Research Article

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Diophantine approximation with one prime, two squares of primes and one *k*th power of a prime

https://doi.org/10.1515/math-2021-0044
received February 14, 2021; accepted May 6, 2021

Abstract: Let $1 < k < 14/5$, $\lambda_1, \lambda_2, \lambda_3$ and λ_4 be non-zero real numbers, not all of the same sign such that λ_1/λ_2 is irrational and let ω be a real number. We prove that the inequality $|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^k - \omega| \le (\max(p_1, p_2^2, p_3^2, p_4^k))^{-\psi(k)+\varepsilon}$ has infinitely many solutions in prime variables p_1, p_2, p_3, p_4 for any $\varepsilon > 0$, where $\psi(k) = \min\left(\frac{1}{14}, \frac{14-5k}{28k}\right)$.

Keywords: Diophantine inequalities, Goldbach-type problems, Hardy-Littlewood method, Davenport-Heilbronn method

MSC 2020: 11D75, 11J25, 11P32, 11P55

1 Introduction

This paper deals with a Diophantine inequality with prime variables involving a prime, two squares of primes and one kth power of a prime. In particular, we prove the following theorem:

Theorem 1. *Assume that* $1 < k < 14/5$, $\lambda_1, \lambda_2, \lambda_3$ *and* λ_4 *be non-zero real numbers, not all of the same sign, that* λ_1/λ_2 *is irrational and let* ω *be a real number. The inequality*

$$|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^k - \omega| \le (\max(p_1, p_2^2, p_3^2, p_4^k))^{-\psi(k)+\varepsilon} \tag{1}$$

has infinitely many solutions in prime variables p_1, p_2, p_3, p_4 *for any* $\varepsilon > 0$, *where*

$$\psi(k) = \min\left(\frac{1}{14}, \frac{14-5k}{28k}\right).$$

Many recent such results are known with various types of assumptions and conclusions. Many of them deal with the number of *exceptional* real numbers ω such that the inequality

$$|\lambda_1 p_1^{k_1} + \cdots + \lambda_r p_r^{k_r} - \omega| \le \eta$$

has no solution in prime variables $p_1, \ldots, p_r$, for small $\eta > 0$ fixed.

Brüdern et al. in [1] dealt with binary linear forms in prime arguments; Cook and Fox in [2] dealt with a ternary form with squares of primes that was improved in terms of approximation by Harman in [3]. Cook in [4] gave a more general description of the problem, later improved by Cook and Harman in [5].

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There are some differences between the results quoted above and our purpose: in our case the value of η does depend on the primes p_j and it will be actually a negative power of the maximum of the p_j while in the papers quoted above η is a small negative power of ω. In their papers, the assumption that the coefficients λ_j are all positive is not a restriction. Moreover, k_j is the same positive integer for all j. Nevertheless, the assumption that λ_1/λ_2 must be irrational is still the heart of the matter.

Vaughan in [6] follows another approach, that is, the same we are using in our article: dealing with a ternary linear form in prime arguments and assuming some more suitable conditions on the λ_j, he proved that there are infinitely many solutions of the problem:

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 - \omega| \le \eta,$$

when η depends on the maximum of the p_j; in his case $\eta = (\max_j p_j)^{-\frac{1}{10}}$. Such result was improved by Baker and Harman in [7] with exponent $-\frac{1}{6}$, by Harman in [8] with exponent $-\frac{1}{5}$ and finally by Matomäki in [9] with exponent $-\frac{2}{9}$.

Languasco and Zaccagnini in [10] and [11] dealt with a ternary problem with a kth power of a prime. In this case, the value of η is a negative power of the maximum of p_j also depending on the parameter k: the idea in this case is to get both the widest k-range and the strongest bound for the approximation.

Li and Wang (see [12]) in 2011 dealt with a quaternary problem with a prime and three square of primes getting as exponent $-\frac{1}{28}$. Languasco and Zaccagnini improved it to $-\frac{1}{18}$ in [13]; Liu and Sun, in turn, improved it to $-\frac{1}{16}$ using the Harman technique in [14]. Finally, Wang and Yao in [15] improved the approximation to the exponent $-\frac{1}{14}$; in this paper, we generalized the problem to a real power $k \in \left(1, \frac{14}{5}\right)$.

2 Outline of the proof

We use a variant of the classical circle method that was introduced by Davenport and Heilbronn in 1946 [16] in order to attack this kind of Diophantine problems. The integration on a circle, or equivalently on the interval [0, 1], is replaced by integration on the whole real line.

Throughout this paper, p and p_i denote prime numbers, $k \ge 1$ is a real number, ε is an arbitrarily small positive number whose value could vary depending on the occurrences and ω is a fixed real number. In order to prove that (1) has infinitely many solutions, it is sufficient to construct an increasing sequence X_n that tends to infinity such that (1) has at least one solution with $\max p_j \in [\delta X_n, X_n]$, with $\delta > 0$ fixed, depending on the choice of λ_j. Let q be a denominator of a convergent to λ_1/λ_2 and let $X_n = X$ (dropping the suffix n) run through the sequence $X = q^{7/3}$. The choice of such X is due to an optimization procedure. Set

$$S_k(\alpha) = \sum_{\delta X \le p^k \le X} \log p\, e(p^k \alpha), \tag{2}$$

$$U_k(\alpha) = \sum_{\delta X \le n^k \le X} e(n^k \alpha), \tag{3}$$

$$T_k(\alpha) = \int_{(\delta X)^{\frac{1}{k}}}^{X^{\frac{1}{k}}} e(\alpha t^k)\, \mathrm{d}t, \tag{4}$$

where $e(\alpha) = e^{2\pi i \alpha}$.

In order to get the best possible estimate we will use the sieve function $\rho(m)$ defined in (5.2) of [17] introduced by Harman and Kumchev and used by Wang and Yao in [15] in the case $k = 2$, which is a non-trivial lower bound for the characteristic function of primes. Such function will allow us to define an exponential function (6) with a different weight:

$$\rho(m) = \psi(m, X^{5/42}) - \sum_{X^{5/42} \le p < X^{1/4}} \psi(m/p, z(p)),$$

where

$$\psi(m,z)=\begin{cases}1 & \text{if } p|m \ \Rightarrow \ p\ge z,\\ 0 & \text{otherwise}\end{cases}$$

and

$$z(p)=\begin{cases}X^{5/28}p^{-1/2} & \text{if } p<X^{1/7},\\ p & \text{if } X^{1/7}\le p\le X^{3/14},\\ X^{5/14}p^{-1} & \text{if } p>X^{3/14}.\end{cases}$$

The choice of the exponents in the definition of ρ (and consequently of z) is an appropriate choice of the function itself having the properties (i)–(v) of [17, Section 3] and constructed with sieve methods. The property we are most interested in about $\rho(m)$ is the estimation (2.3) of [15]:

$$\sum_{m\in I}\rho(m)=\ell|I|(\log X)^{-1}+O(X^{1/2}(\log X)^{-2}), \tag{5}$$

where $\ell>0$ is an absolute constant and I is any subinterval of $[(\delta X)^{1/2},X^{1/2}]$. The exact value of the constant ℓ is the same given by the constant δ defined in (4.4) of [17]: it is expressed in terms of integrals of Buchstab's function in [17, Section 5], from numerical calculation $\ell>9/10$. With this premises we define the following exponential function:

$$\widetilde{S}_2(\alpha)=\sum_{\delta X\le m^2\le X}\rho(m)e(m^2\alpha). \tag{6}$$

We will approximate S_k with T_k and U_k and $\widetilde{S}_2$ with T_2.

By the Prime Number Theorem and first derivative estimates for trigonometric integrals we have

$$S_k(\alpha)\ll X^{\frac{1}{k}},\quad \widetilde{S}_2(\alpha)\ll X^{\frac{1}{2}},\quad T_k(\alpha)\ll_{k,\delta} X^{\frac{1}{k}-1}\min(X,|\alpha|^{-1}), \tag{7}$$

where $k\ge 1$ and $\delta>0$ are real numbers.

Moreover, the Euler summation formula implies that, for $k\ge 1$,

$$T_k(\alpha)-U_k(\alpha)\ll 1+|\alpha|X. \tag{8}$$

We also need a continuous function that we will use to detect the solutions of (1), so we introduce

$$\widehat{K}_\eta(\alpha):=\max\{0,\eta-|\alpha|\},\quad \text{where } \eta>0,$$

whose inverse Fourier transform is

$$K_\eta(\alpha)=\left(\frac{\sin(\pi\alpha\eta)}{\pi\alpha}\right)^2$$

for $\alpha\ne 0$ and, by continuity, $K_\eta(0)=\eta^2$. It vanishes at infinity like $|\alpha|^{-2}$ and in fact it is trivial to prove that

$$K_\eta(\alpha)\ll\min(\eta^2,|\alpha|^{-2}). \tag{9}$$

The original works of Davenport-Heillbronn in [16] and later Vaughan in [6] and [18] approximate directly the difference $|S_k(\alpha)-T_k(\alpha)|$, estimating it with $O(1)$ using the Euler summation formula. Brüdern et al. (see [1]) improved these estimations taking the L^2-norm of $|S_k(\alpha)-T_k(\alpha)|$ leading to significantly better conditions and to have a wider major arc compared to the original DH approach. In fact, setting the generalized version of the Selberg integral, which was first used by Languasco-Setitmi for the case $k=2$ in [19],

$$\mathcal{J}_k(X,h)=\int_X^{2X}\left(\theta\left((x+h)^{\frac{1}{k}}\right)-\theta\left(x^{\frac{1}{k}}\right)-\left((x+h)^{\frac{1}{k}}-x^{\frac{1}{k}}\right)\right)^2\mathrm{d}x,$$

where θ is the Chebyshev Theta function,

$$\theta(x) = \sum_{p \le x} \log p,$$

we have the following lemmas.

Lemma 1. [10, Lemma 1] *Let $k \ge 1$ be a real number. For $0 < Y < \frac{1}{2}$, we have*

$$\int_{-Y}^{Y} |S_k(\alpha) - U_k(\alpha)|^2 \mathrm{d}\alpha \ll_k \frac{X^{\frac{2}{k}-2}\log^2 X}{Y} + Y^2 X + Y^2 \mathcal{J}_k\left(X, \frac{1}{2Y}\right).$$

Lemma 2. [10, Lemma 2] *Let $k \ge 1$ be a real number and ε be an arbitrarily small positive constant. There exists a positive constant $c_1(\varepsilon)$, which does not depend on k, such that*

$$\mathcal{J}_k(X, h) \ll_k h^2 X^{\frac{2}{k}-1} \exp\left(-c_1\left(\frac{\log X}{\log\log X}\right)^{\frac{1}{3}}\right)$$

uniformly for $X^{1-\frac{5}{6k}+\varepsilon} \le h \le X$.

2.1 Setting the problem

Let

$$\mathcal{P}(X) = \{(p_1, p_2, p_3, p_4) : \delta X < p_1 < X,\ \ \delta X < p_2^2, p_3^2 < X,\ \ \delta X < p_4^k < X\}$$

and let us define

$$\mathcal{I}(\eta, \omega, \mathfrak{X}) = \int_{\mathfrak{X}} S_1(\lambda_1\alpha)\widetilde{S}_2(\lambda_2\alpha)S_2(\lambda_3\alpha)S_k(\lambda_4\alpha)K_\eta(\alpha)e(-\omega\alpha)\,\mathrm{d}\alpha,$$

where $\mathfrak{X}$ is a measurable subset of $\mathbb{R}$.

From the construction of $\rho(m)$ follows that, if $\omega(m)$ is the characteristic function of the set of primes,

$$\rho(m) \le \omega(m).$$

Then, from the definitions of $S_j(\lambda_i\alpha)$ and $\widetilde{S}_2(\lambda_2\alpha)$, and performing the Fourier transform for $K_\eta(\alpha)$, we get

$$\begin{aligned}\mathcal{I}(\eta, \omega, \mathbb{R}) &= \sum_{p_i \in \mathcal{P}(X)} \log p_1 \rho(m_2) \log p_3 \log p_4 \cdot (\max(0, \eta - |\lambda_1 p_1 + \lambda_2 m_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^k - \omega|)) \\ &\le \eta (\log X)^3 \mathcal{N}(X),\end{aligned}$$

where $\mathcal{N}(X)$ denotes the number of solutions of the inequality (1) with $(p_1, p_2, p_3, p_4) \in \mathcal{P}(X)$. In other words, $\mathcal{I}(\eta, \omega, \mathbb{R})$ provides a lower bound for the quantity we are interested in; therefore, it is sufficient to prove that $\mathcal{I}(\eta, \omega, \mathbb{R}) > 0$.

We now decompose $\mathbb{R}$ into subsets such that $\mathbb{R} = \mathcal{M} \cup m \cup t$ where $\mathcal{M}$ is the major arc, m is the minor arc (or intermediate arc) and t is the trivial arc. The decomposition is the following:

$$\mathcal{M} = \left[-\frac{P}{X}, \frac{P}{X}\right], \quad m = \left[\frac{P}{X}, R\right] \cup \left[-R, -\frac{P}{X}\right], \quad t = \mathbb{R} \backslash (\mathcal{M} \cup m),$$

so that $\mathcal{I}(\eta, \omega, \mathbb{R}) = \mathcal{I}(\eta, \omega, \mathcal{M}) + \mathcal{I}(\eta, \omega, m) + \mathcal{I}(\eta, \omega, t)$.

The parameters $P = P(X) > 1$ and $R = R(X) > 1/\eta$ are chosen later (see (13) and (16)) as well as $\eta = \eta(X)$, that, as we explained before, we would like to get a small negative power of $\max p_j$ (and so of X, see (24)).

We are expecting to have on $\mathcal{M}$ the main term with the right order of magnitude without any special hypothesis on the coefficients λ_j. It is necessary to prove that $\mathcal{I}(\eta, \omega, m)$ and $\mathcal{I}(\eta, \omega, t)$ are both $o(\mathcal{I}(\eta, \omega, \mathcal{M}))$: the contribution from the trivial is "tiny" with respect to the main term. The real problem is on the minor arc where we will need the full force of the hypothesis on the λ_j and the theory of continued fractions.

Remark: From now on, anytime we use the symbol $\ll$ or $\gg$ we drop the dependence of the approximation from the constants λ_j, δ and k.

2.2 Lemmas

In this paper, we will also use Lemmas 3-4-10 of [20] and (2.5) of [15] that allow us to have an estimation of mean value of $|S_k(\alpha)|^4$ and $|\widetilde{S}_2(\alpha)|^4$:

Lemma 3. [20, Lemma 3] *Let $\varepsilon > 0$ fixed, $k > 1$, $\gamma > 0$ and let $\mathcal{A}(X^{1/k}; k; \gamma)$ denote the number of solutions of the inequality*

$$|n_1^k + n_2^k - n_3^k - n_4^k| < \gamma, \quad X^{1/k} \le n_1, n_2, n_3, n_4 \le 2X^{1/k}.$$

Then

$$\mathcal{A}(X^{1/k}; k; \gamma) \ll (\gamma X^{4/k-1} + X^{2/k}) X^{\varepsilon}.$$

Lemma 4. [20, Lemma 4] *Let $k > 1$, $\tau > 0$. We have*

$$\int_{-\tau}^{\tau} |S_k(\alpha)|^4 \mathrm{d}\alpha \ll (\tau X^{2/k} + X^{4/k-1}) X^{\varepsilon} \ll \max(\tau X^{2/k+\varepsilon}, X^{4/k-1+\varepsilon}).$$

Lemma 5. [20, Lemma 10]

$$\int_{m} |S_k(\lambda\alpha)|^4 K_\eta(\alpha) \mathrm{d}\alpha \ll \eta X^{\varepsilon} \cdot \max(X^{2/k}, X^{4/k-1}).$$

Finally, we will use the following Lemma.

Lemma 6. *Let $S_k(\alpha)$ and $\widetilde{S}_2(\alpha)$ defined as in* (2) *and* (6), *respectively, and m be the minor arc. Then we have*

$$\int_0^1 |S_1(\alpha)|^2 \mathrm{d}\alpha \ll X \log X, \quad \int_m |S_1(\alpha)|^2 K_\eta(\alpha) \mathrm{d}\alpha \ll \eta X \log X,$$

$$\int_0^1 |S_2(\alpha)|^4 \mathrm{d}\alpha \ll X \log^2 X, \quad \int_m |S_2(\alpha)|^4 K_\eta(\alpha) \mathrm{d}\alpha \ll \eta X \log^2 X,$$

$$\int_0^1 |\widetilde{S}_2(\alpha)|^4 \mathrm{d}\alpha \ll X (\log X)^c, \quad \int_m |\widetilde{S}_2(\alpha)|^4 K_\eta(\alpha) \mathrm{d}\alpha \ll \eta X (\log X)^c.$$

Proof. The first two statements come directly from the Prime Number Theorem, the second two estimations are based on Satz 3 of [21, p. 94] and the last two refer to (2.5) of [15]. □

3 The major arc

Let us start from the major arc and the computation of the main term. We replace all S_k and $\widetilde{S}_2$ defined in (2) and (6) with the corresponding T_k defined in (4). This replacement brings up some errors that we must estimate by means of Lemma 1, the Cauchy-Schwarz and the Hölder inequalities. We write

$$\begin{aligned}
\mathcal{I}(\eta,\omega,\mathcal{M}) &= \int_{\mathcal{M}} S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_2(\lambda_3\alpha)S_k(\lambda_4\alpha)K_\eta(\alpha)e(-\omega\alpha)\,\mathrm{d}\alpha \\
&= \ell(\log X)^{-1}\int_{\mathcal{M}} T_1(\lambda_1\alpha)T_2(\lambda_2\alpha)T_2(\lambda_3\alpha)T_k(\lambda_4\alpha)K_\eta(\alpha)e(-\omega\alpha)\,\mathrm{d}\alpha \\
&\quad + \int_{\mathcal{M}} (S_1(\lambda_1\alpha)-T_1(\lambda_1\alpha))\widetilde{S}_2(\lambda_2\alpha)T_2(\lambda_3\alpha)T_k(\lambda_4\alpha)K_\eta(\alpha)e(-\omega\alpha)\,\mathrm{d}\alpha \\
&\quad + \int_{\mathcal{M}} S_1(\lambda_1\alpha)(\widetilde{S}_2(\lambda_2\alpha)-\ell(\log X)^{-1}T_2(\lambda_2\alpha))T_2(\lambda_3\alpha)T_k(\lambda_4\alpha)K_\eta(\alpha)e(-\omega\alpha)\,\mathrm{d}\alpha \\
&\quad + \int_{\mathcal{M}} S_1(\lambda_1\alpha)\widetilde{S}_2(\lambda_2\alpha)(S_2(\lambda_3\alpha)-T_2(\lambda_3\alpha))T_k(\lambda_4\alpha)K_\eta(\alpha)e(-\omega\alpha)\,\mathrm{d}\alpha \\
&\quad + \int_{\mathcal{M}} S_1(\lambda_1\alpha)\widetilde{S}_2(\lambda_2\alpha)S_2(\lambda_3\alpha)(S_k(\lambda_4\alpha)-T_k(\lambda_4\alpha))K_\eta(\alpha)e(-\omega\alpha)\,\mathrm{d}\alpha \\
&= J_1+J_2+J_3+J_4+J_5.
\end{aligned}$$

Since the computations for J_2 are similar to, but simpler than, the corresponding ones for J_3, J_4 and J_5, we will leave it to the reader.

3.1 Main term: lower bound for J_1

As the reader might expect the main term is given by the summand J_1.

Let $H(\alpha) = T_1(\lambda_1\alpha)T_2(\lambda_2\alpha)T_2(\lambda_3\alpha)T_k(\lambda_4\alpha)K_\eta(\alpha)e(-\omega\alpha)$ so that

$$J_1 = \ell(\log X)^{-1}\int_{\mathbb{R}} H(\alpha)\,\mathrm{d}\alpha + O\left((\log X)^{-1}\int_{P/X}^{+\infty}|H(\alpha)|\,\mathrm{d}\alpha\right).$$

Using inequalities (9) and (7),

$$\int_{P/X}^{+\infty}|H(\alpha)|\,\mathrm{d}\alpha \ll X^{-1}X^{\frac{1}{k}-1}\eta^2\int_{P/X}^{+\infty}\frac{\mathrm{d}\alpha}{\alpha^4} \ll X^{\frac{1}{k}+1}\eta^2P^{-3} = o\left(X^{\frac{1}{k}+1}\eta^2\right)$$

provided that $P\to+\infty$. Let $D = [\delta X, X]\times[(\delta X)^{\frac{1}{2}}, X^{\frac{1}{2}}]^2\times[(\delta X)^{\frac{1}{k}}, X^{\frac{1}{k}}]$; we have

$$\begin{aligned}
\int_{\mathbb{R}} H(\alpha)\,\mathrm{d}\alpha &= \int_D\cdots\int\int_{\mathbb{R}} e((\lambda_1t_1+\lambda_2t_2^2+\lambda_3t_3^2+\lambda_4t_4^k-\omega)\alpha)K_\eta(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\,\mathrm{d}t_4 \\
&= \int_D\cdots\int \max(0,\eta-|\lambda_1t_1+\lambda_2t_2^2+\lambda_3t_3^2+\lambda_4t_4^k-\omega|)\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\,\mathrm{d}t_4.
\end{aligned}$$

Apart from trivial changes of sign, there are essentially three cases:

1. $\lambda_1>0$, $\lambda_2>0$, $\lambda_3>0$, $\lambda_4<0$;
2. $\lambda_1>0$, $\lambda_2>0$, $\lambda_3<0$, $\lambda_4<0$;
3. $\lambda_1>0$, $\lambda_2<0$, $\lambda_3<0$, $\lambda_4<0$.

We deal with the second case, cases 1 and 3 being similar. Let us perform the following change of variables: $u_1 = t_1-\frac{\omega}{\lambda_1}$, $u_2 = t_2^2$, $u_3 = t_3^2$, $u_4 = t_4^k$, so that the set D becomes essentially $[\delta X, X]^4$. Let us define

$D' = [\delta X, (1-\delta)X]^4$ for large X, as a subset of D. The Jacobian determinant of the change of variables above is $\frac{1}{4k} u_2^{-\frac{1}{2}} u_3^{-\frac{1}{2}} u_4^{\frac{1}{k}-1}$. Then

$$J_1 \gg (\log X)^{-1} \int_{\mathbb{R}} H(\alpha)\,\mathrm{d}\alpha = (\log X)^{-1} \int_{D'} \cdots \int \max(0, \eta - |\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4|) \frac{\mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}u_3 \mathrm{d}u_4}{u_2^{\frac{1}{2}} u_3^{\frac{1}{2}} u_4^{1-\frac{1}{k}}}$$
$$\gg (\log X)^{-1} X^{\frac{1}{k}-2} \int_{D'} \cdots \int \max(0, \eta - |\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4|)\,\mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}u_3 \mathrm{d}u_4.$$

Now, for $j = 1, 2, 3$ let $a_j = \frac{4|\lambda_4|}{|\lambda_j|}$, $b_j = \frac{3}{2} a_j$ and $\mathcal{D}_j = [a_j \delta X, b_j \delta X]$; if $u_j \in \mathcal{D}_j$, then

$$\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 \in [2|\lambda_4|\delta X, 8|\lambda_4|\delta X]$$

so that, for every choice of (u_1, u_2, u_3) the interval

$$[a, b] = \left[\frac{1}{|\lambda_4|}(-\eta + (\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3)),\ \frac{1}{|\lambda_4|}(\eta + (\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3)) \right]$$

is contained in $[\delta X, (1-\delta)X]$. In other words, for $u_4 \in [a, b]$ the values of $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4$ cover the whole interval $[-\eta, \eta]$. Hence, for any $(u_1, u_2, u_3) \in \mathcal{D}_1 \times \mathcal{D}_2 \times \mathcal{D}_3$ we have

$$\int_{\delta X}^{(1-\delta)X} \max(0, \eta - |\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4|)\,\mathrm{d}u_4 = |\lambda_4|^{-1} \int_{-\eta}^{\eta} \max(0, \eta - |u|)\,\mathrm{d}u \gg \eta^2.$$

Finally,

$$J_1 \gg (\log X)^{-1} \eta^2 X^{\frac{1}{k}-2} \iiint_{\mathcal{D}_1 \times \mathcal{D}_2 \times \mathcal{D}_3} \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}u_3 \gg \eta^2 X^{\frac{1}{k}-2} X^3 = \eta^2 X^{\frac{1}{k}+1} (\log X)^{-1},$$

which is the expected lower bound.

3.2 Bound for J_3

From partial summation on (6) we get

$$\widetilde{S}_2(\lambda_2 \alpha) = \int_{(\delta X)^{\frac{1}{2}}}^{X^{\frac{1}{2}}} e(\lambda t^2 \alpha)\,\mathrm{d}\left(\sum_{\substack{m_2 \le t \\ m_2 \in [(\delta X)^{1/2}, X^{1/2}]}} \rho(m_2) \right),$$

then

$$\widetilde{S}_2(\lambda_2 \alpha) - \ell (\log X)^{-1} T_2(\lambda_2 \alpha) \ll X^{\frac{1}{2}} (\log X)^{-2} (1 + |\alpha| X).$$

Finally, J_3 can be estimated as follows:

$$J_3 \ll \eta^2 \int_{\mathcal{M}} |T_1(\lambda_1 \alpha)| |\widetilde{S}_2(\lambda_2 \alpha) - \ell (\log X)^{-1} T_2(\lambda_2 \alpha)| |T_2(\lambda_3 \alpha)| |T_k(\lambda_4 \alpha)|\,\mathrm{d}\alpha$$
$$\ll \eta^2 X^{\frac{1}{2}} (\log X)^{-2} \int_0^{1/X} |T_1(\lambda_1 \alpha)| |T_2(\lambda_3 \alpha)| |T_k(\lambda_4 \alpha)|\,\mathrm{d}\alpha$$
$$+ \eta^2 X^{\frac{3}{2}} (\log X)^{-2} \int_{1/X}^{P/X} \alpha |T_1(\lambda_1 \alpha)| |T_2(\lambda_3 \alpha)| |T_k(\lambda_4 \alpha)|\,\mathrm{d}\alpha = o\left(\eta^2 X^{\frac{1}{k}+1} (\log X)^{-1} \right)$$

using (7).

3.3 Bound for J_4

Using the triangle inequality,

$$\begin{aligned} J_4 &= \int_{\mathcal{M}} S_1(\lambda_1\alpha)\widetilde{S}(\lambda_2\alpha)(S_2(\lambda_3\alpha) - T_2(\lambda_3\alpha))T_k(\lambda_4\alpha)K_\eta(\alpha)e(-\omega\alpha)\mathrm{d}\alpha \\ &\ll \eta^2 \int_{\mathcal{M}} |S_1(\lambda_1\alpha)||\widetilde{S}(\lambda_2\alpha)||S_2(\lambda_3\alpha) - T_2(\lambda_3\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha \\ &\leq \eta^2 \int_{\mathcal{M}} |S_1(\lambda_1\alpha)||\widetilde{S}(\lambda_2\alpha)|S_2(\lambda_3\alpha) - U_2(\lambda_3\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha \\ &\quad + \eta^2 \int_{\mathcal{M}} |S_1(\lambda_1\alpha)||\widetilde{S}(\lambda_2\alpha)||U_2(\lambda_3\alpha) - T_2(\lambda_3\alpha)||T_k(\lambda_4\alpha)|\mathrm{d}\alpha = \eta^2(A_4 + B_4), \end{aligned}$$

where $U_2(\lambda_3\alpha)$ is given by (3).

Using (7) and the Cauchy-Schwarz inequality,

$$A_4 \ll X^{\frac{1}{2}}X^{\frac{1}{k}} \int_{\mathcal{M}} |S_1(\lambda_1\alpha)||S_2(\lambda_3\alpha) - U_2(\lambda_3\alpha)|\mathrm{d}\alpha \ll X^{\frac{1}{2}}X^{\frac{1}{k}} \left(\int_{\mathcal{M}} |S_1(\lambda_1\alpha)|^2\mathrm{d}\alpha\right)^{\frac{1}{2}} \left(\int_{\mathcal{M}} |S_2(\lambda_3\alpha) - U_2(\lambda_3\alpha)|^2\mathrm{d}\alpha\right)^{\frac{1}{2}}.$$

Using Lemmas 1 and 2 (these will give us some conditions on P but the choice we will make in (13) makes all things work), we have

$$A_4 \ll X^{\frac{1}{2}+\frac{1}{k}}(X\log X)^{\frac{1}{2}}(\log X)^{-\frac{A}{2}} = X^{1+\frac{1}{k}}(\log X)^{\frac{1}{2}-\frac{A}{2}} = o\left(X^{\frac{1}{k}+1}\right)$$

as long as $A > 1$. Again using (7) and (8),

$$\begin{aligned} B_4 &\ll X^{\frac{1}{k}} \int_{\mathcal{M}} |S_1(\lambda_1\alpha)||\widetilde{S}(\lambda_2\alpha)||U_2(\lambda_3\alpha) - T_2(\lambda_3\alpha)|\mathrm{d}\alpha \\ &\ll X^{\frac{1}{k}} \int_0^{\frac{1}{X}} |S_1(\lambda_1\alpha)||\widetilde{S}(\lambda_2\alpha)|\mathrm{d}\alpha + X^{\frac{1}{k}+1} \int_{\frac{1}{X}}^{P/X} \alpha|S_1(\lambda_1\alpha)||\widetilde{S}(\lambda_2\alpha)|\mathrm{d}\alpha. \end{aligned}$$

Recalling that $|\alpha| \leq \frac{P}{X}$ on $\mathcal{M}$ and using the Hölder inequality, trivial bounds and Lemma 6, we have

$$\begin{aligned} B_4 &\ll XX^{\frac{1}{2}}X^{\frac{1}{k}}\frac{1}{X} + X^{\frac{1}{k}+1}\left(\int_{1/X}^{P/X} |S_1(\lambda_1\alpha)|^2\mathrm{d}\alpha\right)^{\frac{1}{2}}\left(\int_{1/X}^{P/X} \alpha^4\mathrm{d}\alpha\right)^{\frac{1}{4}}\left(\int_{1/X}^{P/X} |\widetilde{S}(\lambda_2\alpha)|^4\mathrm{d}\alpha\right)^{\frac{1}{4}} \\ &\ll X^{\frac{1}{2}+\frac{1}{k}} + X^{\frac{3}{2}+\frac{1}{k}}(\log X)^{\frac{1}{2}}\left(\frac{P}{X}\right)^{\frac{5}{4}}X^{\frac{1}{4}}(\log X)^{\frac{c}{4}} = X^{\frac{1}{2}+\frac{1}{k}}P^{\frac{5}{4}}(\log X)^{\frac{1}{2}+\frac{c}{4}}. \end{aligned}$$

Since we must have $P^{\frac{5}{4}} = o(X^{\frac{1}{2}}(\log X)^{-\frac{3}{2}-\frac{c}{4}})$, it follows that

$$P \leq X^{\frac{2}{5}-\varepsilon} \tag{10}$$

is sufficient for our purpose.

3.4 Bound for J_5

In order to provide an estimation for J_5, we use (9),

$$J_5 \ll \eta^2 \int_{\mathcal{M}} |S_1(\lambda_1\alpha)||S_2(\lambda_2\alpha)||S_2(\lambda_3\alpha)||S_k(\lambda_4\alpha) - T_k(\lambda_4\alpha)|\mathrm{d}\alpha.$$

The two terms are equivalent; then we consider only one of them

$$\begin{aligned}
J_5 &\ll \eta^2 \int_{\mathcal{M}} |S_1(\lambda_1\alpha)||\widetilde{S}_2(\lambda_2\alpha)||S_2(\lambda_3\alpha)|S_k(\lambda_4\alpha) - T_k(\lambda_4\alpha)|\mathrm{d}\alpha \\
&\ll \eta^2 \int_{\mathcal{M}} |S_1(\lambda_1\alpha)||\widetilde{S}_2(\lambda_2\alpha)||S_2(\lambda_3\alpha)|S_k(\lambda_4\alpha) - U_k(\lambda_4\alpha)|\mathrm{d}\alpha \\
&\quad + \eta^2 \int_{\mathcal{M}} |S_1(\lambda_1\alpha)||\widetilde{S}_2(\lambda_2\alpha)||S_2(\lambda_3\alpha)|U_k(\lambda_4\alpha) - T_k(\lambda_4\alpha)|\mathrm{d}\alpha = \eta^2(A_5 + B_5).
\end{aligned}$$

Using trivial estimates,

$$A_5 \ll X \int_{\mathcal{M}} |\widetilde{S}_2(\lambda_2\alpha)||S_2(\lambda_3\alpha)|S_k(\lambda_4\alpha) - U_k(\lambda_4\alpha)|\mathrm{d}\alpha,$$

then using the Hölder inequality, for any fixed $A > 2$ by Lemmas 1 and 2 we have

$$\begin{aligned}
A_5 &\ll X\left(\int_{\mathcal{M}} |\widetilde{S}_2(\lambda_2\alpha)|^4\right)^{\frac{1}{4}}\left(\int_{\mathcal{M}} |S_2(\lambda_3\alpha)|^4\mathrm{d}\alpha\right)^{\frac{1}{4}}\left(\int_{\mathcal{M}} |S_k(\lambda_4\alpha) - U_k(\lambda_4\alpha)|^2\mathrm{d}\alpha\right)^{\frac{1}{2}} \\
&\ll XX^{\frac{1}{4}}(\log X)^{\frac{c}{4}} X^{\frac{1}{4}}(\log X)^{\frac{1}{2}} \frac{P}{X}\mathcal{J}_k\left(X, \frac{X}{P}\right)^{\frac{1}{2}} \ll_A X^{1+\frac{1}{k}}(\log X)^{\frac{1}{2}+\frac{c}{4}-\frac{A}{2}} = o\left(X^{\frac{1}{k}+1}\right),
\end{aligned}$$

provided that $\frac{X}{P} \geq X^{1-\frac{5}{6k}+\varepsilon}$ (condition of Lemma 2), that is,

$$P \leq X^{\frac{5}{6k}-\varepsilon}. \tag{11}$$

Now we turn to B_5: by (8) we have

$$B_5 \ll \int_0^{1/X} |S_1(\lambda_1\alpha)||\widetilde{S}_2(\lambda_2\alpha)||S_2(\lambda_3\alpha)|\mathrm{d}\alpha + X \int_{1/X}^{P/X} \alpha|S_1(\lambda_1\alpha)||\widetilde{S}_2(\lambda_2\alpha)||S_2(\lambda_3\alpha)|\mathrm{d}\alpha.$$

Using trivial estimates and Lemma 6

$$\begin{aligned}
B_5 &\ll \left(\int_0^{1/X} |S_1(\lambda_1\alpha)|^2\mathrm{d}\alpha\right)^{\frac{1}{2}}\left(\int_0^{1/X} |\widetilde{S}_2(\lambda_2\alpha)|^4\mathrm{d}\alpha\right)^{\frac{1}{4}}\left(\int_0^{1/X} |S_2(\lambda_3\alpha)|^4\mathrm{d}\alpha\right)^{\frac{1}{4}} \\
&\quad + X\frac{P}{X}\left(\int_{1/X}^{P/X} |S_1(\lambda_1\alpha)|^2\mathrm{d}\alpha\right)^{\frac{1}{2}}\left(\int_{1/X}^{P/X} |\widetilde{S}_2(\lambda_2\alpha)|^4\mathrm{d}\alpha\right)^{\frac{1}{4}}\left(\int_{1/X}^{P/X} |S_2(\lambda_3\alpha)|^4\mathrm{d}\alpha\right)^{\frac{1}{4}} \\
&\ll (X\log X)^{\frac{1}{2}}(X\log^c X)^{\frac{1}{4}}(X\log^2 X)^{\frac{1}{4}} + P(X\log X)^{\frac{1}{2}}(X\log^c X)^{\frac{1}{4}}(X\log^2 X)^{\frac{1}{4}} \\
&= X(\log X)^{1+\frac{c}{4}} + PX(\log X)^{1+\frac{c}{4}}.
\end{aligned}$$

Then we need

$$P = o\left(X^{\frac{1}{k}-\varepsilon}\right). \tag{12}$$

Collecting all the bounds for P, that is, (10), (11), (12) we can take

$$P \le \min\left(X^{\frac{2}{5}-\varepsilon}, X^{\frac{5}{6k}-\varepsilon}\right). \tag{13}$$

In fact, if we consider (10) and (11), we should choose the most restrictive condition between the two: if $k \le \frac{25}{12}$, $P = X^{\frac{2}{5}-\varepsilon}$, otherwise, if $\frac{25}{12} < k < \frac{14}{5}$, $P = X^{\frac{5}{6k}-\varepsilon}$.

4 The trivial arc

From the trivial bound for $S_k(\lambda_4\alpha)$, we see that

$$\begin{aligned} |\mathcal{I}(\eta,\omega,t)| &\ll \int_{R}^{+\infty} |S_1(\lambda_1\alpha)\widetilde{S}_2(\lambda_2\alpha)S_2(\lambda_3\alpha)S_k(\lambda_4\alpha)K_\eta(\alpha)|\mathrm{d}\alpha \\ &\ll X^{\frac{1}{k}}\left(\int_{R}^{+\infty}|S_1(\lambda_1\alpha)|^2K_\eta(\alpha)\,\mathrm{d}\alpha\right)^{\frac{1}{2}}\left(\int_{R}^{+\infty}|\widetilde{S}_2(\lambda_2\alpha)|^4K_\eta(\alpha)\,\mathrm{d}\alpha\right)^{\frac{1}{4}}\left(\int_{R}^{+\infty}|S_2(\lambda_3\alpha)|^4K_\eta(\alpha)\,\mathrm{d}\alpha\right)^{\frac{1}{4}} \\ &\ll X^{\frac{1}{k}}\left(\int_{R}^{+\infty}\frac{|S_1(\lambda_1\alpha)|^2}{\alpha^2}\mathrm{d}\alpha\right)^{\frac{1}{2}}\left(\int_{R}^{+\infty}\frac{|\widetilde{S}_2(\lambda_2\alpha)|^4}{\alpha^2}\mathrm{d}\alpha\right)^{\frac{1}{4}}\left(\int_{R}^{+\infty}\frac{|S_2(\lambda_3\alpha)|^4}{\alpha^2}\mathrm{d}\alpha\right)^{\frac{1}{4}} \\ &= X^{\frac{1}{k}}C_1^{\frac{1}{2}}C_2^{\frac{1}{4}}C_3^{\frac{1}{4}}. \end{aligned}$$

Using the Prime Number Theorem and the periodicity of $S_1(\alpha)$, we have

$$C_1 = \int_{R}^{+\infty}\frac{|S_1(\lambda_1\alpha)|^2}{\alpha^2}\mathrm{d}\alpha \ll \int_{|\lambda_1|R}^{+\infty}\frac{|S_1(\alpha)|^2}{\alpha^2}\mathrm{d}\alpha \ll \sum_{n\ge|\lambda_1|R}\frac{1}{(n-1)^2}\int_{n-1}^{n}|S_1(\alpha)|^2\mathrm{d}\alpha \ll \frac{X\log X}{|\lambda_1|R}. \tag{14}$$

Using Lemma 6 we can estimate both C_2 and C_3, for brevity we leave C_3 to the reader,

$$C_2 = \int_{R}^{+\infty}\frac{|\widetilde{S}_2(\lambda_2\alpha)|^4}{\alpha^2}\mathrm{d}\alpha \ll \int_{|\lambda_2|R}^{+\infty}\frac{|\widetilde{S}_2(\alpha)|^4}{\alpha^2}\mathrm{d}\alpha \ll \sum_{n\ge|\lambda_2|R}\frac{1}{(n-1)^2}\int_{n-1}^{n}|\widetilde{S}_2(\alpha)|^4\mathrm{d}\alpha \ll \frac{X(\log X)^c}{|\lambda_2|R}. \tag{15}$$

Collecting (14) and (15),

$$|\mathcal{I}(\eta,\omega,t)| \ll X^{\frac{1}{k}}\left(\frac{X\log X}{R}\right)^{\frac{1}{2}}\left(\frac{X(\log X)^c}{R}\right)^{\frac{1}{4}}\left(\frac{X\log^2 X}{R}\right)^{\frac{1}{4}} \ll \frac{X^{1+\frac{1}{k}}(\log X)^{1+\frac{c}{4}}}{R}.$$

Hence, remembering that $|\mathcal{I}(\eta,\omega,t)|$ must be $o\left(\eta^2X^{\frac{1}{k}+1}\right)$, i.e., of the main term, the choice

$$R = \frac{(\log X)^{1+c}}{\eta^2} \tag{16}$$

is admissible.

5 The minor arc

In [15], Section 4 it is proven that the measure of the set where $|S_1(\lambda_1\alpha)|^{\frac{1}{2}}$ and $|\widetilde{S}_2(\lambda_2\alpha)|$ are both large for $\alpha \in \mathfrak{m}$ is small, exploiting the fact that the ratio λ_1/λ_2 is irrational.

Lemma 7. [22, Theorem 3.1] *Let α be a real number and a, q be positive integers satisfying $(a, q) = 1$ and $\left|\alpha - \frac{a}{q}\right| < \frac{1}{q^2}$. Then*

$$S_1(\alpha) \ll \left(\frac{X}{\sqrt{q}} + \sqrt{Xq} + X^{\frac{4}{5}}\right)\log^4 X.$$

We now state some considerations about Lemma 7:

Corollary 1. [14, Corollary 2.7] *Suppose that $X \geq Z \geq X^{1-\frac{1}{5}+\varepsilon}$ and $|S_1(\lambda_1\alpha)| > Z$. Then there are coprime integers $(a, q) = 1$ satisfying*

$$1 \leq q \leq \left(\frac{X^{1+\varepsilon}}{Z}\right)^2, \quad |q\lambda_1\alpha - a| \ll \left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z}\right)^2.$$

Lemma 8. [15, Lemma 1] *Suppose that $X^{\frac{1}{2}} \geq Z \geq X^{\frac{1}{2}-\frac{1}{14}+\varepsilon}$ and $|\widetilde{S}_2(\lambda_2\alpha)| > Z$. Then there are coprime integers $(a, q) = 1$ satisfying*

$$1 \leq q \leq \left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z}\right)^4, \quad |q\lambda_2\alpha - a| \ll X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z}\right)^4.$$

Let us now split m into two subsets $\tilde{m}$ and $m^* = m \backslash \tilde{m}$. In turn $\tilde{m} = m_1 \cup m_2$, where

$$m_1 = \{\alpha \in m : |S_1(\lambda_1\alpha)| \leq X^{1-\frac{1}{7}+\varepsilon}\},$$
$$m_2 = \{\alpha \in m : |\widetilde{S}_2(\lambda_2\alpha)| \leq X^{\frac{1}{2}-\frac{1}{14}+\varepsilon}\}.$$

Using the Hölder inequality, Lemmas 5–6 and the definition of m_1, we obtain

$$\begin{aligned}
|\mathcal{I}(\eta, \omega, m_1)| &\ll \int_{m_1} |S_1(\lambda_1\alpha)||\widetilde{S}_2(\lambda_2\alpha)||S_2(\lambda_3\alpha)||S_k(\lambda_4\alpha)|K_\eta(\alpha)\,\mathrm{d}\alpha \\
&\ll \left(\max_{\alpha\in m_1}|S_1(\lambda_1\alpha)|\right)^{\frac{1}{2}}\left(\int_{m_1}|S_1(\lambda_1\alpha)|^2K_\eta(\alpha)\,\mathrm{d}\alpha\right)^{1/4}\left(\int_{m_1}|\widetilde{S}_2(\lambda_2\alpha)|^4K_\eta(\alpha)\,\mathrm{d}\alpha\right)^{1/4} \\
&\quad\times\left(\int_{m_1}|S_2(\lambda_3\alpha)|^4K_\eta(\alpha)\,\mathrm{d}\alpha\right)^{1/4}\left(\int_{m_1}|S_k(\lambda_4\alpha)|^4K_\eta(\alpha)\,\mathrm{d}\alpha\right)^{1/4} \\
&\ll X^{\frac{3}{7}+\varepsilon}(\eta X\log X)^{\frac{1}{4}}(\eta X\log^c X)^{\frac{1}{4}}(\eta X\log^2 X)^{\frac{1}{4}}\left(\eta X^\varepsilon\max(X^{\frac{2}{k}}, X^{\frac{4}{k}-1})\right)^{\frac{1}{4}} \\
&= \eta X^{\frac{33}{28}+2\varepsilon}\max(X^{\frac{1}{2k}}, X^{\frac{1}{k}-\frac{1}{4}}).
\end{aligned} \tag{17}$$

Using the Hölder inequality, Lemmas 5, 6 and the definition of m_2, we obtain

$$\begin{aligned}
|\mathcal{I}(\eta, \omega, m_2)| &\ll \int_{m_2} |S_1(\lambda_1\alpha)||\widetilde{S}_2(\lambda_2\alpha)||S_2(\lambda_3\alpha)||S_k(\lambda_4\alpha)|K_\eta(\alpha)\,\mathrm{d}\alpha \\
&\ll \max_{\alpha\in m_2}|\widetilde{S}_2(\lambda_2\alpha)|\left(\int_{m_2}|S_1(\lambda_1\alpha)|^2K_\eta(\alpha)\,\mathrm{d}\alpha\right)^{1/2}\left(\int_{m_2}|S_2(\lambda_3\alpha)|^4K_\eta(\alpha)\,\mathrm{d}\alpha\right)^{1/4}\left(\int_{m_2}|S_k(\lambda_4\alpha)|^4K_\eta(\alpha)\,\mathrm{d}\alpha\right)^{1/4} \\
&\ll X^{\frac{3}{7}+\varepsilon}(\eta X\log X)^{\frac{1}{2}}(\eta X\log^2 X)^{\frac{1}{4}}\left(\eta X^\varepsilon\max(X^{\frac{2}{k}}, X^{\frac{4}{k}-1})\right)^{\frac{1}{4}} \\
&= \eta X^{\frac{33}{28}+2\varepsilon}\max(X^{\frac{1}{2k}}, X^{\frac{1}{k}-\frac{1}{4}}).
\end{aligned} \tag{18}$$

Both (17) and (18) must be $o\left(\eta^2 X^{1+\frac{1}{k}}\right)$, consequently it is clear that for $1 < k < 2$, η is a negative power of X independently from the value of k and $\psi(k) = \frac{1}{14}$. Then we have the following most restrictive condition for $k \geq 2$:

$$\eta = \infty(X^{-\psi(k)+\varepsilon}),$$

where $\psi(k) = \frac{14-5k}{28k}$. We use the notation $f = \infty(g)$ for $g = o(f)$.

It remains to discuss the set m^* in which the following bounds hold simultaneously

$$|S_1(\lambda_1\alpha)| > X^{\frac{6}{7}+\varepsilon}, \quad |\widetilde{S}_2(\lambda_2\alpha)| > X^{\frac{3}{7}+\varepsilon}, \quad X^{-\frac{3}{5}} < |\alpha| \leq \frac{\log^2 X}{\eta^2} = R.$$

Following the dyadic dissection argument as in [3] we divide m^* into disjoint sets $E(Z_1, Z_2, y)$ in which, for $\alpha \in E(Z_1, Z_2, y)$, we have

$$Z_1 < |S_1(\lambda_1\alpha)| \leq 2Z_1, \quad Z_2 < |\widetilde{S}_2(\lambda_2\alpha)| \leq 2Z_2, \quad y < |\alpha| \leq 2y,$$

where $Z_1 = 2^{k_1}X^{\frac{6}{7}+\varepsilon}$, $Z_2 = 2^{k_2}X^{\frac{3}{7}+\varepsilon}$ and $y = 2^{k_3}X^{-\frac{3}{4}-\varepsilon}$ for some non-negative integers k_1, k_2, k_3.

It follows that the number of disjoint sets are, at the most, $\ll \log^3 X$. Let us define $\mathcal{A}$ as a shorthand for the set $E(Z_1, Z_2, y)$; we have the following result about the Lebesgue measure of $\mathcal{A}$ following the same lines of Lemma 6 in [23]:

Lemma 9. *Let $\varepsilon > 0$. We have $\mu(\mathcal{A}) \ll yX^{\frac{18}{7}+3\varepsilon}Z_1^{-2}Z_2^{-4}$, where $\mu(\cdot)$ denotes the Lebesgue measure.*

Proof. If $\alpha \in \mathcal{A}$, by Corollary 1 and Lemma 8 there are coprime integers (a_1, q_1) and (a_2, q_2) such that

$$1 \leq q_1 \ll \left(\frac{X^{1+\varepsilon}}{Z_1}\right)^2, \quad |q_1\lambda_1\alpha - a_1| \ll \left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_1}\right)^2, \tag{19}$$

$$1 \leq q_2 \ll \left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\right)^4, \quad |q_2\lambda_2\alpha - a_2| \ll X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\right)^4. \tag{20}$$

We remark that $a_1a_2 \neq 0$ otherwise we would have $\alpha \in \mathcal{M}$. In fact, if $a_1 = 0$, recalling the definitions of Z_1 and (19), we get

$$|\alpha| \ll q_1^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_1}\right)^2 \ll X^{-5/7}$$

otherwise, if $a_2 = 0$ recalling the definitions of Z_2 and (20)

$$|\alpha| \ll q_2^{-1}X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\right)^4 \ll X^{-5/7}.$$

It means that on the minor arc

$$|\alpha| \gg X^{-\frac{5}{7}+\varepsilon}. \tag{21}$$

Now, we can further split m^* into sets $I(Z_1, Z_2, y, Q_1, Q_2)$ where, on each set, $Q_j \leq q_j \leq 2Q_j$. In the opposite direction, for a given quadruple a_1, q_1, a_2, q_2, the inequalities (19)–(20) define a subset of α of length

$$\mu(I) \ll \min\left(Q_1^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_1}\right)^2, Q_2^{-1}X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\right)^4\right).$$

By taking the geometric mean we can write

$$\mu(I) \ll Q_1^{-\frac{1}{2}}Q_2^{-\frac{1}{2}}X^{-\frac{1}{2}}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_1}\right)\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\right)^2 \ll \frac{X^{1+3\varepsilon}}{Q_1^{\frac{1}{2}}Q_2^{\frac{1}{2}}Z_1Z_2^2}. \tag{22}$$

Now we need a lower bound for $Q_1^{\frac{1}{2}}Q_2^{\frac{1}{2}}$: by (19) and (20)

$$\left|a_2q_1\frac{\lambda_1}{\lambda_2}-a_1q_2\right| = \left|\frac{a_2}{\lambda_2\alpha}(q_1\lambda_1\alpha-a_1)-\frac{a_1}{\lambda_2\alpha}(q_2\lambda_2\alpha-a_2)\right|$$
$$\ll q_2|q_1\lambda_1\alpha-a_1|+q_1|q_2\lambda_2\alpha-a_2|$$
$$\ll Q_2\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_1}\right)^2+Q_1X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\right)^4.$$

Remembering that $Q_1 \ll \left(\frac{X^{1+\varepsilon}}{Z_1}\right)^2$, $Q_2 \ll \left(\frac{X^{\frac{1}{2}+\varepsilon}}{Z_2}\right)^4$, $Z_1 \gg X^{\frac{6}{7}+\varepsilon}$, $Z_2 \gg X^{\frac{3}{7}+\varepsilon}$, we have

$$\left|a_2q_1\frac{\lambda_1}{\lambda_2}-a_1q_2\right| \ll \left(\frac{X^{\frac{1}{2}+\varepsilon}}{X^{\frac{3}{7}+\varepsilon}}\right)^4\left(\frac{X^{\frac{1}{2}+\varepsilon}}{X^{\frac{6}{7}+\varepsilon}}\right)^2+\left(\frac{X^{1+\varepsilon}}{X^{\frac{6}{7}+\varepsilon}}\right)^2X^{-1}\left(\frac{X^{\frac{1}{2}+\varepsilon}}{X^{\frac{3}{7}+\varepsilon}}\right)^4 \ll \frac{X^{2+4\varepsilon}X^{1+2\varepsilon}}{X^{\frac{12}{7}+4\varepsilon}X^{\frac{12}{7}+2\varepsilon}} \ll X^{-\frac{3}{7}-6\varepsilon} < \frac{1}{4q}. \tag{23}$$

We recall that $X = q^{7/3}$. Hence by (23), Legendre's law of best approximation for continued fractions implies that $|a_2q_1| \geq q$ and by the same argument, for any pair α, α' having distinct associated products a_2q_1 (see Watson [24]),

$$|a_2(\alpha)q_1(\alpha)-a_2(\alpha')q_1(\alpha')| \geq q;$$

thus, by the pigeon-hole principle, there is at most one value of a_2q_1 in the interval $[rq,(r+1)q)$ for any positive integer r. Hence, a_2q_1 determines a_2 and q_1 to within X^ε possibilities (from the bound for the divisor function) and consequently also a_2q_1 determines a_1 and q_2 to within X^ε possibilities from (23).

Hence, we obtain a lower bound for q_1q_2, remembering that $Q_j \leq q_j \leq 2Q_j$:

$$q_1q_2 = a_2q_1\frac{q_2}{a_2} \gg \frac{rq}{|\alpha|} \gg rqy^{-1}$$

for the quadruple under consideration. As a consequence, we obtain from (22), that the total length of the part of the subset $I(Z_1, Z_2, y, Q_1, Q_2)$, with $a_2q_1 \in [rq,(r+1)q)$, is

$$\mu(I) \ll X^{1+3\varepsilon}Z_1^{-1}Z_2^{-2}r^{-\frac{1}{2}}q^{-\frac{1}{2}}y^{\frac{1}{2}}.$$

Now, we sum on every interval to get an upper bound for the measure of $\mathcal{A}$:

$$\mu(\mathcal{A}) \ll X^{1+3\varepsilon}Z_1^{-1}Z_2^{-2}q^{-\frac{1}{2}}y^{\frac{1}{2}}\sum_{1\leq r\ll q^{-1}yX^{4+6\varepsilon}Z_1^{-2}Z_2^{-4}} r^{-\frac{1}{2}}.$$

By standard estimation we obtain

$$\sum_{1\leq r\ll q^{-1}yX^{4+6\varepsilon}Z_1^{-2}Z_2^{-4}} r^{-\frac{1}{2}} \ll (q^{-1}yX^{4+6\varepsilon}Z_1^{-2}Z_2^{-4})^{\frac{1}{2}},$$

then

$$\mu(\mathcal{A}) \ll yX^{3+6\varepsilon}Z_1^{-2}Z_2^{-4}q^{-1} \ll yX^{3+6\varepsilon}Z_1^{-2}Z_2^{-4}X^{-\frac{3}{7}} \ll yX^{\frac{18}{7}+6\varepsilon}Z_1^{-2}Z_2^{-4}.$$

This concludes the proof of the lemma. □

Using Lemma 9, we finally are able to get a bound for $\mathcal{I}(\eta,\omega,\mathcal{A})$:

$$\begin{aligned}|\mathcal{I}(\eta,\omega,\mathcal{A})| &\ll \int_{\mathcal{A}} |S_1(\lambda_1\alpha)||\widetilde{S}_2(\lambda_2\alpha)||S_2(\lambda_3\alpha)||S_k(\lambda_4\alpha)|K_\eta(\alpha)\,\mathrm{d}\alpha \\ &\ll \left(\int_{\mathcal{A}} |S_1(\lambda_1\alpha)\widetilde{S}_2(\lambda_2\alpha)|^2K_\eta(\alpha)\,\mathrm{d}\alpha\right)^{\frac{1}{2}}\left(\int_{\mathcal{A}} |S_2(\lambda_3\alpha)|^4K_\eta(\alpha)\,\mathrm{d}\alpha\right)^{\frac{1}{4}}\left(\int_{\mathcal{A}} |S_k(\lambda_4\alpha)|^4K_\eta(\alpha)\,\mathrm{d}\alpha\right)^{\frac{1}{4}}\end{aligned} \tag{24}$$

$$
\begin{aligned}
&\ll \left(\min\left(\eta^2, \frac{1}{y^2}\right)\right)^{\frac{1}{2}} ((Z_1 Z_2)^2 \mu(\mathcal{A}))^{\frac{1}{2}} (\eta X \log^2 X)^{\frac{1}{4}} \left(\eta X^{\varepsilon} \max\left(X^{\frac{2}{k}}, X^{\frac{4}{k}-1}\right)\right)^{\frac{1}{4}} \\
&\ll \eta Z_2^{-1} X^{\frac{9}{7}+2\varepsilon} X^{\frac{1}{4}+2\varepsilon} \max\left(X^{\frac{1}{2k}}, X^{\frac{1}{k}-\frac{1}{4}}\right) \ll \eta X^{\frac{6}{7}+2\varepsilon} X^{\frac{1}{4}+2\varepsilon} \max\left(X^{\frac{1}{2k}}, X^{\frac{1}{k}-\frac{1}{4}}\right) \\
&\ll \eta X^{\frac{31}{28}+4\varepsilon} \max\left(X^{\frac{1}{2k}}, X^{\frac{1}{k}-\frac{1}{4}}\right),
\end{aligned}
$$

so $\eta = \infty\left(\max\left(X^{-\frac{1}{14}}, X^{-\frac{14-5k}{28k}+\varepsilon}\right)\right)$ is the optimal choice.

Acknowledgements: I wish to thank my PhD supervisor Prof. Alessandro Zaccagnini for his professional guidance and valuable support and Prof. Alessandro Languasco for his fruitful suggestions which improved my article. This is a part of my PhD dissertation. I also thank the referees for several suggestions.

Conflict of interest: Author states no conflict of interest.

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