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On twisted A-harmonic sums and Carlitz finite zeta values ☆

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A R T I C L E   I N F O

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To the memory of David Goss

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A B S T R A C T

Twisted A-harmonic sums are partial sums of a class of zeta values introduced by the first author. We prove some new identities for such sums and we deduce properties of analogues of finite zeta values in the framework of the Carlitz module. In the theory of finite multiple zeta values as introduced by Kaneko and Zagier, finite zeta values are all zero and there is no known non-zero finite multiple zeta value. In the Carlitz ian setting the phenomenology is different as we can deduce, from our results, the irrationality of certain finite zeta values.

1. Introduction

Let $A = \mathbb{F}_q[\theta]$ be the ring of polynomials in an indeterminate θ with coefficients in $\mathbb{F}_q$ the finite field with q elements and characteristic p, and let K be the fraction field of A. We consider variables $t_1, \ldots, t_s$ over K and we write $\underline{t}_s$ for the family of variables

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$(t_1, \ldots, t_s)$. We write $\mathbb{C}_\infty$ for the completion $\widehat{K_\infty^{ac}}$ of an algebraic closure K_∞^{ac} of the completion K_∞ of K at the infinite place corresponding to $\frac{1}{\theta}$. In this paper, we are interested in the zeta-values

$$\zeta_A(n; s) := \sum_{a \in A^+} \frac{a(t_1) \cdots a(t_s)}{a^n}, \quad n > 0, \quad s \geq 0,$$

with A^+ the set of monic polynomials of A, converging in the *Tate algebra*

$$\mathbb{T}_s = \mathbb{C}_\infty \widehat{\otimes}_{\mathbb{F}_q} \mathbb{F}_q[\underline{t}_s]$$

in the variables $\underline{t}_s$ and with coefficients in $\mathbb{C}_\infty$ (with the trivial valuation over $\mathbb{F}_q[\underline{t}_s]$), introduced in [11] and studied, for example, in [2–4]. We choose once and for all a $(q-1)$-th root of $-\theta$ in $\mathbb{C}_\infty$. Let

$$\omega(t) = (-\theta)^{\frac{1}{q-1}} \prod_{i \geq 0} \left(1 - \frac{t}{\theta^{q^i}}\right)^{-1},$$

be the Anderson–Thakur function, in $\mathbb{T} = \mathbb{T}_1$ the Tate algebra in the variable $t = t_1$ with coefficients in $\mathbb{C}_\infty$ (see [1] for one of the first papers in which this function was studied and [3] for a more recent treatise of its basic properties). Let us also consider the fundamental period $\widetilde{\pi} \in \mathbb{C}_\infty$ of the $\mathbb{F}_q[t]$-linear Carlitz exponential $\exp_C : \mathbb{T} \to \mathbb{T}$ (so that $\omega(t) = \exp_C(\frac{\widetilde{\pi}}{\theta - t})$, as in [3]). We have the following result (see [2,4]).

Theorem 1 *(B. Anglès and the first author). For $s \equiv 1 \pmod{q-1}$ and $s > 1$, there exists a polynomial $\lambda_{1,s} \in A[\underline{t}_s]$ such that*

$$\zeta_A(1; s) = \frac{\widetilde{\pi} \lambda_{1,s}}{\omega(t_1) \cdots \omega(t_s)}.$$

There is no general explicit formula for $\lambda_{1,s}$ in terms of s but it can be proved that this polynomial is of exact degree equal to $\frac{s-q}{q-1}$ in θ, and $\mathbb{B}_s := (-1)^{\frac{s-1}{q-1}} \lambda_{1,s}$ is monic. For $s = 1$ Theorem 1 does not hold in the above strong form, but there is an explicit formula (see [11]):

$$\zeta_A(1; 1) = \frac{\widetilde{\pi}}{(\theta - t)\omega(t)}.$$

These formulas, reminiscent of the classical Euler's formulas for $\zeta(2k)$ (for $k > 0$) played an important role in the arithmetic of function fields yielding various analytic applications among which an alternate proof of Herbrand–Ribet–Taelman theorem for the Carlitz module in [4], or more recently, a proof of a function field analogue of a folkloric conjecture for the reduction modulo prime numbers of Bernoulli numbers, in [5].

One of the reasons for which the polynomials $\lambda_{1,s}$ contain so much arithmetic information is that for $s \equiv 1 \pmod{q-1}$ with $s > 1$, $\lambda_{1,s}$ is a generator of the Fitting ideal of the class $A[\underline{t}_s]$-module of an appropriate twist of the Carlitz module (see [4, §5]), inspired by Taelman's theory [16] (note that a similar twist of the multiplicative group $\mathbb{G}_m$ is presently not known). We come now to the content of the present paper.

For $d \geq 0$ an integer, we denote by $A^+(d)$ the set of monic polynomials of A of degree d. We first define the *twisted power sum of level s, degree d, and exponent n*

$$S_d(n;s) = \sum_{a \in A^+(d)} \frac{a(t_1)\cdots a(t_s)}{a^n} \in K[\underline{t}_s],$$

where, for a polynomial $a = \sum_i a_i\theta^i \in A$ with $a_i \in \mathbb{F}_q$ and a variable t, $a(t)$ denotes the polynomial $\sum_i a_i t^i$. Then, we define *twisted A-harmonic sums* as above, which are the sums:

$$F_d(n;s) = \sum_{i=0}^{d-1} S_i(n;s) \in K[\underline{t}_s], \quad n \in \mathbb{Z}, s \in \mathbb{N}, d \in \mathbb{N} \setminus \{0\}.$$

We are mainly interested in the case $n = 1$ and $s \equiv 1 \pmod{q-1}$ (just as in Theorem 1, the case $n = 1$ turns out to be the most relevant). We denote by $b_i(t)$ the product $(t-\theta)\cdots(t-\theta^{q^{i-1}}) \in A[t]$ if $i > 0$, and we set $b_0(t) = 1$. We also write $m = \lfloor \frac{s-1}{q-1} \rfloor$ (the brackets denote the integer part so that m is the biggest integer $\leq \frac{s-1}{q-1}$). We set

$$\Pi_{s,d} = \frac{b_{d-m}(t_1)\cdots b_{d-m}(t_s)}{l_{d-1}} \in K[\underline{t}_s], \quad d \geq \max\{1, m\}.$$

The main purpose of the present paper is to show the following result.

Theorem 2. *For all integers $s \geq 1$, such that $s \equiv 1 \pmod{q-1}$, there exists a non-zero rational fraction $\mathbb{H}_s \in K(\underline{t}_s)(Y)$ such that, for all $d \geq m$, the following identity holds:*

$$F_d(1;s) = \Pi_{s,d}\mathbb{H}_s(\theta^{q^{d-m}}).$$

If $s = 1$, we have the explicit formula

$$\mathbb{H}_1 = \frac{1}{t_1 - \theta}.$$

Further, if $s = 1 + m(q-1)$ for an integer $m > 0$, then the fraction $\mathbb{H}_s$ is a polynomial of $A[\underline{t}_s][Y]$ with the following properties:

(1) For all i, $\deg_{t_i}(\mathbb{H}_s) = m - 1$,
(2) $\deg_Y(\mathbb{H}_s) = \frac{q^m-1}{q-1} - m$.

The polynomial $\mathbb{H}_s$ is uniquely determined by these properties.

Theorem 2 can be seen as a "finite sum analogue" of Theorem 1. It is easy to see, by taking the limit $n \to \infty$ in the Tate algebra $\mathbb{T}_s$, that Theorem 2 implies Theorem 1. But Theorem 2 contains more information; the process of convergence at the infinite place which takes us to Theorem 1 from Theorem 2 suppresses various information encoded in the formula of Theorem 2.

It would be nice to understand the arithmetic meaning of the polynomials $\mathbb{H}_s$. For instance, it is desirable to know if they are related in some way to Taelman's class modules just as the polynomials $\lambda_{1,s}$ (as shown in [4]). However, the methods introduced in the present paper, essentially elementary and based on polynomial interpolation theory, do not seem to be sufficient to give clarity to this question.

1.1. Some consequences and applications

We list below a few consequences of our Theorem 2.

1.1.1. Finite zeta values

Theorem 2 opens a way to the study of *finite zeta and multi-zeta values*. Let us first shortly review the classical setting. We consider, following Zagier, the ring

$$\mathcal{A}_{\mathbb{Q}} = \frac{\displaystyle\prod_p \frac{\mathbb{Z}}{p\mathbb{Z}}}{\displaystyle\bigoplus_p \frac{\mathbb{Z}}{p\mathbb{Z}}},$$

the product and the direct sum running over the prime numbers p ([1]). Two elements $(a_p)_p$ and $(b_p)_p \in \mathcal{A}_{\mathbb{Q}}$ are equal if and only if $a_p = b_p$ for all but finitely many p. The ring $\mathcal{A}_{\mathbb{Q}}$ is not a domain. However, there is a natural injective ring homomorphism

$$\mathbb{Q} \to \mathcal{A}_{\mathbb{Q}}$$

defined by sending $r \in \mathbb{Q}$ to the class modulo $\oplus_p \frac{\mathbb{Z}}{p\mathbb{Z}}$ of the sequence of its reductions mod p, well defined for almost all p (that is, for all but finitely many p). Therefore, $\mathcal{A}_{\mathbb{Q}}$ is a $\mathbb{Q}$-algebra. This algebra is the main recipient for the theory of *finite multiple zeta values*, as in Kaneko's [9]. For example, for all $k > 0$, the finite zeta value of exponent k trivially vanishes in $\mathcal{A}_{\mathbb{Q}}$:

$$\zeta_{\mathcal{A}}(k) := \left(\sum_{n=1}^{p-1} \frac{1}{n^k} \pmod p\right)_p = 0 \in \mathcal{A}_{\mathbb{Q}}.$$

[1] It can be considered as a residue ring of the rational adèles $\mathbb{A}_{\mathbb{Q}}$. Indeed, there is a natural ring epimorphism $\mathbb{A}_{\mathbb{Q}} \to \mathcal{A}_{\mathbb{Q}}$ sending an adèle $(x_p)_p$ to the well defined residue $(x_p \pmod p))_p$.

More generally, the finite multiple zeta values are defined, for $k_1, \dots, k_r \in \mathbb{Z}$, by

$$\zeta_{\mathcal{A}}(k_1, \dots, k_r) = \left(\sum_{0<n_1<\cdots<n_k<p} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \pmod p \right)_p \in \mathcal{A}_{\mathbb{Q}}.$$

We apply Theorem 2 to show that certain variants of finite zeta values, introduced just below, are non-zero. We denote by $\boldsymbol{F}_s$ the field $\mathbb{F}_q(\underline{t}_s)$, so that $\boldsymbol{F}_0 = \mathbb{F}_q$. As an analogue of the ring $\mathcal{A}_{\mathbb{Q}}$, we consider the ring

$$\mathcal{A}_s := \frac{\displaystyle\prod_P \frac{\boldsymbol{F}_s[\theta]}{P\boldsymbol{F}_s[\theta]}}{\displaystyle\bigoplus_P \frac{\boldsymbol{F}_s[\theta]}{P\boldsymbol{F}_s[\theta]}},$$

where the product and the direct sum run over the irreducible monic polynomials of A. Ordering the indices of the variables $t_1, \dots, t_s$ by size induces embeddings $\mathcal{A}_0 \hookrightarrow \mathcal{A}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{A}_s$ and in the following, we are viewing the rings $\mathcal{A}_s$ embedded one in the other as above. Let K^{1/p^∞} be the perfect closure of K, that is, the subfield of an algebraic closure K^{ac} of K whose elements x are such that $x^{p^i} \in K$ for some i (note that this is equal to the subfield of K^{ac} whose elements are the x such that $x^{q^i} \in K$ for some i). There is a natural embedding (described in §2.2)

$$K^{1/p^\infty} \otimes_{\mathbb{F}_q} \boldsymbol{F}_s \xrightarrow{\iota} \mathcal{A}_s.$$

Let $P \in A$ be an irreducible monic polynomial of degree d in θ; we extend the P-adic valuation v_P of K to $K \otimes_{\mathbb{F}_q} \boldsymbol{F}_s$ by setting it to be the trivial valuation on $\boldsymbol{F}_s$. Then, $v_P(S_i(n; s)) \geq 0$ for all $0 \leq i < d$ so that $v_P(F_d(n; s)) \geq 0$ for all $n \in \mathbb{Z}$ and $s \in \mathbb{N}$. In particular, we have the *finite zeta value of level s and exponent n*,

$$Z_{\mathcal{A}}(n; s) := \left(F_{\deg_\theta(P)}(n; s) \pmod P\right)_P \in \mathcal{A}_s.$$

We consider the following elements,

$$\widehat{\pi} := \left(-\frac{1}{P'}\right)_P \in \mathcal{A}_0^\times, \quad \widehat{\omega}(t) := \left(\frac{1}{P(t)}\right)_P \in \mathcal{A}_1^\times, \text{ and } \quad \widehat{\Pi}_{n,s} := \frac{\widehat{\pi}^n}{\widehat{\omega}(t_1) \cdots \widehat{\omega}(t_s)} \in \mathcal{A}_s^\times,$$

where we write $t = t_1$ if $s = 1$ and n, s are integers such that $n \equiv s \pmod{q-1}$ (if R is a unitary ring, $R^\times$ denotes the group of invertible elements of R). Here, the dash $'$ denotes the derivative with respect to θ in K. It is easy to show that they are indeed units of the respective rings.

Our application of Theorem 2 to finite zeta values is the following result.

Theorem 3. *Assume that* $s \equiv 1 \pmod{q-1}$. *Then, there exists a non-zero explicit element* $\mu_{1,s} \in K^{1/p^\infty} \otimes_{\mathbb{F}_q} \boldsymbol{F}_s$ *such that*

$$Z_{\mathcal{A}}(1;s) = \widehat{\Pi}_{1,s}\iota(\mu_{1,s}).$$

In particular, $Z_{\mathcal{A}}(1;s) \in \mathcal{A}_s^\times$ *and furthermore, it is irrational, in the sense that it does not belong to* $\iota(K) \otimes_{\mathbb{F}_q} \boldsymbol{F}_s$.

Further basic properties of $\widehat{\pi}$ and $\widehat{\omega}$ are given in §2.2.1; for example, we will prove, as a particular case of Theorem 17 that $\widehat{\pi}$ is "irrational", i.e. it does not belong to $\iota(K)$. The above irrationality result is deduced from a density estimate for irreducible elements of A with a certain 'backwards digit expansion' due to Hayes. We are not aware of other similar irrationality results in the literature. Moreover, it seems challenging to prove that $\widehat{\pi}$ is transcendental.

1.1.2. Lower coefficients of $\mathbb{H}_s$

We can deduce, from Theorem 2 (see Remark 15):

Corollary 4. *For* $s \geq q$ *and* $s \equiv 1 \pmod{q-1}$, $-\lambda_{1,s}$ *is the leading coefficient of* $\mathbb{H}_s$ *as a polynomial in* Y.

What can be said about the lower coefficients of $\mathbb{H}_s$ from Theorem 2? With $\mu = \frac{q^m-1}{q-1} - m$, we write

$$\mathbb{H}_s = \sum_{i=0}^{\mu} \mathbb{D}_i Y^i \in A[\underline{t}_s][Y].$$

We discuss analytic formulas involving the coefficients $\mathbb{D}_i \in A[\underline{t}_s]$. We recall that $\mathbb{H}_s$ satisfies, by Theorem 2,

$$\frac{F_d(1;s)}{\Pi_{s,d}} = \frac{l_{d-1}F_d(1;s)}{b_{d-m}(t_1)\cdots b_{d-m}(t_s)} = \mathbb{H}_s(\theta^{q^{d-m}}), \tag{1}$$

for all $d \geq m$. Since $\mu = 0$ for $s = q$, we can restrict our attention to the case $s \geq 2q-1$ in this part.

We set, for all $d \geq m$ ([2]),

$$\Gamma_d := \frac{\prod_{i \geq d}\left(1 - \frac{\theta}{\theta^{q^i}}\right)}{\prod_{i \geq d-m}\prod_{j=1}^{s}\left(1 - \frac{t_j}{\theta^{q^i}}\right)} \in \mathbb{T}_s(K_\infty).$$

[2] Here, $\mathbb{T}_s(K_\infty)$ denotes the subring of the Tate algebra $\mathbb{T}_s$ whose elements are formal series in $\underline{t}_s$ with coefficients in K_∞.

In Lemma 19 below, we will define a sequence of polynomials $\Gamma_{s,r} \in A[\underline{t}_s][Y]$ monic of degree $\mu - r$ in Y related to the series expansion of a certain interpolating series of the product Γ_d. On the basis of this sequence of explicitly computable polynomials, we have the following result.

Theorem 5. *For all r such that $0 \leq r \leq \mu - 1$, we have:*

$$\mathbb{D}_r = -\lim_{d\to\infty}\left(\frac{\omega(t_1)\cdots\omega(t_s)}{\widetilde{\pi}}\Gamma_{s,r}(\theta^{q^{d-m}})F_d(1;s) + \sum_{i=1}^{\mu-r}\mathbb{D}_{i+r}\theta^{iq^{d-m}}\right).$$

It is easy to deduce, from Theorem 5, an explicit identity for the coefficient $\mathbb{D}_{\mu-1}$ (if $s \geq 2q-1$). With $r = \mu - 1$, Theorem 5 and the simple identity $\Gamma_{s,\mu-1} = Y + \sum_{i=1}^{s} t_i$ imply the next result.

Corollary 6. *There exists a polynomial $\nu_{1,s} \in A[\underline{t}_s]$ such that*

$$\lim_{d\to\infty}\theta^{q^{d-m}}\sum_{i\geq d}S_i(1;s) = \frac{\widetilde{\pi}\nu_{1,s}}{\omega(t_1)\cdots\omega(t_s)},$$

and

$$\mathbb{D}_{\mu-1} = \nu_{1,s} - (t_1 + \cdots + t_s)\lambda_{1,s}.$$

1.1.3. Twisted power sums

If $s = 0$ in $S_d(n,s)$, we recover the power sums already studied by several authors; see Thakur's [17] and the references therein. For general s these sums have been the object of study, for example, in the papers [2,6]. We recall that, in [1, (3.7.4)], Anderson and Thakur proved, for all $n \geq 1$, that there exists a unique polynomial $H_n \in A[Y]$ (with Y an indeterminate) of degree in Y which is at most $\frac{nq}{q-1}$, such that, for all $d \geq 0$,

$$S_d(n;0) = \frac{H_n(\theta^{q^d})}{\Pi_n l_d^n},$$

where Π_n is the n-th *Carlitz factorial* (see Goss' [7, Chapter 9]) and l_d denotes $(-1)^d$ times the least common multiple of all polynomials of degree d; explicitly, l_d is given by the product $(\theta - \theta^q)\cdots(\theta - \theta^{q^d}) \in A$, for $d \geq 1$, and $l_0 := 1$. These investigations have also been generalized by F. Demeslay in his Ph. D. thesis [6] to the sums $S_i(n;s)$ for any value of $s \geq 0$. F. Demeslay recently proved that, for all $n \geq 1$ and $s' \geq 0$, there exists a unique rational fraction $Q_{n,s'}(\underline{t}_{s'}, Y) \in K(\underline{t}_{s'}, Y)$ such that, for some fixed integer $r \geq 0$ (depending on n and s) and for all d,

$$S_d(n;s') = l_d^{-n}b_d(t_1)\cdots b_d(t_{s'})Q_{n,s'}(\underline{t}_{s'}, \theta^{q^{d-r}}), \tag{2}$$

hence providing a complement to [1, (3.7.4)]. Demeslay in fact proves several other properties related to the sums $S_d(n;s')$, and we refer the reader to his thesis [6] for the details of his results.

We consider an integer $s \geq q$ such that $s \equiv 1 \pmod{q-1}$. We set $m = \frac{s-1}{q-1}$, and we choose an integer s' such that $0 \leq s' < s$. We set $\mathbb{H}_{s,s'}$ to be the coefficient of $t_{s'+1}^{m-1} \cdots t_s^{m-1}$ in $\mathbb{H}_s$. It is a polynomial of $A[\underline{t}_{s'}][Y]$. It is easy to see that the coefficient of $t_1^{m-1} \cdots t_s^{m-1}$ in $\mathbb{H}_s$ is one. Hence, the coefficient of $t_1^{m-1} \cdots t_{s'}^{m-1}$ in $\mathbb{H}_{s,s'}$ is also equal to one.

Theorem 7. *Let $s = 1 + m(q-1)$, with $m \geq 1$, and let $0 \leq s' < s$. For each $d \geq m-1$, we have that*

$$S_d(1;s') = l_d^{-1} \prod_{i=1}^{s'} b_{d+1-m}(t_i) \mathbb{H}_{s,s'}(\theta^{q^{d+1-m}}).$$

This result also implies similar, but less decipherable formulas for general twisted power sums $S_d(n;s')$ that we do not mention here.

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2. Proofs

Basic notation.

- $\lfloor x \rfloor$: largest rational integer n such that $n \leq x$, with x real number.
- $\log_q(x)$: logarithm in base q of the real number $x > 0$.
- $\ell_q(n)$: sum of the digits of the expansion in base q of the non-negative integer n.
- m: the integer $\lfloor \frac{s-1}{q-1} \rfloor$ for $s \geq 1$.
- $[n]$: The polynomial $\theta^{q^n} - \theta \in A$ with $n > 0$ integer. We also set $[0] := 1$.
- l_n: The polynomial of A defined recursively by $l_0 = 1$ and $l_n = -[n]l_{n-1}$ with $n > 0$.
- D_n: the polynomials of A defined by $D_0 = 1$ and $D_n = [n]D_{n-1}^q$.
- b_n: the polynomial $(Y - \theta) \cdots (Y - \theta^{q^{n-1}}) \in A[Y]$ if $n > 0$ and $b_0 = 1$, for $n > 0$.
- $A(i)$: for $i \geq 1$ an integer, the $\mathbb{F}_q$-vector space of polynomials of A of degree $< i$ in θ. We also set $A(0) = \{0\}$.
- $\boldsymbol{F}_s$: the field $\mathbb{F}_q(\underline{t}_s) = \mathbb{F}_q(t_1, \ldots, t_s)$.

2.1. Proof of Theorem 2

We first consider the separate case of $s = 1$ and we set $t = t_1$; in fact, this case is straightforward. In $K[t]$, for all $d \geq 1$,

$$F_d(1;1) = \sum_{i=0}^{d-1} \sum_{a \in A^+(i)} a^{-1} a(t) = \frac{b_d(t)}{(t-\theta) l_{d-1}}, \tag{3}$$

see for instance [3]. This proves Theorem 2 with $s = 1$, $m = 0$ and $\mathbb{H}_1 = \frac{1}{t-\theta} \in K(t)$.

In all the rest of this subsection, we suppose that $s \geq q$ is an integer such that $s \equiv 1 \pmod{q-1}$, and we recall that we have set $m = \frac{s-1}{q-1}$. We proceed in several steps: in §2.1.1 we analyze the dependence in the degree d of the relations and the occurrence of the variable Y, in §2.1.2 we study the interpolation properties of certain series and conclude the proof of the first part of Theorem 2, namely, that $\mathbb{H}_s$ exists and belongs to $K(Y)[\underline{t}_s]$. Finally, in §2.1.3, we conclude the proof of Theorem 2 by using a density argument for the topology of Zariski.

2.1.1. Existence of relations with simple dependence in the degree

We shall write, for a new indeterminate z

$$E_i = E_i(z) := D_i^{-1} \prod_{a \in A(i)} (z - a) \in K[z],$$

where $A(i)$ denotes the $\mathbb{F}_q$-vector space of polynomials of A whose degree is strictly less than i in θ. In Goss' [7, Theorem 3.1.5], the reader can find a proof of the following formula, due to Carlitz:

$$E_i(z) = \sum_{j=0}^{i} \frac{z^{q^j}}{D_j l_{i-j}^{q^j}}, \quad i \geq 0.$$

From this formula the following can be easily deduced:

Proposition 8. *The following properties hold, for all $i \geq 0$.*

(1) The polynomial E_i is $\mathbb{F}_q$-linear of degree q^i in z, and $E_i(\theta^i) = 1$.
(2) For all $a \in A(i)$ and all $i \geq 0$ we have

$$\left.\frac{E_i(z)}{z-a}\right|_{z=a} = \frac{d}{dz} E_i = \frac{1}{l_i}.$$

(3) For all $i \geq 0$, we have that $E_i^q = E_i + [i+1] E_{i+1}$. □

Next, if n is a non-negative integer and if $n = n_0 + n_1 q + \cdots + n_r q^r$ is its base-q expansion (with the digits $n_0, \ldots, n_r \in \{0, \ldots, q-1\}$), we shall consider the polynomial

$$G_n = E_0^{n_0} \cdots E_r^{n_r} \in K[z],$$

so that $G_0 = 1$. By Proposition 8, (1), $\deg_z(G_n) = n$ so that every polynomial $Q \in K[z]$ can be written, in a unique way, as a finite sum

$$Q = \sum_{n \geq 0} c_n G_n, \quad c_n \in K.$$

Lemma 9. *Let $\underline{j} := (j_1, \ldots, j_r)$ be an r-tuple of non-negative integers. There exist polynomials $\{c_{\underline{j},i}\}_{i \geq 0} \subset A[Y]$, all but finitely many of which are non-zero, such that for each non-negative integer n we have*

$$E_{n+j_1} E_{n+j_2} \cdots E_{n+j_r} = \sum_{i \geq 0} c_{\underline{j},i}(\theta^{q^n}) G_{iq^n}.$$

Proof. We proceed by induction on the number of entries r of the tuple $\underline{j} = (j_1, j_2, \ldots, j_r)$. After reindexing, we may also assume that $0 \leq j_1 \leq j_2 \leq \cdots \leq j_r$.

If no q consecutive entries are equal, i.e. $j_{k+1} = j_{k+2} = \cdots = j_{k+q}$ does not hold for any k, we have

$$E_{n+j_1} E_{n+j_2} \cdots E_{n+j_r} = G_{q^n(q^{j_1} + q^{j_2} + \cdots + q^{j_r})},$$

and we can set $c_{\underline{j}, \sum q^{j_i}}(Y) = 1$ in this case.

So, assume that we have $j_{k+1} = j_{k+2} = \cdots = j_{k+q}$, for some $0 \leq k \leq r - q$. Grouping these E_i's together and applying (3) of Proposition 8 above, we obtain

$$\begin{aligned} E_{n+j_1} E_{n+j_2} \cdots E_{n+j_r} =& E_{n+j_1} \cdots E_{n+j_k} (E_{n+j_{k+1}})^q E_{n+j_{k+q+1}} \cdots E_{n+j_r} \\ =& E_{n+j_1} \cdots E_{n+j_k} E_{n+j_{k+1}} E_{n+j_{k+q+1}} \cdots E_{n+j_r} \\ &+ [n + j_{k+1} + 1] E_{n+j_1} \cdots E_{n+j_k} E_{n+j_{k+1}+1} E_{n+j_{k+q+1}} \cdots E_{n+j_r}. \end{aligned} \tag{4}$$

Now, both

$$e_1 := E_{n+j_1} \cdots E_{n+j_k} E_{n+j_{k+1}} E_{n+j_{k+q+1}} \cdots E_{n+j_r} \text{ and}$$
$$e_2 := E_{n+j_1} \cdots E_{n+j_k} E_{n+j_{k+1}+1} E_{n+j_{k+q+1}} \cdots E_{n+j_r},$$

which occur in the previous displayed line, both come from tuples with $r - (q-1)$ many entries, namely

$$\underline{j}_1 := (j_1, \ldots, j_{k+1}, j_{k+q+1}, \ldots, j_r) \quad \text{and} \quad \underline{j}_2 := (j_1, \ldots, j_k, j_{k+1} + 1, j_{k+q+1}, \ldots, j_r)$$

and hence by induction we deduce the existence of sets of polynomials $\{c_{\underline{j}_1,i}(Y)\}_{i\geq 0}$ and $\{c_{\underline{j}_2,i}(Y)\}_{i\geq 0}$ in $A[Y]$, all but finitely many of which are non-zero, such that

$$e_1 = \sum_{i\geq 0} c_{\underline{j}_1,i}(\theta^{q^n})G_{iq^n} \text{ and } e_2 = \sum_{i\geq 0} c_{\underline{j}_2,i}(\theta^{q^n})G_{iq^n},$$

for all $n \geq 0$.

Returning with this to (4), we obtain

$$E_{n+j_1}E_{n+j_2}\cdots E_{n+j_r} = \sum_{i\geq 0}\Big(c_{\underline{j}_1,i}(Y) + (Y^{q^{j_{k+1}+1}} - \theta)c_{\underline{j}_2,i}(Y)\Big)_{Y=\theta^{q^n}} G_{iq^n}.$$

So we let $c_{\underline{j},i}(Y) := c_{\underline{j}_1,i}(Y) + (Y^{q^{j_{k+1}+1}} - \theta)c_{\underline{j}_2,i}(Y)$ in this case. □

2.1.2. Interpolation properties

Recall that $A(d)$ denotes the $\mathbb{F}_q$-vector space of the polynomials of A of degree strictly less than d in θ. We consider, for $d \geq 1$, the element

$$\psi_{s,d} := \sum_{a\in A(d)} \frac{a(t_1)\cdots a(t_s)}{z-a} \in \boldsymbol{F}_s \otimes_{\mathbb{F}_q} K(z).$$

By Proposition 8 properties (1) and (2),

$$N_{s,d} := l_d E_d \psi_{s,d}$$

is the unique polynomial of $K[z,\underline{t}_s]$ of degree $< q^d$ in z such that the associated map $\mathbb{C}_\infty \to \mathbb{C}_\infty[\underline{t}_s]$ which sends z to the polynomial $N_{s,d}(z)$ interpolates the map

$$A(d) \ni a \mapsto a(t_1)\cdots a(t_s) \in \mathbb{F}_q[\underline{t}_s].$$

In [14], the second author found several explicit formulas for the sums $\psi_s = \psi_{s,\infty} = \lim_{d\to\infty}\psi_{s,d}$ for small values of s and in [12] the two authors of the present paper improved qualitatively the previous results without any restriction on s. In particular, the following formula holds, with $t = t_1$, valid for $d \geq 1$:

$$N_{1,d} = \sum_{j=0}^{d-1} E_j(z)b_j(t) \in K[z,t]. \tag{5}$$

We also set $M_{s,d} = \prod_{i=1}^{s}(N_{1,d})_{t=t_i}$ and notice that, since $\deg_z(N_{1,d}) = q^{d-1}$ by (1) of Proposition 8, the degree in z of $M_{s,d}$ is equal to sq^{d-1}. We can thus write, for $d \geq 1$:

$$\begin{aligned}M_{s,d}&=\sum_{\underline{i}\leq d-1}E_{i_1}(z)\cdots E_{i_s}(z)b_{i_1}(t_1)\cdots b_{i_s}(t_s)\\&=\sum_{\underline{i}\leq d-1}\sum_{0\leq j\leq sq^{d-1}}\epsilon_{\underline{i},j}G_j(z)b_{i_1}(t_1)\cdots b_{i_s}(t_s)\end{aligned}$$

where, we expand

$$E_{i_1}\cdots E_{i_s}=\sum_j\epsilon_{\underline{i},j}G_j, \tag{6}$$

with $\epsilon_{\underline{i},j}\in K$, and where, $\underline{i}\leq n$ stands for the inequalities $i_j\leq n$ for $j=1,\ldots,s$ (and similarly for $\underline{i}\geq n$).

Since both $M_{s,d}$ and $N_{s,d}$ have the same interpolation property (they interpolate the map $a\mapsto a(t_1)\cdots a(t_s)$ over $A(d)$), the polynomial $M_{s,d}-N_{s,d}$ vanishes for all $z=a\in A(d)$. This means that E_d divides $M_{s,d}-N_{s,d}$. But $\deg_z(N_{s,d})<q^d=\deg_z E_d$ from which we deduce that, for $d\geq 1$,

$$M_{s,d}-N_{s,d}=\sum_{\underline{i}\leq d-1}\sum_{q^d\leq j\leq sq^{d-1}}\epsilon_{\underline{i},j}G_j(z)b_{i_1}(t_1)\cdots b_{i_s}(t_s),$$

and, in particular,

$$N_{s,d}=\sum_{\underline{i}\leq d-1}\sum_{0\leq j\leq q^d-1}\epsilon_{\underline{i},j}G_j(z)b_{i_1}(t_1)\cdots b_{i_s}(t_s).$$

Now we notice that, for $s\geq 2$, we have that $(M_{s,d}/E_d)_{z=0}=0$. On one hand, we have (recall that $s\equiv 1\pmod{q-1}$):

$$\left(\frac{M_{s,d}-N_{s,d}}{l_dE_d}\right)_{z=0}=-\psi_{s,d}(0)=-\sum_{a\in A(d)}\frac{a(t_1)\cdots a(t_s)}{-a}=-F_d(1;s).$$

On the other hand we note that, for all $j>0$, if $\ell_q(j)\neq 1$, then $(G_j/l_dE_d)_{z=0}=0$, and if $\ell_q(j)=1$, then, for some $k\geq 0$, $j=q^k$ and $G_j=E_k$. Further, we have

$$\left(\frac{E_k}{l_dE_d}\right)_{z=0}=\frac{D_d}{l_dD_k}\frac{\prod_{0\neq a\in A(k)}a}{\prod_{0\neq a\in A(d)}a}=\frac{1}{l_k},$$

by [7, §3.2] or by (2) of Proposition 8. Thus, we obtain

$$\begin{aligned}-F_d(1;s)&=\sum_{d\leq k\leq d-1+\lfloor\log_q(s)\rfloor}l_k^{-1}\sum_{\underline{i}\leq d-1}\epsilon_{\underline{i},q^k}b_{i_1}(t_1)\cdots b_{i_s}(t_s)\\&=\sum_{\underline{i}\leq d-1}b_{i_1}(t_1)\cdots b_{i_s}(t_s)\sum_{d\leq k\leq d-1+\lfloor\log_q(s)\rfloor}l_k^{-1}\epsilon_{\underline{i},q^k}.\end{aligned}$$

By [2, Proposition 10] we see that, for all $i = 1, \ldots, s$ and all $d \geq m$, if $r = 0, \ldots, d - m - 1$, then $F_d(1; s)|_{t_i=\theta^{q^r}} = 0$ (if $d = m$ there is nothing to prove). Since the family of polynomials $(b_{i_1}(t_1)\cdots b_{i_s}(t_s))_{\underline{i}\geq 0}$ is a basis of the K-vector space $K[\underline{t}_s]$, this means that, for all $d \geq m$,

$$-F_d(1; s) = \sum_{d-m\leq \underline{i}\leq d-1} b_{i_1}(t_1)\cdots b_{i_s}(t_s) \sum_{d\leq k\leq d+\lfloor \log_q(s)\rfloor -1} l_k^{-1}\epsilon_{\underline{i},q^k}.$$

We can rewrite this identity as follows, for all $d \geq m$:

$$-F_d(1; s) = \sum_{h=0}^{\lfloor \log_q(s)\rfloor -1} \frac{1}{l_{d+h}} \times \sum_{0\leq j_1,\ldots,j_s\leq m} \epsilon_{(d-m+j_1,\ldots,d-m+j_s),q^{d+h}} b_{d-m+j_1}(t_1)\cdots b_{d-m+j_s}(t_s), \quad (7)$$

and this puts us in the hypotheses of Lemma 9, with $n = d-m$, $r = s$ and $\underline{j} = (j_1, \ldots, j_s)$. Thus, we see from (6) and the aforementioned lemma that

$$\epsilon_{(d-m+j_1,\ldots,d-m+j_s),q^{d+h}} = c_{\underline{j},q^{h+m}}|_{Y=\theta^{q^{d-m}}}.$$

Finally, for each h and $\underline{j}$, as above, we have

$$\frac{l_{d-1}}{\prod_{i=1}^s b_{d-m}(t_i)} \frac{\prod_{i=1}^s b_{d-m+j_i}(t_i)}{l_{d+h}} = \left.\frac{\prod_{i=1}^s (t_i - Y)(t_i - Y^q)\cdots(t_i - Y^{q^{j_i-1}})}{(\theta - Y^{q^m})(\theta - Y^{q^{m+1}})\cdots(\theta - Y^{q^{m+h}})}\right|_{Y=\theta^{q^{d-m}}}.$$

Thus, letting $w_{\underline{j},s} := \frac{\prod_{i=1}^s (t_i-Y)(t_i-Y^q)\cdots(t_i-Y^{q^{j_i-1}})}{(\theta-Y^{q^m})(\theta-Y^{q^{m+1}})\cdots(\theta-Y^{q^{m+h}})}$, we obtain

$$\frac{l_{d-1}}{\prod_{i=1}^s b_{d-m}(t_i)} F_d(1; s) = -\sum_{h=0}^{\lfloor \log_q(s)\rfloor -1} \sum_{0\leq j_1,\ldots,j_s\leq m} (c_{\underline{j},q^{h+m}} w_{\underline{j},s})|_{Y=\theta^{q^{d-m}}}, \quad (8)$$

completing the proof of the first part of Theorem 2, namely, that $\mathbb{H}_s$ exists, and is a rational fraction of $K(Y)[\underline{t}_s]$.

2.1.3. Conclusion of the proof of Theorem 2

Proposition 10. *Assume that $s \geq q$ and $s \equiv 1 \pmod{q-1}$. Then, for all $d \geq m$, we have that*

$$\mathbb{H}_s(\theta^{q^{d-m}}) = H_{s,d},$$

where $H_{s,d}$ is a non-zero polynomial of $A[\underline{t}_s]$ of degree $m - 1 = \frac{s-q}{q-1}$ in t_i for all $i = 1, \ldots, s$.

Proof. We note that $l_{d-1}F_d(1;s) \in A[\underline{t}_s]$ and that, for $d \geq m$, the polynomial $l_{d-1}F_d(1;s)$ is divisible by $b_{d-m}(t_1)\cdots b_{d-m}(t_s)$ in virtue of [2, Proposition 10].

One easily calculates, for each $1 \leq i \leq s$, that the degree in t_i of $F_d(1;s)$ equals $d-1$, and the degree of $b_{d-m}(t_i)$ in t_i equals $d-m$. Since $F_d(1;s)/\Pi_{s,d} \in A[\underline{t}_s]$, the degree in t_i of this ratio is $d-1-(d-m) = m-1$. $\square$

Remark 11. The degree of $H_{s,d}$ in θ can be easily computed, for all $d \geq m$, from the arguments in the proof of [2, Proposition 11]. We obtain for this degree

$$\delta_{s,d} := \frac{q^d - q}{q-1} - s\frac{q^{d-m}-1}{q-1} = m - 1 + \mu q^{d-m}$$

in θ, where $\mu = \frac{q^m-1}{q-1} - m$. To verify the displayed formula for $\delta_{s,d}$, write $s = m(q-1)+1$ and observe that $m \geq 1$ because $s \geq q$.

Integrality of $\mathbb{H}_s$. We need to show now that $\mathbb{H}_s$ is a polynomial in Y. We need an elementary fact.

Lemma 12. *Let $\mathbb{U} = \mathbb{U}(Y)$ be a polynomial of $A[\underline{t}_s][Y]$ and $M \geq 0$ and integer such that, for all d big enough,*

$$\frac{\mathbb{U}(\theta^{q^d})}{\theta^{q^{d+M}} - \theta} \in A[\underline{t}_s].$$

Then, $\mathbb{U} = (Y^{q^M} - \theta)\mathbb{V}$ with $\mathbb{V} \in A[Y][\underline{t}_s]$.

Proof. If $\mathbb{U}$ has degree $< q^M$ in Y, then, for all d big enough, $\deg_\theta(\mathbb{U}(\theta^{q^d})) \leq C + q^d(q^M - 1)$ with C a constant depending on $\mathbb{U}$. Therefore, as d tends to infinity,

$$\deg_\theta\left(\frac{\mathbb{U}(\theta^{q^d})}{\theta^{q^{d+M}} - \theta}\right) \leq C + q^d(q^M-1) - q^{d+M} \to -\infty,$$

which implies that $\mathbb{U}(\theta^{q^d}) = 0$ for all d big enough, and $\mathbb{U} = 0$ identically because the set $\{\theta^{q^d}; d \geq d_0\} \subset \mathbb{C}_\infty$ is Zariski-dense for all d_0. This proves the Lemma in this case.

Now, if $\mathbb{U}$ has degree in Y which is $\geq q^M$, we can write, by euclidean division (the polynomial $Y^{q^M} - \theta$ is monic in Y), $\mathbb{U} = (Y^{q^M} - \theta)\mathbb{V} + \mathbb{W}$ with $\mathbb{V}, \mathbb{W} \in A[Y][\underline{t}_s]$ and $\deg_Y(\mathbb{W}) < q^M$. Hence, by the first part of the proof, $\mathbb{W} = 0$. $\square$

First we show that $\mathbb{H}_s \in A[\underline{t}_s][Y]$. Indeed, by (8), we can write

$$\mathbb{H}_s(Y) = \frac{\mathbb{U}(Y)}{(\theta - Y^{q^m})\cdots(\theta - Y^{q^{m+\kappa_0}})},$$

where $\kappa_0 = \lfloor \log_q(s) \rfloor - 1$ and $\mathbb{U}$ is a polynomial in $A[Y][\underline{t}_s]$. By Proposition 10, $\mathbb{H}_s(\theta^{q^j}) \in A[\underline{t}_s]$, for all $j \geq 0$, and we deduce that $\frac{\mathbb{U}(\theta^{q^j})}{(\theta-\theta^{q^{j+k}})} \in A[\underline{t}_s]$, for each $m \leq k \leq m + \kappa_0$ and all $j \geq 0$. By Lemma 12, we conclude that $\frac{\mathbb{U}(Y)}{(\theta - Y^{q^k})} \in A[\underline{t}_s, Y]$, for each $m \leq k \leq m + \kappa_0$. Since the polynomials $\theta - Y^{q^m}, \theta - Y^{q^{m+1}}, \ldots, \theta - Y^{q^{m+\kappa_0}}$ are relatively prime, it follows that $\mathbb{H}_s \in A[Y][\underline{t}_s]$, as claimed.

We end the proof of Theorem 2 by verifying the quantitative data using Proposition 10. We set $\nu = m - 1$ (note that $\mu = 0$ in case $s = q$), so that $\deg_{t_i}(H_{s,d}) = \nu$, for all $d \geq m$. Writing $\mathbb{H}_s = \sum_{\underline{i}} c_{\underline{i}}(Y)\underline{t}^{\underline{i}}$ with $c_{\underline{i}}(Y) \in A[Y]$ and $H_{s,d} = \sum_{\underline{i}} c'_{\underline{i},d}\underline{t}^{\underline{i}}$ with $c'_{\underline{i},d} \in A$ ([3]), we have

$$c_{\underline{i}}(\theta^{q^{d-m}}) = c'_{\underline{i},d},$$

for all $d \geq m$ and for all $\underline{i}$. This means that $\deg_{t_i}(\mathbb{H}_s) = \deg_{t_i}(H_{s,d}) = \nu$ for all $d \geq m$ and $i = 1, \ldots, s$. This confirms the data on the degree in t_i for all i. Now that we know $\mathbb{H}_s \in A[\underline{t}_s, Y]$, to complete the proof of the theorem we only need to verify that the degree of $\mathbb{H}_s$ in Y is equal to $\frac{q^m-1}{q-1} - m$.

The degree in Y of $\mathbb{H}_s$. We choose a root $(-\theta)^{\frac{1}{q-1}}$ of $-\theta$ and we set:

$$\widetilde{\pi}_d := \theta(-\theta)^{\frac{1}{q-1}} \prod_{i=1}^{d-1} \left(1 - \frac{\theta}{\theta^{q^i}}\right)^{-1} \in (-\theta)^{\frac{1}{q-1}} K^\times,$$

$$\omega_d(t) := (-\theta)^{\frac{1}{q-1}} \prod_{i=0}^{d-1} \left(1 - \frac{t}{\theta^{q^i}}\right)^{-1} \in (-\theta)^{\frac{1}{q-1}} K(t)^\times.$$

Then, in $\mathbb{T}_s$,

$$\lim_{d\to\infty} \frac{\widetilde{\pi}_d}{\omega_{d-m}(t_1)\cdots\omega_{d-m}(t_s)} = \frac{\widetilde{\pi}}{\omega(t_1)\cdots\omega(t_s)}.$$

We note that $\deg_\theta(\widetilde{\pi}_d) = \frac{q}{q-1}$ and $\deg_\theta(\omega_d(t)) = \frac{1}{q-1}$.

We recall that we have set $\delta_{s,d} = \frac{q^d-q}{q-1} - s\frac{q^{d-m}-1}{q-1}$ and that we have defined μ so that the identity $\delta_{s,d} = m - 1 + \mu q^{d-m}$ holds.

Lemma 13. *We have, in $K[\underline{t}_s]$, that:*

$$\frac{\widetilde{\pi}_d}{\omega_{d-m}(t_1)\cdots\omega_{d-m}(t_s)} = -(-\theta)^{\delta_{s,d}-m+1}\frac{b_{d-m}(t_1)\cdots b_{d-m}(t_s)}{l_{d-1}}, \quad d \geq m.$$

[3] We are adopting multi-index notations, so that, if $\underline{i} = (i_1, \ldots, i_s) \in \mathbb{N}^s$, then $\underline{t}^{\underline{i}} = t_1^{i_1} \cdots t_s^{i_s}$.

Proof. We note that

$$\begin{aligned}\widetilde{\pi}_d &= \theta(-\theta)^{\frac{1}{q-1}}\prod_{i=1}^{d-1}\frac{-\theta^{q^i}}{\theta-\theta^{q^i}}\\ &= -((-\theta)(-\theta)^{\frac{q^d-q}{q-1}}(-\theta)^{\frac{1}{q-1}})l_{d-1}^{-1}\\ &= -(-\theta)^{\frac{q^d}{q-1}}l_{d-1}^{-1},\end{aligned}$$

and that

$$\begin{aligned}\omega_d(t) &= (-\theta)^{\frac{1}{q-1}}\prod_{i=0}^{d-1}\left(\frac{-\theta^{q^i}}{t-\theta^{q^i}}\right)\\ &= (-\theta)^{\frac{1}{q-1}}(-\theta)^{\frac{q^d-1}{q-1}}b_d(t)^{-1}\\ &= (-\theta)^{\frac{q^d}{q-1}}b_d(t)^{-1},\end{aligned}$$

so that

$$\begin{aligned}\frac{\widetilde{\pi}_d}{\omega_{d-m}(t_1)\cdots\omega_{d-m}(t_s)} &= -(-\theta)^{\frac{q^d}{q-1}}l_{d-1}^{-1}(-\theta)^{-\frac{sq^{d-m}}{q-1}}b_{d-m}(t_1)\cdots b_{d-m}(t_s)\\ &= -(-\theta)^{\frac{q^d}{q-1}-\frac{sq^{d-m}}{q-1}}\frac{b_{d-m}(t_1)\cdots b_{d-m}(t_s)}{l_{d-1}}\\ &= -(-\theta)^{\delta_{s,d}-m+1}\frac{b_{d-m}(t_1)\cdots b_{d-m}(t_s)}{l_{d-1}},\end{aligned}$$

because $\frac{q^d}{q-1}-\frac{sq^{d-1}}{q-1}=\delta_{s,d}+\frac{q}{q-1}-\frac{s}{q-1}=\delta_{s,d}+\frac{q-s}{q-1}=\delta_{s,d}-m+1$. □

The following lemma concludes the proof of Theorem 2 and supplies the degree in Y of $\mathbb{H}_s$.

Lemma 14. *We have, in $\mathbb{T}_s$:*

$$\lim_{d\to\infty}\theta^{-q^{d-m}\mu}\mathbb{H}_s(\theta^{q^{d-m}})=-\lambda_{1,s}.$$

Hence, the degree in Y of $\mathbb{H}_s$ equals $\mu:=\frac{q^m-1}{q-1}-m$.

Proof. By Theorem 1, $\lambda_{1,s}\in A[\underline{t}_s]$, which is known to be a polynomial of degree $m-1=\frac{s-q}{q-1}$ in θ, and by Lemma 13:

$$\lambda_{1,s}=\frac{\omega(t_1)\cdots\omega(t_s)}{\widetilde{\pi}}\lim_{d\to\infty}F_d(1;s)$$

$$= \lim_{d\to\infty} \frac{\omega_{d-m}(t_1)\cdots\omega_{d-m}(t_s)}{\widetilde{\pi}_d} \frac{b_{d-m}(t_1)\cdots b_{d-m}(t_s)}{l_{d-1}} \mathbb{H}_s(\theta^{q^{d-m}})$$
$$= -\lim_{d\to\infty} (-\theta)^{m-1-\delta_{s,d}} \mathbb{H}_s(\theta^{q^{d-m}})$$
$$= -\lim_{d\to\infty} \theta^{-q^{d-m}\mu} \mathbb{H}_s(\theta^{q^{d-m}})$$

Since we have shown that $\mathbb{H}_s$ is a polynomial in Y, the claim on the degrees is now clear. □

Remark 15. From the above, Corollary 4 follows at once.

2.2. Proof of Theorem 3

We first need to give the complete definition of the embedding ι in the statement of Theorem 3. The ring $\mathcal{A}_s$ of the introduction is easily seen to be an $\boldsymbol{F}_s$-algebra (use the diagonal embedding of $\boldsymbol{F}_s$ in $\mathcal{A}_s$). The map $x \mapsto x^q$ induces an $\boldsymbol{F}_s$-linear automorphism of $\boldsymbol{F}_s[\theta]/P\boldsymbol{F}_s[\theta]$, hence, component-wise, an $\boldsymbol{F}_s$-linear automorphism of $\mathcal{A}_s$ that we denote again by τ. There is a natural injective ring homomorphism

$$K \otimes_{\mathbb{F}_q} \boldsymbol{F}_s \xrightarrow{\iota} \mathcal{A}_s$$

uniquely defined by sending $r \in K$ to the sequence of its reductions mod P, well defined for almost all primes P (that is, irreducible monic). Now, ι extends to $K^{1/p^\infty} \otimes_{\mathbb{F}_q} \boldsymbol{F}_s$ in a unique way by setting, for $r \in K^{1/p^\infty} \otimes_{\mathbb{F}_q} \boldsymbol{F}_s$,

$$\iota(r) = \tau^{-w}(\iota(\tau^w(r))), \quad w \gg 0.$$

Let P be an irreducible polynomial of A^+. We observe that, for $d = \deg_\theta(P)$, $l_{d-1} = (\theta-\theta^q)\cdots(\theta-\theta^{q^{d-1}}) \equiv \frac{P(t)}{t-\theta}\Big|_{t=\theta} \equiv P'(\theta) \pmod{P}$. Also, $b_d(t) = (t-\theta)\cdots(t-\theta^{q^{d-1}}) \equiv P(t) \pmod{P}$. Therefore:

$$\Pi_{s,d} \equiv \frac{P(t_1)\cdots P(t_s)}{P'(\theta)\prod_{i=1}^s\prod_{j=1}^m(t_i-\theta^{q^{d-j}})}$$
$$\equiv \frac{P(t_1)\cdots P(t_s)}{P'(\theta)\prod_{i=1}^s\prod_{j=1}^m(t_i-\theta^{q^{-j}})} \pmod{P}.$$

Also, we have that $\mathbb{H}_s(\theta^{q^{d-m}}) \equiv \mathbb{H}_s(\theta^{q^{-m}}) \pmod{P}$.

We set

$$\mu_{1,s} := -\frac{\mathbb{H}_s(\theta^{q^{-m}})}{\prod_{i=1}^s\prod_{j=1}^m(t_i-\theta^{q^{-j}})} \in K^{1/p^\infty} \otimes_{\mathbb{F}_q} \boldsymbol{F}_s,$$

and Theorem 2 gives $Z_{\mathcal{A}}(1;s) = \widehat{\Pi}_{1,s}\iota(\mu_{1,s})$.

We tackle the non-vanishing of $\iota(\mu_{1,s})$. Let $d \geq 0$ be an integer, define the polynomial $\Psi_d(X) := \frac{X^{q^d}-X}{X^q-X}$, of degree $q^d - q$. Then, modulo $\Psi_d(X)$, the powers $1, X, X^2, \ldots, X^{q^d-q-1} \in \mathbb{F}_q[X]$ are linearly independent over $\mathbb{F}_q$. We set now, for $m \geq 1$ and for d big enough, $w := \mu q^{d-m} + m - 1$, and we observe that, again for d big enough, $0 \leq w < q^d - q$. Hence, the images of $1, \theta, \ldots, \theta^w$ in the ring $\mathbb{F}_q[\underline{t}_s, \theta]/(\Psi_d)$ are $\mathbb{F}_q$-linearly independent (where $\Psi_d = \Psi_d(\theta)$) and a polynomial $H \in \mathbb{F}_q[\underline{t}_s, \theta]$ of degree $\leq w$ in θ is zero modulo Ψ_d if and only if it is identically zero. Now, one easily shows using the degree in Y of $\mathbb{H}_s$ that for all d sufficiently large, $-H_{s,d} = -\mathbb{H}_s(\theta^{q^{d-m}})$ is a monic polynomial in θ of degree w. In particular, the image of $-H_{s,d}$ in $\mathbb{F}_q[\underline{t}_s, \theta]/(\Psi_d)$ is non-zero.

We now end the proof of the non-vanishing of $\mu_{1,s}$, equivalent to the non-vanishing of $\iota(\mu_{1,s})$, via a proof by contradiction. Let us suppose that for all d big enough, and for any P irreducible monic polynomial of A of degree d, we have $H_{s,d} \equiv 0 \pmod{P}$. In particular, this occurs for all large enough prime numbers $d = \varpi$. Now,

$$\Psi_\varpi = \prod_{P;\deg_\theta(P)=\varpi} P,$$

and the reduction of $H_{s,\varpi}$ modulo Ψ_ϖ is zero, giving the contradiction. Observe also that $\iota(\mu_{1,s}) \in \mathcal{A}_s^\times$.

2.2.1. Irrationality properties

We proceed to show the irrationality property of Theorem 3 to complete its proof. We study a few additional properties of the elements of $\mathcal{A}_1$

$$\widehat{\pi} = \left(-\frac{1}{P'}\right)_P, \quad \widehat{\omega}(t) = \left(\frac{1}{P(t)}\right)_P$$

that we present here as some kind of finite analogues of the elements $\widetilde{\pi}/(-\theta)^{\frac{1}{q-1}}$ and $\omega(t)/(-\theta)^{\frac{1}{q-1}}$. In this way the ratio $\frac{\widehat{\pi}}{\widehat{\omega}}$ can be viewed as a close analogue of the ratio $\frac{\widetilde{\pi}}{\omega(t)}$.

The transcendence over K of $\widetilde{\pi}$ and the transcendence of ω over $K(t)$ can be proved in a variety of ways (see for example the techniques of Papanikolas' [10]). From this, we immediately deduce that $\widetilde{\pi}$ and ω are algebraically independent over the field $K(t)$. We presently do not know if $\widehat{\pi}$ and $\widehat{\omega}$ are algebraically independent over $K^{1/p^\infty} \otimes_{\mathbb{F}_q} \boldsymbol{F}_1$. Nevertheless, we can prove that $\widehat{\pi}$ is irrational (in other words, in $\mathcal{A}_0$, $\widehat{\pi} \notin \iota(K)$; see the case $s = 0$ in Theorem 17 below).

To continue, we invoke the following result of Hayes [8], strengthening Artin's analogue of the Prime Number Theorem for the field K.

Fix $d \geq 1$. For $1 \leq k \leq d$, choose $(\alpha_1, \ldots, \alpha_k) \in \mathbb{F}_q^k$ and relatively prime polynomials $f, g \in A$. Denote by $\sharp(d)$ the cardinality of the set of primes $P \in A$ of degree $d \geq k$ such that $P \equiv g \pmod{f}$ and

$$\deg_\theta(P - \theta^d - \alpha_1\theta^{d-1} - \cdots - \alpha_k\theta^{d-k}) < d - k.$$

Theorem 16 *(Hayes). We have, for fixed k and for all d large,*

$$\sharp(d) = \frac{q^{d-k}}{d\Phi(f)} + O\left(\frac{q^{\vartheta d}}{d}\right),$$

with $0 \leq \vartheta < 1$, and where Φ is the function field analogue of Euler's φ-function relative to the ring A. ([4])

The term $O\left(\frac{q^{\vartheta d}}{d}\right)$ can be made more explicit, but we only use in our proof that $\sharp(d)$ tends to infinity as d tends to infinity. We apply Theorem 16 to prove the next:

Theorem 17. *For all $s \geq 0$, $\widehat{\Pi}_{1,s} = \frac{\widehat{\pi}}{\widehat{\omega}(t_1)\cdots\widehat{\omega}(t_s)} \in \mathcal{A}_s$ is irrational, that is, does not belong to $\iota(K) \otimes_{\mathbb{F}_q} \boldsymbol{F}_s$.*

In particular, $\widehat{\pi} \notin \iota(K) \subset \mathcal{A}_0$.

Proof. We deduce from Theorem 16 that for all $n \geq 1$ fixed, there exist infinitely many $\delta > n$ and infinitely many irreducible elements $P \in A^+$ of the form

$$P = \theta^{p\delta} + \theta^{p(\delta-n)+1} + \cdots \tag{9}$$

(recall that p is the characteristic of $\mathbb{F}_q$). We denote by $\mathcal{P}_n$ the (infinite) set of primes P of the form (9).

Let us suppose, by contradiction, that the statement of the Theorem is false. Then, we can find relatively prime polynomials $a, b \in A[\underline{t}_s]$ and, for all $n \geq 1$ and for all but finitely many $P \in \mathcal{P}_n$, a polynomial $Q_P \in A[\underline{t}_s]$, such that

$$bP(t_1)\cdots P(t_s) - aP'(\theta) = Q_P P. \tag{10}$$

We can already exclude the case $ab = 0$ which immediately yields a contradiction. Setting $\alpha = \deg_\theta(a) \geq 0$ and $\beta = \deg_\theta(b) \geq 0$ and considering $P \in \mathcal{P}_n$, we can transcribe the above identities (10) in the inequality

$$\deg_\theta(Q_P) \leq \max\{\beta - \delta p, \alpha + (\delta - n)p - \delta p\} = \max\{\beta - \delta p, \alpha - np\}.$$

Choosing $n \geq 1$ such that $np > \alpha$, we see that $Q_P = 0$ for all but finitely many $P \in \mathcal{P}_n$. Hence, $bP(t_1)\cdots P(t_s) = aP'(\theta)$ for all but finitely many $P \in \mathcal{P}_n$ which implies $a = b = 0$ and a contradiction. □

[4] For the definition and the basic properties of Φ, see Rosen, [15, Chapter 1].

2.2.2. End of proof of Theorem 3

We are now in condition to finish the proof of Theorem 3. We have seen that $Z_{\mathcal{A}}(1;s) = \widehat{\Pi}_{1,s}\iota(\mu_{1,s})$ and that $\iota(\mu_{1,s}) \neq 0$. Moreover, $\widehat{\Pi}_{1,s}$ is irrational by Theorem 17 and we are done. □

It seems more difficult to deduce the following conjecture from Hayes' Theorem 16:

Conjecture 18. *$\widehat{\pi}$ is transcendental over K.*

2.3. Proof of Theorem 5

We recall that we have set:

$$\Gamma_d = \frac{\prod_{i\geq d}\left(1-\frac{\theta}{\theta^{q^i}}\right)}{\prod_{i\geq d-m}\prod_{j=1}^{s}\left(1-\frac{t_j}{\theta^{q^i}}\right)} \in \mathbb{T}_s(K_\infty).$$

We need the following result where we suppose that $s = 1 + m(q-1)$ with $m > 0$.

Lemma 19. *For any integer r with $0 \leq r \leq \mu$ there exists a polynomial*

$$\Gamma_{s,r} \in A[\underline{t}_s, \theta][Y],$$

monic of degree $\mu - r$ in Y, such that, for all $d \geq m$,

$$\theta^{(\mu-r)q^{d-m}}\Gamma_d = \Gamma_{s,r}(\theta^{q^{d-m}}) + w_d, \tag{11}$$

where $(w_d)_{d\geq m}$ is a sequence of elements of $\mathbb{T}_s(K_\infty)$ which tends to zero for the Gauss norm as d tends to infinity.

Proof. Let $f(Y) = 1 + \sum_{i\geq 1} f_i Y^{-i}$ be an element of $\mathbb{F}_q[\underline{t}_s][[Y^{-1}]]^\times$ and $g(Y) = 1 + \sum_{i\geq 1} g_i Y^{-i}$ an element of $A[[Y^{-1}]]$. Then, for all $k \geq 0$, we can decompose in a unique way:

$$Y^k g(Y) f(Y)^{-1} = H_0(Y) + H_1(Y), \tag{12}$$

where $H_0(Y) \in A[\underline{t}_s][Y]$ has degree k in Y and $H_1(Y) \in \frac{1}{Y}A[\underline{t}_s][[\frac{1}{Y}]]$. We suppose that g converges for $Y = y \in \mathbb{T}_s$ with $\|y\| > 1$ where we recall that $\|\cdot\|$ denotes the Gauss norm of the Tate algebra $\mathbb{T}_s$ (hence, the convergence takes place in a subset of $\mathbb{T}_s$). Of course f and f^{-1} also converge at such elements y. If now $(y_d)_d$ is a sequence of elements of $\mathbb{T}_s$ with $\|y_d\| > 1$ and such that $\|y_d\| \to \infty$ as $d \to \infty$, then, $\|H_1(y_d)\| \to 0$.

We set, for all $d \geq 0$, $y_d = \theta^{q^d}$. We also set:

$$f(Y) := \prod_{i\geq 0}\prod_{j=1}^{s}\left(1-\frac{t_j}{Y^{q^i}}\right) = \prod_{j=1}^{s}\sum_{n_j\geq 0}{}^{'}(-t_j)^{\ell_q(n_j)}Y^{-n_j} \in \mathbb{F}_q[\underline{t}_s][[Y^{-1}]]^{\times},$$
$$g(Y) := \prod_{i\geq 0}\left(1-\frac{\theta}{Y^{q^{i+m}}}\right) = \sum_{n\geq 0}{}^{'}(-\theta)^{\ell_q(n)}Y^{-q^m n} \in A[[Y^{-1}]],$$

where the sum is restricted to the integers n which have, in their expansion in base q, only $0,1$ as digits. Then, g converges for any choice $Y = y \in \mathbb{T}_s$ with $\|y\| > 1$. Writing $h(Y) = g(Y)f(Y)^{-1} \in A[\underline{t}_s][[Y]]$, we have

$$h(y_{d-m}) = \Gamma_d, \quad \forall d \geq m,$$

hence $y_{d-m}^{\mu-r}\Gamma_d = H_0(y_{d-m}) + \underbrace{H_1(y_{d-m})}_{w_d\to 0}$. The identity (11) of the lemma now follows by using (12) with $k = \mu - r$ and the polynomial $\Gamma_{s,r}$ is easily seen to be monic of the claimed degree. □

Remark 20. For example, a simple computation shows that $\Gamma_{s,1} = Y + t_1 + \cdots + t_s$ if $m > 0$.

We can prove Theorem 5. We suppose that $d \geq m$ and that $m \geq 1$. We recall from Lemma 13 that

$$\begin{aligned}\frac{1}{\Pi_{s,d}} &= -(-\theta)^{\delta_{s,d}-m+1}\frac{\omega_{d-m}(t_1)\cdots\omega_{d-m}(t_s)}{\widetilde{\pi}_d}\\ &= -(-\theta)^{m-1+\mu q^{d-m}-m+1}\frac{\omega_{d-m}(t_1)\cdots\omega_{d-m}(t_s)}{\widetilde{\pi}_d}\\ &= -\theta^{\mu q^{d-m}}\frac{\omega_{d-m}(t_1)\cdots\omega_{d-m}(t_s)}{\widetilde{\pi}_d}.\end{aligned}$$

After explicit expansion of (1), we write

$$\Pi_{s,d}^{-1}F_d(1;s) - \sum_{i=r+1}^{\mu}\mathbb{D}_i\theta^{iq^{d-m}} = \mathbb{D}_r\theta^{rq^{d-m}} + \sum_{j=0}^{r-1}\mathbb{D}_j\theta^{jq^{d-m}}.$$

Dividing both sides by $\theta^{rq^{d-m}}$ we deduce that

$$\mathbb{D}_r = -\theta^{(\mu-r)q^{d-m}}\frac{\omega_{d-m}(t_1)\cdots\omega_{d-m}(t_s)}{\widetilde{\pi}_d}F_d(1;s) - \sum_{i=1}^{\mu-r}\mathbb{D}_{i+r}\theta^{iq^{d-m}} + u_d,$$

with u_d a sequence of elements of $\mathbb{T}_s(K_\infty)$ tending to zero. We can rewrite this as

$$\mathbb{D}_r = -\theta^{(\mu-r)q^{d-m}}\frac{\omega(t_1)\cdots\omega(t_s)}{\widetilde{\pi}}\Gamma_d F_d(1;s) - \sum_{i=1}^{\mu-r}\mathbb{D}_{i+r}\theta^{iq^{d-m}} + u_d,$$

$$= -\Gamma_{s,r}(\theta^{q^{d-m}})\frac{\omega(t_1)\cdots\omega(t_s)}{\widetilde{\pi}}F_d(1;s) - \sum_{i=1}^{\mu-r}\mathbb{D}_{i+r}\theta^{iq^{d-m}} + v_d,$$

for another sequence of elements v_d of $\mathbb{T}_s(K_\infty)$ tending to zero as $d \to \infty$, by using (11) of Lemma 19. The theorem follows. □

Remark 21. Note that in contrast, in the classical setting of tail series of Riemann's zeta values, we have a much more transparent, almost trivial situation. Indeed, with $\psi^{(m)}(z)$ the function $\frac{d^m}{dz^m}\frac{\Gamma'(z)}{\Gamma(z)}$ (Γ being Euler's function), one sees easily the formula

$$\frac{\psi^{(m)}(n)}{(-1)^{m+1}m!} = \sum_{k\geq n}\frac{1}{k^{m+1}}, \quad m \geq 1.$$

Now, since $\lim_{n\to\infty}\psi^{(m)}(n)n^m = (m-1)!$ with this range of m, we reach the well known trivial limit

$$\lim_{n\to\infty} n^{m-1}\sum_{k\geq n}\frac{1}{k^m} = \frac{1}{m-1}, \quad m \geq 2.$$

But in our function field case, additional complexity is introduced by the presence of the variables t_i, and it looks hard to give an arithmetic significance to the coefficients $\mathbb{D}_r$ but the leading coefficient of $\mathbb{H}_s$.

2.4. Proof of Theorem 7

For $a \in A^+(i)$ with $i \geq 0$ and for j an index between 1 and s', we can write $a(t_j) = t_j^i + b(t_j)$ where $b \in \mathbb{F}_q[t]$ is a polynomial with degree in t strictly smaller than i (depending on a). Hence, we can write, with $G_{s,d}$ a polynomial of $K[\underline{t}_s]$ such that for all $j = 1, \ldots, s'$, $\deg_{t_j}(G_{s,d}) < d-1$:

$$F_{s,d} = \sum_{i=0}^{d-1}\sum_{a\in A^+(i)}\frac{a(t_1)\cdots a(t_s)}{a}$$

$$= \sum_{a\in A^+(d-1)}\frac{a(t_1)\cdots a(t_{s'})t_{s'+1}^{d-1}\cdots t_s^{d-1}}{a} + G_{s,d}$$

$$= t_{s'+1}^{d-1}\cdots t_s^{d-1}S_{d-1}(1;s') + G_{s,d}.$$

In particular, $S_{d-1}(1,s') \in K[\underline{t}_{s'}]$ is the coefficient of $t_{s'+1}^{d-1}\cdots t_s^{d-1}$ of the polynomial $F_{s,d} \in K[\underline{t}_s]$. Now, by Theorem 2 we see that $S_{d-1}(1,s')$ is the coefficient of $t_{s'+1}^{d-1}\cdots t_s^{d-1}$

in the polynomial $\Pi_{s,d}\mathbb{H}_s|_{Y=\theta^{q^{d-m}}}$ (for all $d \geq m$, with $s = m(q-1)+1$). Since the coefficient of $t_{s'+1}^{d-m} \cdots t_s^{d-m}$ in $b_{d-m}(t_1) \cdots b_{d-m}(t_s)$ is equal to $b_{d-m}(t_1) \cdots b_{d-m}(t_{s'})$, we deduce that $S_{d-1}(1, s')$ is equal to $\Pi_{s',s,d} = \Pi_{s,d}/\prod_{i=s'+1}^{s} b_{d-m}(t_i)$ times the coefficient of $t_{s'+1}^{m-1} \cdots t_s^{m-1}$ of $\mathbb{H}_s|_{Y=\theta^{q^{d-m}}}$ which is $\mathbb{H}_{s,s'}|_{Y=\theta^{q^{d-m}}}$. $\square$

3. Examples

We give some formulas without proof. If $s = q$, then $m = 1$, and it can be proved that $\mathbb{H}_q = -1$ and $\lambda_{1,s} = 1$. If $s = 2q-1$, we have that $m = 2$, $\mu = q-1$ and the following formula holds, when $q > 2$:

$$\mathbb{H}_{2q-1} = \prod_{i=1}^{2q-1}(t_i - Y) + (Y^q - \theta)e_{q-1}(t_1 - Y, \ldots, t_s - Y) \in A[\underline{t}_s], \tag{13}$$

where the polynomials e_i are the elementary symmetric polynomials that, in the variables $T_1, \ldots, T_s$, is defined by:

$$\prod_{i=1}^{s}(X - T_i) = X^s + \sum_{j=1}^{s}(-1)^j X^{s-j} e_j(T_1, \ldots, T_s).$$

Developing (13) we obtain, again in the case $q > 2$:

$$\mathbb{H}_{2q-1} = \sum_{i=0}^{q-1}(-1)^i(e_{2q-1-i} - \theta e_{q-1-i})Y^i$$

with $e_j = e_j(\underline{t}_s)$, from which it is easy to see that $\mathbb{H}_{2q-1}$ has the following partial degrees: $\deg_{t_i}(\mathbb{H}_{2q-1}) = 1$ for all i, $\deg_\theta(\mathbb{H}_{2q-1}) = 1$ and $\deg_Y(\mathbb{H}_{2q-1}) = \mu = q-1$ in agreement with the Theorem 2. Furthermore, the coefficient of Y^μ is equal to $-\mathbb{B}_{2q-1} = -\lambda_{1,2q-1} = -\theta + e_q(t_1, \ldots, t_{2q-1})$ (see the examples in [4]).

Also, computing the coefficient of the appropriate monomial in $t_1^{d-1}, \ldots, t_s^{d-1}$ as in §2.4 from the formula (13), it is easy to deduce the formulas

$$S_{d-1}(1; s) = \frac{b_{d-1}(t_1) \cdots b_{d-1}(t_s)}{l_{d-1}}$$

for $d \geq 1$ and $s = 0, \ldots, q-1$. Further, we compute easily, for $d \geq 2$:

$$S_{d-1}(1, q) = \frac{b_{d-2}(t_1) \cdots b_{d-2}(t_q)}{l_{d-1}} \left(\prod_{i=1}^{q}(t_i - \theta^{q^{d-2}}) + \theta^{q^{d-1}} - \theta \right),$$

which agrees with [13, Corollary 4.1.8]. The analysis of the case $q = 2$ is similar but we will not describe it here. For $s = 3q-2$ we have $m = 3$ and $\mu = q^2 + q - 2$. We refrain from displaying here the explicit formulas we obtain.

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