pm 1\}$ *and let* $\varepsilon \in \{\pm 1\}$, $\mu \in \{0, 1\}$ *be the corresponding indices. Let* $\mu \leq \lambda$ *be partitions in* $\mathcal{P}_{\varepsilon,\mu}(n)$ *and let* σ^0 *the sequence of ab-diagrams of definition 6.1.7 defined on* μ. *Also, for each* $i = 1, \dots, t$, *let* d_i *be the list of* d_i^a *as and* d_i^b *bs as defined by (6.1.6). Then, for each* $\tau \in \Lambda_{(\varepsilon_0,\varepsilon_1)}$ *such that* $\pi(\tau_1) = \mu$, *one has*

$$\tau_i \in \mathrm{aug}_{\varepsilon_{i-1},\varepsilon_i,\sigma_i^0}(d_i) \;\forall i = 1, \dots, t.$$

Chapter 7

Normality proofs

In this chapter we will make use of theorem 5.3.2 in order to prove the normality of Z and of $\overline{C_\lambda}$. We only need to establish (5.3.1), but it will not always be the case as not each conjugacy class C_λ has normal closure.

In each case, we will perform an analysis of the strata Z_τ in order to decide whether $Z_\tau \subseteq Z^0$ is smooth or the inequality $\dim Z^0 \geq \dim Z_\tau - 2$ holds.

7.1 Normality of conjugacy classes for GL_n

Proposition 7.1.1. *Let $\lambda \in \mathcal{P}(n)$, let $Z = Z^{(\lambda)}$ be the variety of definition 5.1.1 and $Z^0 \subseteq Z$ the open subvariety of corollary 5.1.6. Then*

$$\dim(Z \setminus Z^0) \leq \dim Z^0 - 2.$$

The proof of this proposition will follow from a few lemma involving the stratification $(Z_\tau)_{\tau\in\Lambda}$ of Z introduced in definition 6.1.3 and the dimension formula of proposition 6.1.6.

Lemma 7.1.2. $\dim Z_{\tau^0} = \dim Z^0$.

Proof. By corollary 5.1.6,

$$\dim Z^0 = \dim M - \dim N = \sum_{i=1}^{t} 2n_{i-1}n_i - \sum_{i=1}^{t-1} n_i^2.$$

By proposition 6.1.6,

$$\dim Z_{\tau^0} = \frac{1}{2}\dim C_{\pi(\tau_1^0)} + \sum_{i=1}^{t}\left(n_{i-1}n_i - \Delta(\tau_i^0)\right) = \frac{1}{2}\dim C_\lambda + \sum_{i=1}^{t} n_{i-1}n_i$$

since $\pi(\tau_1^0) = \lambda^{(0)} = \lambda$ by definition of τ^0 and, for each $i = 1, \ldots, t$, τ_i^0 has no rows starting with b and therefore $\Delta(\tau_i^0) = 0$. By proposition 2.5.1 and by (2.2.1) (and recalling that $n_i = |\lambda^{(i)}|$)

$$\dim C_\lambda = n_0^2 - \sum_{i=1}^{t}\left(\widehat{\lambda}_i\right)^2 = n_0^2 - \sum_{i=1}^{t}\left(n_{i-1} - n_i\right)^2.$$

Therefore

$$\begin{aligned}
\dim Z_{\tau^0} =& \frac{1}{2}\left(n_0^2 - \sum_{i=1}^{t}(n_{i-1}-n_i)^2\right) + \sum_{i=1}^{t} n_{i-1}n_i \\
=& \frac{1}{2}\left(n_0^2 - \sum_{i=1}^{t}(n_{i-1}^2+n_i^2)\right) + 2\sum_{i=1}^{t} n_{i-1}n_i \\
=& \frac{1}{2}\left(n_0^2 - n_0^2 - 2\sum_{i=1}^{t-1} n_i^2 - n_t^2\right) + 2\sum_{i=1}^{t} n_{i-1}n_i \\
=& 2\sum_{i=1}^{t} n_{i-1}n_i - \sum_{i=1}^{t-1} n_i^2,
\end{aligned}$$

since $n_t = 0$. □

Lemma 7.1.3. *For each $\tau \in \Lambda$ such that $\tau \neq \tau^0$, $\dim Z_\tau < \dim Z_{\tau^0}$.*

Proof. We compute directly the difference $\dim Z_{\tau^0} - \dim Z_\tau$ using proposition 6.1.6 and proposition 2.5.1:

$$\begin{aligned}
\dim Z_{\tau^0} - \dim Z_\tau =& \left(\frac{1}{2}\dim C_\lambda + \sum_{i=1}^{t} n_{i-1}n_i\right) + \\
&- \left(\frac{1}{2}\dim C_{\pi(\tau_1)} + \sum_{i=1}^{t}(n_{i-1}n_i - \Delta(\tau_i))\right) \\
=& \frac{1}{2}\left(\dim C_\lambda - \dim C_{\pi(\tau_1)}\right) + \sum_{i=1}^{t}\Delta(\tau_i) \\
\geq& \frac{1}{2}\left(\dim C_\lambda - \dim C_{\pi(\tau_1)}\right) \geq 0
\end{aligned}$$

since $\pi(\tau_1) \leq \lambda$ as $\Theta(Z_\tau) = C_{\pi(\tau_1)}$ by proposition 6.1.5, $\Theta(Z_\tau) \subseteq \overline{C_\lambda}$ by proposition 5.1.3. However, the last inequality is strict as $\pi(\tau_1) \neq \lambda = \pi(\tau_1^0)$ for $\tau \neq \tau^0$ by proposition 6.1.9. □

Lemma 7.1.4. *If $\dim Z_\tau = \dim Z_{\tau^0} - 1$ then $Z_\tau \subseteq Z^0$.*

Proof. As in the previous lemma, for $\tau \neq \tau^0$,

$$\dim Z_{\tau^0} - \dim Z_\tau = \frac{1}{2}\left(\dim C_\lambda - \dim C_{\pi(\tau_1)}\right) + \sum_{i=1}^{t}\Delta(\tau_i) \geq \frac{1}{2}\cdot 2 + 0 = 1$$

since $\dim C_\nu$ is even for every $\nu \vdash n$ by theorem 1.4.11. Let $\mu := \pi(\tau_1)$. As we have equality, it must happen that $\Delta(\tau_i) = 0$ for each $i = 1, \ldots, t$ and $\dim C_\lambda - \dim C_\mu = 2$.

The condition $\Delta(\tau_i) = 0$ implies that there cannot be both an *ab*-string of τ_i formed by a single a and another by a single b. Otherwise $a_1 > 0$, $b_1 > 0$ and $\Delta(\tau_i) \geq a_1 b_1 > 0$.

The condition $\dim C_\lambda - \dim C_\mu = 2$ implies that the degeneration $\mu < \lambda$ must be minimal, otherwise we may find a conjugacy class corresponding to a partition $\mu < \nu < \lambda$ which would contradict the parity of the dimension by theorem 1.4.11.

Moreover, by lemma 3.1.7, the reduced degeneration $\mu' < \lambda'$ must be of type (m), $(m-1,1)$ and not of type $(2,1^{m-2})$, (1^m) with $m > 2$, otherwise, by proposition 3.1.6,

$$\dim C_\lambda - \dim C_\mu = \dim C_{\lambda'} - \dim C_{\mu'} = (m^2-(m-1)^2-1^2)-(m^2-m^2) = 2m-2 > 4.$$

Therefore, the degeneration $\mu < \lambda$ is obtained by moving down a single box of λ from a row j to the next row $j+1$.

Let $\tau \in \Lambda$ be a sequence of ab-diagrams such that $\pi(\tau_1) = \mu$. For each $i = 1, \ldots, t$ we prove that each row of τ_i starts with a or ends with a, assuming that it cannot have both a single a and a single b by the previous argument. As $\pi(\tau_1) = \mu$, $\tau_i \in \operatorname{aug}_{\sigma_i^0}(d_i)$ for some list of letters d_i.

By (6.1.6), $d_{\lambda_{j+1}} = (b)$, $d_i = (a,b)$ for $\lambda_{j+1} < i < \lambda_j$, $d_{\lambda_j} = (a)$ and d_i is empty otherwise. Therefore, each ab-diagram in $\operatorname{aug}_{\sigma_i^0}(d_i)$ with $i < \lambda_{j+1}$ or $i \geq \lambda_j$ has only rows starting and ending with a, since that is true for σ_i^0. In the other cases, the single $b \in d_i$ cannot be placed both at the start at the end of an ab-string, unless it forms a new row alone. However, for $i > \lambda_{j+1}$, also the single $a \in d_i$ must be placed in a new row alone (as every row in σ_i^0 starts and ends with b), but we assumed this not to be the case. Finally, if $i = \lambda_{j+1}$, there is already a line of σ_i^0 formed with a single a, by the minimality of the degeneration $\mu < \lambda$.

We found that each row of τ_i starts with a or ends with a. By lemma 4.3.4, for each $z \in Z_\tau$, the map A_i is surjective or the map B_i is injective. In any case, $z \in Z^0$ and we are done. □

Proof of proposition 7.1.1. By proposition 6.1.10 and by lemma 7.1.2, one gets that the special stratum Z_{τ^0} is open in Z^0. Since $Z \setminus Z^0$ is a union of finitely many subvarieties Z_τ, one may compute its dimension by taking the maximum of $\dim Z_\tau$ among $\tau \in \Lambda$ such that $Z_\tau \subseteq Z \setminus Z^0$. The conclusion follows by lemma 7.1.3 and by lemma 7.1.4. □

Corollary 7.1.5. *Let $\lambda \in \mathcal{P}(n)$ be any partition. Then $\overline{C_\lambda}$ is normal.*

Proof. Put together proposition 7.1.1 and theorem 5.3.2. □

7.2 Conjugacy classes with not normal closure

In this section we describe all the conjugacy classes $C_{\varepsilon,\eta,\lambda}$ such that $\overline{C_{\varepsilon,\eta,\lambda}}$ is not normal. The main tools for our analysis is proposition 3.3.3, which we are finally able to prove.

Proof of proposition 3.3.3. One may prove proposition 3.3.3 by resolving separately the cases *erasing rows* and *erasing columns*. The former case is given by proposition 3.3.8 and by corollary 7.1.5. The latter is given by proposition 4.4.9. □

Proposition 3.3.3 suggests we should look at degenerations $\mu < \lambda$ which are irreducible in the sense of definition 3.1.5. We first inspect minimal irreducible degenerations. Those are listed in table 3.1. Then we use proposition 3.3.3 to obtain a characterization of the partitions λ such that $\overline{C_{\varepsilon,\eta,\lambda}}$ is not normal.

Theorem 7.2.1. *Let $\varepsilon \in \{\pm 1\}$, $\eta = 0$ and let $\lambda = (\lambda_1, \dots, \lambda_h) \in \mathcal{P}_{\varepsilon,0}(n)$. Suppose there is $1 \leq i \leq h-1$ such that*

$$(\lambda_i, \lambda_{i+1}, \lambda_{i+2}, \lambda_{i+3}) = (k+2m, k+2m, k, k)$$

for some integers $k \geq 0$, $m > 0$ such that k is even if $\varepsilon = 1$ and k is odd if $\varepsilon = -1$ (assuming $\lambda_j = 0$ if $j > h$). Then $\overline{C_{\varepsilon,0,\lambda}}$ is not normal.

Proof. Let $\mu \vdash n$ be the partition defined by $\mu_j := \lambda_j$ if $j \notin \{i, i+1, i+2, i+3\}$ and

$$(\mu_i, \mu_{i+1}, \mu_{i+2}, \mu_{i+3}) := (k+2m-1, k+2m-1, k+1, k+1).$$

Then clearly $\mu \in \mathcal{P}_{\varepsilon,0}(n)$ and $\mu < \lambda$. Moreover, if one reduces (λ, μ) by erasing the first $i-1$ common rows and the first k columns, one obtains partitions $\lambda' = (2m, 2m)$, $\mu' = (2m-1, 2m-1, 1, 1)$ that belongs to $\mathcal{P}_{1,0}(4m)$ (since $(-1)^k = \varepsilon$), such that the degeneration (λ', μ') has type e according to table 3.1b.

By theorem 2.4.3, the partition $\lambda' = (2m, 2m)$ correspond to a not connected K_ε-conjugacy class. As remarked at the start of chapter 3, $\overline{C_{1,0,\lambda'}}$ is connected, since it is a cone. Therefore, the closure of the two connected components in $C_{1,0,\lambda'}$ are the two irreducible components of $\overline{C_{1,0,\lambda'}}$. Therefore, $\mathrm{Sing}(\overline{C_{1,0,\lambda'}}, 0)$ is not normal. In fact, $\mathrm{Sing}(\overline{C_{1,0,\lambda'}}, X)$ is not normal for every X lying in the intersection of those two irreducible components. By proposition 3.2.4, there is a connecting path P between $C_{1,0,\lambda'}$ and $C_{1,0,\mu'}$. The action of $\mathbb{Z}/2 \simeq O_{4m}/SO_{4m}$ on P must send each point $P(t)$, for $t \neq 0$, to a point $P'(t)$ belonging to the connected component of $C_{1,0,\lambda'}$ not containing $P(t)$ and must fix $P(0)$, since $C_{1,0,\lambda'}$ is not connected but $C_{1,0,\mu'}$ is connected. Therefore $X := P(0)$ lies in the intersection of the two irreducible componenets of $\overline{C_{1,0,\lambda'}}$ and in $C_{1,0,\mu'}$. Hence the singularity $\mathrm{Sing}(\overline{C_{1,0,\lambda'}}, C_{1,0,\mu'})$ is not normal. By proposition 3.3.3, the singularity $\mathrm{Sing}(\overline{C_{\varepsilon,0,\lambda}}, C_{\varepsilon,0,\mu})$ is not normal as well. □

Theorem 7.2.2. *Let $\varepsilon = 1$, $\eta = 1$ and $\lambda \in \mathcal{P}_{1,1}(n) = P(n)$. Suppose there is $1 \leq i \leq h$ such that $\lambda_i - \lambda_{i+1} > 1$ (assuming $\lambda_{i+1} = 0$). Then $\overline{C_{1,1,\lambda}}$ is not normal.*

Proof. Let $m := \lambda_i - \lambda_{i+1} \geq 2$. Let $\mu \vdash n$ be the partition defined by $\mu_j := \lambda_j$ if $j \notin \{i, i+1\}$, $\mu_i := \lambda_{i+1} + m - 1$, $\mu_{i-1} := \lambda_{i+1} + 1$. Then clearly $\mu < \lambda$. Moreover if one reduces (λ, μ) by erasing the first $i-1$ common rows and the first λ_{i+1} columns, one obtains partitions $\lambda' = (m)$, $\mu' = (m-1, 1)$ such that the degeneration (λ', μ') has type x according to table 3.1c.

Since $\lambda' = (m)$ correspond to the regular nilpotent conjugacy class and $\mu = (m-1, 1)$ correspond to the subregular conjugacy class, by theorem 3.3.9,

$$\mathrm{Sing}(\overline{C_{1,1,\lambda}}, C_{1,1,\mu}) = \mathrm{Sing}(\{x^m - y^2 = 0\}, 0),$$

but the curve $\{x^m - y^2 = 0\}$ is not normal in 0 if $m > 1$. Therefore, the singularity $\mathrm{Sing}(\overline{C_{1,1,\lambda'}}, C_{1,1,\mu'})$ is not normal. By proposition 3.3.3 the singularity $\mathrm{Sing}(\overline{C_{1,0,\lambda}}, C_{1,0,\mu})$ is not normal as well. □

7.3 Condition for normality of conjugacy classes for O_n and Sp_{2n}

In the remaining part of this chapter we fix $\varepsilon \in \{\pm 1\}$, $\eta \in \{0,1\}$, $\varepsilon_0 := \varepsilon$, $\varepsilon_1 := \varepsilon(-1)^{1+\eta}$. We also denote the variety $Z_{(\varepsilon_0,\varepsilon_1)}$ by Z and the strata $Z_{(\varepsilon_0,\varepsilon_1),\tau}$ by Z_τ in order to simplify the notation.

Proposition 7.3.1. $Z_{\tau^0} = Z^0$. *In particular* $\dim Z_{\tau^0} = \dim Z^0$.

Proof. As Z is pointwise fixed by the involution σ, for each

$$(A_1, B_1, \dots, A_t, B_t) \in Z$$

and for each $i = 1, \dots, t$, we have $B_i = A_i^*$, hence $\operatorname{rk} B_i = \operatorname{rk} A_i$ by lemma 4.4.3. In particular, A_i has maximal rank if and only if B_i has maximal rank. Therefore, one has

$$Z^0 = \{(A_1, B_1, \dots, A_t, B_t) \in Z : \text{ both } A_i, B_i \text{ have maximal rank } \forall i = 1, \dots, t\}.$$

As $n_{i-1} > n_i$ for each $i = 1, \dots, t$, the only possibility for the points in Z^0 is that A_i is surjective and B_i is injective. By lemma 4.3.4, this is ensured just by some condition on the ab-diagram τ_i of such points. In particular one gets that each ab-string of τ_i starts and ends with a.

We prove that $\tau_i = \tau_i^0$ for each $i = 1, \dots, t$ if each ab-string of τ_i for $i = 1, \dots, t$ starts and ends with a. Let $\mu = \pi(\tau_1)$ and suppose that $\mu \neq \lambda$, so that $\mu < \lambda$. By lemma 6.2.9, $\tau_i \in \operatorname{aug}_{\varepsilon_{i-1},\varepsilon_i,\sigma_i^0}(d_i)$. By (3.1.1) there is a first column j such that $\widehat{\mu}_j \neq \widehat{\lambda}_j$ and one must have $\widehat{\mu}_j > \widehat{\lambda}_j$. By (6.1.6), d_j is not empty and consists of only bs, therefore each ab-diagram $\delta \in \operatorname{aug}_{\sigma_j^0}(d_j)$ must have at least one ab-string starting or ending with b. This contradiction shows that $\mu = \lambda$. In particular $\Theta(Z_\tau) \subseteq C_\lambda$, thus one may conclude by proposition 6.2.7. □

Remark. One might actually obtain $\dim Z_{\tau^0} = \dim Z^0$ by comparing corollary 5.2.6 with proposition 6.2.4 (if $\varepsilon_0 = -\varepsilon_1$) or proposition 6.2.5 (if $\varepsilon_0 = \varepsilon_1$).

Lemma 7.3.2. *Let* $\tau \in \Lambda$. *Then*

$$\dim Z_{\tau^0} - \dim Z_\tau \geq \frac{1}{2}\left(\dim C_\lambda - \dim C_{\pi(\tau_1)}\right).$$

Proof. By proposition 6.2.4

$$\begin{aligned}
\dim Z_{\tau^0} - \dim Z_\tau = & \left(\frac{1}{2}\dim C_\lambda + \frac{1}{2}\sum_{i=1}^{t} n_{i-1}n_i\right) + \\
& - \left(\frac{1}{2}\dim C_{\pi(\tau_1)} + \frac{1}{2}\sum_{i=1}^{t}\left(n_{i-1}n_i - \Delta(\tau_i)\right)\right) \\
= & \frac{1}{2}\left(\dim C_\lambda - \dim C_{\pi(\tau_1)}\right) + \frac{1}{2}\sum_{i=1}^{t}\Delta(\tau_i) \\
\geq & \frac{1}{2}\left(\dim C_\lambda - \dim C_{\pi(\tau_1)}\right).
\end{aligned}$$

□

Lemma 7.3.3. *Let $\eta = 0$ and let $\lambda \in \mathcal{P}_{\varepsilon,0}(n)$. Let $Z = Z^{(\lambda)}$ be the variety of definition 5.2.1. Then*

$$\dim(Z \setminus Z^0) < \dim Z^0.$$

Proof. By proposition 7.3.1 $Z_{\tau^0} = Z^0$. Since $Z \setminus Z_{\tau^0}$ is a union of finitely many subvarieties Z_τ, one may compute its dimension by taking the maximum of $\dim Z_\tau$ among $\tau \in \Lambda$ such that $\tau \neq \tau^0$. By proposition 6.2.7, $\pi(\tau_1) < \pi(\tau_1^0) = \lambda$ for such τ. The conclusion follows by lemma 7.3.2 (and the fact that $\dim C_{\pi(\tau_1)} < \dim C_\lambda$). □

Even if we have not gained the stronger inequality (5.3.1), we might still obtain it for a certain class of partitions $\lambda \in \mathcal{P}_{\varepsilon,0}(n)$ as soon as there is no minimal degeneration $\mu < \lambda$ of codimension 2 in $\overline{C_{\varepsilon,0,\lambda}}$.

Proposition 7.3.4. *In the same settings of lemma 7.3.3, suppose, additionally, that there is no minimal degeneration $\mu < \lambda$ such that $\dim C_{\varepsilon,0,\mu} = \dim C_{\varepsilon,0,\lambda} - 2$. Then*

$$\dim(Z \setminus Z^0) \leq \dim Z^0 - 2.$$

For any conjugacy class $C \in \mathfrak{g}_{\varepsilon,0}$, we denote by $\tilde{C}$ the complement in $\overline{C}$ of the union of all conjugacy classes of codimension at least 4. $\tilde{C}$ is the union of C with the codimension 2 classes

$$\tilde{C} = C \cup \bigcup_i C_i \quad \text{such that } \operatorname{codim}_{\overline{C}} C_i = 2,$$

in particular it is open in $\overline{C}$.

Proposition 7.3.5. *1. Every regular function on $\tilde{C}$ extends to $\overline{C}$.*

2. $\overline{C}$ is normal if and only if $\tilde{C}$ is normal.

Proof.
1. By lemma 7.3.3 and by proposition 5.3.1, Z is an affine Cohen-Macaulay variety; in particular its regular functions ring has property S_2. By lemma 7.3.2, $\operatorname{codim}_Z(\overline{C} - \tilde{C}) \geq 2$. Let $f \in \mathcal{O}(\tilde{C})$. By proposition 1.2.17, the map $F := f \circ \Theta \in \mathcal{O}(\Theta^{-1}(\tilde{C}))$ extends to Z. Since F is invariant on $\Theta^{-1}(\tilde{C})$, so must be the extension on Z. This implies that F defines a regular function on C.

2. By theorem 1.2.15, $\overline{C}$ is normal if and only if it has properties R_1 and S_2. Property R_1 is granted since $\operatorname{codim}_{\overline{C}}(\overline{C} \setminus C) \geq 2$. We have property S_2 if and only if every regular function on C extends to $\overline{C}$. By *1.*, every regular function on $\tilde{C}$ extends to $\overline{C}$, so the normality of $\overline{C}$ is reduced to the normality of $\tilde{C}$.

□

Lemma 7.3.6. *Let $\mu < \lambda$. Then* $\operatorname{Sing}(\overline{C_\lambda}, C_\mu)$ *is normal if degeneration (λ, μ) has type a, b, c, f, g, h, according to table 3.1.*

Proof. Let $\mu' < \lambda'$ be the irreducible degeneration obtained by erasing rows and columns from $\mu < \lambda$. By proposition 3.3.3, $\operatorname{Sing}(\overline{C_\lambda}, C_\mu)$ is normal if and only if $\operatorname{Sing}(\overline{C'_\lambda}, C'_\mu)$ is normal.

If (λ', μ') has type f, g or h, then $C_{\lambda'}$ is the minimal nilpotent conjugacy class and $C_{\mu'}$ is its only degeneration, the class of 0. Therefore, by proposition 7.3.4 and by theorem 5.3.2, Z and $\overline{C_{\lambda'}}$ are normal.

If (λ', μ') has type a, b or c, then $C_{\lambda'}$ is the regular nilpotent conjugacy class and $\overline{C_{\lambda'}}$ is the nilpotent cone of $\mathfrak{g}_{\varepsilon,0}$. Hence it is normal by theorem 1.4.10. □

The degenerations of type e have already been discussed: by theorem 7.2.1, those degenerations are not normal. It remains to prove that the degenerations of type d are normal. This has been proven by Kraft and Procesi after a meticulous analysis of the degenerations of types d and e. We only report the main result.

Proposition 7.3.7 ([KP82, Proposition 15.4 (b)]). *Let $\lambda = (2n+1, 2n+1)$. Then $\overline{C_{-1,0,\lambda}} \subseteq \mathfrak{g}_{-1,0}$ is normal.*

For any conjugacy class $C = C_\lambda \in \mathfrak{g}_{\varepsilon,0}$, let C_e be the union of C with the codimension 2 classes $C_\mu \subseteq \overline{C}$ such that $\mu < \lambda$ is a degeneration of type e.

Theorem 7.3.8. *Let $C \subseteq \mathfrak{g}_{\varepsilon,0}$ be a conjugacy class.*

1. *Let C_e be as above. Then every regular function on C_e extends to a regular function on $\overline{C}$.*

2. *$\overline{C}$ is normal if and only if C has no degenerations of type e (table 3.1b).*

Proof. 1. Let $C'_e := C \cup (\tilde{C} \setminus C_e)$. Then C_e and C'_e are open in $\tilde{C}$ and $\tilde{C} = C_e \cup C'_e$. By lemma 7.3.6 and by proposition 7.3.7, C'_e is normal. Let $f \in \mathcal{O}(C_e)$. Then the restriction $f|_C$ may be extended to C'_e, therefore f may be extended to $\tilde{C}$. Then the conclusion follows by proposition 7.3.5, *1.*

2. If $\overline{C}$ is normal, then we must have $C = C_e$ by theorem 7.2.1. Conversely, if $C = C_e$, every regular function on C extends to $\tilde{C}$ by *1.*, hence $\tilde{C}$ is normal and the conclusion follows by proposition 7.3.5, *2.*

□

7.4 Normality of symmetric conjugacy classes for Sp_{2n}

Proposition 7.4.1. *Let $\lambda \in \mathcal{P}_{-1,1}(n)$, let $Z = Z^{(\lambda)}_{(-1,-1)}$ be the variety of definition 5.2.1 and $Z^0 \subseteq Z$ be the open subvariety of corollary 5.2.6. Then*

$$\dim(Z \setminus Z^0) \leq \dim Z^0 - 2.$$

Lemma 7.4.2. *Let $\tau \in \Lambda$. Then*

$$\dim Z_{\tau^0} - \dim Z_\tau \geq \frac{1}{2}\left(\dim C_\lambda - \dim C_{\pi(\tau_1)}\right).$$

Proof. By proposition 6.2.5

$$\begin{aligned}\dim Z_{\tau^0} - \dim Z_\tau = &\left(\frac{1}{2}\dim C_\lambda + \sum_{i=1}^{t}\left(\frac{1}{2}n_{i-1}n_i - \frac{\varepsilon}{4}(n_{i-1}+n_i)\right) - \frac{1}{4}\sum_{i=1}^{t} o(\tau_i^0)\right) + \\ &-\left(\frac{1}{2}\dim C_{\pi(\tau_1)} + \sum_{i=1}^{t}\left(\frac{1}{2}n_{i-1}n_i - \frac{\varepsilon}{4}(n_{i-1}+n_i)\right) + \right.\\ &\left. + \sum_{i=1}^{t}\left(-\frac{1}{4}o(\tau_i) - \frac{1}{2}\Delta(\tau_i)\right)\right)\end{aligned}$$

$$
\begin{aligned}
&=\frac{1}{2}\left(\dim C_\lambda - \dim C_{\pi(\tau_1)}\right) + \sum_{i=1}^{t}\left(\frac{1}{4}o(\tau_i) - \frac{1}{4}o(\tau_i^0) + \frac{1}{2}\Delta(\tau_i)\right)\\
&\geq\frac{1}{2}\left(\dim C_\lambda - \dim C_{\pi(\tau_1)}\right) + \frac{1}{4}\sum_{i=1}^{t}\left(o(\tau_i) - o(\tau_i^0)\right)
\end{aligned}
$$

We claim that $o(\tau_i) \geq o(\tau_i^0)$ for each $i = 1, \dots, t$. Let A_j (resp. B_j) the number of as (resp. bs) in the j-th ab-string of τ_i. It is clear that $|A_j - B_j| \leq 1$ and we have equality exactly if the j-th row has odd length. Since $\sum_j A_j = n_{i-1}$ and $\sum_j B_j = n_i$ by lemma 6.2.1, one gets

$$
o(\tau_i) = \sum_j |A_j - B_j| \geq \left|\sum_j (A_j - B_j)\right| = n_{i-1} - n_i = \widehat{\lambda}_i.
$$

On the other hand, each ab-string of τ_i^0 has odd length by definition. The claim follows as τ_i^0 has as many rows as $\lambda^{(i-1)}$ that is $\widehat{\lambda^{(i-1)}}_1 = \widehat{\lambda}_i$. □

Lemma 7.4.3. *Let* $\mu \in \mathcal{P}_{-1,1}(n)$*. Then* $\dim C_{-1,1,\mu}$ *is a multiple of* 4*.*

Proof. Since $\mu \in \mathcal{P}_{-1,1}(n)$, one may find a partition $\nu \in \mathcal{P}(n/2)$ such that $\mu = \nu \oplus \nu$; in particular, for each column $i = 1, \dots, \mu_1$, one has $\widehat{\mu}_i = 2\widehat{\nu}_i$. By proposition 2.5.5 and by proposition 2.5.1, one gets

$$
\dim C_{-1,1,\mu} = \frac{1}{2}\dim C_\mu = 2\dim C_\nu.
$$

As $\dim C_\nu$ is even by theorem 1.4.11, the conclusion follows. □

Proof of proposition 7.4.1. By proposition 7.3.1 $Z_{\tau^0} = Z^0$. Since $Z \setminus Z_{\tau^0}$ is a union of finitely many subvarieties Z_τ, one may compute its dimension by taking the maximum of $\dim Z_\tau$ among $\tau \in \Lambda$ such that $\tau \neq \tau^0$. By proposition 6.2.7, $\pi(\tau_1) < \pi(\tau_1^0) = \lambda$ for such τ. The conclusion follows by lemma 7.4.2 and by lemma 7.4.3. □

7.5 Condition for normality of symmetric conjugacy classes for O_n

7.5.1 s-step condition

In this section we define a condition which plays a pivotal role in the subsequent study of the nilpotent symmetric orbits.

Definition 7.5.1. Given a partition $\lambda = (\lambda_1, \dots, \lambda_h)$, we say that λ satisfies the *s-step condition* if for every $i = 1, \dots, h$ we have

$$
\lambda_i \leq \lambda_{i+1} + s
$$

with the convention that $\lambda_{h+1} = 0$.

We want to highlight some simple properties of this definition. First of all, s_1-step condition implies s_2-step condition for each $s_2 \geq s_1$. Secondly, if λ satisfies the s-step condition, then if we remove the first row or the first column from λ, the resulting partition still satisfies the s-step condition.

Example. The single line partition (n) satisfies the n-step condition, but not the $(n-1)$-step condition. The triangular partition $(n, n-1, \dots, 1)$ satisfies the 1-step condition. The *2-skew* triangular partition $(2n, 2(n-1), \dots, 2)$ satisfies the 2-step condition (but not the 1-step condition).

7.5.2 Differences between partitions

In this section we study some quantitative ways to address the difference between two partitions. Let $\lambda, \mu \in P(n)$ be two partitions of n such that $\lambda > \mu$.

We consider two other ways of comparing partitions. Given a partition λ, for each box B in the Young diagram of λ we define its column number $c(B)$ (resp. row number $r(B)$) by counting the columns (resp. rows) starting from the leftmost column (resp. uppermost row). For example, given $\lambda = (7, 2, 2, 1)$ we have:

$c(-)$:

1	2	3	4	5	6	7
1	2					
1	2					
1						

$r(-)$:

1	1	1	1	1	1	1
2	2					
3	3					
4						

Remark. By summing up over the rows (resp. the columns) of λ, we have:

$$\sum_{B\in\lambda} c(B) = \sum_{i=1}^{h} \binom{\lambda_i + 1}{2} \quad \left(\text{resp. } \sum_{B\in\lambda} r(B) = \sum_{i=1}^{t} \binom{\widehat{\lambda}_i + 1}{2}\right). \tag{7.5.1}$$

Definition 7.5.2. Given $\lambda \geq \mu$ as before, we define:

$$c(\lambda, \mu) = \sum_{B\in\lambda} c(B) - \sum_{B\in\mu} c(B),$$

where B is selected among the boxes in the Young diagrams of λ and μ and, similarly,

$$r(\lambda, \mu) = \sum_{B\in\mu} r(B) - \sum_{B\in\lambda} r(B).$$

Example. Let $\lambda = (7, 2, 2, 1)$ and $\mu = (5, 3, 1, 1, 1, 1)$. We compute

$$c(\lambda, \mu) = (28 + 3 + 3 + 1) - (15 + 6 + 1 + 1 + 1 + 1) = 10,$$
$$r(\lambda, \mu) = (21 + 3 + 3 + 1 + 1) - (10 + 6 + 1 + 1 + 1 + 1 + 1) = 8.$$

Remark. It is always guaranteed that $c(\lambda, \mu) \geq 0$ and $r(\lambda, \mu) \geq 0$ for every $\lambda \geq \mu$. We even have that

$$\lambda = \mu \iff c(\lambda, \mu) = 0 \iff r(\lambda, \mu) = 0. \tag{7.5.2}$$

In fact we have that $c(\lambda, \mu) \geq 0$ is equivalent to

$$\sum_i \binom{\lambda_i + 1}{2} \geq \sum_i \binom{\mu_i + 1}{2}. \tag{7.5.3}$$

As the function $n \mapsto \binom{n+1}{2}$ is convex, (7.5.3) is given by the Karamata's inequality ([BB12, Chapter 1, §28]).

Remark. Given partitions $\lambda \geq \mu \geq \nu$, by definition of c and r we have:

$$c(\lambda, \nu) = c(\lambda, \mu) + c(\mu, \nu), \tag{7.5.4}$$

$$r(\lambda, \nu) = r(\lambda, \mu) + r(\mu, \nu). \tag{7.5.5}$$

Remark. For every pair of partitions $\lambda \geq \mu$ we have that

$$c(\lambda, \mu) \geq q(\lambda, \mu). \tag{7.5.6}$$

In fact, when $q(\lambda, \mu) = 1$, by (7.5.2) we get (7.5.6). Otherwise, by lemma 3.1.3, we have a sequence of partitions $\lambda^0, \dots, \lambda^{q(\lambda,\mu)}$. For each pair $(\lambda^i, \lambda^{i+1})$, we have $c(\lambda^i, \lambda^{i+1}) \geq 1$, therefore the conclusion follows by summing up each inequalities by (7.5.4).

Lemma 7.5.3. *Let λ be a partition satisfying the s-step condition and $\mu < \lambda$ be another partition. Then*

$$s \cdot r(\lambda, \mu) \geq c(\lambda, \mu) + q(\lambda, \mu). \tag{7.5.7}$$

Moreover, (7.5.7) holds strictly if there exists a column index c such that $\widehat{\mu}_c > \widehat{\lambda}_c + 1$ or a row index r such that $\mu_r > \lambda_r + 1$.

Proof. Let $q = q(\lambda, \mu)$. By lemma 3.1.3, we obtain a sequence

$$\lambda = \lambda^0 > \cdots > \lambda^q = \mu,$$

with $q(\lambda^i, \lambda^{i+1}) = 1$ for $i = 0, \dots, q-1$.

We want to prove

$$s \cdot r(\lambda^i, \lambda^{i+1}) \geq c(\lambda^i, \lambda^{i+1}) + 1 \tag{7.5.8}$$

for all $i = 0, \dots, q-1$. Once (7.5.8) is proved, the conclusion will follow by summing up each of those inequalities by (7.5.5) and (7.5.4).

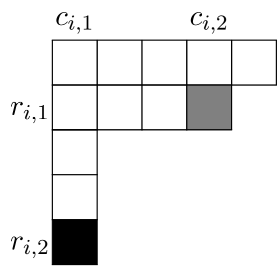
We start by noticing that the s-step condition implies that, for every pair of row indices $r_1 < r_2$,

$$\lambda_{r_1} - \lambda_{r_2} \leq s(r_2 - r_1).$$

Fix r_1 to be an index row such that $\lambda_{r_1} \geq \mu_{r_1}$ and fix r_2 to be an index row such that $\lambda_{r_2} \leq \mu_{r_2}$. Lemma 3.1.3 asserts that $\lambda^i_{r_j}$ is a monotone sequence for both $j = 1, 2$; therefore, for such r_1, r_2, we get $\lambda^i_{r_1} - \lambda^i_{r_2} \leq \lambda_{r_1} - \lambda_{r_2}$, so we get

$$\lambda^i_{r_1} - \lambda^i_{r_2} \leq s(r_2 - r_1). \tag{7.5.9}$$

As $q(\lambda^i, \lambda^{i+1}) = 1$, we already observed that the Young diagrams of λ^i, λ^{i+1} have only one box in different positions. This means that there exist exactly two columns $c_{i,1} < c_{i,2}$ which are not equal between λ^i, λ^{i+1}. Similarly, there exist exactly two rows indices $r_{i,1} < r_{i,2}$.



We have $c(\lambda^i, \lambda^{i+1}) = c_{i,2} - c_{i,1}$ and $r(\lambda^i, \lambda^{i+1}) = r_{i,2} - r_{i,1}$. Moreover, by definition of λ^i and λ^{i+1}, we have $\lambda^i_{r_{i,1}} = c_{i,2}$ and $\lambda^i_{r_{i,2}} = c_{i,1} - 1$. Therefore, by (7.5.9), we get

$$s \cdot r(\lambda^i, \lambda^{i+1}) = s(r_{i,2} - r_{i,1}) \geq \lambda^i_{r_{i,1}} - \lambda^i_{r_{i,2}} = c_{i,2} - c_{i,1} + 1 = c(\lambda^i, \lambda^{i+1}) + 1.$$

Finally, we consider the additional hypothesis of the existence of a column index c such that $\widehat{\mu}_c > \widehat{\lambda}_c + 1$ or a row index r such that $\mu_r > \lambda_r + 1$. We are going to prove that there exists an i such that (7.5.9) holds strictly.

Let

$$I_{\mathrm{cols}}(c) = \left\{ i \in \{0, \ldots, q-1\} : \widehat{\lambda^{i+1}_c} > \widehat{\lambda^i_c} \right\}.$$

As $q(\lambda^i, \lambda^{i+1}) = 1$, we have $|\widehat{\lambda^{i+1}}_c - \widehat{\lambda^i}_c| \leq 1$ for each i. Therefore, if there exists c such that $\widehat{\mu}_c > \widehat{\lambda}_c + 1$, we have $|I_{\mathrm{cols}}(c)| \geq 2$. Let j be the minimum in $I_{\mathrm{cols}}(c)$. We show that (7.5.9) holds strictly for the rows $r_{i,1}, r_{i,2}$ for every $i \in I_{\mathrm{cols}}(c)$ such that $i \neq j$. Indeed, as $\lambda^i_{r_{i,2}} = c = \lambda^j_{r_{j,2}}$, we have

$$\begin{aligned} &\lambda^i_{r_{i,1}} - \lambda^i_{r_{i,2}} \leq \lambda^j_{r_{i,1}} - \lambda^j_{r_{j,2}} \leq \lambda_{r_{i,1}} - \lambda_{r_{j,2}} \leq \\ &\leq s(r_{j,2} - r_{i,1}) \leq s(r_{i,2} - 1 - r_{i,1}) < s(r_{i,2} - r_{i,1}). \end{aligned}$$

In a similar fashion, let

$$I_{\mathrm{rows}}(r) = \left\{ i \in \{0, \ldots, q-1\} : \lambda^{i+1}_r > \lambda^i_r \right\}.$$

If there exists r such that $\mu_r > \lambda_r + 1$, then $|I_{\mathrm{rows}}(r)| \geq 2$ and we take $j \in I_{\mathrm{rows}}(r)$ to be the minimum and $i \in I_{\mathrm{rows}}(r)$ to be another index. We have $\lambda^i_{r_{i,2}} > \lambda^j_{r_{i,2}} = \lambda_{r_{i,2}}$, so we get

$$\lambda^i_{r_{i,1}} - \lambda^i_{r_{i,2}} < \lambda_{r_{i,1}} - \lambda_{r_{i,2}} \leq s(r_{i,2} - r_{i,1}).$$

In both cases (7.5.9) holds strictly for at least an i. Therefore (7.5.8) holds strictly for this i, and (7.5.7) must hold strictly as a consequence. Thus, the lemma is proven. □

Example. Let

$$\lambda = (6, 4, 2), \quad \mu = (5, 3, 2, 1, 1), \quad \nu = (5, 3, 3, 1)$$

be three partitions of $n = 12$. We have that $\mu < \lambda$ and $\nu < \lambda$; moreover, in the pair (λ, μ) the first column differs by 2 boxes, while no column in (λ, ν) differs by more than 1 box.

Since λ satisfies the 2-step condition, we can compute each term involved in (7.5.7). For (λ, μ) we have: $2 \cdot 6 \geq 8 + 2$ (which holds strictly), while for (λ, ν) we have: $2 \cdot 4 \geq 6 + 2$.

7.5.3 Inequalities on the dimensions

In this section we will use the tools introduced in section 7.5.2 to effectively compute the dimensions of the strata in Z.

Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of $n := |\lambda|$ with $t := \lambda_1$ columns. We want to compare the dimension of Z_{τ^0} with the dimension of each other stratum Z_τ with $\tau \in \Lambda^{(\lambda)}$. Let $\mu = \pi(\tau)$, so that $\mu \leq \lambda$. The aim of this section is to give some sufficient combinatorial conditions on the pair of partitions (λ, μ) in order to secure a bound on the difference $\dim Z_{\tau^0} - \dim Z_\tau$.

The aim of the current section is to prove proposition 7.5.4 and proposition 7.5.8. In order to introduce the stronger inequality in proposition 7.5.8, we need an in-depth study of the combinatorics of Λ, so we prefer to introduce it later.

Proposition 7.5.4. *Let $\lambda \geq \mu$ be two partitions and let $Z = Z^{(\lambda)}$ be the variety built from λ. Let Z_{τ^0} be the only stratum of Z with $\pi(\tau_1^0) = \lambda$ and let Z_τ be a stratum with $\tau = (\tau_1, \ldots, \tau_t)$ such that $\pi(\tau_1) = \mu$.*

For each real number x, we have that:

$$2r(\lambda, \mu) - c(\lambda, \mu) - q(\lambda, \mu) \geq 4x \quad \implies \quad \dim Z_{\tau^0} - \dim Z_\tau \geq x.$$

The proof will proceed as follows: we take the dimension formula of the strata of Z given by (6.2.5) then we will estimate each term on the difference. Each of the following lemmas deals with one of the terms.

Lemma 7.5.5. *Let $\lambda \geq \mu$ be two partitions of n. Let $t = \lambda_1$ be the number of columns of λ. Then*

$$\sum_{i=1}^{t} \left(\widehat{\mu}_i^2 - \widehat{\lambda}_i^2 \right) = 2r(\lambda, \mu).$$

(We allow $\widehat{\mu}_i = 0$ for each $i > \mu_1$.)

Proof. For every integer m we have: $m^2 = 2\binom{m+1}{2} - m$. So, by (7.5.1), we get:

$$\sum_{i=1}^{t} \left(\widehat{\mu}_i^2 - \widehat{\lambda}_i^2 \right) = \sum_{i=1}^{t} \left(2\binom{\widehat{\mu}_i + 1}{2} - \widehat{\mu}_i - 2\binom{\widehat{\lambda}_i + 1}{2} + \widehat{\lambda}_i \right) = 2r(\lambda, \mu) - n + n.$$

□

In order to carry on the computation, we examine carefully all the strata Z_τ with $\tau \neq \tau^0$. We do so by grouping together the $\tau \in \Lambda$ which can appear in $\Theta^{-1}(C_\mu)$, for fixed μ. We already know that for every $\tau = (\tau_1, \ldots, \tau_t)$ such that $\Theta(Z_\tau) = C_\mu$, we have that $\pi(\tau_1) = \mu$.

The conditions $\tau \in \Lambda$ and $\pi(\tau_1) = \mu$ place important combinatorial restrictions in the choice of the ab-diagrams $\tau_1, \ldots, \tau_t$. We are going to describe them in the following paragraphs.

Let $Z^{(\mu)}$ be the variety Z built from the partition μ (rather than the partition λ). We will denote by σ^0 the unique stratum in $\Lambda^{(\mu)}$ such that $\Theta(Z^{(\mu)}_{\sigma^0}) = C_\mu$. As in the case of τ^0, every ab-diagram σ_i^0 has only rows starting and ending by a, in particular they all have odd length, so:

$$\sum_{i=1}^{t} o(\sigma_i^0) = n = \sum_{i=1}^{t} o(\tau_i^0). \tag{7.5.10}$$

The sequence of ab-diagrams σ^0 does not belong to $\Lambda^{(\lambda)}$ because the integers n_i computed from μ are different from those computed from λ.

Nonetheless, for every $\tau \in \Lambda^{(\lambda)}$ such that $\pi(\tau_1) = \mu$, the condition $\rho(\tau_{i-1}) = \pi(\tau_i)$ implies that each τ_i can be obtained from σ_i^0 by adding an adequate amount of a's and b's letters. For example, if $i = 0$ we need some b's, possibly.

For each $i = 1, \dots, t$, let d_i be the list of a's and b's we need to add to σ_i^0 in order to obtain τ_i. Let d_i^a (resp. d_i^b) be the number of a's (resp. b's) in d_i. We remark that d_i^a and d_i^b depend only on (λ, μ) and not on the particular τ. In fact, we can compute d_i^a by taking the differences

$$d_i^a = \sum_{j=i}^{t} \left(\widehat{\lambda}_j - \widehat{\mu}_j \right); \tag{7.5.11}$$

and therefore we have $d_i^b = d_{i+1}^a$.

Example. Let $\tau \in \Lambda^{(\lambda)}$ and let $\sigma^0 \in \Lambda^{(\mu)}$. Suppose that the i-th diagrams are the following:

$$\sigma_i^0 = \begin{matrix} aba \\ aba \\ b \end{matrix}\ ; \qquad \tau_i = \begin{matrix} ababa \\ aba \\ ba \\ ab \end{matrix}\ .$$

Then, we have $d_i = (a, a, a, b, b)$, $d_i^a = 3$, $d_i^b = 2$.

Given an ab-diagram δ and a list d of a's and b's, we define $\mathrm{aug}_\delta(d)$ to be the set of ortho-symmetric ab-diagrams obtainable from δ by adding the letters in d.

Example. Let $d = (a, b)$ and

$$\delta = \begin{matrix} aba \\ aba \\ b \end{matrix}\ .$$

Then $\mathrm{aug}_\delta(d)$ has three elements:

$$\mathrm{aug}_\delta(d) = \left\{ \begin{matrix} ababa \\ aba \\ b \end{matrix}\ , \quad \begin{matrix} aba \\ aba \\ bab \end{matrix}\ , \quad \begin{matrix} aba \\ aba \\ a \\ b \\ b \end{matrix} \right\}.$$

Lemma 7.5.6. *Let δ^0 be an ab-diagram with rows starting and ending only with a. Let d be a list of d^a a's and d^b b's. Let $\delta \in \mathrm{aug}_{\delta^0}(d)$. Then the following holds:*

$$o(\delta) - 2\Delta(\delta) - o(\delta^0) \leq \max\{d^a, d^b\}.$$

Proof. The ab-diagram δ is obtained from δ^0 by adding d^a a's and d^b b's.

Let L (resp. S) be the set of rows of δ longer than 1 (resp. of length 1) built using only letters in d. As $o(\delta)$ counts some of the rows in δ, we compare the difference $o(\delta) - o(\delta^0)$ with the number of rows built only with letters in d and we get $o(\delta) - o(\delta^0) \leq |L| + |S|$.

Let S_a (resp. S_b) the rows of S starting with a (resp. b), so that $S = S_a \sqcup S_b$. Recall that a_i (resp. b_i) is the number of rows of length i starting with a (resp. b); in this particular ab-diagram δ we have $|S_b| = b_1$ and $|S_a| \leq a_1$.

$$\begin{aligned} o(\delta) - o(\delta^0) - 2\Delta(\delta) \leq & |L| + |S| - 2\sum_{i \text{ odd}} a_i b_i \\ \leq & |L| + |S| - 2a_1 b_1 \\ \leq & |L| + |S_a| + |S_b| - 2|S_a||S_b| \\ \leq & |L| + \max\{|S_a|, |S_b|\}. \end{aligned}$$

If $|S_a| \geq |S_b|$, then $|L| + |S_a| \leq d_i^a$ because each row in L or in S_a contains at least one a. If the opposite holds true, then $|L| + |S_b| \leq d_i^b$ for the same reason. Thus the lemma is proven. □

We can introduce a small, but very important, improvement on lemma 7.5.6.

Lemma 7.5.7. *In the same setting of lemma 7.5.6, if we furthermore assume that $d = (b)$, that is $d^a = 0$ and $d^b = 1$, and that δ^0 has l rows of length 1, we have the stronger equality:*

$$o(\delta) - 2\Delta(\delta) - o(\delta^0) = \max\{d^a, d^b\} - 2l = 1 - 2l.$$

Proof. As all the rows in δ^0 have odd length, we cannot extend exactly one row with letter b, otherwise the resulting ab-diagram would not be ortho-symmetric. So we must place b on a new row.

In this case, $o(\delta) - o(\delta^0) = 1$ and $\Delta(\delta) = a_1 b_1 = l \cdot 1$. □

Using this lemma we will be able to introduce the previously announced stronger version of proposition 7.5.4, namely:

Proposition 7.5.8. *Let $\lambda \geq \mu$ be two partitions and let $Z = Z^{(\lambda)}$ be the variety built from λ. Let Z_{τ^0} be the only stratum of Z with $\pi(\tau_1^0) = \lambda$ and let Z_τ be a stratum with $\tau = (\tau_1, \ldots, \tau_t)$ such that $\pi(\tau_1) = \mu$. As in proposition 7.5.4, we suppose that there exists a real number x such that:*

$$2r(\lambda, \mu) - c(\lambda, \mu) - q(\lambda, \mu) \geq 4x.$$

If there exists an index i such that $d_i = (b)$ and σ_i^0 has l rows of length one, then:

$$\dim Z_{\tau^0} \geq \dim Z_\tau + x + l/2.$$

The proof of proposition 7.5.8 will be given together with the proof of proposition 7.5.4 at the end of this section.

We want an estimate of the sum of the terms $\max\{d_i^a, d_i^b\}$ depending only on λ, μ.

Lemma 7.5.9. *Let $\lambda \geq \mu$ be partitions, so that we can define d_i^a, d_i^b for all $i = 1, \ldots, t$. Then*

$$\sum_{i=1}^{t} \max\{d_i^a, d_i^b\} \leq c(\lambda, \mu) + q(\lambda, \mu). \tag{7.5.12}$$

Proof. First we prove the claim for $q(\lambda, \mu) = 1$.

In this case, μ is obtained from λ by moving down a single box B. Let us call c_1 (resp. c_2) the column where B lies in μ (resp. λ); so $c_1 < c_2$. By (7.5.11), one gets $d_i^a = 1$ for each $c_1 < i \leq c_2$, and 0 otherwise. Moreover one also gets that $d_i^b = 1$ for each $c_1 \leq i < c_2$ and 0 otherwise. Therefore $\max\{d_i^a, d_i^b\} = 1$ for all $c_1 \leq i \leq c_2$ and 0 otherwise. Therefore

$$\sum_{i=1}^{t} \max\{d_i^a, d_i^b\} = c_2 - c_1 + 1,$$

while

$$c(\lambda, \mu) = c_2 - c_1,$$

so

$$\sum_{i=1}^{t} \max\{d_i^a, d_i^b\} = c(\lambda, \mu) + 1 \tag{7.5.13}$$

and the conclusion holds in this case.

In the general case, let $q := q(\lambda, \mu)$ so, by lemma 3.1.3, we get a sequence of partitions

$$\lambda = \lambda^0 > \lambda^1 > \cdots > \lambda^q = \mu$$

such that $q(\lambda^j, \lambda^{j+1}) = 1$ for each $j = 0, \ldots, q-1$. By (7.5.13), for every j, we have

$$\sum_{i=1}^{t} \max\{d_i^a, d_i^b\}(\lambda^j, \lambda^{j+1}) = c(\lambda^j, \lambda^{j+1}) + 1$$

and, summing up over j,

$$\sum_{i=1}^{t} \sum_{j=0}^{q-1} \max\{d_i^a, d_i^b\}(\lambda^j, \lambda^{j+1}) = \sum_{j=0}^{q-1} c(\lambda^j, \lambda^{j+1}) + q.$$

By (7.5.4), the function c is additive on the pairs $(\lambda^j, \lambda^{j+1})$, therefore the right hand side is equal to $c(\lambda, \mu) + q$. Also d_i^a and d_i^b are additive functions on the pairs $(\lambda^j, \lambda^{j+1})$ (as it follows from (7.5.11)), therefore

$$\begin{aligned}\sum_{j=0}^{q-1} \max\{d_i^a, d_i^b\}(\lambda^j, \lambda^{j+1}) &= \sum_{j=0}^{q-1} \max\left\{d_i^a(\lambda^j, \lambda^{j+1}), d_i^b(\lambda^j, \lambda^{j+1})\right\} \\ &\geq \max\left\{\sum_{j=0}^{q-1} d_i^a(\lambda^j, \lambda^{j+1}), \sum_{j=0}^{q-1} d_i^b(\lambda^j, \lambda^{j+1})\right\} \\ &= \max\left\{d_i^a(\lambda, \mu), d_i^b(\lambda, \mu)\right\} = \max\{d_i^a, d_i^b\}\end{aligned}$$

and the conclusion follows. □

We are finally able to prove both proposition 7.5.4 and proposition 7.5.8.

Proof of proposition 7.5.4 and proposition 7.5.8. We start from the dimension formula of the strata (6.2.5). We recall that $\Delta(\tau^0) = 0$ as no ab-diagram of τ^0 has rows starting with b. Therefore

$$\begin{aligned}
4(\dim Z_{\tau^0} - \dim Z_\tau - x) &= 2(\dim C_{\pi(\tau_1^0)} - \dim C_{\pi(\tau_1)}) + \\
&\quad + \left(o(\tau^0) - 2\Delta(\tau^0)\right) - (o(\tau) - 2\Delta(\tau)) - 4x \\
&= 2(\dim C_\lambda - \dim C_\mu) + o(\tau^0) - o(\tau) + 2\Delta(\tau) - 4x \\
&= \sum_{i=1}^{t} \left(\widehat{\mu}_i^2 - \widehat{\lambda}_i^2\right) + \sum_{i=1}^{t} \left(o(\tau_i^0) - o(\tau_i) + 2\Delta(\tau_i)\right) - 4x \\
\text{(by lemma 7.5.5)} &= 2r(\lambda,\mu) + \sum_{i=1}^{t} \left(o(\tau_i^0) - o(\tau_i) + 2\Delta(\tau_i)\right) - 4x \\
\text{(by (7.5.10))} &= 2r(\lambda,\mu) + \sum_{i=1}^{t} \left(o(\sigma_i^0) - o(\tau_i) + 2\Delta(\tau_i)\right) - 4x \\
&\geq 2r(\lambda,\mu) - 4x + \\
&\quad - \sum_{i=1}^{t} \max_{\tau_i \in \mathrm{aug}_{\sigma_i^0}(d_i)} \left(o(\tau_i) - 2\Delta(\tau_i) - o(\sigma_i^0)\right) \\
\text{(by lemma 7.5.6)} &\geq 2r(\lambda,\mu) - \sum_{i=1}^{t} \max\{d_i^a, d_i^b\} - 4x \\
\text{(by lemma 7.5.9)} &\geq 2r(\lambda,\mu) - c(\lambda,\mu) - q(\lambda,\mu) - 4x.
\end{aligned}$$

Therefore $\dim Z_{\tau^0} - \dim Z_\tau \geq x$ is guaranteed as soon as $2r(\lambda,\mu) - c(\lambda,\mu) - q(\lambda,\mu) \geq 4x$.

In order to obtain the sharper result of proposition 7.5.8, it is enough to use lemma 7.5.7 in place of lemma 7.5.6 in the second to last step. □

7.5.4 Complete intersection conditions

In this section we want to give conditions under which we can make sure that Z is a complete intersection variety.

Proposition 7.5.10. *Let λ be a partition and recall that $Z = \Phi^{-1}(0) \subseteq M$. We have*

$$\mathrm{codim}_M(Z_{\tau^0}) = \dim N.$$

Proof. In order to prove the equality of the required dimensions, we will work with the dimension formula given by (6.2.5). We are going to prove that

$$\dim Z_{\tau^0} = \dim M - \dim N. \tag{7.5.14}$$

We immediately have:

$$\dim M = \sum_{i=1}^{t} \dim L(V_{i-1}, V_i) = \sum_{i=1}^{t} n_{i-1} n_i$$

and

$$\dim N = \sum_{i=1}^{t-1} \dim \mathfrak{p}(V_i) = \sum_{i=1}^{t-1} \frac{1}{2} n_i^2 + \sum_{i=1}^{t-1} \frac{1}{2} n_i,$$

so

$$\dim M - \dim N = \sum_{i=1}^{t} n_{i-1} n_i - \frac{1}{2} \sum_{i=1}^{t-1} n_i^2 - \frac{1}{2} \sum_{i=1}^{t-1} n_i.$$

On the other hand, we have:

$$\dim Z_{\tau^0} = \frac{1}{2} \dim C_{\pi(\tau_1^0)} + \sum_{i=1}^{t} \left(\frac{1}{2} n_{i-1} n_i - \frac{\varepsilon}{4} (n_{i-1} + n_i) \right) + \\ + \sum_{i=1}^{t} \left(\frac{1}{4} o(\tau_i^0) - \frac{1}{2} \Delta(\tau_i^0) \right).$$

We have that $\pi(\tau_1^0) = \lambda$; we then use proposition 2.5.5.

We also have that $o(\tau_i^0)$ is precisely the number of rows of τ_i^0 (as they all have odd length), the number of rows of τ_i^0 is equal to the number of rows of $\lambda^i := \pi(\tau_i^0)$, the partition λ^i has clearly $\widehat{\lambda}_1^i$ rows and $\widehat{\lambda}_1^i = \widehat{\lambda}_i$, that is the i-th column of λ.

Finally, we have that $\Delta(\tau_i^0) = 0$, because there are no rows in τ_i^0 starting with b.

Therefore:

$$\dim Z_{\tau^0} = \frac{1}{4} \left(n_0^2 - \sum_{i=1}^{t} \widehat{\lambda}_i^2 \right) + \sum_{i=1}^{t} \left(\frac{1}{2} n_{i-1} n_i - \frac{\varepsilon}{4} (n_{i-1} + n_i) \right) + \frac{1}{4} \sum_{i=1}^{t} \widehat{\lambda}_i;$$

we recall that $\widehat{\lambda}_i = n_{i-1} - n_i$ and that $n_0 = n = |\lambda|$, $n_t = 0$; so

$$\begin{aligned} \dim Z_{\tau^0} =& \frac{1}{4} \left(n_0^2 - \sum_{i=1}^{t} (n_{i-1} - n_i)^2 \right) + \\ &+ \frac{1}{2} \sum_{i=1}^{t} n_{i-1} n_i - \frac{1}{4} n_0 - \frac{1}{2} \sum_{i=1}^{t-1} n_i + \frac{1}{4} \sum_{i=1}^{t} n_{i-1} - n_i \\ =& - \frac{1}{2} \sum_{i=1}^{t-1} n_i^2 + \frac{1}{2} \sum_{i=1}^{t-1} n_{i-1} n_i + \frac{1}{2} \sum_{i=1}^{t} n_{i-1} n_i - \frac{1}{2} \sum_{i=1}^{t-1} n_i \\ =& \dim M - \dim N. \end{aligned}$$

□

One of the remaining steps to conclude the complete intersection of Z is the following lemma, which can finally be proved using lemma 7.5.3 and propositions 7.5.4 and 7.5.8.

Lemma 7.5.11. *If the partition λ satisfies the 2-step condition and $\tau \in \Lambda$ is a string of ab-diagrams different from τ^0, then*

$$\dim Z_{\tau^0} > \dim Z_{\tau}.$$

Proof. Let $\mu = \pi(\tau)$, so in particular $\mu < \lambda$, as $\tau \neq \tau^0$. Therefore lemma 7.5.3 implies

$$2r(\lambda, \mu) \geq c(\lambda, \mu) + q(\lambda, \mu) \tag{7.5.15}$$

and, by combining it with proposition 7.5.4 we immediately get

$$\dim Z_{\tau^0} \geq \dim Z_{\tau}.$$

So our concern is reduced to obtain a strict inequality. We will gain the strict inequality either by proving that (7.5.15) holds strictly for the pair (λ, μ) or by using proposition 7.5.8 with $l > 0$.

Let c be the first column such that $\widehat{\mu}_c > \widehat{\lambda}_c$ and let $r = \widehat{\mu}_c$. This means that the first $c-1$ columns of λ and μ are the same and $\mu_r > \lambda_r$. On the basis of c and r, we distinguish three cases.

$\widehat{\mu}_c - \widehat{\lambda}_c \geq 2$: in this case we can use lemma 7.5.3 with the column c to obtain that

$$2r(\lambda, \mu) > c(\lambda, \mu) + q(\lambda, \mu).$$

$\widehat{\mu}_c - \widehat{\lambda}_c = 1$, $\mu_r - \lambda_r \geq 2$: we can still use lemma 7.5.3, this time with the row r, to obtain that

$$2r(\lambda, \mu) > c(\lambda, \mu) + q(\lambda, \mu).$$

$\widehat{\mu}_c - \widehat{\lambda}_c = 1$, $\mu_r - \lambda_r = 1$: in this case we can use proposition 7.5.8 with $i = c$ and $l \geq 1$. Indeed, the following two facts are immediately seen: by (7.5.11), $d_c = (b)$; the ab-diagram σ_c^0 has at least one row equal to a single a, namely the $\widehat{\mu}_c$-th row.

□

We finally have the main result of this section, which is the consequence of lemma 7.5.11 and proposition 5.3.1.

Proposition 7.5.12. *If λ satisfies the 2-step condition, then Z is reduced and complete intersection.*

Remark. In this case, the inequality (5.3.2) is not true. In fact, one may show that as soon as λ is not 3-step, (5.3.2) fails to be true.

Remark. If one allows n to vary, there is no lower bound for the difference $\dim Z_{\tau^0} - \dim Z_{\tau}$.

7.5.5 Normality conditions

Lemma 7.5.13. *If the partition λ satisfies the 1-step condition, then*

$$\dim(Z \setminus Z_{\tau^0}) \leq \dim Z - 2.$$

Proof. Let $\tau \in \Lambda$ be such that $\tau \neq \tau^0$. Thus $\mu := \pi(\tau_1) < \lambda$. We need to prove that

$$\dim Z_{\tau^0} - \dim Z_\tau \geq 2.$$

By lemma 7.5.3, $r(\lambda, \mu) \geq c(\lambda, \mu) + q(\lambda, \mu)$ and hence

$$2r(\lambda, \mu) - c(\lambda, \mu) - q(\lambda, \mu) \geq 4x$$

where $x = \frac{1}{4}\left(c(\lambda, \mu) + q(\lambda, \mu)\right)$. By proposition 7.5.4, we get

$$\dim Z_{\tau^0} - \dim Z_\tau \geq x,$$

so we are done if $x > 1$.

Let us assume that $x \leq 1$. For simplicity of notation, we put $c := c(\lambda, \mu)$ and $q := q(\lambda, \mu)$. We claim that there are only four cases left: $q = c = 2$ and $q = 1$, $c \leq 3$. Indeed, by (7.5.2), $\lambda > \mu$ if and only if both $c \geq 1$ and $q \geq 1$. By (7.5.6), we also have $c \geq q$. Since $4x = c + q \leq 4$, then $q < 3$. Indeed, $q \geq 3 \Rightarrow c \geq 3 \Rightarrow c + q \geq 6$. If $q = 2$, then $c \leq 2$ by the same reasoning. If $q = 1$, then $c = 4x - q \leq 4 - 1$.

We analyse the four cases separately and, in each case, we find an index i such that $d_i = (b)$, the ab-diagram σ_i^0 has l rows of length one and l is big enough so that we can conclude the proof by proposition 7.5.8 as follows:

$$\dim Z_{\tau^0} - \dim Z_\tau \geq x + \frac{l}{2} = \frac{1}{4}(c + q) + \frac{l}{2} > 1.$$

$q = 2$, $c = 2$ There are two boxes B, B' which are lowered from λ in order to obtain μ. Let i' be the column of B in λ and i be the column of B in μ. As $c = 2$, $i = i' - 1$. By (7.5.11), the list $d_i = (b)$. In σ_i^0 the rows with indices $\widehat{\mu}_i = \widehat{\lambda}_i + 1$, $\widehat{\lambda}_i$ and $\widehat{\lambda}_i - 1$ have length one. Indeed, since λ is 1-step, λ does not have two different columns with the same height. Thus, σ_i^0 has $l \geq 3$ rows of length one. We found that $\frac{1}{4}(c + q) + \frac{l}{2} \geq \frac{5}{2} > 1$.

$q = 1$, $c = 3$ There is a single box B which is lowered from λ in order to obtain μ. Let i' be the column of B in λ and i be the column of B in μ. As $c = 3$, $i = i' - 3$. By (7.5.11), the list $d_i = (b)$. In σ_i^0 the rows with indices $\widehat{\mu}_i$, $\widehat{\mu}_i - 1$ have length one. Thus, σ_i^0 has $l \geq 2$ rows of length one. We found that $\frac{1}{4}(c + q) + \frac{l}{2} \geq 2 > 1$.

$q = 1$, $c = 2$ Similar to the previous case, there is a single box B moving from column $i' = i + 2$ to column i; $d_i = (b)$; the rows with indices $\widehat{\mu}_i$, $\widehat{\mu}_i - 1$ have length one; so σ_i^0 has $l \geq 2$ rows of length one. We found that $\frac{1}{4}(c + q) + \frac{l}{2} \geq \frac{7}{4} > 1$.

$q = 1$, $c = 1$ There is a single box B moving from column $i' = i + 1$ to column i. Similar to the case $q = 2$, $c = 2$, $d_i = (b)$, the rows with indices $\widehat{\mu}_i = \widehat{\lambda}_i + 1$, $\widehat{\lambda}_i$ and $\widehat{\lambda}_i - 1$ have length one and σ_i^0 has $l \geq 3$ rows of length one. We found that $\frac{1}{4}(c + q) + \frac{l}{2} \geq 2 > 1$.

□

The final result is a consequence of lemma 7.5.13, proposition 7.3.1 and theorem 5.3.2.

Theorem 7.5.14. *If the partition λ satisfies the 1-step condition, then Z and $\overline{C_\lambda}$ are normal.*

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