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ANDERSON-STARK UNITS FOR $\mathbb{F}_q[\theta]$

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ABSTRACT. We investigate the arithmetic of special values of a new class of L-functions recently introduced by the second author. We prove that these special values are encoded in some particular polynomials which we call Anderson-Stark units. We then use these Anderson-Stark units to prove that L-functions can be expressed as sums of polylogarithms.

CONTENTS

1. INTRODUCTION

A major theme in the arithmetic theory of global function fields is the study of the arithmetic properties of special values of D. Goss L-functions. A typical example of such a function is given by the Carlitz-Goss zeta function $\zeta_A(.)$, where $A = \mathbb{F}_q[\theta]$ is the polynomial ring in the variable θ with coefficients in a finite field $\mathbb{F}_q$. Its special values are given by the following formula:

$$\forall n \geq 1,\, \zeta_A(n) = \sum_{a \in A_+} \frac{1}{a^n} \in K_\infty,$$

where A_+ is the set of monic elements in A and $K_\infty = \mathbb{F}_q((\frac{1}{\theta}))$. In 1990, G. Anderson and D. Thakur proved the following fundamental result ([AT90, Theorem 3.8.3]): for $n \geq 1$, there exists $z_n \in \mathrm{Lie}(C^{\otimes n})(K_\infty)$ such that $\exp_n(z_n) \in C^{\otimes n}(A)$, and

$$\Gamma_n \zeta_A(n) = e_n(z_n),$$

where $\exp_n$ is the exponential map associated to the nth tensor power of the Carlitz module $C^{\otimes n}$, $e_n(z_n)$ is the last coordinate of $z_n \in K_\infty^n$, and $\Gamma_n \in A$ is the Carlitz factorial (we refer the reader to [BP] for the basic properties of $C^{\otimes n}$). This result has recently been generalized by M. A. Papanikolas in [Pap] who proved a log-algebraicity theorem for $C^{\otimes n}$ in the spirit of the work of G. Anderson in [And96].

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M. A. Papanikolas applies this log-algebraicity theorem to obtain remarkable explicit formulas for a large class of special values of D. Goss Dirichlet L-functions. Observe that the t-motive associated to the t-module $C^{\otimes n}$ can be understood as the following object: $(A[t], \tau)$, where t is an indeterminate over $K = \mathbb{F}_q(\theta)$, and $\tau : A[t] \to A[t]$ is the $\mathbb{F}_q[t]$-linear map defined as follows:

$$\tau\left(\sum_{k\geq 0} a_k t^k\right) = (t-\theta)^n \left(\sum_{k\geq 0} a_k^q t^k\right),$$

where $a_k \in A$.

Let $s \geq 1$ be an integer and let $t_1, \ldots, t_s$ be s indeterminates over K. Consider the following object: $(A[t_1, \ldots, t_s], \tau)$ where $\tau : A[t_1, \ldots, t_s] \to A[t_1, \ldots, t_s]$ is the morphism of $\mathbb{F}_q[t_1, \ldots, t_s]$-modules, semi-linear[1] with respect to $\tau_0 : A \to A, x \mapsto x^q$, given by:

$$\tau\left(\sum_{i_1,\ldots,i_s\in\mathbb{N}} a_{i_1,\ldots,i_s} t_1^{i_1}\cdots t_s^{i_s}\right) = (t_1-\theta)\cdots(t_s-\theta)\left(\sum_{i_1,\ldots,i_s\in\mathbb{N}} a^q_{i_1,\ldots,i_s} t_1^{i_1}\cdots t_s^{i_s}\right),$$

where $a_{i_1,\ldots,i_s} \in A$. Note that we have a natural morphism of $\mathbb{F}_q[t_1, \ldots, t_s]$-algebras

$$\phi : A[t_1, \ldots, t_s] \to \mathrm{End}_{\mathbb{F}_q[t_1,\ldots,t_s]} A[t_1, \ldots, t_s]$$

given by $\phi_\theta = \theta + \tau$. Let $\mathbb{T}_s(K_\infty)$ be the Tate algebra in the variables $t_1, \ldots, t_s$, with coefficients in K_∞. Then τ extends naturally to a continuous morphism of $\mathbb{F}_q[t_1, \ldots, t_s]$-modules on $\mathbb{T}_s(K_\infty)$. The second author introduced (see [Pel12], [Per14b], [Per14a], [AP15]) for integers $N \in \mathbb{Z}$ and $s \geq 0$ the L-series

$$L(N, s) = \sum_{a\in A_+} \frac{a(t_1)\cdots a(t_s)}{a^N},$$

which converges in $\mathbb{T}_s(K_\infty)$. If z is another indeterminate, we also set

$$L(N, s, z) = \sum_{d\geq 0} z^d \sum_{\substack{a\in A_+ \\ \deg_\theta a = d}} \frac{a(t_1)\ldots a(t_s)}{a^N} \in K[t_1, \ldots, t_s][[z]].$$

These series converge at $z = 1$ in $\mathbb{T}_s(K_\infty)$ and we have the equality

$$L(N, s) = L(N, s, z) \mid_{z=1} .$$

Our main goal in this article is the study of the arithmetic properties of the $L(N, s, z), N \in \mathbb{Z}$. Let us give a brief description of our principal results.

We let τ act on $K[t_1, \ldots, t_s][[z]]$ by

$$\tau(\sum_{k\geq 0} f_k z^k) = \sum_{k\geq 0} \tau(f_k) z^k,$$

where $f_k \in K[t_1, \ldots, t_s]$. The exponential function associated to ϕ is defined by

$$\exp_\phi = \sum_{i\geq 0} \frac{1}{D_i}\tau^i,$$

[1] We signal here to avoid confusion that in the rest of the article τ will denote more generally a morphism semi-linear with respect to τ_0.

where $D_0 = 1$, and for $i \geq 1$, $D_i = (\theta^{q^i} - \theta)D_{i-1}^q$. We also set

$$\exp_{\phi,z} = \sum_{i\geq 0} \frac{z^i}{D_i}\tau^i.$$

A formulation of the s-variable version of Anderson's log-algebraicity theorem is (see Theorem 4.6 and Proposition 5.4)

$$\exp_{\phi,z}(L(1,s,z)) \in A[t_1,\dots,t_s,z]$$

from which we also deduce that, in $\mathbb{T}_s(K_\infty)$,

$$\exp_\phi(L(1,s)) \in A[t_1,\dots,t_s].$$

This s-variable version has been proved in [APTR16] as a consequence of a class formula. We give here a more direct proof, close to Anderson's original proof in [And96].

The special elements $\exp_{\phi,z}(L(1,s,z))$ and $\exp_\phi(L(1,s))$ play the role of Stark units in our context. Let us give an example, for $1 \leq s \leq q-1$, by Proposition 5.4 we have the following equality in $\mathbb{T}_s$:

$$L(1,s) = \log_\phi(1),$$

where $\log_\phi = \sum_{i\geq 0} \frac{1}{l_i}\tau^i$ is the Carlitz logarithm, $l_0 = 1$ and for $i \geq 1$, $l_i = (\theta - \theta^{q^i})l_{i-1}$. We define for $N > 0$, the Nth "polylogarithm"

$$\log_{\phi,N,z} = \sum_{i\geq 0} \frac{z^i}{l_i^N}\tau^i.$$

Set $b_0(t) = 1$, and for $r \geq 1$, $b_r(t) = \prod_{k=0}^{r-1}(t - \theta^{q^k})$. Let $N \geq 1$ be an integer and let $r \geq 1$ be the unique integer such that $q^r \geq N > q^{r-1}$. We can then prove (see Theorem 6.2 for the precise statement) that there exists a finite set of completely explicit elements $h_j \in A[t_1,\dots,t_s,z]$, $0 \leq j \leq d$, that are built from the "unit" $\exp_{\phi,z}(L(1,n+q^r-N,z))$, such that

$$l_{r-1}^{q^r-N} b_r(t_1)\cdots b_r(t_n)L(N,n,z) = \sum_{j=0}^{d} \theta^j \log_{\phi,N,z}(h_j).$$

The paper is organized as follows: we first (§3) introduce a Banach space $\mathbb{B}_s$ which is a completion of an s-variable polynomial ring for a norm similar to the one considered by Anderson in [And96]. The study of different natural Carlitz actions on $\mathbb{B}_s$ allows us to endow $\mathbb{B}_s$ with an action of the Tate algebra $\mathbb{T}_s$ and to translate some statements on $\mathbb{B}_s$ into statements on $\mathbb{T}_s$. We then (§4) prove the s-variable log-algebraicity theorem, following the lines of Anderson's proof in [And96], and establish some properties of the special polynomials. We also state two "converses" to the log-algebraicity theorem (Propositions 4.16 and 4.17). In the next section (§5) we translate the preceding results in $\mathbb{T}_s$, so that the L-functions $L(1,s,z)$ and $L(1,s)$ appear naturally. The last section (§6) is devoted to the proof that $L(N,n,z)$ can be expressed as a sum of polylogarithms.

2. Notation

Let $\mathbb{F}_q$ be a finite field with q elements, where q is a power of a prime p, θ an indeterminate over $\mathbb{F}_q$, $A = \mathbb{F}_q[\theta]$, $A^* = A\backslash\{0\}$ and $K = \mathbb{F}_q(\theta)$. The set of monic elements (respectively of degree $j \geq 0$) of A is denoted by A_+ (respectively $A_{+,j}$). Let v_∞ be the valuation on K given by $v_\infty(\frac{a}{b}) = \deg_\theta b - \deg_\theta a$. We identify with $\mathbb{F}_q((\frac{1}{\theta}))$ the completion K_∞ of K with respect to v_∞. Let $\mathbb{C}_\infty$ be the completion of an algebraic closure of K_∞. Then v_∞ extends uniquely to a valuation on $\mathbb{C}_\infty$, still denoted by v_∞, and we set for all $\alpha \in \mathbb{C}_\infty$, $|\alpha|_\infty = q^{-v_\infty(\alpha)}$. The algebraic closures of K and $\mathbb{F}_q$ in $\mathbb{C}_\infty$ will be denoted by $\overline{K}$ and $\overline{\mathbb{F}_q}$.

Let τ denote an operator which we let act as the Frobenius on $\mathbb{C}_\infty$: for all $\alpha \in \mathbb{C}_\infty$, $\tau(\alpha) = \alpha^q$. If R is a ring endowed with an action of τ (for instance, R a subring of $\mathbb{C}_\infty$ stable under τ), then we denote by $R[[\tau]]$ the ring of formal series in τ with coefficients in R subject to the commutation rule: for all $r \in R, \tau.r = \tau(r).\tau$. We also denote by $R[\tau]$ the subring of $R[[\tau]]$ of polynomials in τ.

The *Carlitz module* is the unique morphism $C_{\boldsymbol{\cdot}} : A \to A[\tau]$ of $\mathbb{F}_q$-algebras determined by $C_\theta = \theta + \tau$. If M is an A-module endowed with a semi-linear endomorphism τ_M ($\forall m \in M, \forall a \in A$, $\tau_M(am) = \tau(a)\tau_M(m)$), then $C_{\boldsymbol{\cdot}}$ induces a new action of A on M; endowed with this action, the A-module M is denoted by $C(M)$.

The *Carlitz exponential* is the formal series

$$\exp_C = \sum_{i\geq 0} \frac{1}{D_i}\tau^i \in K[[\tau]],$$

where $D_0 = 1$ and for $i \geq 1, D_i = (\theta^{q^i} - \theta)D_{i-1}^q$. The evaluation $\exp_C : \mathbb{C}_\infty \to \mathbb{C}_\infty$; $x \mapsto \exp_C(x) = \sum_{i\geq 0} \frac{1}{D_i}\tau^i(x)$ defines an entire $\mathbb{F}_q$-linear function on $\mathbb{C}_\infty$ and $\ker(\exp_C : \mathbb{C}_\infty \to \mathbb{C}_\infty) = \tilde{\pi}A$ where $\tilde{\pi}$ is the Carlitz period defined by (see [Gos96, Chapter 3])

$$\tilde{\pi} = \sqrt[q-1]{\theta - \theta^q} \prod_{i\geq 1} \left(1 - \frac{\theta^{q^i} - \theta}{\theta^{q^{i+1}} - \theta}\right) \in \sqrt[q-1]{-\theta}(\theta + \mathbb{F}_q[[\frac{1}{\theta}]]).$$

The *Carlitz logarithm* is the formal series

$$\log_C = \sum_{i\geq 0} \frac{1}{l_i}\tau^i \in K[[\tau]],$$

where $l_0 = 1$ and for $i \geq 1, l_i = (\theta - \theta^{q^i})l_{i-1}$. It satisfies in $K[[\tau]]$ the equality $\log_C . \exp_C = 1$. It defines a function $x \mapsto \log_C(x)$ on $\mathbb{C}_\infty$ converging for $v_\infty(x) > -\frac{q}{q-1}$. Moreover, if $v_\infty(x) > -\frac{q}{q-1}$, then $v_\infty(x) = v_\infty(\exp_C(x)) = v_\infty(\log_C(x))$ and $\exp_C \circ \log_C(x) = x = \log_C \circ \exp_C(x)$. We have the formal identities in $K[[\tau]]$ for all $a \in A$:

$$\exp_C a = C_a \exp_C \quad \text{and} \quad \log_C C_a = a \log_C .$$

The identity $\exp_C(ax) = C_a(\exp_C(x))$ holds for all $x \in \mathbb{C}_\infty, a \in A$.

The set of A-torsion points of $C(\mathbb{C}_\infty)$ is denoted by $\Lambda_C \subset C(\overline{K})$. Let $a \in A$ with $\deg_\theta a > 0$, the a-torsion points are precisely the elements $\exp_C(\frac{b\tilde{\pi}}{a}) \in \overline{K}$ with $b \in A$ and $\deg_\theta b < \deg_\theta a$. Therefore, $\Lambda_C = \exp_C(K\tilde{\pi})$. Since $\exp_C$ is continuous for the topology defined by v_∞, the closure of Λ_C in $\mathbb{C}_\infty$ is the compact set

$$\mathfrak{K} = \overline{\Lambda_C} = \exp_C(K_\infty\tilde{\pi}) = \exp_C\left(\frac{1}{\theta}\mathbb{F}_q[[\frac{1}{\theta}]]\tilde{\pi}\right) = \sqrt[q-1]{-\theta}\mathbb{F}_q[[\frac{1}{\theta}]],$$

where the last equality comes from the facts that $\tilde{\pi} \in \sqrt[q-1]{-\theta}(\theta + \mathbb{F}_q[[\frac{1}{\theta}]])$, and that for $\lambda \in \mathbb{F}_q^*$ and $n \geq 1$, $\exp_C\left(\frac{\lambda\tilde{\pi}}{\theta^n}\right) \equiv \frac{\lambda\tilde{\pi}}{\theta^n} \mod \frac{\tilde{\pi}}{\theta^{n+1}}\mathbb{F}_q[[\frac{1}{\theta}]]$. It is customary to consider $K_\infty\tilde{\pi}$ as an analogue of the imaginary line; the compact $\mathfrak{K}$ is then an analogue of the unit circle. We remark that $\exp_C$ and $\log_C$ define reciprocal automorphisms of $\mathfrak{K}$.

3. Some functional analysis

3.1. **General settings.** Let $s \geq 1$ be a fixed integer and let $\mathbf{X} = (X_1, \ldots, X_s)$ be a set of indeterminates over $\mathbb{C}_\infty$. We want to consider polynomials $F \in \mathbb{C}_\infty[\mathbf{X}]$ as polynomial functions on $\mathfrak{K}^s$. Thus we introduce the following norm on $\mathbb{C}_\infty[\mathbf{X}]$:

$$\|F\| = \max\{|F(x_1, \ldots, x_s)|_\infty \;\; x_1, \ldots, x_s \in \mathfrak{K}\}.$$

Since $\mathfrak{K}$ is compact and infinite, this is a well-defined, ultrametric norm of $\mathbb{C}_\infty$-algebra. (In particular, for all $F, G \in \mathbb{C}_\infty[\mathbf{X}]$, $\|FG\| \leq \|F\|\|G\|$. Moreover, $\|F\| = 0 \Rightarrow F = 0$ is a consequence of the fact that $\mathfrak{K}$ is infinite.)

If $\mathbf{i} = (i_1, \ldots, i_s)$ where the $i_j \geq 0$ are integers, then we write $\mathbf{X}^\mathbf{i}$ for $X_1^{i_1} \ldots X_s^{i_s}$ and $|\mathbf{i}| = i_1 + \cdots + i_s$.

Lemma 3.1.

(1) *If* $\mathbf{i} \in \mathbb{N}^s$, *then* $\|\mathbf{X}^\mathbf{i}\| = q^{\frac{|\mathbf{i}|}{q-1}}$.

(2) *Write for* $n \geq 1$, $\lambda_{\theta^n} = \exp_C(\frac{\tilde{\pi}}{\theta^n}) \in \Lambda_C$ *and let* $W \subset \Lambda_C$ *be the* $\mathbb{F}_q$*-vector space spanned by the* λ_{θ^n}, $n \geq 1$. *Then* W *is dense in* $\mathfrak{K}$. *In particular, for all* $F \in \mathbb{C}_\infty[\mathbf{X}]$,

$$\|F\| = \sup\{|F(\mathbf{x})|_\infty \;\; \mathbf{x} \in \Lambda_C^s\} = \sup\{|F(\mathbf{x})|_\infty \;\; \mathbf{x} \in W^s\}.$$

Proof.

(1) This is a consequence of the fact that if $a, b \in A^*$ with $\deg_\theta a < \deg_\theta b$, then

$$v_\infty\left(\exp_C\left(\frac{a\tilde{\pi}}{b}\right)\right) = v_\infty\left(\frac{a\tilde{\pi}}{b}\right) = \deg_\theta b - \deg_\theta a - \frac{q}{q-1} \geq \frac{-1}{q-1}.$$

(2) This follows from the fact that the $\mathbb{F}_q$-vector space spanned by the $\frac{1}{\theta^n}$ for $n \geq 1$ is $\frac{1}{\theta}\mathbb{F}_q[\frac{1}{\theta}]$ which is dense in $\frac{1}{\theta}\mathbb{F}_q[[\frac{1}{\theta}]]$. □

Remark 3.2. Note that the norm $\|.\|$ is not multiplicative. We shall give an example in the one variable case. We have

$$\|C_\theta(X)\| = \|X\| = q^{\frac{1}{q-1}}$$

but $C_\theta(X) = \prod_{\lambda \in \mathbb{F}_q}\left(X - \lambda\exp_C(\frac{\tilde{\pi}}{\theta})\right)$, where for all $\lambda \in \mathbb{F}_q$, $\|X - \lambda\exp_C(\frac{\tilde{\pi}}{\theta})\| = q^{\frac{1}{q-1}}$.

Since Λ_C is the torsion set of $C(\mathbb{C}_\infty)$, it is naturally endowed with the Carlitz action of A: if $x \in \Lambda_C, a \in A$, then $C_a(x) \in \Lambda_C$, which extends by continuity to $\mathfrak{K}$. Thus, we get a natural action of the multiplicative monoid of A on the polynomial functions on $\mathfrak{K}^s$:

$$\forall F(\mathbf{X}) \in \mathbb{C}_\infty[\mathbf{X}], \forall a \in A \;\; a * F(\mathbf{X}) = F(C_a(X_1), \ldots, C_a(X_s)). \tag{3.1}$$

This action is a generalisation of Anderson's construction ([And96, §3.2]) to our settings. Observe that since for all $a \in A^*, C_a : \Lambda_C \to \Lambda_C$ is surjective, this action is isometric with respect to the norm $\|\cdot\|$.

3.2. **The one variable case.** Set $L = K_\infty(\tilde{\pi})$ and $\pi = \frac{\theta}{\tilde{\pi}}$. Since $v_\infty(\pi) = \frac{1}{q-1}$, the valuation ring of L is

$$O_L = \mathbb{F}_q[[\pi]] = \sum_{k=0}^{q-2} \pi^k \mathbb{F}_q[[\frac{1}{\theta}]],$$

and its maximal ideal is

$$\mathfrak{P}_L = \pi O_L.$$

Recall that since $K_\infty = A \oplus \frac{1}{\theta}\mathbb{F}_q[[\frac{1}{\theta}]]$, we have

$$\exp_C(\tilde{\pi}K_\infty) = \frac{1}{\pi}\mathbb{F}_q[[\frac{1}{\theta}]] \subset \mathfrak{P}_L^{-1}.$$

Let $N \in \mathbb{N} = \{0, 1, \dots\}$ and let $N = \sum_{i=0}^r N_i q^i$, with for all i, $0 \leq N_i \leq q-1$, be its base-q decomposition. Set $l_q(N) = \sum_{i=0}^r N_i$. We define the polynomial $G_N(X)$ by

$$G_N(X) = \pi^{l_q(N)} \left(\prod_{i=0}^r (\theta^i * X)^{N_i} \right) = \pi^{l_q(N)} \left(\prod_{i=0}^r C_{\theta^i}(X)^{N_i} \right) \in L[X].$$

Lemma 3.3.

(1) *The set $\{G_N(X), N \geq 0\}$ generates $L[X]$ as an L-vector space.*
(2) *For $N \in \mathbb{N}$, we have:*

$$\|G_N(X)\| = 1.$$

Proof.

(1) It follows from the fact that for all $N \geq 0$, $\deg_X(G_N(X)) = N$.
(2) We remark that for all $\lambda \in \Lambda_C$, $v_\infty(G_N(\lambda)) \geq 0$ and that if $\alpha \in \mathbb{F}_q^*$ and $\lambda_{\theta+\alpha} = \exp_C\left(\frac{\tilde{\pi}}{\theta+\alpha}\right)$, then $v_\infty(G_N(\lambda_{\theta+\alpha})) = 0$. □

If $\beta = (\beta_i)_{i \geq 1}$ is a sequence of elements in $\mathbb{F}_q$, we set

$$\lambda(\beta) = \sum_{i \geq 1} \beta_i \exp_C\left(\frac{\tilde{\pi}}{\theta^i}\right) \in \mathfrak{P}_L^{-1}.$$

Note that if we set $\mu(\beta) = \sum_{i \geq 1} \frac{\beta_i}{\theta^i} \in K_\infty$, then we have $\lambda(\beta) = \exp_C(\tilde{\pi}\mu(\beta))$.

Lemma 3.4. *Let $\beta = (\beta_i)_{i \geq 1}$ be a sequence of elements in $\mathbb{F}_q$, and let $N = \sum_{i=0}^r N_i q^i$ be a nonnegative integer written in base q. Then*

$$G_N(\lambda(\beta)) \equiv \prod_{i=0}^r \beta_{i+1}^{N_i} \mod \mathfrak{P}_L.$$

Proof. Observe that

$$\exp_C\left(\frac{\tilde{\pi}}{\theta}\right) \equiv \frac{1}{\pi} \mod O_L.$$

Thus, for $j \geq 0$,

$$\pi C_{\theta_j}(\lambda(\beta)) \equiv \beta_{j+1} \mod \mathfrak{P}_L,$$

whence the result. □

Lemma 3.5. *Let k, r be two integers such that $r \geq 1$ and $1 \leq k \leq q^r$, let $\alpha_1, \ldots, \alpha_k \in \mathbb{F}_q^*$ and let $N_1, \ldots, N_k$ be k distinct integers in $\{0, \ldots, q^r - 1\}$. Write $N_i = \sum_{j=0}^{r-1} n_{i,j} q^i$ in base q. Then, there exists $\beta_1, \ldots, \beta_r \in \mathbb{F}_q$ such that*

$$\sum_{i=1}^{k} \alpha_i \prod_{j=0}^{r-1} \beta_{j+1}^{n_{i,j}} \neq 0$$

with the convention that $0^0 = 1$.

Proof. We proceed by induction on r.

If $r = 1$, then $k \leq q$ and for $1 \leq j \leq k$, $N_i = n_{i,1} \in \{0, \ldots, q-1\}$. Since the N_i's are distinct, the polynomial $\sum_{i=1}^{k} \alpha_i X^{N_i}$ is not divisible by $X^q - X$, and this implies the assertion of the lemma in this case.

We assume now that the lemma is proved for all integers less than $r - 1 \geq 1$, and we also assume that at least one N_i is $\geq q^{r-1}$. We define an equivalence relation over the set $\{1, \ldots, k\}$: for all $1 \leq i, i' \leq k$, $i \sim i'$ if and only if $n_{i,j} = n_{i',j}$ for all $1 \leq j \leq r-2$ (that is, if $N_i \equiv N_{i'} \mod q^{r-1}$). We denote by $I_1, \ldots, I_t$ the equivalence classes and if $i \in I_m$ we define for $1 \leq j \leq r-2$, $n_j^{(m)} = n_{i,j}$ the common value. Let $\beta_1, \ldots, \beta_r \in \mathbb{F}_q^*$; then

$$\sum_{i=1}^{k} \alpha_i \prod_{j=0}^{r-1} \beta_{j+1}^{n_{i,j}} = \sum_{m=1}^{t} \left(\sum_{i \in I_m} \alpha_i \beta_r^{n_{i,r-1}} \right) \prod_{j=0}^{r-2} \beta_{j+1}^{n_j^{(m)}}.$$

Now, by the case $r = 1$, we can find β_r such that the sum $\sum_{i \in I_1} \alpha_i \beta_r^{n_{i,r-1}}$ is not zero and we can apply the induction hypothesis to conclude the proof. $\square$

Let $K_\infty \subseteq E \subseteq \mathbb{C}_\infty$ be a field complete with respect to $|\cdot|_\infty$, and $\mathbb{B}(E)$ denote the completion of $E[X]$ with respect to $\|\cdot\|$.

Theorem 3.6. *The family $\{G_N(X), N \geq 0\}$ forms an* orthonormal basis *of the E-Banach space $\mathbb{B}(E)$, that is:*

(i) *any $F \in \mathbb{B}(E)$ can be written in a unique way as a convergent series $F = \sum_{N \geq 0} f_N G_N(X)$ with $f_N \in E, N \geq 0$, and $\lim_{N \to \infty} f_N = 0$;*
(ii) *if F is written as above, then $\|F\| = \max_{N \geq 0} |f_N|_\infty$.*

Proof. It is enough to prove the above properties (i) and (ii) for $F \in E[X]$. Note that property (i) is a consequence of the fact that $\deg_X G_N(X) = N$ for all $N \geq 0$. Let us prove property (ii). It is enough to consider $F = \sum_{i=0}^{r} x_r G_{N_i}$ with for all $0 \leq i \leq r$, $v_\infty(x_i) = 0$. We are reduced to proving that $\|F\| = 1$, that is, $\|F\| \geq 1$ since we already know the converse inequality; and the existence of $\lambda \in \Lambda_C$ such that $v_\infty(F(\lambda)) = 0$ is a consequence of Lemmas 3.4 and 3.5. $\square$

3.3. **The multivariable case.** Let $s \geq 1$ be an integer, we define for a field $K_\infty \subseteq E \subseteq \mathbb{C}_\infty$ complete with respect to $|\cdot|_\infty$, $\mathbb{B}_s(E)$ to be the completion of $E[\mathbf{X}]$ with respect to $\|\cdot\|$. We write also for short $\mathbb{B}_s = \mathbb{B}_s(\mathbb{C}_\infty)$. Observe that for $N_1, \ldots, N_s \in \mathbb{N}$, we have

$$\|G_{N_1}(X_1) \cdots G_{N_s}(X_s)\| = 1.$$

Theorem 3.7. *Let $L \subseteq E \subseteq \mathbb{C}_\infty$ be complete with respect to $|\cdot|_\infty$. Then the family*

$$\{G_{N_1}(X_1)\cdots G_{N_s}(X_s), N_1,\dots,N_s \in \mathbb{N}\}$$

forms an orthonormal basis of the E-Banach space $\mathbb{B}_s(E)$, that is:

(i) *any $F \in \mathbb{B}_s(E)$ can be written in a unique way as the sum of a summable family*

$$F = \sum_{(N_1,\dots,N_s)\in\mathbb{N}^s} f_{N_1,\dots,N_s} G_{N_1}(X_1)\cdots G_{N_s}(X_s)$$

with $f_{N_1,\dots,N_s} \in E$ for all $N_1,\dots,N_s \in \mathbb{N}$, and $f_{N_1,\dots,N_s}$ goes to 0 with respect to the Fréchet filter;[2]

(ii) *if F is written as above, then $\|F\| = \max\{|f_{N_1,\dots,N_s}|_\infty, N_1,\dots,N_s \in \mathbb{N}\}$.*

Proof. We proceed by induction on $s \geq 1$. The case $s = 1$ is the statement of Theorem 3.6. Assume now that $s \geq 2$ and that the theorem is true for $s-1$. It will be enough to prove (i) and (ii) for polynomials, and (i) is still an easy consequence of $\deg_X G_N(X) = N$ for all $N \geq 0$. Write a polynomial

$$F = \sum_{i=0}^{r} \alpha_r G_{N_i}(X_s) \in E[\mathbf{X}], \text{ where } \forall 1 \leq i \leq r, \alpha_i \in E[X_1,\dots,X_{s-1}].$$

Write for $1 \leq i \leq r$, the polynomial

$$\alpha_i = \sum_{i_1,\dots,i_{s-1}} \alpha^{(i)}_{i_1,\dots,i_{s-1}} G_{i_1}(X_1)\cdots G_{i_{s-1}}(X_s)$$

with $\alpha^{(i)}_{i_1,\dots,i_{s-1}} \in E$. Then the induction hypothesis shows that for all i:

$$\|\alpha_i\| = \max\left\{\left|\alpha^{(i)}_{i_1,\dots,i_{s-1}}\right|_\infty, i_1,\dots,i_{s-1} \in \mathbb{N}\right\}.$$

Thus

$$\|F\| \leq \max_{1\leq i\leq r} \|\alpha_i\| = \max\left\{\left|\alpha^{(i)}_{i_1,\dots,i_{s-1}}\right|_\infty, i_1,\dots,i_{s-1}, i \in \mathbb{N}\right\}.$$

Let $1 \leq i_0 \leq r$ be such that $\|\alpha_{i_0}\| = \max_{1\leq i\leq r}\|\alpha_i\|$, to prove the converse inequality, we will find $\lambda_1,\dots,\lambda_s \in \mathfrak{K}^s$ such that $|F(\lambda_1,\dots,\lambda_s)|_\infty = \|\alpha_{i_0}\|$. Let $\lambda_1,\dots,\lambda_{s-1} \in \mathfrak{K}^{s-1}$ such that

$$|\alpha_{i_0}(\lambda_1,\dots,\lambda_{s-1})|_\infty = \|\alpha_{i_0}\|.$$

Then, by the case $s = 1$,

$$\|F(\lambda_1,\dots,\lambda_{s-1},X_s)\| = \max|\alpha_{i_0}(\lambda_1,\dots,\lambda_{s-1})|_\infty = \|\alpha_{i_0}\|.$$

Therefore, we can find $\lambda \in \mathfrak{K}$ such that $|F(\lambda_1,\dots,\lambda_{s-1},\lambda)|_\infty = \|\alpha_{i_0}\|$, proving that $\|F\| = \|\alpha_{i_0}\|$ and the theorem. ☐

For all $N = \sum_{i=0}^{r} N_i q^i \geq 0$, define

$$H_N(X) = \left(\prod_{i=0}^{r} (\theta^i * X)^{N_i}\right) = \pi^{-l_q(N)} G_N(X) \in K_\infty[X].$$

[2]We recall that, here, this just means that $\lim_{N_1+\cdots+N_s\to\infty} f_{N_1,\dots,N_s} = 0$.

Then the H_N's generate $K_\infty[X]$ and $\|H_N(X)\| = q^{\frac{l_q(N)}{q-1}}$. If E does not contain L, in particular if $E = K_\infty$, then G_N has no longer coefficients in E and there might not exist an orthonormal basis of $\mathbb{B}(E)$. However, Theorem 3.7 still implies the following corollary.

Corollary 3.8. *Let $K_\infty \subseteq E \subseteq \mathbb{C}_\infty$ be complete with respect to $|\cdot|_\infty$. Then the family*

$$\{H_{N_1}(X_1)\cdots H_{N_s}(X_s), N_1, \ldots, N_s \in \mathbb{N}\}$$

forms an orthogonal basis *of the E-Banach space $\mathbb{B}_s(E)$, that is:*

(i) *any $F \in \mathbb{B}_s(E)$ can be written in a unique way as the sum of a summable family*

$$F = \sum_{(N_1,\ldots,N_s)\in\mathbb{N}^s} f_{N_1,\ldots,N_s} H_{N_1}(X_1)\cdots H_{N_s}(X_s)$$

with $f_{N_1,\ldots,N_s} \in E$ for all $N_1, \ldots, N_s \in \mathbb{N}$, and $|f_{N_1,\ldots,N_s}|_\infty q^{\frac{l_q(N_1)+\cdots+l_q(N_s)}{q-1}}$ goes to 0 with respect to the Fréchet filter;

(ii) *if F is written as above, then*

$$\begin{aligned}\|F\| &= \max\{|f_{N_1,\ldots,N_s} H_{N_1}(X_1)\cdots H_{N_s}(X_s)|_\infty, N_1, \ldots, N_s \in \mathbb{N}\}\\ &= \max\{|f_{N_1,\ldots,N_s}|_\infty q^{\frac{l_q(N_1)+\cdots+l_q(N_s)}{q-1}}, N_1, \ldots, N_s \in \mathbb{N}\}.\end{aligned}$$

3.4. **The Carlitz action.** In this section, $K_\infty \subseteq E \subseteq \mathbb{C}_\infty$ is a field complete with respect to $|\cdot|_\infty$. Note that the action $*$ of A on $E[\mathbf{X}]$ defined in (3.1) satisfies that for all $a \in A^*$, the map $F \mapsto a * F$ is an isometry on $E[\mathbf{X}]$. Thus, the action $*$ extends to an action, still denoted $*$, of A on $\mathbb{B}_s(E)$, such that for all $a \in A^*$, the map $F \mapsto a * F$ is an isometry on $\mathbb{B}_s(E)$.

Now, instead of considering the simultaneous action of A on each of the X_j, we will separate this action into actions on a single variable X_j, namely, for $1 \leq j \leq s$, $F \in \mathbb{B}_s(E)$ and $a \in A$, we set:

$$a *_j F(\mathbf{X}) = F(X_1, \ldots, X_{j-1}, C_a(X_j), X_{j+1}, \ldots, X_s). \tag{3.2}$$

This is still an action of monoid, but if we restrict this action to the set of polynomials in $E[\mathbf{X}]$ which are $\mathbb{F}_q$-linear in the variable X_j, the action $*_j$ induces a structure of A-module. Thus we define:

$$E[\mathbf{X}]^{\text{lin}} = \{F \in E[\mathbf{X}]; F \text{ is linear with respect to each of the variables } X_1, \ldots, X_s\}$$

which is the sub-E-vector space of $E[\mathbf{X}]$ spanned by the monomials $X_1^{q^{i_1}} \cdots X_s^{q^{i_s}}$, $i_1, \ldots, i_s \in \mathbb{N}$. Since the actions $*_j$ and $*_i$ commute and commute with the linear action of E, $E[\mathbf{X}]^{\text{lin}}$ has a structure of module over $E \otimes_{\mathbb{F}_q} A^{\otimes s}$, that is, if $t_1, \ldots, t_s$ are new indeterminates, we identify $E \otimes_{\mathbb{F}_q} A^{\otimes s}$ with $E[t_1, \ldots, t_s]$ and $E[\mathbf{X}]^{\text{lin}}$ has a structure of $E[t_1, \ldots, t_s]$-module given by

$$\forall 1 \leq j \leq s, \quad t_j.F(X_1, \ldots, X_s) = F(X_1, \ldots, X_{j-1}, C_\theta(X_j), X_{j+1}, \ldots, X_s). \tag{3.3}$$

We write $\mathbf{t}$ for the set of variable $t_1, \ldots, t_s$ and if $\mathbf{i} = (i_1, \ldots, i_s) \in \mathbb{N}^s$, $\mathbf{t}^{\mathbf{i}} = t_1^{i_1} \cdots t_s^{i_s}$.

The action defined by formula (3.3) extends to an action on $E[\mathbf{X}]$, turning $E[\mathbf{X}]$ into an $E[\mathbf{t}]$-algebra. We define the subordinate norm $\|.\|_\infty$ on $E[\mathbf{t}]$ by

$$\|f\|_\infty = \sup_{F \in E[\mathbf{X}]\setminus\{0\}} \frac{\|f.F\|}{\|F\|}.$$

Lemma 3.9. *Let $f \in E[\mathbf{t}]$, $f = \sum_{\mathbf{i}} f_{\mathbf{i}} \mathbf{t}^{\mathbf{i}}$. Then for all $F \in E[\mathbf{X}] \setminus \{0\}$,*

$$\|f\|_\infty = \max_{\mathbf{i}} |f_{\mathbf{i}}|_\infty = \frac{\|f.F\|}{\|F\|}.$$

Remark 3.10. The lemma says in particular that the norm $\|\cdot\|_\infty$ coincides with the Gauss norm on $E[\mathbf{t}]$, which is known to be multiplicative. This property also follows easily from the lemma.

Proof of the lemma. Write $F = \sum_{N_1,\ldots,N_s} F_{N_1,\ldots,N_s} H_{N_1}(X_1) \cdots H_{N_s}(X_s)$ and $M = \max_{\mathbf{i}} |f_{\mathbf{i}}|_\infty$. Note that for all $N \geq 1$ and for all $1 \leq i \leq s$, $t_i.H_N(X_i) = H_{qN}(X_i)$. Since $l_q(N) = l_q(qN)$, we deduce from Corollary 3.8 that $\|f.F\| \leq M\|F\|$.

Conversely, consider $(N_{1,0}, \ldots, N_{s,0})$ the index, minimal for the lexicographic ordering on $\mathbb{N}^s$, such that

$$|F_{N_{1,0},\ldots,N_{s,0}}|_\infty q^{\frac{l_q(N_{1,0})+\cdots+l_q(N_{s,0})}{q-1}} = \|F\|$$

and $\mathbf{i}_0 = (i_{1,0}, \ldots, i_{s,0})$ the index, minimal for the lexicographic ordering on $\mathbb{N}^s$, such that $M = |f_{\mathbf{i}_0}|_\infty$. Then, the coefficient of

$$H_{q^{i_{1,0}} N_{1,0}}(X_1) \cdots H_{q^{i_{s,0}} N_{s,0}}(X_s)$$

in the expansion of $f.F$ in the basis of Corollary 3.8 is equal to

$$f_{\mathbf{i}_0} F_{N_{1,0},\ldots,N_{s,0}} \; + \; \text{terms of lower norm},$$

whence the result. □

We define

- $\mathbb{B}_s^{\mathrm{lin}}(E)$ the adherence of $E[\mathbf{X}]^{\mathrm{lin}}$ in $\mathbb{B}_s(E)$,
- $\mathbb{T}_s(E)$ the completion of $E[\mathbf{t}]$ for the Gauss norm $\|.\|_\infty$.

Recall that $\mathbb{T}_s(E)$ is the standard Tate algebra in s variables over E (see [FvdP04, §II.1.]), that is, the algebra of formal series $\sum_{\mathbf{i} \in \mathbb{N}^s} f_{\mathbf{i}} \mathbf{t}^{\mathbf{i}}$ with $f_{\mathbf{i}} \in E$ going to zero with respect to the Fréchet filter. The action of $E[\mathbf{t}]$ extends naturally to an action of $\mathbb{T}_s(E)$ on $\mathbb{B}_s(E)$ and on $\mathbb{B}_s^{\mathrm{lin}}(E)$.

Lemma 3.11.

(1) *The family $\{H_{q^{n_1}}(X_1) \cdots H_{q^{n_s}}(X_s), n_1, \ldots, n_s \in \mathbb{N}\}$ forms an orthogonal basis of elements of the same norm $q^{\frac{s}{q-1}}$ of the E-Banach space $\mathbb{B}_s^{\mathrm{lin}}(E)$.*

(2) *The map $\left\{ \begin{array}{ccc} \mathbb{T}_s(E) & \to & \mathbb{B}_s(E) \\ f & \mapsto & f.(X_1 \cdots X_s) \end{array} \right.$ is injective with, for all $f \in \mathbb{T}_s(E)$,*

$$\|f.(X_1 \cdots X_s)\| = q^{\frac{s}{q-1}} \|f\|_\infty.$$

(3) *$E[\mathbf{X}]^{\mathrm{lin}} = E[\mathbf{t}].X_1 \cdots X_s$.*

(4) *$\mathbb{B}_s^{\mathrm{lin}}(E) = \mathbb{T}_s(E).X_1 \cdots X_s$.*

Proof. Since for all $1 \leq i \leq s$ and all $n \geq 0$, $H_{q^n}(X_i)$ is an $\mathbb{F}_q$-linear polynomial of degree q^n, the family $\{H_{q^{n_1}}(X_1) \cdots H_{q^{n_s}}(X_s), n_1, \ldots, n_s \in \mathbb{N}\}$ forms a basis of $E[\mathbf{X}]^{\mathrm{lin}}$ and the first assertion follows from Corollary 3.8. The relation $t_i^n.X_i = H_{q^n}(X_i)$ then implies the other assertions. □

As a consequence, the map $f \mapsto f.X_1 \cdots X_s$ defines, up to the normalisation constant $q^{\frac{s}{q-1}}$, an isometric immersion of $\mathbb{T}_s(E)$ into $\mathbb{B}_s(E)$. Writing $A[\mathbf{X}]^{\mathrm{lin}} = A[\mathbf{X}] \cap E[\mathbf{X}]^{\mathrm{lin}}$, we have the following lemma.

Lemma 3.12. *Let $f \in E[\mathbf{t}]$. Then $f.(X_1 \cdots X_s) \in A[\mathbf{X}]^{\mathrm{lin}}$ if, and only if, $f \in A[\mathbf{t}]$. In particular, $A[\mathbf{X}]^{\mathrm{lin}} = A[\mathbf{t}].X_1 \cdots X_s$.*

Proof. It is clear that if $f \in A[\mathbf{t}]$, then $f.(X_1 \cdots X_s) \in A[\mathbf{X}]$. Note that, since

$$t_1^{i_1} \cdots t_s^{i_s}.H_{N_1}(X_1) \cdots H_{N_s}(X_s) = H_{q^{i_1}N_1}(X_1) \cdots H_{q^{i_s}N_s}(X_s),$$

a consequence of Corollary 3.8 is that $\mathbb{B}_s(E)$ is a torsion-free $\mathbb{T}_s(E)$-module. Then, the converse is an easy consequence of the fact that $t_i.X_i$ is a monic polynomial in $A[X_i]$. □

4. Multivariable log-algebraicity

4.1. **The log-algebraicity theorem.** Let Z be another indeterminate over $\mathbb{C}_\infty$. We let τ act on $\mathbb{C}_\infty[\mathbf{X}][[Z]]$ (or in the one variable case on $\mathbb{C}_\infty[X][[Z]]$) via $\tau(F) = F^q$.

Let $F \in A[X]$; we form the series

$$\sum_{d \geq 0} Z^{q^d} \sum_{a \in A_{+,d}} \frac{a * F}{a} \in K[X][[Z]]$$

and take $\exp_C$ of this series which makes sense in $K[X][[Z]]$. Anderson's log-algebraicity theorem [And96, Theorem 3] for A then states the following.

Theorem 4.1 (Anderson)**.** *For all $F \in A[X]$,*

$$\exp_C \left(\sum_{d \geq 0} Z^{q^d} \sum_{a \in A_{+,d}} \frac{a * F}{a} \right) \in A[X, Z].$$

The aim of this section is to give a multivariable generalisation of this result. But first, let us give a simple proof of Theorem 4.1 in the case of $F = X$ and $Z = 1$.

Example 4.2. Write $X = \exp_C Y$, where $Y = \log_C X \in K[[X]]$. Then $a * X = \exp_C(aY) = \sum_{j \geq 0} \frac{a^{q^j} Y^{q^j}}{D_j}$. Thus,

$$\begin{aligned} \sum_{d \geq 0} Z^{q^d} \sum_{a \in A_{+,d}} \frac{a * X}{a} &= \sum_{d \geq 0} Z^{q^d} \sum_{a \in A_{+,d}} \sum_{j \geq 0} \frac{a^{q^j - 1} Y^{q^j}}{D_j} \\ &= \sum_{j \geq 0} \frac{Y^{q^j}}{D_j} \sum_{d \geq 0} Z^{q^d} \sum_{a \in A_{+,d}} a^{q^j - 1}. \end{aligned}$$

But one can evaluate at $Z = 1$ since (see [Gos96, Example 8.13.9]) $\sum_{a \in A_{+,d}} a^{q^j - 1} = 0$ for $d \gg j$, and moreover $\sum_{d \geq 0} \sum_{a \in A_{+,d}} a^{q^j - 1} = 0$ for all $j > 0$ while this sum equals 1 when $j = 0$. Therefore, we get

$$\sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a * X}{a} = Y = \log_C X.$$

Lemma 4.3. *If $F \in A[\mathbf{X}]$ satisfies $\|F\| \leq 1$, then $F \in \mathbb{F}_q$.*

Proof. If $\lambda_1, \ldots, \lambda_s \in \Lambda_C$, then $F(\lambda_1, \ldots, \lambda_s)$ is integral over A, and the condition $\|F\| \leq 1$ implies that for all $\lambda_1, \ldots, \lambda_s \in \Lambda_C$, $F(\lambda_1, \ldots, \lambda_s) \in \overline{\mathbb{F}}_q$. But $\mathbb{F}_q$ is algebraically closed in $K(\lambda_1, \ldots, \lambda_s)$(see [Ros02, Corollary to Theorem 12.14]), so that $F(\lambda_1, \ldots, \lambda_s) \in \mathbb{F}_q$. Now, for any $\lambda_1, \ldots, \lambda_{s-1} \in \Lambda_C$, the polynomial

$F(\lambda_1, \ldots, \lambda_{s-1}, X_s)$ takes at least one value infinitely many times. An easy induction on s then implies that F is constant, that is $F \in \mathbb{F}_q$. $\square$

We define an action of the multiplicative monoid A^* over $\mathbb{C}_\infty[\mathbf{X}][[Z]]$ by letting for $F(\mathbf{X}, Z) \in \mathbb{C}_\infty[\mathbf{X}][[Z]]$ and $a \in A^*$:

$$a * F = F\left(C_a(X_1), \ldots, C_a(X_s), Z^{q^{\deg_\theta a}}\right).$$

Observe that $\exp_C$ gives rise to a well-defined endomorphism of $\mathbb{C}_\infty[\mathbf{X}][[Z]]$ and that

$$\exp_C(K[\mathbf{X}][[Z]]) \subset K[\mathbf{X}][[Z]].$$

Let $F \in \mathbb{C}_\infty[\mathbf{X}]$; following Anderson, we set for $k < 0$:

$$L_k(F) = Z_k(F) = 0$$

and for $k \geq 0$,

$$\begin{aligned} L_k(F) &= \sum_{a \in A_{+,k}} \frac{a * F}{a} \in \mathbb{C}_\infty[\mathbf{X}], \\ Z_k(F) &= \sum_{j \geq 0} \frac{L_{k-j}(F)^{q^j}}{D_j} \in \mathbb{C}_\infty[\mathbf{X}]. \end{aligned}$$

Define, moreover,

$$\begin{aligned} l(F, Z) &= \sum_{a \in A_+} \frac{a * (FZ)}{a} = \sum_{k \geq 0} Z^{q^k} L_k(F) \in \mathbb{C}_\infty[\mathbf{X}][[Z]], \\ \mathfrak{L}(F, Z) &= \exp_C(l(F, Z)) = \sum_{k \geq 0} Z_k(F) Z^{q^k} \in \mathbb{C}_\infty[\mathbf{X}][[Z]]. \end{aligned}$$

Lemma 4.4. *Let $F \in \mathbb{C}_\infty[\mathbf{X}]$ and $k \geq 0$.*

(1) $\|L_k(F)\| \leq \|F\| q^{-k}$,
(2) $\|Z_k(F)\| \leq \max_{0 \leq j \leq k} \|F\|^{q^j} q^{-kq^j}$.

Proof. This comes from the definitions and the fact that for all $a \in A^*$, $\|a * F\| = \|F\|$. $\square$

We call a monic irreducible polynomial of A a *prime* of A. Let P be a prime of A. Let $F \in K[\mathbf{X}]$ and let I be a finite subset of $\mathbb{N}^s$ such that $F = \sum_{\mathbf{i} \in I} \alpha_{\mathbf{i}} \mathbf{X}^{\mathbf{i}} \in K[\mathbf{X}]$, Let v_P be the P-adic valuation on K normalized by $v_P(P) = 1$, we set

$$v_P(F) = \inf\{v_P(\alpha_{\mathbf{i}}), \mathbf{i} \in I\}.$$

Recall that we have for $F, G \in K[\mathbf{X}]$, and $\lambda \in K$:

- $v_P(F+G) \geq \inf(v_P(F), v_P(G))$, $v_P(FG) = v_P(F) + v_P(G)$,
- $v_p(\lambda F) = v_P(\lambda) + v_P(F)$,
- $v_P(F) = +\infty$ if and only if $F = 0$.

Lemma 4.5. *Let P be a prime of A. Let $F \in K[\mathbf{X}]$ be such that $v_P(F) \geq 0$. Then for all $k \geq 0$, $v_P(Z_k(F)) \geq 0$.*

Proof. The proof is essentially the same as [And96, Proposition 6]. We recall it because some details will be needed in the proof of Proposition 4.18.

Set $A_{(P)} = \{x \in K, v_P(x) \geq 0\}$. Let d be the degree of P, we have in $A[\tau]$: $C_P \equiv \tau^d \mod PA[\tau]$. We prove that if $G = \sum_{k\geq 0} G_k Z^{q^k} \in K[\mathbf{X}][[Z]]$ satisfies

$$(C_P - P*)(G) \in PA_{(P)}[\mathbf{X}][[Z]],$$

then for all $k \geq 0, G_k \in A_{(P)}[\mathbf{X}]$. Set $G_k = 0$ if $k < 0$ and write $C_P = \sum_{i=0}^d [P]_i \tau^i$, where $[P]_0 = P$, $[P]_i \in PA$ if $i < d$ and $[P]_d = 1$. We have $(C_P - P*)(G) = \sum_{k\geq 0} H_k Z^{q^k}$ with for all $k \geq 0$,

$$\begin{aligned} H_k &= \sum_{i=0}^{d} [P]_i \tau^i (G_{k-i}) - P * G_{k-d} \\ &= PG_k + \sum_{i=1}^{d-1} [P]_i \tau^i (G_{k-i}) + \tau^d (G_{k-d}) - P * G_{k-d} \in PA_{(P)}[\mathbf{X}]. \end{aligned}$$

In particular, $H_0 = PG_0 \in PA_{(P)}[\mathbf{X}]$ so that $G_0 \in A_{(P)}[\mathbf{X}]$. Now, by induction on k, if we know that $G_{k-i} \in A_{(P)}[\mathbf{X}]$ for $i = 1, \ldots, d$, then $\tau^d(G_{k-d}) - P * G_{k-d} \in PA_{(P)}[\mathbf{X}]$ and we deduce that $G_k \in PA_{(P)}[\mathbf{X}]$.

Defining $l^*(F, Z) = \sum_{a\in A_+, P\nmid a} \frac{a*(FZ)}{a} \in A_{(P)}[\mathbf{X}][[Z]]$, we have

$$\begin{aligned} l(F,Z) &= \sum_{a\in A_+, P|a} \frac{a*(FZ)}{a} + \sum_{a\in A_+, P\nmid a} \frac{a*(FZ)}{a} \\ &= \sum_{a\in A_+} \frac{(aP)*(FZ)}{aP} + l^*(F,Z) = \frac{1}{P}\left(P * l(F,Z)\right) + l^*(F,Z) \end{aligned}$$

which yields the relation

$$Pl(F,Z) - P * l(F,Z) = Pl^*(F,Z).$$

Note that the action $*$ commutes with τ, and thus with $\exp_C$, thus if we apply $\exp_C$, we get

$$(C_P - P*)\left(\mathfrak{L}(F,Z)\right) = \exp_C(Pl^*(F,Z)) = \sum_{j\geq 0} \frac{P^{q^j}}{D_j} l^*(F,Z)^{q^j}$$

and since for all $j \geq 0$, $v_P(\frac{P^{q^j}}{D_j}) \geq 1$, we get $(C_P - P*)\left(\mathfrak{L}(F,Z)\right) \in PA_{(P)}[\mathbf{X}][[Z]]$ whence $\mathfrak{L}(F,Z) \in A_{(P)}[\mathbf{X}][[Z]]$. $\square$

We can now state and prove the multivariable log-algebraicity theorem.

Theorem 4.6. *Let $F \in A[\mathbf{X}]$. Then*

$$\mathfrak{L}(F,Z) = \exp_C\left(\sum_{a\in A_+} \frac{a*(FZ)}{a}\right) \in A[\mathbf{X}, Z].$$

Proof. By Lemma 4.5, for all $k \geq 0$, $Z_k(F) \in A[\mathbf{X}]$. If $k_0 \geq 0$ is the smallest integer such that $\|F\| \leq q^{k_0}$, then by Lemma 4.4, for all $k > k_0$, $\|Z_k(F)\| < 1$. Therefore, Lemma 4.3 shows that $Z_{k_0}(F) \in \mathbb{F}_q$ and for all $k > k_0$, $Z_k(F) = 0$. $\square$

The previous theorem can also be obtained as a consequence of a class formula for a Drinfeld module on a Tate algebra (see [APTR16]).

4.2. **The special polynomials.** If $s \geq 1$ is an integer, we define the special polynomial:

$$\mathbb{S}_s = \mathbb{S}_s(\mathbf{X}, Z) = \mathfrak{L}(X_1 \cdots X_s, Z) \in A[\mathbf{X}, Z].$$

Let us recall that Anderson's special polynomials are the one variable polynomials $S_m(X, Z) = \mathfrak{L}(X^m, Z)$. We recover $S_m(X, Z)$ from $\mathbb{S}_m$ by specializing each of the $X_j, 1 \leq j \leq m$, to X. We establish in this section some properties of the polynomials $\mathbb{S}_s$.

The following proposition is used to compute explicitly the polynomial $\mathfrak{L}(F, Z)$.

Proposition 4.7.

(1) *The polynomial $\mathbb{S}_s(\mathbf{X}, Z)$ is $\mathbb{F}_q$-linear with respect to each of the variables $X_1, \ldots, X_s, Z$; in particular, $\mathbb{S}_s(\mathbf{X}, Z)$ is divisible by $X_1 \cdots X_s Z$.*

(2) *If $r \in \{1, \ldots, q-1\}$ satisfies $s \equiv r \mod q-1$, then*

$$\deg_Z \mathbb{S}_s \leq q^{\frac{s-r}{q-1}}.$$

In particular, if $1 \leq s \leq q-1$, we have

$$\mathbb{S}_s = X_1 \cdots X_s Z.$$

Proof. The first assertion is obvious. By Lemmas 4.3, 4.4 and 4.5, $Z_k(X_1 \cdots X_s) \in \mathbb{F}_q$ if $k \geq \frac{s}{q-1}$. But since $X_1 \cdots X_s$ divides $Z_k(X_1 \cdots X_s)$, we get $Z_k(X_1 \cdots X_s) = 0$ for $k \geq \frac{s}{q-1}$. The last part comes from the congruence

$$\mathbb{S}_s \equiv X_1 \cdots X_s Z \mod Z^q.$$

□

Corollary 4.8. *Let $s, k_1, \ldots, k_s \geq 1$ be integers such that $\sum_{j=1}^s k_j \leq q-1$ and let $a_{1,1}, \ldots, a_{1,k_1}, \ldots, a_{s,1}, \ldots, a_{s,k_s} \in A$. Set*

$$G = (a_{1,1} * X_1) \cdots (a_{1,k_1} * X_1) \cdots (a_{s,1} * X_s) \cdots (a_{s,k_s} * X_s) \in A[\mathbf{X}].$$

Then

$$\mathfrak{L}(G, Z) = GZ.$$

Proof. It is sufficient to consider the case where $k_j = 1$ for all $1 \leq j \leq s$ since we obtain the general case by specializing variables. The action $*_j$ of A (defined in (3.2)) satisfies for all $a \in A$, $F \in A[\mathbf{X}]$,

$$a *_j (\mathfrak{L}(F, Z)) = \mathfrak{L}(a *_j F, Z).$$

The corollary follows then from the relation $\mathfrak{L}(X_1 \cdots X_s, Z) = X_1 \cdots X_s Z$ since $s \leq q-1$. □

Any $\mathbb{F}_q$-linear combination F of polynomials of the above form still satisfies the equality $\mathfrak{L}(F, Z) = FZ$. We can ask whether there are other polynomials satisfying this relation. In fact, Proposition 4.19 below assures that if $\mathfrak{L}(F, Z) = FZ$, then $F \in A[\mathbf{X}]$, so we can ask more generally the following question.

Question 4.9. Describe the set of the $F \in A[\mathbf{X}]$ such that $\mathfrak{L}(F, Z) = FZ$.

Lemma 4.10. *Let $s \geq 1$. Then $\mathbb{S}_s(\mathbf{X}, 1) = 0$ if, and only if, $s \geq 2$ and $s \equiv 1 \mod q-1$.*

Proof. First, suppose $s \geq 2$ and $s \equiv 1 \mod q-1$.

Let $a \in A$. Recall (see [AT90, Section 3.4]) that from the relation $C_a(X) = \exp_C(a \log_C(X))$, we deduce that we can write

$$C_a(X) = \sum_{k=0}^{\deg_\theta a} \psi_k(a) X^{q^k},$$

where $\psi_k(x) \in A[x]$ is an $\mathbb{F}_q$-linear polynomial of degree q^k, which vanishes exactly at the polynomials $x \in A$ of degree less than k. Thus

$$a * (X_1 \cdots X_s) = \sum_{k_1,\ldots,k_s \geq 0} \psi_{k_1}(a) \cdots \psi_{k_s}(a) X_1^{q^{k_1}} \cdots X_s^{q^{k_s}},$$

where the right-hand side is a finite sum.

We deduce that $\mathbb{S}_s(\mathbf{X}, Z)$ is equal to

$$\sum_{n\geq 0} Z^{q^n} \sum_{d=0}^{n} D_{n-d}^{-1} \sum_{k_1,\ldots,k_s\geq 0} \sum_{a\in A_{+,d}} \left(\frac{\psi_{k_1}(a)\cdots\psi_{k_s}(a)}{a}\right)^{q^{n-d}} X_1^{q^{k_1+n-d}} \cdots X_s^{q^{k_s+n-d}}$$

and by Theorem 4.6, this is a polynomial. Now, note that $\sum_{a\in A_{+,d}} \frac{\psi_{k_1}(a)\cdots\psi_{k_s}(a)}{a}$ is a linear combination (with coefficients depending only on $k_1, \ldots, k_s, r_1, \ldots, r_s$ and independent on d) of sums of the form $\sum_{a\in A_{+,d}} a^{q^{r_1}+\cdots+q^{r_s}-1}$ with, for all $1 \leq j \leq s$, $0 \leq r_j \leq k_j$. According to [Gos96, Lemma 8.8.1], this sum vanishes for $d > \frac{q^{r_1}+\cdots+q^{r_s}-1}{q-1}$. Thus the coefficient of $X_1^{q^{m_1}} \cdots X_s^{q^{m_s}}$ in $\mathbb{S}_s(\mathbf{X}, 1)$ is a linear combination of (finite) sums of the form $\sum_{a\in A_+} a^{q^d(q^{r_1}+\cdots+q^{r_s}-1)}$. But since $s \equiv 1 \mod q-1$, $q^d(q^{r_1}+\cdots+q^{r_s}-1) \equiv 0 \mod q-1$, and since $s \geq 2$, $q^d(q^{r_1}+\cdots+q^{r_s}-1) \neq 0$. Thus, by [Gos96, Example 8.13.9], all the sums $\sum_{a\in A_+} a^{q^d(q^{r_1}+\cdots+q^{r_s}-1)}$ vanish, that is, $\mathbb{S}_s(\mathbf{X}, 1) = 0$.

Conversely, the coefficient of $X_1 \cdots X_s$ in $\mathbb{S}_s(\mathbf{X}, 1)$ is $\sum_{a\in A_+} a^{s-1}$ which is congruent to 1 modulo $\theta^q - \theta$ if $s = 1$ or $s \not\equiv 1 \mod q-1$, so $\mathbb{S}_s(\mathbf{X}, 1)$ does not vanish. □

Remark 4.11. Thakur used similar arguments in [Tha04, §8.10] to obtain explicit log-algebraicity formulas.

Example 4.12. We already know that $\mathbb{S}_s(\mathbf{X}, Z) = X_1 \cdots X_s Z$ if $1 \leq s \leq q-1$. Using Proposition 4.7 and Lemma 4.10, we easily see that

$$\mathbb{S}_q(\mathbf{X}, Z) = X_1 \cdots X_q Z - X_1 \cdots X_q Z^q.$$

For $q \geq 3$, a computation leads to

$$\mathbb{S}_{q+1}(\mathbf{X}, Z) = X_1 \cdots X_{q+1} Z - X_1 \cdots X_{q+1}(X_1^{q-1} + \cdots + X_{q+1}^{q-1}) Z^q.$$

Lemma 4.13. *Let $s \geq 1$,*

(1) *for all integer $k \geq \frac{s}{q-1}$, the sum $\sum_{a\in A_{+,k}} a(t_1)\cdots a(t_{s-1})$ vanishes, so that $L(0, s-1) = \sum_{k\geq 0} \sum_{a\in A_{+,k}} a(t_1)\cdots a(t_{s-1}) \in \mathbb{F}_q[\mathbf{t}]$,*

(2) *$\mathbb{S}_s(\mathbf{X}, 1) \equiv (L(0, s-1).(X_1 \cdots X_{s-1})) X_s \mod X_s^q$.*

Proof. For all $k \geq 0$, $L_k(X_1 \cdots X_s) = \sum_{a\in A_{+,k}} \frac{a*(X_1\cdots X_s)}{a}$ can be viewed as a polynomial in X_s, with no constant term, and since $C_a(X_s) \equiv aX_s \mod X_s^q$, we have

$$\begin{aligned} Z_k(X_1\cdots X_s) &\equiv L_k(X_1\cdots X_s) \equiv \sum_{a\in A_{+,k}} \frac{a*(X_1\cdots X_s)}{a} \mod X_s^q \\ &\equiv \sum_{a\in A_{+,k}} a*(X_1\cdots X_{s-1})\frac{aX_s}{a} \mod X_s^q \\ &\equiv X_s\sum_{a\in A_{+,k}} a*(X_1\cdots X_{s-1}) \mod X_s^q. \end{aligned}$$

But Proposition 4.7 tells that $Z_k(X_1\cdots X_s)=0$ if $k\geq \frac{s}{q-1}$. Now note that

$$\sum_{a\in A_{+,k}} a*(X_1\cdots X_{s-1}) = \sum_{a\in A_{+,k}} a(t_1)\cdots a(t_{s-1}).(X_1\cdots X_{s-1}).$$

We deduce then the first point from Lemma 3.11 and the evaluation at $Z=1$,

$$L(0,s-1).(X_1\cdots X_{s-1})X_s \equiv \sum_{k\geq 0} Z_k(X_1\cdots X_s) \equiv \mathbb{S}_s(X,1) \mod X_s^q$$

gives the second point. □

Note that the first point of the above lemma is also a consequence of [Gos96, Lemma 8.8.1] (see also [AP15, Lemma 30] and [AP14, Lemma 4]).

Lemma 4.14. *Let $s\geq 1$. If there exists $b,c\in A$ and $r\in X_s\mathbb{C}_\infty[\mathbf{X}]$ such that*

$$C_b(r)=C_c(\mathbb{S}_s(\mathbf{X},1)),$$

then b divides c in A and $r=C_{\frac{c}{b}}(\mathbb{S}_s(\mathbf{X},1))$.

Proof. We first prove that r has coefficients in A. We will use the fact that $\exp_C$ and $\log_C$ define reciprocal bijections of $X_sK[X_1,\dots,X_{s-1}][[X_s]]$ satisfying for all $F\in X_sK[X_1,\dots,X_{s-1}][[X_s]]$ and $a\in A$, $\log_C(C_a(F))=a\log_C(F)$ and $\exp_C(aF)=C_a(\exp_C(F))$. Thus $C_b(r)=\exp_C(b\log_C(r))$ and $C_c(\mathbb{S}_s(\mathbf{X},1))=\exp_C(c\log_C(\mathbb{S}_s(\mathbf{X},1)))$. We deduce that $r=\exp_C(\frac{c}{b}\mathbb{S}_s(\mathbf{X},1))\in X_sK[\mathbf{X}]$. But $C_b(X)$ is monic up to a unit in $\mathbb{F}_q^*$, and $A[\mathbf{X}]$ is integrally closed. Thus the fact that $C_b(r)\in A[\mathbf{X}]$ implies that $r\in A[\mathbf{X}]$.

Write now $r\equiv X_sr_1 \mod X_s^2$ with $r_1\in A[X_1,\dots,X_{s-1}]$. Then $C_b(r)\equiv X_sbr_1 \mod X_s^2$ and by Lemma 4.13,

$$C_c(\mathbb{S}_s(\mathbf{X},1))\equiv cX_s\left(L(0,s-1).(X_1\cdots X_{s-1})\right) \mod X_s^2,$$

thus $r_1=\frac{c}{b}L(0,s-1).(X_1\cdots X_{s-1})$. Since $r_1\in A[X_1,\dots,X_{s-1}]$, Lemma 3.12 assures that $\frac{c}{b}L(0,s-1)\in A[t_1,\dots,t_{s-1}]$. But $L(0,s-1)\in\mathbb{F}_q[t_1,\dots,t_{s-1}]$. We obtain that b divides c in A and that $r=\exp_C(\frac{c}{b}\mathbb{S}_s(\mathbf{X},1))=C_{\frac{c}{b}}(\mathbb{S}_s(\mathbf{X},1))$. □

Set $\mathfrak{R}=\bigcup_{s\geq 1}\mathbb{C}_\infty[X_1,\dots,X_s]$ and let $\mathfrak{F}$ be the sub-A-module of $C(\mathfrak{R})$ generated by the polynomials $\mathbb{S}_s(X_1,\dots,X_s,1)$, $s\geq 1$. Set

$$\sqrt{\mathfrak{F}}=\{r\in\mathfrak{R},\exists a\in A^*,C_a(r)\in\mathfrak{F}\}.$$

Theorem 4.15.

$$\sqrt{\mathfrak{F}}=\mathfrak{F}+C(\Lambda_C).$$

Proof. The inclusion $\mathfrak{F} + C(\Lambda_C) \subset \sqrt{\mathfrak{F}}$ is clear.

Let $r \in \sqrt{\mathfrak{F}}$. Then there exists $n \geq 1$ such that $r \in \mathbb{C}_\infty[X_1, \ldots, X_n]$ and there exist $a \in A^*$, $a_1, \ldots, a_n \in A$ such that

$$C_a(r) = \sum_{m=1}^{n} C_{a_m}(\mathbb{S}_m(X_1, \ldots, X_m, 1)). \tag{4.1}$$

We now prove by induction on $n \geq 1$ that $r \in \mathfrak{F} + C(\Lambda_C)$.

In the case $n = 1$, equation (4.1) reduces to

$$C_a(r) = C_c(\mathbb{S}_1(X_1, 1))$$

with $c \in A$ and $r \in \mathbb{C}_\infty[X_1]$. The constant term of r is then in $C(\Lambda_C)$ and we can therefore assume $r \in X_1\mathbb{C}_\infty[X_1]$. The result in this case is then just the one of Lemma 4.14.

We suppose now $n > 1$ and that the result is proved for all $k \leq n - 1$. We can assume that $a_n \neq 0$ and $\mathbb{S}_n(X_1, \ldots, X_n, 1) \neq 0$, that is, $n \not\equiv 1 \mod q - 1$. Write $r = \sum_{i=0}^{d} r_i(X_1, \ldots, X_{n-1})X_n^i$, with $d > 0$. Then equation (4.1) evaluated at $X_n = 0$ yields

$$C_a(r_0(X_1, \ldots, X_{n-1})) = \sum_{m=1}^{n-1} C_{a_m}\mathbb{S}_m(X_1, \ldots, X_m, 1)$$

and the induction hypothesis assures that $r_0(X_1, \ldots, X_{n-1}) \in \mathfrak{F} + C(\Lambda_C)$. Thus we can assume $r_0 = 0$ and, for some $c \in A$,

$$C_a(r) = C_c(\mathbb{S}_n(X_1, \ldots, X_n, 1)).$$

Again, we are reduced to the result proved in Lemma 4.14. □

4.3. **Converses of the** log**-algebraicity theorem.** Let $\overline{A}$ be the integral closure of A in $\overline{K}$. The log-algebraicity theorem asserts that if $F \in A[\mathbf{X}]$, then $\mathfrak{L}(F, Z) \in A[\mathbf{X}, Z]$. We will prove in this section conversely that, if $F \in \mathbb{C}_\infty[\mathbf{X}]$ and $\mathfrak{L}(F, Z)$ belongs to $\mathbb{C}_\infty[\mathbf{X}, Z]$ or to $\overline{A}[\mathbf{X}][[Z]] \otimes_A K$, then necessarily, $F \in A[\mathbf{X}]$.

If P is a prime of A, $\overline{A}_{(P)}$ denotes the ring of elements of $\overline{K}$ that are P-integral.

Lemma 4.16. *Let $x \in A$ such that for infinitely many primes P,*

$$x^{q^d} \equiv x \pmod{P^p},$$

where d is the degree of P. Then $x \in A^p$.

Proof. Let $F \in A \setminus A^p$, then $F' \neq 0$, where F' denotes the derivative of F with respect to the variable θ. Then $F^{q^d} - F \equiv (\theta^{q^d} - \theta)F' \mod P^2$, so that for all primes P not dividing F', $v_P(F^{q^d} - F) = 1$. □

Lemma 4.17.

(1) *Let $\alpha \in \overline{A}$ such that for all but finitely many primes P of A,*

$$\alpha^{q^d} \equiv \alpha \pmod{P\overline{A}},$$

where d is the degree of P. Then $\alpha \in A$.

(2) *Let $\alpha \in \overline{K}$ such that for all but finitely many primes P of A,*

$$\alpha^{q^d} \equiv \alpha \pmod{P\overline{A}_{(P)}},$$

where d is the degree of P. Then $\alpha \in K$.

Proof.

(1) First we assume that α is separable over K. Set $F = K(\alpha)$ and let O_F be the integral closure of A in F. For a prime P not dividing the discriminant of $A[\alpha]$, we have:

$$O_F \otimes_A A_P = A[\alpha] \otimes_A A_P,$$

where A_P is the P-adic completion of A. Therefore, for all but finitely many primes P, we have:

$$\forall x \in O_F,\, x^{q^d} \equiv x \pmod{PO_F}.$$

This implies that all but finitely many primes P of A are totally split in F. By the Čebotarev density theorem (see for example [Neu99, Chapter VII, Section 13]), this implies that $F = K$ and thus $\alpha \in A$.

In general there exists a minimal integer $m \geq 0$ such that α^{p^m} is separable over K. If $m \geq 1$, then $x = \alpha^{p^m} \in A$ and for all but finitely many primes P of A:

$$x^{q^d} \equiv x \pmod{P^{p^m} A}.$$

Therefore $\alpha^{p^{m-1}} \in A$ by Lemma 4.16. We deduce that $\alpha \in A$.

(2) Let $b \in A \setminus \{0\}$ such that $x = b\alpha \in \overline{A}$. Then by the first assertion of the lemma, $x \in A$. Therefore $\alpha \in K$. □

Proposition 4.18. *For all $s \geq 1$, if $\mathbf{X} = (X_1, \ldots, X_s)$, then*

$$\left\{F \in \mathbb{C}_\infty[\mathbf{X}]; \mathfrak{L}(F,Z) \in \overline{A}[\mathbf{X}][[Z]] \otimes_A K\right\} = A[\mathbf{X}].$$

Proof. Let $F \in \mathbb{C}_\infty[\mathbf{X}]$ such that $\mathfrak{L}(F,Z) \in \overline{A}[\mathbf{X}][[Z]] \otimes_A K$, *i.e.*, there exists $b \in A \setminus \{0\}$ such that $b\mathfrak{L}(F,Z) \in \overline{A}[\mathbf{X}][[Z]]$. Since $\mathfrak{L}(F,Z) \equiv FZ \pmod{Z^q}$, we get $F \in \overline{K}[\mathbf{X}]$. Let P be a prime of A of degree d not dividing b. Then by the proof of Lemma 4.5,

$$\mathfrak{L}(F,Z) \in \overline{A}_{(P)}[\mathbf{X}][[Z]] \quad \text{and} \quad (C_P - P*)\left(\mathfrak{L}(F,Z)\right) \in P\overline{A}_{(P)}[\mathbf{X}][[Z]]$$

and since $C_P \equiv \tau^d \mod PA[\tau]$, the coefficient of Z^{q^d} in $(C_P - P*)\left(\mathfrak{L}(F,Z)\right)$ is congruent to $F^{q^d} - P * F \mod P\overline{A}_{(P)}[\mathbf{X}][[Z]]$. Therefore

$$F(X_1^{q^d}, \ldots, X_s^{q^d}) \equiv F^{q^d} \mod P\overline{A}_{(P)}[\mathbf{X}].$$

Thus, by Lemma 4.17, we get $F \in K[\mathbf{X}]$. Now select $c \in A \setminus \{0\}$ such that $cF \in A[\mathbf{X}]$. Then by Theorem 4.6:

$$C_c(\mathfrak{L}(F,Z)) \in A[X_1, \ldots, X_s, Z].$$

Therefore $\mathfrak{L}(F,Z) \in A[\mathbf{X}][[Z]] \otimes_A K$ is integral over $A[\mathbf{X}][[Z]]$. But $A[\mathbf{X}][[Z]]$ is integrally closed (see[Bou64, Chapitre 5, Proposition 14]) thus $\mathfrak{L}(F,Z) \in A[\mathbf{X}][[Z]]$ and this implies that $F \in A[\mathbf{X}]$ since $\mathfrak{L}(F,Z) \equiv FZ \mod Z^q$. We then have the direct inclusion, the equality follows by Theorem 4.6. □

We remark that if we only suppose that $\mathfrak{L}(F,Z) \in \overline{K}[\mathbf{X}][[Z]]$, then the result no longer holds, for instance $F = \frac{X}{\theta} \in K[X] \setminus A[X]$ and $\mathfrak{L}(F,Z) \in K[X][[Z]]$. Note that the above proposition implies that $\mathfrak{L}^{-1}(\overline{K}[\mathbf{X}, Z]) = A[\mathbf{X}]$. In fact we have:

Proposition 4.19.

$$\{F \in \mathbb{C}_\infty[\mathbf{X}]; \mathfrak{L}(F,Z) \in \mathbb{C}_\infty[\mathbf{X}, Z]\} = A[\mathbf{X}].$$

Proof. Recall that if $\mathbf{i} = (i_1, \dots, i_s) \in \mathbb{N}^s$, then $\mathbf{X}^{\mathbf{i}} = X_1^{i_1} \cdots X_s^{i_s}$. If $F \in \mathbb{C}_\infty[\mathbf{X}]$, write $F = \sum_{\mathbf{i}} \alpha_{\mathbf{i}} \mathbf{X}^{\mathbf{i}}$ and define $\mathbf{deg}(F) \in \mathbb{N}_+^s \cup \{\pm\infty\}$ to be the maximum for the lexicographic ordering of the exponents $\mathbf{i}$ such that $\alpha_{\mathbf{i}} \neq 0$. Now, if $F = \sum_{k \geq 1} F_k Z^k \in Z\mathbb{C}_\infty[\mathbf{X}][[Z]]$, where for all k, $F_k \in \mathbb{C}_\infty[\mathbf{X}]$, then we define the relative degree of F, $\mathrm{rdeg}(F) \in \mathbb{R}_+^s \cup \{\pm\infty\}$, to be

$$\mathrm{rdeg}(F) = \begin{cases} -\infty & \text{if } F = 0, \\ \sup_{k \geq 1} \left(\dfrac{\mathbf{deg}(F_k)}{k} \right) \in \mathbb{R}_+^s \cup \{+\infty\} & \text{otherwise,} \end{cases}$$

where the supremum is still relative to the lexicographic ordering on $\mathbb{R}_+^s$ and is well defined in $\mathbb{R}_+^s \cup \{+\infty\}$. Note the following properties of rdeg: if $F, G \in Z\mathbb{C}_\infty[\mathbf{X}][[Z]]$ and ψ is an $\mathbb{F}_q$-linear power series in $\mathbb{C}_\infty[[T]]$, then

- $\mathrm{rdeg}(F + G) \leq \max(\mathrm{rdeg}(F), \mathrm{rdeg}(G))$ with equality if $\mathrm{rdeg}(F) \neq \mathrm{rdeg}(G)$,
- $\mathrm{rdeg}(F^q) = \mathrm{rdeg}(F)$,
- $\mathrm{rdeg}(\psi(F)) \leq \mathrm{rdeg}(F)$,
- if $\psi \neq 0$, for $k \geq 1$ and $\mathbf{i} \in \mathbb{N}^s$, $\mathrm{rdeg}(\psi(\mathbf{X}^{\mathbf{i}} Z^k)) = \frac{\mathbf{i}}{k}$,
- if $F = \sum_{k \geq 1} F_k Z^k$ is such that there exists infinitely many indices k_j such that $\mathbf{deg}(F_{k_j}) = k_j \,\mathrm{rdeg}(F)$ (in particular $F \notin A[\mathbf{X}, Z]$) and $\mathrm{rdeg}(F) > \mathrm{rdeg}(G)$, then $F + G \notin A[\mathbf{X}, Z]$.

For the last property, if we write $G = \sum_{k \geq 1} G_k Z^k$, then $F + G = \sum_{k \geq 1} (F_k + G_k) Z^k$ with for all j, $\mathbf{deg}(F_{k_j} + G_{k_j}) = k_j \,\mathrm{rdeg}(F)$ so that $F_{k_j} + G_{k_j} \neq 0$ and $F + G \notin A[\mathbf{X}, Z]$.

Now let $\mathbf{i} \in \mathbb{N}^s$. For $k \geq 0$, we have

$$L_k(\mathbf{X}^{\mathbf{i}}) = \frac{\mathbf{X}^{q^k \mathbf{i}}}{l_k} + G_{k,\mathbf{i}},$$

where $G_{k,\mathbf{i}} \in K[\mathbf{X}]$ satisfies $\mathbf{deg}(G_{k,\mathbf{i}}) < q^k \mathbf{i}$. Thus

$$\mathfrak{L}(\mathbf{X}^{\mathbf{i}}, Z) = \mathbf{X}^{\mathbf{i}} Z + F_{\mathbf{i}}, \tag{4.2}$$

where $F_{\mathbf{i}} \in Z^q \mathbb{C}_\infty[\mathbf{X}, Z]$ has relative degree $\mathrm{rdeg}(F_{\mathbf{i}}) < \mathbf{i}$.

Fix $\alpha \in \mathbb{C}_\infty$; then

$$C_\alpha(T) = \exp_C(\alpha \log_C(T)) \in \mathbb{C}_\infty[[T]]$$

is an $\mathbb{F}_q$-linear power series, and $\mathbb{C}_\alpha(T) \in \mathbb{C}_\infty[T]$ if and only if $\alpha \in A$ (see [Gos96, Chapter 3]).

Now let $F \in \mathbb{C}_\infty[\mathbf{X}] \backslash A[\mathbf{X}]$; we want to prove that $\mathfrak{L}(F, Z) \notin \mathbb{C}_\infty[\mathbf{X}, Z]$. By Theorem 4.6, we can suppose $F = \sum_{\mathbf{i}} \alpha_{\mathbf{i}} \mathbf{X}^{\mathbf{i}}$ with for all $\mathbf{i}$ such that $\alpha_{\mathbf{i}} \neq 0$, $\alpha_{\mathbf{i}} \notin A$. Then equation (4.2) gives

$$\mathfrak{L}(F, Z) = \sum_{\mathbf{i}} C_{\alpha_{\mathbf{i}}} \left(\mathfrak{L}(\mathbf{X}^{\mathbf{i}}, Z) \right) = \sum_{\mathbf{i}} C_{\alpha_{\mathbf{i}}} \left(\mathbf{X}^{\mathbf{i}} Z \right) + C_{\alpha_{\mathbf{i}}} \left(F_{\mathbf{i}} \right).$$

If $\mathbf{i}_0 = \mathbf{deg}(F)$, then we deduce that

$$\mathfrak{L}(F, Z) = C_{\alpha_{\mathbf{i}_0}} \left(\mathbf{X}^{\mathbf{i}_0} Z \right) + G$$

with $\mathrm{rdeg}(G) < \mathbf{i}_0$. Since $C_{\alpha_{\mathbf{i}_0}}(\mathbf{X}^{\mathbf{i}_0} Z) \notin \mathbb{C}_\infty[\mathbf{X}, Z]$ has infinitely many terms of relative degree $\mathbf{i}_0$, we have $\mathfrak{L}(F, Z) \notin \mathbb{C}_\infty[\mathbf{X}, Z]$. ☐

5. Multivariable L-functions

5.1. **Frobenius actions.** Let $K_\infty \subseteq E \subseteq \mathbb{C}_\infty$ be a field complete with respect to $|\cdot|_\infty$. Observe that if $n \geq 0$ and $1 \leq i \leq s$, then

$$H_{q^n}(X_i)^q = C_{\theta^n}(X_i)^q = C_{\theta^{n+1}}(X_i) - \theta C_{\theta^n}(X_i) = (t_i - \theta).H_{q^n}(X_i).$$

Thus we define the following action of τ on $\mathbb{T}_s(E)$:

$$\forall f = \sum_{\mathbf{i}} f_{\mathbf{i}} \mathbf{t}^{\mathbf{i}} \in \mathbb{T}_s(E), \quad \tau(f) = (t_1 - \theta) \cdots (t_s - \theta) \sum_{\mathbf{i}} f_{\mathbf{i}}^q \mathbf{t}^{\mathbf{i}}$$

and we get for all $f \in \mathbb{T}_s(E)$ the equality in $\mathbb{B}_s^{\mathrm{lin}}(E)$:

$$\tau(f.(X_1 \cdots X_s)) = \tau(f).(X_1 \cdots X_s).$$

We then define on $\mathbb{T}_s(E)$ the operator φ which will be a Frobenius acting only on coefficients, namely:

$$\forall f = \sum_{\mathbf{i}} f_{\mathbf{i}} \mathbf{t}^{\mathbf{i}} \in \mathbb{T}_s(E), \quad \varphi(f) = \sum_{\mathbf{i}} f_{\mathbf{i}}^q \mathbf{t}^{\mathbf{i}},$$

so that on $\mathbb{T}_s(E)$, we have $\tau = (t_1 - \theta) \cdots (t_s - \theta)\varphi$. Moreover, for $d \geq 1$, if we define, $b_d(t) = (t - \theta)(t - \theta^q) \cdots (t - \theta^{q^{d-1}})$, then for all $f \in \mathbb{T}_s(E)$,

$$\tau^d(f) = b_d(t_1) \cdots b_d(t_s)\varphi^d(f).$$

We also set $b_0(t) = 1$ so that the above relation still holds for $d = 0$. Note that for all $f, g \in \mathbb{T}_s(E)$, and $d \geq 0$,

$$\tau^d(fg) = \tau^d(f)\varphi^d(g).$$

Observe moreover that

$$\forall f \in \mathbb{T}_s(E),\ \forall d \geq 0, \quad \|\varphi^d(f)\|_\infty = \|f\|_\infty^{q^d} \text{ and } \|\tau^d(f)\|_\infty = q^{s\frac{q^d-1}{q-1}} \|f\|_\infty^{q^d}.$$

We deduce that $\exp_C = \sum_{j\geq 0} \frac{1}{D_j}\tau^j$ is defined on $\mathbb{T}_s(E)$ and that for all $f \in \mathbb{T}_s(E)$, we have in $\mathbb{B}_s^{\mathrm{lin}}(E)$:

$$\exp_C(f.(X_1 \cdots X_s)) = \exp_C(f).(X_1 \cdots X_s). \tag{5.1}$$

We now extend the action of $E[\mathbf{t}]$ on $E[\mathbf{X}]$ to an action of $E[\mathbf{t}][[z]]$ on $E[\mathbf{X}][[Z]]$ via

$$\left(\sum_{k\geq 0} f_k(\mathbf{t})z^k\right) . \left(\sum_{n\geq 0} F_n(\mathbf{X})Z^n\right) = \sum_{k\geq 0}\sum_{n\geq 0} \left(f_k(\mathbf{t}).F_n(\mathbf{X})\right) Z^{nq^k}$$

and we let τ act on Z via $\tau(Z) = Z^q$. Since $\tau(Z) = z.Z$, we define on $E[\mathbf{t}][[z]]$ the operator τ_z by, for all $f = \sum_{k\geq 0}\sum_{\mathbf{i}} f_{\mathbf{i}}\mathbf{t}^{\mathbf{i}}z^k \in E[\mathbf{t}][[z]]$,

$$\tau_z(f) = z(t_1 - \theta) \cdots (t_s - \theta) \sum_{k\geq 0}\sum_{\mathbf{i}} f_{\mathbf{i}}^q \mathbf{t}^{\mathbf{i}} z^k = \sum_{k\geq 0} z^{k+1}\tau\left(\sum_{\mathbf{i}} f_{\mathbf{i}}^q \mathbf{t}^{\mathbf{i}}\right).$$

Thus if we extend φ by

$$\forall f = \sum_{k\geq 0}\sum_{\mathbf{i}} f_{\mathbf{i}}\mathbf{t}^{\mathbf{i}}z^k \in E[\mathbf{t}][[z]], \quad \varphi(f) = \sum_{k\geq 0}\sum_{\mathbf{i}} f_{\mathbf{i}}^q \mathbf{t}^{\mathbf{i}} z^k,$$

we get for all $f = \sum_{k\geq 0} f_k z^k \in E[\mathbf{t}][[z]]$ and $d \geq 0$, $\tau_z^d(f) = z^d b_d(t_1) \cdots b_d(t_s)\varphi^d(f)$. By construction, if $f = E[\mathbf{t}][[z]]$, then $f.(X_1 \cdots X_s Z) \in E[\mathbf{X}][[Z]]$ and for all $d \geq 0$,

$$\tau^d(f.(X_1 \cdots X_s Z)) = \tau_z^d(f).(X_1 \cdots X_s Z).$$

We then have an operator $\exp_C = \sum_{j\geq 0} \frac{1}{D_j}\tau^j$ on $ZE[\mathbf{X}][[Z]]$ and an operator $\exp_z = \sum_{j\geq 0} \frac{1}{D_j}\tau_z^j$ on $E[\mathbf{t}][[z]]$ such that for all $f \in E[\mathbf{t}][[z]]$,

$$\exp_C(f.(X_1\cdots X_s Z)) = \exp_z(f).(X_1\cdots X_s Z). \tag{5.2}$$

A similar property holds for $\log_C = \sum_{j\geq 0} \frac{1}{l_j}\tau^j$ and $\log_z = \sum_{j\geq 0} \frac{1}{l_j}\tau_z^j$:

$$\log_C(f.(X_1\cdots X_s Z)) = \log_z(f).(X_1\cdots X_s Z), \tag{5.3}$$

where $\log_z$ and $\exp_z$ define the reciprocal bijection of $E[\mathbf{t}][[z]]$.

We now state compatibility results for evaluations at $Z = 1$ and $z = 1$.

Lemma 5.1. *Let $F(\mathbf{X}, Z) = \sum_{n\geq 0} F_n(\mathbf{X})Z^n \in E[\mathbf{X}][[Z]]$ with $F_n(\mathbf{X}) \in E[\mathbf{X}]$ and $\lim_{n\to\infty} \|F_n\| = 0$ for all $n \geq 0$, and let $f = \sum_{k\geq 0} f_k z^k \in E[\mathbf{t}][[z]]$ with $f_k \in E[\mathbf{t}]$ and $\lim_{k\to\infty} \|f_k\|_\infty = 0$ for all $k \geq 0$. Then F and $f.F$ converge in $\mathbb{B}_s(E)$ at $Z = 1$, f converges in $\mathbb{T}_s(E)$ at $z = 1$, and we have the following equality in $\mathbb{B}_s(E)$:*

$$(f.F(\mathbf{X}, Z))_{|Z=1} = f(\mathbf{t}, 1).F(\mathbf{X}, 1).$$

Proof. The convergence of F at $Z = 1$ and of f at $z = 1$ are obvious, the convergence of $f.F$ follows from the equality $\|f_k.F_n\| = \|f_k\|_\infty \|F_n\|$ from Lemma 3.9. Finally, both sides of the equality are equal to $\sum_{k\geq 0}\sum_{n\geq 0} f_k.F_n$. □

Lemma 5.2. *Let $\eta = \sum_{n\geq 0} \eta_n \tau_z^n \in E[[\tau_z]]$ and $\eta^1 = \sum_{n\geq 0} \eta_n \tau^n \in E[[\tau]]$, let $f = \sum_{k\geq 0} f_k z^k \in E[\mathbf{t}][[z]]$ with $\lim_{k\to\infty} \|f_k\|_\infty = 0$, write $M = \sup_{k\geq 0} \|f_k\|_\infty$ and suppose $\lim_{n\to\infty} |\eta_n|(q^{\frac{s}{q-1}} M)^{q^n} = 0$; finally write $g(\mathbf{t}, z) = \eta(f(\mathbf{t}, z)) = \sum_{n\geq 0} \eta_n \tau_z^n(f) \in E[\mathbf{t}][[z]]$. Then f and g converge in $\mathbb{T}_s(E)$ at $z = 1$ and we have the following equality in $\mathbb{T}_s(E)$:*

$$\eta^1(f(\mathbf{t}, 1)) = g(\mathbf{t}, 1).$$

Proof. The convergence of f is obvious. Both sides of the above equality are easily seen to be equal to

$$\sum_{n\geq 0, k\geq 0} \eta_n \tau^n(f_k(\mathbf{t}))$$

which is the sum of a summable family in $\mathbb{T}_s(E)$. This gives at once the convergence of both sides of the equality and the desired identity. □

Now define

$$E[\mathbf{X}][[Z]]^{\text{lin}} = \{F \in E[\mathbf{X}][[Z]]; F \text{ is linear with respect to each of } X_1, \ldots, X_s, Z\}.$$

Lemma 5.3.

(1) *The map*

$$\begin{cases} E[\mathbf{t}][[z]] & \to & E[\mathbf{X}][[Z]], \\ f & \mapsto & f.(X_1\cdots X_s Z) \end{cases}$$

is injective with image $E[\mathbf{X}][[Z]]^{\text{lin}}$,

(2) *$f \in E[\mathbf{t}][[z]]$ satisfies $f.(X_1\cdots X_s Z) \in A[\mathbf{X}][Z]$ if, and only if, $f \in A[\mathbf{t}][z]$.*

Proof. The first point is an immediate consequence of Lemma 3.11 and the second one a consequence of Lemma 3.12. □

5.2. **Anderson-Stark units.** We define for all integers $N \in \mathbb{Z}$, $s \geq 1$:

$$L(N,s,z) = \sum_{d\geq 0} z^d \sum_{a\in A_{+,d}} \frac{a(t_1)\dots a(t_s)}{a^N} \in K[\mathbf{t}][[z]]$$

and

$$L(N,s) = \sum_{d\geq 0} \sum_{a\in A_{+,d}} \frac{a(t_1)\dots a(t_s)}{a^N} \in \mathbb{T}_s(K_\infty),$$

where $L(N,s,z) \in A[\mathbf{t},z]$ if $N \leq 0$ because of Lemma 4.13. We also define the operator Γ:

$$\forall F \in \mathbb{B}_s(\mathbb{C}_\infty), \quad \Gamma(F) = L(1,s).F \in \mathbb{B}_s(\mathbb{C}_\infty).$$

We will refer to Γ as to *Anderson's operator*. Note that $L(1,s)$ has norm 1, so that Γ is an isometry of $\mathbb{B}_s(\mathbb{C}_\infty)$, in particular, $\|\Gamma(X_1\cdots X_s)\| = q^{\frac{s}{q-1}}$. We now define

$$\sigma_s(\mathbf{t},z) = \exp_z(L(1,s,z)).$$

We then have the following.

Proposition 5.4.

(1) $L(1,s,z) = \log_z(\sigma_s(\mathbf{t},z))$,
(2) $L(1,s,z).X_1\cdots X_sZ = \mathfrak{L}(X_1\cdots X_s,Z) = \log_C(\mathbb{S}_s(\mathbf{X},Z))$,
(3) $\sigma_s(\mathbf{t},z).X_1\cdots X_sZ = \mathbb{S}_s(\mathbf{X},Z)$ *and* $\sigma_s(\mathbf{t},z) \in A[\mathbf{t},z]$.

Proof. The first point and equality $L(1,s,z).X_1\cdots X_sZ = \mathfrak{L}(X_1\cdots X_s,Z)$ are clear. The equality $L(1,s,z).X_1\cdots X_sZ = \log_C(\mathbb{S}_s(\mathbf{X},Z))$ comes from equation (5.3). Equation (5.2) shows that $\sigma_s(\mathbf{t},z).X_1\cdots X_sZ = \mathbb{S}_s(\mathbf{X},Z)$ and the fact that $\sigma_s(\mathbf{t},z) \in A[\mathbf{t},z]$ is a consequence of Lemma 5.3 and Theorem 4.6. □

We call the special polynomial $\sigma_s(\mathbf{t},z)$ the *Anderson-Stark unit of level s*.
The evaluation at $Z=1$ leads to the following.

Proposition 5.5.

(1) $\exp_C(\Gamma(X_1\cdots X_s)) = \mathbb{S}_s(\mathbf{X},1)$,
(2) *if* $s<q$, *then* $\Gamma(X_1\cdots X_s) = \log_C(X_1\cdots X_s)$.

Proof. Lemma 5.2 shows that $\sigma_s(\mathbf{t},1) = \exp_C(L(1,s))$, and equation (5.1) yields to the first point. For the second point, we remark that if $s<q$, then $\mathbb{S}_s(\mathbf{X},Z) = X_1\cdots X_sZ$ so that

$$\mathfrak{L}(X_1\cdots X_s,Z) = \log_C(\mathbb{S}_s(\mathbf{X},Z)) = \sum_{n\geq 0} \frac{(X_1\cdots X_sZ)^{q^n}}{l_n}$$

but $\|X_1\cdots X_s\| = q^{\frac{s}{q-1}} < q^{\frac{q}{q-1}}$ so that

$$\Gamma(X_1\cdots X_s) = \sum_{n\geq 0} \frac{(X_1\cdots X_s)^{q^n}}{l_n} = \log_C(X_1\cdots X_s)$$

converges in $\mathbb{B}_s(K_\infty)$. □

We can recover properties of σ_s from the ones of $\mathbb{S}_s$.

Proposition 5.6.

(1) $\deg_z(\sigma_s(\mathbf{t},z)) \leq \frac{s-1}{q-1}$,
(2) $z-1$ *divides* $\sigma_s(\mathbf{t},z)$ *if, and only if,* $s \equiv 1 \mod q-1$ *and* $s>1$.

Proof. The first point comes from Proposition 4.7 and the second one from Lemma 4.10. □

Note that in practice, formulas for σ_s are more manageable and easier to compute than the formulas for $\mathbb{S}_s$. Compare the following example with Example 4.12.

Example 5.7. If $1 \leq s \leq q-1$, then $\sigma_s = 1$. The next two polynomials are: $\sigma_q = 1-z$ and $\sigma_{q+1} = 1-(t_1-\theta)\cdots(t_{q+1}-\theta)z$.

In the spirit of Lemma 4.13, we can recover the values $L(N,s,z)$ for $N \leq 0$ from the polynomials $\mathbb{S}_s(\mathbf{X},Z)$:

Theorem 5.8. *For all* $N \geq 0$ *and* $s \geq 1$,

$$L(-N,s,z).(X_1\cdots X_s) = \frac{d}{dX_{s+1}}\cdots\frac{d}{dX_{s+N+1}}\mathbb{S}_{s+N+1}(X_1,\ldots,X_{s+N+1},Z).$$

Proof. Since $\frac{d}{dX}(a*X)=a$, we have

$$\frac{d}{dX_{s+1}}\cdots\frac{d}{dX_{s+N+1}}Z_k(X_1\cdots X_{s+N+1}) = \sum_{a\in A_{+,k}} a*(X_1\cdots X_s)a^N$$

which gives the result. □

6. Special L-values

The purpose of this section is to express the series $L(N,s,z)$ as sums of polylogarithms. The idea here is to use the fact that if we evaluate $t_{n+1},\ldots,t_s$ at θ in $\varphi^r(L(1,s,z)) = L(q^r,s,z)$, we just obtain $L(q^r+n-s,n,z)$.

If P is a polynomial in a variable among $t,t_1,\ldots,$ or t_s, we will write P^φ for $\varphi(P)$.

Lemma 6.1. *For all integers* $k \geq 0$ *and* $r \geq 0$, *we have:*

(1) $b_{k+r}(t) = \varphi^k(b_r(t))b_k(t) = \varphi^r(b_k(t))b_r(t)$,
(2) *if* $r \geq 1$, $\varphi^r(b_k(t)) = \varphi^k(b_r(t))\frac{b_k(t)}{b_r(t)} = \varphi^k(b^\varphi_{r-1}(t))\frac{b^\varphi_k(t)}{b^\varphi_{r-1}(t)}$,
(3) $b^\varphi_k(\theta) = l_k$.

Proof. The verification of these identities is left to the reader. □

We start from the identity of the first point of Proposition 5.4, and we write $\sigma_s(\mathbf{t},z) = \sum_{i=0}^m \sigma_{s,i}(\mathbf{t})z^i$:

$$\begin{aligned} L(1,s,z) &= \log_z(\sigma_s(\mathbf{t},z)) \\ &= \sum_{k\geq 0}\frac{1}{l_k}\tau_z^k(\sigma_s(\mathbf{t},z)) = \sum_{k\geq 0}\sum_{i=0}^m \frac{z^{k+i}}{l_k}\tau^k(\sigma_{s,i}(\mathbf{t})) \\ &= \sum_{k\geq 0}\sum_{i=0}^m \frac{z^{k+i}}{l_k}b_k(t_1)\cdots b_k(t_s)\varphi^k(\sigma_{s,i}(\mathbf{t})). \end{aligned}$$

If we apply φ^r on both sides, we get by Lemma 6.1:

$$L(q^r,s,z)=\sum_{k\geq 0}\sum_{i=0}^{m}\frac{z^{k+i}}{l_k^{q^r}}\varphi^r(b_k(t_1)\cdots b_k(t_s))\varphi^{k+r}(\sigma_{s,i}(\mathbf{t}))$$
$$=\sum_{i=0}^{m}\sum_{k\geq 0}\frac{z^{k+i}}{l_k^{q^r}}\frac{b_k(t_1)\cdots b_k(t_n)b_k^{\varphi}(t_{n+1})\cdots b_k^{\varphi}(t_s)}{b_r(t_1)\cdots b_r(t_n).b_{r-1}^{\varphi}(t_{n+1})\cdots b_{r-1}^{\varphi}(t_s)}\varphi^k(U_{s,i}),$$

where $U_{s,i}=b_r(t_1)\cdots b_r(t_n)b_{r-1}^{\varphi}(t_{n+1})\cdots b_{r-1}^{\varphi}(t_s)\varphi^r(\sigma_{s,i}(\mathbf{t}))$.

Write $U_{s,i}=\sum_{i_{n+1},\ldots,i_s}f_{i_{n+1},\ldots,i_s}t_{n+1}^{i_{n+1}}\cdots t_s^{i_s}$ with $f_{i_{n+1},\ldots,i_s}\in A[t_1,\ldots,t_n]$ and for $j\geq 0$, $g_{i,j}=\sum_{i_{n+1}+\cdots+i_s=j}f_{i_{n+1},\ldots,i_s}$ so that $U_{s,i}$ evaluated at $t_{n+1}=\cdots=t_s=\theta$ is the polynomial $\sum_{j\geq 0}\theta^j g_{i,j}$. We now evaluate $L(q^r,s,z)$ at $t_{n+1}=\cdots=t_s=\theta$ and we write $N=q^r-s+n$:

$$L(N,n,z)\quad=\quad\sum_{j\geq 0}\theta^j\sum_{i=0}^{m}\sum_{k\geq 0}\frac{z^{k+i}}{l_k^N l_{r-1}^{s-n}}\frac{b_k(t_1)\cdots b_k(t_n)}{b_r(t_1)\cdots b_r(t_n)}\varphi^k(g_{i,j}).$$

Now write $\log_{N,z}=\sum_{k\geq 0}z^k\frac{b_k(t_1)\cdots b_k(t_n)}{l_k^N}\varphi^k=\sum_{k\geq 0}\frac{1}{l_k^N}\tau_z^k$. Then

$$L(N,n,z)\quad=\quad\frac{1}{l_{r-1}^{s-n}b_r(t_1)\cdots b_r(t_n)}\sum_{j\geq 0}\theta^j\log_{N,z}\left(\sum_{i=0}^{m}z^i g_{i,j}\right).$$

We have proved the next theorem.

Theorem 6.2. *For all integers $N\in\mathbb{Z}$, $n\geq 1$ and $r\geq 1$ such that $q^r\geq N$, there exist an integer $d\geq 0$, and for $0\leq j\leq d$, polynomials $h_j\in A[t_1,\ldots,t_s,z]$ such that*

$$L(N,n,z)=\frac{1}{l_{r-1}^{q^r-N}b_r(t_1)\cdots b_r(t_n)}\sum_{j=0}^{d}\theta^j\log_{N,z}(h_j).$$

Now denote for $N\in\mathbb{Z}$, $\log_N=\sum_{k\geq 0}\frac{1}{l_k^N}\tau^k$ the *Nth Carlitz polylogarithm.*

Corollary 6.3. *For all integers $N\in\mathbb{Z}$, $n\geq 1$ and $r\geq 1$ such that $q^r\geq N$, there exist an integer $d\geq 0$, and for $0\leq j\leq d$, polynomials $H_j\in A[X_1,\ldots,X_n,Z]^{\mathrm{lin}}$ such that*

$$L(N,n,z).\left(X_1^{q^r}\cdots X_n^{q^r}Z\right)=\frac{1}{l_{r-1}^{q^r-N}}\sum_{j=0}^{d}\theta^j\log_N(H_j).$$

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