On a class of fully nonlinear elliptic equations in dimension two

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Abstract: We study existence and asymptotic behavior of radial positive solutions of some fully nonlinear equations involving Pucci's extremal operators in dimension two. In particular we prove the existence of a positive solution of a fully nonlinear version of the Liouville equation in the plane. Moreover for the $\mathcal{M}_{\lambda,\Lambda}^{-}$ operator, we show the existence of a critical exponent and give bounds for it.

1 Introduction

In this paper we study positive radial solutions for the fully nonlinear Lane Emden equation driven by Pucci's extremal operators:

$$-\mathcal{M}_{\lambda,\Lambda}^{\pm}(D^2u) = u^p \qquad \text{in } \Omega \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is either the ball B_R of radius $R > 0$, centered at the origin or the whole plane $\mathbb{R}^2$. In the case of the ball we will impose the homogeneous Dirichlet boundary condition.
We recall that for a $\mathcal{C}^2$ function in $\mathbb{R}^N$, $N \geq 2$, the Pucci's operators are defined by:

$$\mathcal{M}_{\lambda,\Lambda}^{+}(D^2u) = \Lambda \sum_{\mu_i>0} \mu_i + \lambda \sum_{\mu_i<0} \mu_i$$

$$\mathcal{M}_{\lambda,\Lambda}^{-}(D^2u) = \lambda \sum_{\mu_i>0} \mu_i + \Lambda \sum_{\mu_i<0} \mu_i$$

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where $0 < \lambda \leq \Lambda$ are the ellipticity constants and $\mu_i = \mu_i(D^2u)$, $i = 1,\ldots,N$ are the eigenvalues of the hessian matrix D^2u. Associated to $\mathcal{M}^{\pm}_{\lambda,\Lambda}$ are dimension like numbers $\tilde{N}_+$ and $\tilde{N}_-$ defined as

$$\tilde{N}_+ = \frac{\lambda}{\Lambda}(N-1) + 1 \qquad \text{and} \ \tilde{N}_- = \frac{\Lambda}{\lambda}(N-1) + 1 \tag{2}$$

These numbers allow to give estimates for the exponent p for which existence or nonexistence of solutions of (1) in B_R or $\mathbb{R}^N$ holds when $N \geq 3$ and $\tilde{N}_+ > 2$, (note that $\tilde{N}_-$ is always larger than two if $N \geq 3$).

Indeed a first result obtained in [5] shows that if $N \geq 3$ and $p \leq \frac{\tilde{N}_{\pm}}{\tilde{N}_{\pm}-2}$ then no nontrivial positive viscosity supersolutions of (1) exist in $\mathbb{R}^N$. Using this result, the existence of positive solutions in bounded domains, not necessarily radial, was proved in [13] for the same range of exponents.

In the radial setting Felmer and Quaas in [9] (see also [10]) provided, for $N \geq 3$, the existence of critical exponents p_+^* for $\mathcal{M}^+_{\lambda,\Lambda}$ and p_-^* for $\mathcal{M}^-_{\lambda,\Lambda}$ satisfying

$$\max\left\{\frac{\tilde{N}_+}{\tilde{N}_+ - 2}, \frac{N+2}{N-2}\right\} < p_+^* < \frac{\tilde{N}_+ + 2}{\tilde{N}_+ - 2} \tag{3}$$

$$\frac{\tilde{N}_- + 2}{\tilde{N}_- - 2} < p_-^* < \frac{N+2}{N-2}$$

which are thresholds for the existence of radial positive solutions of (1) in the ball or in $\mathbb{R}^N$.

More precisely they proved that existence in B_R holds if and only if $p < p_{\pm}^*$, while existence in $\mathbb{R}^N$ holds if and only if $p \geq p_{\pm}^*$. In the last case a complete classification of the solutions was given, according to the decay at infinity. Recently, the same kind of results has been obtained in [12] for more general equations with an alternative proof.

In dimension N=2, the situation for the two operators $\mathcal{M}^+_{\lambda,\Lambda}, \mathcal{M}^-_{\lambda,\Lambda}$ is different. Indeed, by (2) we have that $\tilde{N}_+ < 2$, while $\tilde{N}_- > 2$, for $\lambda < \Lambda$. In the first case the result of [5] still implies the nonexistence of nontrivial solutions in $\mathbb{R}^N$ for the equation (1) for $\mathcal{M}^+_{\lambda,\Lambda}$ for every $p \in (0,\infty)$.

As a consequence a positive radial solution of

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}D^2u = u^p & \text{in } B_R \\ u > 0 & \text{in } B_R \\ u = 0 & \text{on } \partial B_R \end{cases} \tag{4}$$

exists for every $0 < p < \infty$ as in [13]. Thus no critical exponent as in (3) can be defined for $\mathcal{M}^+_{\lambda,\Lambda}$, in dimension 2.

Instead in the case of the operator $\mathcal{M}_{\lambda,\Lambda}^-$ the number $\tilde{N}_- - 2$ is still positive which suggests that a critical exponent p_-^* as in higher dimensions could exist, though, any upper bound for it as in (3), could not be given, since $N = 2$.

In this paper we obtain new results for (1) in $\mathbb{R}^2$, but of different kind for each one of the two operators.

In the case of $\mathcal{M}_{\lambda,\Lambda}^+$, when a unique radial positive solution of (4) in the ball exists for every $p > 1$(see Theorem 2.1), we study the asymptotic behavior of u_p as $p \to \infty$. This is done by analyzing the rescaled function

$$z_p = z_p(r) = \frac{p}{u_p(0)}(u(\varepsilon_p r) - u_p(0)), \tag{5}$$

for $r = |x|$, and

$$\varepsilon_p^{-2} = p \cdot u_p(0)^{p-1},$$

as in the semilinear case ([1],[8]). As a byproduct, we obtain the existence of a radial solution of the "fully nonlinear Liouville equation" in the plane. As far as we know, this is the first existence result for this equation.
More precisely we get:

Theorem 1.1. *The function z_p converges in $\mathcal{C}_{loc}^2(\mathbb{R}^2)$ to a radial function z which satisfies:*

$$-\mathcal{M}_{\lambda,\Lambda}^+(D^2 z)(x) = e^z. \tag{6}$$

Moreover z is negative, radially decreasing and, as a function of $|x|$, changes concavity only once at $|x| = R_0 = 2\sqrt{2\lambda}$. Finally:

$$z(x) = \log\left(\frac{1}{(1 + \frac{1}{8\lambda}|x|^2)^2}\right) \ in \ B_{R_0}.$$

In the case of the operator $\mathcal{M}_{\lambda,\Lambda}^-$, we are able to prove that in dimension $N = 2$, a critical exponent p_-^* having similar features as the one for $N \geq 3$ indeed exists and we provide bounds for it.

As mentioned before, to get an upper bound for p_-^* is not obvious since the corresponding estimate in higher dimensions in (3) blows up when $N = 2$.

We get it through the existence of another relevant exponent, denoted by $\tilde{p}_-$, which is responsible for the existence or lack of periodic orbits for a related dynamical system that we study, following the approach of [12].

As it will be made clear in Section 3, the periodic orbits of this auxiliary dynamical system are related to the nonexistence of solutions in the ball, and possibly allow the existence of entire oscillating radial solutions for (1).

Referring to Definition 3.5 for fast, slow, or pseudo-slow decaying solutions we state the main result for the operator $\mathcal{M}_{\lambda,\Lambda}^-$.

Theorem 1.2 (Critical Exponent). *Let the dimension N be two, there are exponents p_-^* and $\tilde{p}_-$ satisfying:*

$$\frac{\tilde{N}_- + 2}{\tilde{N}_- - 2} < p_-^* \leq \tilde{p}_- \leq \frac{\tilde{N}_- + 2}{\tilde{N}_- - 2} + \frac{4}{\tilde{N}_- - 2 + \frac{\lambda}{\Lambda}(\tilde{N}_- - 2)^2} \tag{7}$$

with $\tilde{N}_-$ as in (2), such that, considering equation (1) for $\mathcal{M}_{\lambda,\Lambda}^-$:

- *i) for $p < p_-^*$ there is no nontrivial radial positive solution of problem (1) in $\mathbb{R}^2$, while, for every $R > 0$ there is a unique positive radial solution in B_R.*

- *ii) if $p = p_-^*$ there is a unique fast decaying radial positive solution of (1) in $\mathbb{R}^2$*

- *iii) for $p_-^* < p < \tilde{p}_-$ there is a unique positive radial solution of (1) in $\mathbb{R}^2$, which may be either pseudo-slow or slow decaying.*

- *iv) for $p > \tilde{p}_-$ there is a unique slow decaying solution of (1) in $\mathbb{R}^2$*

- *v) for $p \geq p_-^*$ there is no positive radial solution of (1) in B_R.*

  In the case of $\mathbb{R}^2$ uniqueness is meant up to scaling.

We recall that, by using the moving plane method, it is proved in [6] that every positive solution of (1) in the ball B_R satisfying $u = 0$ on ∂B_R is radial. Thus, by Theorem 1.2, we have that such a solution in B_R exists if and only if $p < p_-^*$.

Then, we could study the asymptotic behavior of u_p, as $p \nearrow p_-^*$, to understand the limit profile of these solutions.

As for the higher dimensional case ([2]) we get:

Theorem 1.3. *Let $N = 2$, p_-^* as in Theorem 1.2 and $\varepsilon > 0$. Then, for the solution u_{p_ε} of (1) (for $\mathcal{M}_{\lambda,\Lambda}^-$) in the ball B_R, with $p_\varepsilon = p_-^* - \varepsilon$, the following statements hold:*

- *i) $M_\varepsilon = u_{p_\varepsilon}(0) \to \infty$*

- *ii) the rescaled function*

$$u_\varepsilon = \frac{1}{u_{p_\varepsilon}(0)} u_{p_\varepsilon}\left(\frac{x}{u_{p_\varepsilon}(0)^{\frac{p_\varepsilon - 1}{2}}}\right)$$

  converges up to a subsequence, to a limit function U in $C_{loc}^2(\mathbb{R}^2)$ where U is the unique positive solution of:

$$-\mathcal{M}_{\lambda,\Lambda}^-(D^2 U)(x) = U^{p_-^*} \qquad in \ \mathbb{R}^2$$

  satisfying $U(0) = 1$

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iii) $u_{p_\varepsilon} \to 0$ *in* $\mathcal{C}^2_{loc}(\bar{B}_R \setminus \{0\})$.

The proof of the previous theorem is similar to that of [2] for the analogous results in higher dimension, though the statement iii) could be obtained more easily analyzing the dynamical system introduced in [12]. Also, the results about the energy invariance of [2] can be easily extended to the two-dimensional case.

Finally, a classification of solutions of (1) singular at the origin similar to the one of Theorem 1.8 of [12] follows in the same way, with obvious changes.

We conclude with some remarks about higher dimensions which are related to our results for $N = 2$.

First when the dimension N is greater than two, we may consider the case where $\tilde{N}_+$ as defined in (2) is smaller than two. Then the results of [5] and [13] still imply that there is, for every $p \in (1, \infty)$, a unique radial positive solution u_p of (4) in the ball B_R.

In particular, the approach we use to treat the two dimensional problem may be immediately applied in this setting producing results analogous to those of Theorem 1.1. Thus we get solutions of the Liouville equation (6) in some higher dimensions. Note that even for the semilinear case, where the Pucci operator $\mathcal{M}^+_{\lambda,\Lambda}$ is replaced by the Laplacian the corresponding equation (6) has different features according to the dimension (see [7], [11] and references therein). It would be interesting to investigate problem (6) in the fully nonlinear setting in all dimensions and also for the operator $\mathcal{M}^-_{\lambda,\Lambda}$.

Concerning the operator $\mathcal{M}^-_{\lambda,\Lambda}$, the approach we have used in Section 3 to estimate p^*_- as in (7) can also be used in higher dimensions to have an estimate of the critical exponent from above better than the one given in (3) and to determine the existence of oscillating solutions. This will be done in a forthcoming paper.

The paper is organized as follows. In Section 2 we study the equation (1) for the operator $\mathcal{M}^+_{\lambda,\Lambda}$ proving Theorem 1.1. Section 3 is devoted to the Pucci Operator $\mathcal{M}^-_{\lambda,\Lambda}$. After recalling several preliminaries about an associated dynamical system we prove a result on the nonexistence of periodic orbits for such a system. This allows to prove Theorem 1.2. We end Section 3 with the proof of Theorem 1.3.

2 The Problem for $\mathcal{M}^+$

2.1 A Preliminary Result

For $p > 1$, we consider the Dirichlet problem (4) and recall known results for it.

Theorem 2.1. *For every $p > 1$ the problem (4) admits a unique solution u_p which is radial, i.e, with an abuse of notation, $u_p(x) = u_p(|x|)$ for $|x| = r \in [0, R]$. Furthermore u_p satisfies:*

i) $u_p(0) = \max u_p$

ii) u_p *is strictly decreasing*

iii) u_p *changes concavity only once at a point* y_p *and is concave around the origin.*

Proof. The existence of a positive solution of (4) for every $p > 1$ derives from the nonexistence of solutions of the analogous equation in $\mathbb{R}^2$ (see [5]). Indeed if entire solutions do not exist, then apriori estimates hold which allow to prove the existence of a solution of (4) as in [13].

The radial symmetry of u_p and i)-ii) have been proved in [6] by the moving plane method. The uniqueness follows by the invariance by scaling of the equation and the uniqueness of the initial value problem for the corresponding ODE.

Finally, iii) can be proved exactly as in [10, Lemma 3.1] using the Emden-Fowler analysis. $\qquad\square$

Remark 2.2. *The uniqueness of the radius* y_p *where* $u''(r) = 0$ *can be obtained also as in [12] analyzing the flow induced by an associated dynamical system. This method also works in dimension two, and we will use it to study the problem for* $\mathcal{M}_{\lambda,\Lambda}^-$.

2.2 Asymptotic behavior of rescaled solutions

In this section we prove Theorem 1.1.

We are interested in the asymptotic behavior of the solution u_p when the exponent p goes to $+\infty$.

Recalling that the parameter ε is defined by

$$\varepsilon_p^{-2} = p \cdot u_p(0)^{p-1}$$

we prove the following preliminary result.

Lemma 2.3. *It holds :*
$$\lim_{p \to +\infty} \varepsilon_p^{-2} = +\infty$$

Proof. The proof is based on the fact that $u_p(0)^{p-1}$ does not converge to 0. Assume otherwise, then for p sufficiently big, we have:

$$u_p(0)^{p-1} < \lambda_1(\mathcal{M}_{\lambda,\Lambda}^+, B_R)$$

where $\lambda_1(\mathcal{M}_{\lambda,\Lambda}^+, B_R)$ is the first eigenvalue of the Pucci operator on B_R with homogeneous Dirichlet boundary conditions.

In particular, this implies that

$$\begin{cases} -\mathcal{M}_{\lambda,\Lambda}^+(D^2 u_p)(x) = u_p(x)^p < \lambda_1(\mathcal{M}_{\lambda,\Lambda}^+, B_R) \cdot u & \text{in } B_R \\ u_p > 0 & \text{in } B_R \\ u_p = 0 & \text{on } \partial B_R \end{cases}$$

This is a contradiction with the characterization of the first eigenvalue. Therefore we conclude that $\lim_{p \to \infty} p \cdot u_p(0)^{p-1} = \infty$.

$\square$

Proposition 2.4. *The rescaled function z_p defined in (5) converges in $C^2_{loc}(\mathbb{R}^2)$, up to a subsequence, to a solution z of the Liouville equation (6).*

Proof. A simple computation gives for $x \in B_{R/\varepsilon_p}$:

$$D^2 z_p(x) = \frac{p}{u_p(0)} \cdot \varepsilon_p^2 \cdot D^2 u_p(\varepsilon_p \cdot x)$$

Therefore,

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 z_p)(x) = \frac{p}{u_p(0)} \cdot \varepsilon_p^2 \cdot \mathcal{M}^+_{\lambda,\Lambda}(D^2 u_p)(\varepsilon_p \cdot x)$$

Since u_p is a solution to Problem (4), we get

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 z_p)(x) = \frac{-p}{u_p(0)} \cdot \varepsilon_p^2 u_p^p(\varepsilon_p \cdot x) = -\left(1 + \frac{z_p}{p}\right)^p$$

Since z_p is nonpositive and has a maximum at 0, where $z_p(0) = 0$, then $-z_p$ is a nonnegative solution of

$$-\mathcal{M}^-_{\lambda,\Lambda}(D^2(-z_p))(x) = -\left(1 + \frac{z_p(x)}{p}\right)^p = f_p(x) \in [-1, 0] \quad \text{in } B_{R/\varepsilon_p} \quad (8)$$

Applying the Harnack inequality given by Theorem 4.3 in [3] we get that z_p is locally uniformly bounded.

We proceed by using the $\mathcal{C}^\alpha$ and $\mathcal{C}^{2,\alpha}$ estimates given in [3] again to get the existence of $\alpha(\lambda, \Lambda)$ and $C(\Lambda, \lambda, R_1)$ such that

$$\sup_{B_{R_1/4}} [z_p]_{\mathcal{C}^\alpha} \leq C.$$

Since

$$\left[1 + \frac{z_p}{p}\right]_{\mathcal{C}^\alpha} \leq [z_p]_{\mathcal{C}^\alpha}.$$

It follows that

$$\sup_{B_{R_1/8}} |z_p|_{\mathcal{C}^{2,\alpha}} \leq C(\Lambda, \lambda, R_1).$$

Thus from the Arzela-Ascoli theorem there is a subsequence z_{p_k} which, for every $\alpha' < \alpha$ locally converges to a limit function z in $\mathcal{C}^{2,\alpha'}$ in B_{R_2}, for a sufficiently small R_2. Since R_1 is arbitrary, the convergence holds locally over the whole plane.

$\square$

2.3 The fully nonlinear Liouville equation

In the previous section, we have shown that, up to a subsequence, the radial functions z_p converge to a function z which solves (6) which is a fully nonlinear version of the Liouville equation. Now we proceed in describing such a function in particular proving that z changes concavity only once.

Proof of Theorem 1.1. The convergence of the rescaled function z_p is just the statement of Proposition 2.4. Hence the existence of a radial solution z of (6) which is negative and decreasing is deduced by that. Now we show that the limit function z changes concavity only once. Thus we consider the only radius y_p where u_p changes concavity (iii) of Theorem 2.1).

After passing to a subsequence, such that $\lim\limits_{p\to\infty}\frac{y_p}{\varepsilon_p}$ exists, we consider the three possible cases:

1. $\lim\limits_{p\to\infty}\frac{y_p}{\varepsilon_p} = \infty$.

2. $\lim\limits_{p\to\infty}\frac{y_p}{\varepsilon_p} = 0$.

3. $\lim\limits_{p\to\infty}\frac{y_p}{\varepsilon_p} = R_0, \qquad 0 < R_0 < \infty$.

Case 1: Since $\lim\limits_{p\to\infty}\frac{y_p}{\varepsilon_p} = \infty$, the limit equation being satisfied by z is

$$\begin{cases} -\lambda\Delta z & = e^z \qquad \text{in } \mathbb{R}^2 \\ z(0) & = 0, \end{cases} \tag{9}$$

because z is concave and decreasing on the whole space. From the classification of solution of the Liouville equation by Chen-Li [4], we know that the solution of (9) is:

$$z(x) = \log\left(\frac{1}{(1 + \frac{1}{8\lambda}|x|^2)^2}\right). \tag{10}$$

We conclude that Case 1 is not possible since such a solution is not concave in the whole space.

Case 2: Since $\lim\limits_{p\to\infty}\frac{y_p}{\varepsilon_p} = 0$, the limit ODE being satisfied by $z(x) = z(|x|)$ is

$$\begin{cases} -\Lambda z''(r) - \lambda\frac{z'(r)}{r} = e^{z(r)} \qquad \text{in } (0,\infty) \\ z(0) = 0, \end{cases} \tag{11}$$

and z is a nonpositive, decreasing and convex function.

First observe that we may rewrite the above equations as

$$-(r^{\frac{\lambda}{\Lambda}}\varphi'(r))' = \frac{r^{\lambda/\Lambda}}{\Lambda}e^{\varphi(r)} \qquad r \in (0,\infty).$$

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Integrating between 0 and r since z is negative so that $e^z \leq 1$ we obtain:

$$r^{\lambda/\Lambda}\left(z'(r)\right) = \int_0^r \frac{-s^{\lambda/\Lambda}}{\Lambda} e^{z(s)} ds \geq \frac{-1}{\Lambda} r^{1+\frac{\lambda}{\Lambda}} \frac{\Lambda}{\lambda + \Lambda}.$$

Dividing by $r^{\lambda/\Lambda}$ we obtain

$$0 \geq z'(r) \geq \frac{-1}{\lambda + \Lambda} \cdot r,$$

and taking the limit as r goes to zero we get $z'(0) = 0$. In particular this implies that z is positive around the origin since $z(0) = 0, z'(0) = 0$ and z is convex. This contradicts the fact that z is negative.

Case 3: Since $\lim_{p \to \infty} \frac{y_p}{\varepsilon_p} = R_0$, the limit equations being satisfied by $z(x) = z(|x|)$ are

$$\begin{cases} -\lambda z''(r) - \lambda \frac{z'(r)}{r} = e^{z(r)} & \text{in } (0, R_0) \\ -\Lambda z''(r) - \lambda \frac{z'(r)}{r} = e^{z(r)} & \text{in } (R_0, \infty) \\ z(0) = 0 \\ z''(R_0) = 0, \end{cases} \tag{12}$$

and z is a nonpositive, decreasing function, concave in $(0, R_0)$ and convex in (R_0, ∞).

As in the previous case, we may repeat the same procedure by exchanging Λ by λ and obtain that $z'(0) = 0$.

Therefore z satisfies the following initial boundary value problem

$$\begin{cases} -\lambda z''(r) - \lambda \frac{z'(r)}{r} = e^{z(r)} & \text{in } (0, R_0) \\ z(0) = 0 \\ z'(0) = 0 \end{cases} \tag{13}$$

Since such a problem admits only one solution it must be given by

$$z(r) = \log\left(\frac{1}{(1 + \frac{1}{8\lambda} r^2)^2}\right)$$

Note that the above function changes concavity only once in $\mathbb{R}^+$ at $2 \cdot \sqrt{2 \cdot \lambda}$. Therefore $R_0 = 2 \cdot \sqrt{2 \cdot \lambda}$. In particular, due to the fact that z is $\mathcal{C}^2_{loc}(\mathbb{R}^2)$, we obtain the following:

- $z(R_0) = -2 \cdot \ln 2$

- $z'(R_0) = -\frac{1}{\sqrt{2\lambda}}$

Then by (12) z satisfies this other initial value problem

$$\begin{cases} \Lambda z''(r) - \lambda \frac{z'(r)}{r} = e^{z(r)} & \text{in } (R_0, +\infty) \\ z(R_0) = -2 \cdot \ln 2 \\ z'(R_0) = -\frac{1}{\sqrt{2\lambda}} \end{cases} \tag{14}$$

There is only one solution to problem (14) and therefore we obtain that z is given by gluing the solutions of the two initial boundary value problems (13) and (14). □

3   The Problem for $\mathcal{M}^-$

We consider the problem

$$\begin{cases} -\mathcal{M}^-_{\lambda, \Lambda}(D^2 u)(x) = u(x)^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \end{cases} \tag{15}$$

where Ω is either $\mathbb{R}^2$ or the ball B_R, centered at the origin, with radius $R > 0$. In the last case we assume:

$$u = 0 \qquad \text{on } \partial\Omega \tag{16}$$

The first step to study the above problem is understanding whether a critical exponent for the existence of solutions of (15) can be defined or not.

To this aim we are going to use the approach of [12] which involves an auxiliary dynamical system. Thus we start by introducing it together with some preliminary results from [12].

3.1   Preliminaries on a Dynamical System

Since we are interested in radial solutions to (15) we write $u = u(r) = u(|x|)$ as an expression of u in radial coordinates. The eigenvalues of $D^2 u$ are the simple eigenvalue $u''(r)$ and $\frac{u'(r)}{r}$ which has multiplicity $(N-1)$ (see [9]).

Thus u satisfies a corresponding ODE from which it is easy to deduce that u is decreasing as long as it is positive, concave in an interval $(0, \tau_0)$ and changes concavity at least once (see [9] or [12])

Hence (15) can be written as:

$$u'' = M(-\Lambda r^{-1} u' - u^p), \tag{17}$$

where

$$M(s) = \begin{cases} \frac{s}{\Lambda} & \text{if } s \leq 0 \\ \frac{s}{\lambda} & \text{if } s > 0 \end{cases}, \tag{18}$$

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Next we introduce the following auxiliary functions:

$$X(t) = -\frac{ru'(r)}{u(r)}, \qquad Z(t) = -\frac{ru(r)^p}{u'(r)} \qquad \text{for } t = \ln r \tag{19}$$

whenever $u(r) \neq 0, u'(r) \neq 0$

Since $u > 0$ and $u' < 0$, we have that the above quantities belong to the first quadrant of the (X, Z) plane.

In the new variables, the equation (17) becomes the following autonomous dynamical system

$$\left(\dot{X}, \dot{Z}\right) = F(X, Z) = (f(X, Z), g(X, Z)) \tag{20}$$

where the dot $\cdot$ stands for derivation with respect to t and f, g are given by:

$$f(X, Z) = \begin{cases} X(X + \frac{Z}{\Lambda}) & \text{if } (X, Z) \in R^+ \\ X(X - (\tilde{N}_- - 2) + \frac{Z}{\lambda}) & \text{if } (X, Z) \in R^- \end{cases}$$

$$g(X, Z) = \begin{cases} Z(2 - pX - \frac{Z}{\Lambda}) & \text{if } (X, Z) \in R^+ \\ Z(\tilde{N}_- - pX - \frac{Z}{\lambda}) & \text{if } (X, Z) \in R^- \end{cases}$$

where the regions R^+ and R^- are:

$$R^+ = \{(X, Z) \mid Z > \Lambda\} \tag{21}$$
$$R^- = \{(X, Z) \mid 0 < Z < \Lambda\} \tag{22}$$

Note that whenever $(X(t), Z(t))$ belongs to the line

$$\ell = \{(X, Z \mid Z = \Lambda)\} \tag{23}$$

then the corresponding solution u of (17) satisfies $u'' = 0$. Hence R^+ and R^- represent, in terms of the new variables (X, Z), the regions of concavity and convexity of u, respectively.

Other important sets which are relevant to study the dynamics induced by (20) are:

$$\ell_1 = \{(X, Z) \mid Z = \lambda(\tilde{N}_- - 2 - X)\} \tag{24}$$

which is the set where $\dot{X} = 0$, and

$$\ell_2 = \{(X, Z) \in R^+ \mid Z = \Lambda(2 - pX)\} \cup \{(X, Z) \in R^- \mid Z = \lambda(\tilde{N}_- - pX)\} \tag{25}$$

which is the set where $\dot{Z} = 0$.

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Then we need to consider the stationary points for (20) and their classification.

We recall that for $p \leq \frac{\tilde{N}_-}{\tilde{N}_- - 2}$ is already known that there exists a positive solution of (15) with Dirichlet boundary conditions in the ball B_R ([13]).

Therefore, to determine the critical exponent for $\mathcal{M}_{\lambda,\Lambda}^-$ only the values of p larger than $\frac{\tilde{N}_-}{\tilde{N}_- - 2}$ are important, hence, we just state all is needed for $p > \frac{\tilde{N}_-}{\tilde{N}_- - 2}$.

Proposition 3.1. *The ODE system (20) admits the following stationary points:*

i) $N_0 = (0, 2\Lambda)$ *which is a saddle point.*

ii) $A_0 = (\tilde{N}_- - 2, 0)$ *which is a saddle point.*

iii) $M_0 = \left(\frac{2}{p-1}, \lambda(\tilde{N}_- - 2 - \frac{2}{p-1}) \right)$ *which is a source for* $\frac{\tilde{N}_-}{\tilde{N}_- - 2} < p < \frac{\tilde{N}_- + 2}{\tilde{N}_- - 2}$, *a center for* $p = \frac{\tilde{N}_- + 2}{\tilde{N}_- - 2}$, *a sink for* $p > \frac{\tilde{N}_- + 2}{\tilde{N}_- - 2}$

iv) $O = (0, 0)$ *which is a saddle point.*

The stationary points and the direction of the vector field F on the relevant sets for (20) are displayed in Figure 1.

3.2 Periodic Orbits

For the sequel it is important to see for which range of the exponent p there are no periodic orbits of (20).

Theorem 3.2. *Let* $0 < \lambda \leq \Lambda$. *Then for the system (20) it holds:*

i) *no periodic orbits exist if* $1 < p < \frac{\tilde{N}_- + 2}{\tilde{N}_- - 2}$.

ii) *there exists* $p_o > \max\{3, \frac{\tilde{N}_- + 2}{\tilde{N}_- - 2}\}$ *such that for* $p > p_o$ *there is no periodic orbit.*

Proof. The statement i) was already proved in Proposition 2.10 of [12]. We include the proof here for the reader's convenience.

Note that a periodic orbit must necessarily contain a stationary point in its interior. In our case, the only one which can be in it is the point M_0. Moreover by the direction of the vector field F (see (20)) on the lines $L_1 = \{(X, Z) \mid Z = 2\Lambda\}$ and $L_2 = \{(X, Z) \mid X = \tilde{N}_- - 2\}$ we deduce that no periodic orbit can intersect them. Hence, a periodic orbit must be contained in $[0, \tilde{N}_- - 2] \times [0, 2\Lambda]$.

Consider $\varphi(X, Z) = X^\alpha Z^\beta$, where $\alpha = \frac{2}{p-1}$ and $\beta = \frac{3-p}{p-1}$. Set $\Phi(X, Z) = \partial_X(\varphi f) + \partial_Z(\varphi g)$, with f and g as in (20), i.e

$$\Phi(X, Z) = \begin{cases} X^\alpha Z^\beta \frac{4}{p-1} & \text{if } (X, Z) \in R^+ \\ -X^\alpha Z^\beta \frac{p(\tilde{N}_- - 2) - (\tilde{N}_- + 2)}{p-1} & \text{if } (X, Z) \in R^- \end{cases}$$

Suppose there is a periodic orbit which encloses a region D. It follows from the Green's area formula that:

$$0 = \int_{\partial D} \varphi\{f dZ - g dX\} = \int_D \Phi dX dZ = \int_{R^+ \cap D} \Phi dX dZ + \int_{R^- \cap D} \Phi dX dZ,$$

where the first equality holds since $dX = f dt$ and $dZ = g dt$.

From this it follows that for $p \neq \frac{\tilde{N}_- + 2}{\tilde{N}_- - 2}$, any periodic orbit must intersect both regions R^+, R^-.

It is clear that, for p satisfying $1 < p < \frac{\tilde{N}_- + 2}{\tilde{N}_- - 2}$, Φ is positive and the above identity is a contradiction. Hence i) holds.

Now assume $p > 3$ (which implies that $\beta < 0$). In the region R^+, we have $\Lambda < Z < 2\Lambda$, which implies $Z^\beta < \Lambda^\beta$. Also, in R^-, $0 < Z < \Lambda$, hence $-Z^\beta < -\Lambda^\beta$. Hence the following bound holds:

$$0 = \int_{R^+ \cap D} \Phi dX dZ + \int_{R^- \cap D} \Phi dX dZ \leq$$

$$\frac{4\Lambda^\beta}{p-1} \int_{R^+ \cap D} X^\alpha dX dZ - \frac{p(\tilde{N}_- - 2) - (\tilde{N}_- + 2)}{p-1} \Lambda^\beta \int_{R^- \cap D} X^\alpha dX dZ \quad (26)$$

In order to conclude the proof it is sufficient to check that the ratio of the areas $|\frac{R^+ \cap D|}{|R^- \cap D|} \nrightarrow \infty$ as p goes to infinity. Indeed if this holds, taking the limit as p goes to infinity we get a contradiction and the result follows.
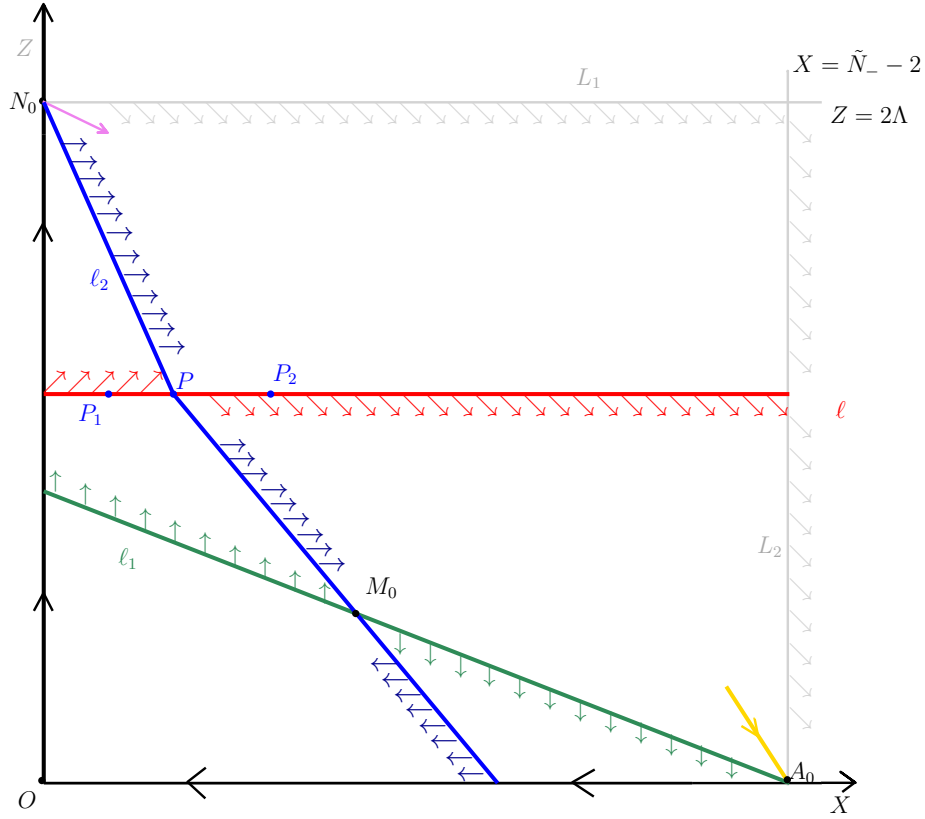
Figure 1: Indicative of the flow for $p > \frac{\tilde{N}_-}{\tilde{N}_- - 2}$

Let $P_1 = (x_1, \Lambda), P_2 = (x_2, \Lambda)$ be two points where the orbit intersects the concavity line ℓ, note that since $P = (\frac{1}{p}, \Lambda)$ is the only point where $\dot{Z}$ is zero, this implies the uniqueness of P_1, P_2. Since $\dot{X} > 0$ in R^+, it follows that $R^+ \cap D$ is contained in the rectangle $Q_1 = [x_1, x_2] \times [\Lambda, 2\Lambda]$. Also since $\dot{X} > 0$ above the line ℓ_1 defined in (24) the rectangle $Q_2 = [x_1, x_2] \times [\lambda(\tilde{N}_- - 2), \Lambda]$ is contained in $R^- \cap D$.

Therefore

$$\frac{|R^+ \cap D|}{|R^- \cap D|} \leq \frac{(x_2 - x_1)\Lambda}{(x_2 - x_1)(\Lambda - \lambda(\tilde{N}_- - 2))} = \frac{\Lambda}{\Lambda - \lambda(\tilde{N}_- - 2)}.$$

Therefore the right hand side cannot converge to $+\infty$ as $p \to \infty$, so the existence of p_o satisfying ii) is guaranteed.

$\square$

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$$\text{Define } \tilde{p}_- = \inf \left\{ p \in (\frac{\tilde{N}_- + 2}{\tilde{N}_- - 2}, \infty) \mid \nexists \text{ periodic orbits } \forall p' > p \right\} \qquad (27)$$

Theorem 3.3 (Bound for $\tilde{p}_-$). *It holds:*

$$\frac{\tilde{N}_- + 2}{\tilde{N}_- - 2} \le \tilde{p}_- \le \frac{\tilde{N}_- + 2}{\tilde{N}_- - 2} + \frac{4}{\tilde{N}_- - 2 + \frac{\lambda}{\Lambda}(\tilde{N}_- - 2)^2}$$

.

Proof. If $p > 3$ and a periodic orbit exists, then, going back to (26), it holds:

$$0 \le \frac{4\Lambda^\beta}{p-1} \int_{R^+ \cap D} X^\alpha dX dZ - \frac{p(\tilde{N}_- - 2) - (\tilde{N}_- + 2)}{p-1} \Lambda^\beta \int_{R^- \cap D} X^\alpha dX dZ$$

Considering the rectangles Q_1 and Q_2 defined in the proof of Theorem 3.2, we have:

$$\int_{R^+ \cap D} X^\alpha dX dZ \le \int_{Q_1} X^\alpha dX dZ \le \frac{\Lambda}{\Lambda - \lambda(\tilde{N}_- - 2)} \int_{Q_2} X^\alpha dX dZ \le$$

$$\frac{\Lambda}{\Lambda - \lambda(\tilde{N}_- - 2)} \int_{R^- \cap D} X^\alpha dX dZ$$

Therefore

$$0 \le \left(\frac{\Lambda}{\Lambda - \lambda(\tilde{N}_- - 2)} \cdot \frac{4\Lambda^\beta}{p-1} - \frac{p(\tilde{N}_- - 2) - (\tilde{N}_- + 2)}{p-1} \Lambda^\beta \right) \int_{R^- \cap D} X^\alpha dX dZ$$

Observing that the term in brackets is negative whenever $p > \frac{\tilde{N}_- + 2}{\tilde{N}_- - 2} + \frac{4}{\tilde{N}_- - 2 - \frac{\lambda}{\Lambda}(\tilde{N}_- - 2)^2}$, which is greater than 3, we get (3.3). $\qquad \square$

3.3 Critical exponent

We recall that for an orbit τ of the dynamical system (20) the set of limit points of $\tau(t)$, as $t \to -\infty$, is usually called α-limit and denoted by $\alpha(\tau)$. Analogously it is defined the ω-limit $\omega(\tau)$ at $+\infty$. With the same proofs as in [12] we have

Lemma 3.4. *For every $p > 1$, any regular solution of (17) satisfying the initial conditions: $u_p(0) = \gamma > 0$, $u_p'(0) = 0$, corresponds to the unique trajectory Γ_p of (20) whose α-limit is the stationary point N_0. Moreover:*

 i) if $u_p = u_p(r)$ is positive $\forall r \in (0, +\infty)$ then Γ_p is bounded and remains in the rectangle $(0, \tilde{N}_- - 2) \times (0, 2\Lambda)$, for all time.

ii) *if there exists R_p such that $u_p(R_p) = 0$ and $u_p > 0$ in $(0, R_p)$, then the corresponding trajectory $\Gamma_p = (X_p, Z_p)$ blows up in finite time T, in particular:*

$$\lim_{t \to T} X_p(t) = +\infty \qquad \lim_{t \to T} Z_p(t) = 0$$

and there exists $t_1 < T$ such that $X_p(t_1) = \tilde{N}_- - 2$.

Proof. See Proposition 2.1, Proposition 3.6 and Proposition 3.9 of [12]. $\quad\square$

We prove now a crucial result about nonexistence of solutions for the problem (15) in the ball. It is the keypoint to define and estimate the critical exponent.

Theorem 3.5. *For $p > \tilde{p}_-$, defined in (27), there are no solutions of (15) in the ball.*

Proof. We recall that $\tilde{p}_- \geq \frac{\tilde{N}_-+2}{\tilde{N}_--2}$. Assume that there is a solution of (15) in the ball for some $p > \tilde{p}_-$. Then we would have a trajectory $\Gamma_p = \Gamma_p(t) = (X_p(t), Z_p(t))$ starting from N_0 such that Γ_p blows up in finite time, after intersecting the line $L_1 = \{(X, Z) \,|\, X = \tilde{N}_- - 2\}$ (see Lemma 3.4). On the other hand, if we consider the unique trajectory τ_p whose ω-limit is A_0 (which is a saddle point for $p > \frac{\tilde{N}_-}{\tilde{N}_--2}$) and backtrack it we should be in either one of the following cases:

a) $\alpha(\tau_p)$ is a stationary point or a periodic orbit around M_0.

b) τ_p blows up backward.

The case a) is not possible. Indeed M_0 is a sink and no periodic orbit exists for $p > \tilde{p}_-$, by definition (27). Moreover N_0 cannot be the α-limit of τ_p because N_0 is a nondegenerate saddle point, so that, only the trajectory Γ_p starts from N_0 and $\omega(\Gamma_p)$ is not A_0 since Γ_p blows up in finite time.
Also case b) is not possible because τ_p is necessarily bounded because it stays in the regions enclosed by the X axis, Z axis, the orbit Γ_p and the line L_2 from which any orbit can only exit in forward time (by the direction of the vector field on L_2), see Figure 2.
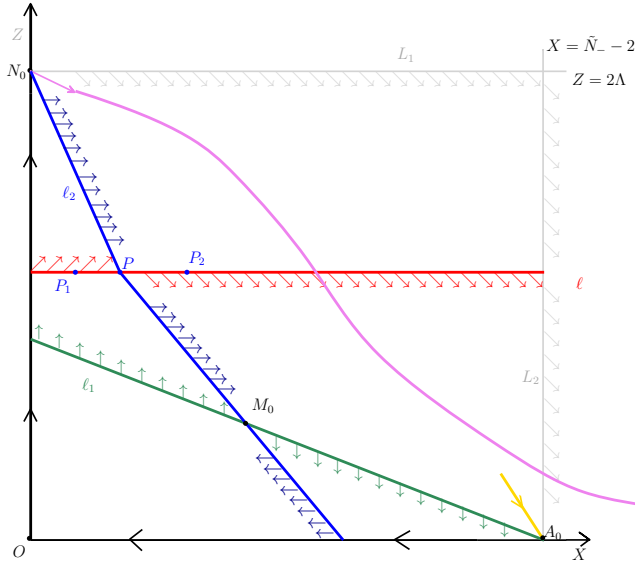
Figure 2:

Thus we have a contradiction which implies that Γ_p cannot correspond to a solution in the ball.

$\square$

Then we consider the set:

$$\mathcal{C} = \{p > 1 \,|\ \text{ a solution of (15) in } B_R \text{ exists.}\} \tag{28}$$

and we observe that $\mathcal{C}$ is nonempty since we have already remarked that for $p \in (1, \frac{\tilde{N}_-}{\tilde{N}_- - 2}]$ (15) has a solution in B_R.

So as in [12] we define:

$$p_-^* = \sup \mathcal{C} < \infty \tag{29}$$

and call it the critical exponent for $\mathcal{M}_{\lambda,\Lambda}^-$.

Before proving Theorem 1.2 we recall the following definitions.

Definition 3.6. *Let u_p be a radial solution of (15) in $\mathbb{R}^2$ and $\alpha = \frac{2}{p-1}$. Then u_p is said to be:*

i) *fast decaying if there exists $c > 0$ such that $\lim\limits_{r \to \infty} r^{\tilde{N}_- - 2} u(r) = c$.*

ii) *slow decaying if there exists $c > 0$ such that $\lim\limits_{r \to \infty} r^{\alpha} u(r) = c$.*

iii) *pseudo slow decaying if there exist constants $0 < c_1 < c_2$ such that*

$$c_1 = \liminf_{r \to \infty} r^{\alpha} u(r) < \limsup_{r \to \infty} r^{\alpha} u(r) = c_2$$

.

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As shown in [12] in terms of the dynamical system (20), a fast decaying, slow decaying or pseudo-slow decaying solution corresponds to an orbit Γ_p such that $\omega(\Gamma_p) = A_0$, or M_0 or a periodic orbit around M_0 respectively.

All the proprieties of the number p_-^* stated in Theorem 1.2 can be proved in the same way as in [12]. We give some indications for the reader's convenience.

Proof of Theorem 1.2. It is not difficult to see that $p_-^* \geq \frac{\tilde{N}_- + 2}{\tilde{N}_- - 2}$ because we already know that for $p \leq \frac{\tilde{N}_-}{\tilde{N}_- - 2}$ a solution of (15) in B_R exists. Moreover for $p \in (\frac{\tilde{N}_-}{\tilde{N}_- - 2}, \frac{\tilde{N}_- + 2}{\tilde{N}_- - 2})$, M_0 is a source, so if the trajectory Γ_p starting from N_0 does not blow up in finite time, producing so a solution in the ball, $\omega(\Gamma_p)$ should be the stationary point A_0, because no periodic orbits exist, by Theorem 3.2. However $\omega(\Gamma_p)$ cannot be A_0, otherwise, the region bounded by Γ_p and the X and Z axis would be an invariant region containing the point M_0 and the trajectories starting at M_0 would have no limit points to reach in forward time.

The proof that $p_-^* \neq \frac{\tilde{N}_- + 2}{\tilde{N}_- - 2}$ is more delicate and can be carried out as in [12] (see Theorem 5.2 and Theorem 4.7 therein). The fact that $p_-^* \leq \tilde{p}_-$ comes from Theorem 3.5, therefore using Theorem 3.3 we complete the estimate (7).

The proprieties iv) and v) derive from the definitions of $\tilde{p}_-$ and p_-^*. In particular we stress that for $p > \tilde{p}_-$ there are no periodic orbits so the only possibility is that the regular trajectory Γ_p starting at N_0 converges to M_0 which means that the corresponding solution u_p of (17) is slow decaying.

Finally i), ii) and iii) come from the proprieties of the critical exponent p_-^*, which can be proved exactly as in [12], see Theorem 5.2, Theorem 4.7, Proposition 4.2 and Corollary 4.4. □

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