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SHAPE PROGRAMMING OF A MAGNETIC ELASTICA

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ABSTRACT. We consider a cantilever beam which possesses a possibly non-uniform permanent magnetization, and whose shape is controlled by an applied magnetic field. We model the beam as a plane elastic curve and we suppose that the magnetic field acts upon the beam by means of a distributed couple that pulls the magnetization towards its direction. Given a list of target shapes, we look for a design of the magnetization profile and for a list of controls such that the shapes assumed by the beam when acted upon by the controls are as close as possible to the targets, in an averaged sense. To this effect, we formulate and solve an optimal design and control problem leading to the minimization of a functional which we study by both direct and indirect methods. In particular, we prove uniqueness of minimizers for sufficiently low intensities of the controlling magnetic fields. To this aim, we use two nested fixed-point arguments relying on the Lagrange-multiplier formulation of the problem, a method which also suggests a numerical scheme.

CONTENTS

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1. Introduction and main results

1.1. **Motivation.** Recent technological developments have made it possible to assemble materials which can convert into deformation, and hence motion, a diversity of energetic inputs in the form of heat, light, chemical agents, electric and magnetic fields. Magnetic actuation, in particular, is contactless, it offers fast response, and it does not affect the surrounding medium by polarization (which is the case of electric actuation). Moreover, an external magnetic field is relatively easy to be controlled both in magnitude and direction [21]. Enhancements in manufacturing processes allow to control with pinpoint accuracy the composition and the texture of an engineering manufact, making it possible to realize polymer composites, the so-called *magnetorheological elastomers* (MRE), obtained by embedding magnetic particles in a soft elastomeric matrix. Originally, MRE were devised to be employed as viscoelastic solids whose mechanical response can be controlled by an applied magnetic field [12]. However, thanks to the compliance of the elastomeric matrix, these materials now find applications in circumstances when large displacements are in need.

These technological advances have been offering several challenges to mechanics and mathematics: for example, the definition of a suitable continuum-mechanical framework [16, 10], stability at both the macroscopic [19] and the microstructural level [22], and the development of *ad hoc* computational techniques [20].

When crafted in the form of thin bodies, such as rods or plates, magnetorheological elastomers display a very large range of motion, so much so that they appear to be a promising technology for the realization of small-scale untethered microrobots which can walk and crawl [14, 23]. This has also stimulated substantial theoretical work concerning *shape programming*, i.e., the design of textures and controls that produce desired shapes, a topic which is becoming increasingly relevant in elasticity (see e.g. [1, 2]). In this respect, shape programming appears to be rather intriguing, even for a simple mechanical model such Euler's Elastica. This is not surprising, since the qualitative and quantitative properties of equilibrium solutions for elastic curves in a diversity of settings is still the object of intense mathematical research (see e.g. [9, 11, 15, 18]).

1.2. **The mechanical system.** We focus on a model problem featuring a *cantilever beam* with *permanent magnetization* under a *spatially-constant magnetic field*, as shown in the following figure.

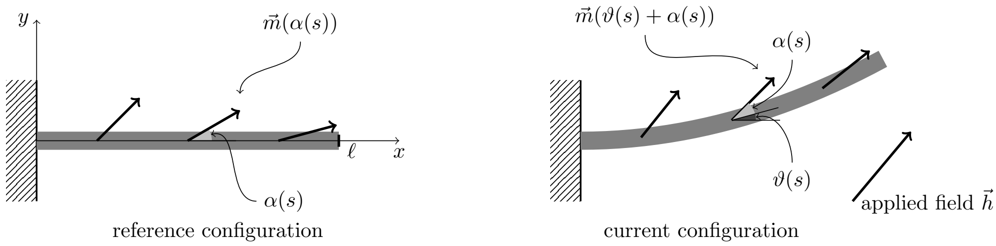

Figure 1. An cantilever beam with a permanent magnetization of uniform intensity and angle $\alpha(s)$ with respect to the tangent.

We model the cantilever beam as a *planar elastica*, and we describe its configuration through the parametric curve $\mathbf{r} : (0, 1) \to \mathbb{R}^2$ defined by

$$\mathbf{r}(s) = \ell \int_0^s \vec{m}(\vartheta(\xi))\, \mathrm{d}\xi, \tag{1.1}$$

where ℓ is the length of the beam, $\vec{m} : \mathbb{R} \to \mathbb{R}^2$ is defined by

$$\vec{m}(v) = (\cos(v), \sin(v)), \tag{1.2}$$

and $\vartheta(s)$ is the *rotation at* s. With slight abuse of language, we shall refer to the function $\vartheta : (0,1) \to \mathbb{R}$ as the *shape of the beam*. Since the beam is clamped, the shape must satisfy the essential boundary condition:

$$\vartheta(0) = 0, \tag{1.3}$$

which holds irrespectively of the loading environment.

The beam has a permanent magnetization per unit length, whose intensity is a constant M_0 (its unit in the S.I. System is ampere·meter^{-2} [Am^{-2}]), and whose orientation with respect to the tangent line is given by a possibly non-uniform *relative angle* $\alpha(s) \in \mathbb{R}$, $s \in (0,1)$. We assume that the relative angle $\alpha(s)$ is not affected by the magnetic field and by the deformation process. Hence the vector field

$$\vec{m}(\vartheta + \alpha) = (\cos(\alpha + \vartheta), \sin(\alpha + \vartheta)) \tag{1.4}$$

is the *orientation of the magnetization* in the typical deformed configuration.

The theory of magnetoelastic rods deduced in [7] by dimension reduction predicts that when a spatially constant magnetic field $\vec{H}$ [Am^{-1}] is applied to the beam, any stable equilibrium configuration must be a local minimizer of the renormalized *magnetoelastic energy*

$$\mathcal{E}(\vartheta) = \int_0^1 \Big(\frac{1}{2}(\vartheta')^2 - \vec{h} \cdot \vec{m}(\vartheta + \alpha)\Big)\, \mathrm{d}s, \tag{1.5}$$

where a dot denotes the scalar product, $\vec{h}$ is the renormalized magnetic field defined by $\vec{h} = \mu_0 \frac{M_0 \ell^2}{S} \vec{H}$, with μ_0 [Hm^{-1}] the magnetic permeability of vacuum and S [Nm^2] is the *bending stiffness*. The vector $\vec{h}$ is dimensionless, since its modulus $|\vec{h}| = (\mu_0 M_0 H \ell)/(S\ell^{-1})$ can be written as the ratio between the *magnetic energy* $\frac{1}{2}\mu_0 M_0 H \ell$ that must be expended to immerse the beam in the magnetic field, and the *elastic energy* S/ℓ that must be stored in the system to impart the curvature ℓ^{-1} to the beam.

1.3. **The state equation.** For $\vec{h} = 0$ the magnetoelastic energy has the unique minimizer $\vartheta = 0$, which corresponds through (1.1) to the straight configuration. As detailed in Section 4 (see Corollary 4.4), for given, arbitrary $\vec{h}$ and α the magnetoelastic energy has at least one minimizer, which furthermore solves the *Euler-Lagrange system*

$$\begin{cases} -\vartheta'' - \vec{h} \cdot D\vec{m}(\alpha + \vartheta) = 0 & \text{in } (0,1), \\ \vartheta(0) = 0, \\ \vartheta'(1) = 0, \end{cases} \tag{P_ϑ}$$

where

$$D\vec{m}(v) = (-\sin v, \cos v), \qquad \text{for all } v \in \mathbb{R} \tag{1.6}$$

is the derivative of the function $\vec{m}$ defined in (1.2); moreover, such minimizer is unique if

$$|\vec{h}| < c_p^{-2}, \tag{1.7}$$

where $c_p = 2/\pi$ is the best constant in the Poincaré-type inequality

$$\int_0^1 v^2 \le c_p^2 \int_0^1 (v')^2 \quad \text{for all } v \in C^1([0,1]) \text{ such that } v(0) = 0. \tag{1.8}$$

The *state equation* (P_ϑ) is a variant of the well-known *e*lastica equation. Given α, (P_ϑ) defines a *solution operator*

$$\Theta_\alpha \ : \quad B(0, c_p^{-2}) \ni \vec{h} \ \longmapsto \ \vartheta = \Theta_\alpha(\vec{h}) \tag{1.9}$$

which maps the *control* $\vec{h}$ into the *state* $\vartheta = \Theta_\alpha(\vec{h})$. The manifold of attainable configurations parametrized by the chart Θ_α is two-dimensional. Thus, one may hope that complex motions, such as for instance those required for applications to microswimmers [4, 3], could be realized, at least with a reasonable approximation, by a judicious choice of a fixed magnetization profile and a time varying magnetic field. The papers [14] and [17] offer experimental evidence of this possibility.

1.4. **The optimal design-control problem.** In this paper we are concerned with the following situation. We are given a list of n prescribed *target shapes*,

$$\overline{\boldsymbol{\vartheta}} = (\bar{\vartheta}_1, \ldots, \bar{\vartheta}_n) : [0,1] \to \mathbb{R}^n,$$

which the beam should ideally attain by applying n *controls*: these are the n magnetic fields

$$\vec{\boldsymbol{h}} = (\vec{h}_1, \ldots, \vec{h}_n) \in \mathbb{R}^{2n},$$

with $\vec{h}_i = (h_{ix}, h_{iy}) \in \mathbb{R}^2$. At our disposal is also a *design*, the magnetization α of the beam. Thus, we look for a design α and a control $\vec{\boldsymbol{h}}$ such that the shapes $\vartheta_i = \Theta_\alpha(\vec{h}_i)$ attained by the beam when applying the magnetic fields $\vec{h}_i$, namely the solutions of

$$\begin{cases} -\vartheta_i'' - \vec{h}_i \cdot D\vec{m}(\alpha + \vartheta_i) = 0 & \text{in } (0,1), \\ \vartheta_i(0) = 0, & \qquad\qquad i = 1, \ldots, n, \\ \vartheta_i'(1) = 0, \end{cases} \tag{P_{ϑ_i}}$$

are "as close as possible" to the targets $\overline{\vartheta}_i$. The precise meaning of "closeness" depends on the choice of the *c*ost functional $\mathcal{C}$, which we define as follows:

$$\mathcal{C}(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) = \frac{1}{2} \sum_{i=1}^{n} \int_0^1 |\vartheta_i - \bar{\vartheta}_i|^2 + \frac{\varepsilon}{2} \int_0^1 |\alpha'|^2 + \frac{\gamma}{2} \sum_{i=1}^{n} |\vec{h}_i|^2. \tag{1.10}$$

where $\varepsilon > 0$ and $\gamma > 0$ are positive parameters.

Remark 1.1 (The cost functional)**.** The choice of the cost functional $\mathcal{C}$ in (1.10) deserves a discussion. The first integral has an obvious interpretation, since we aim at minimizing the distance between the n attained shapes ϑ_i and the n target shapes $\overline{\vartheta}_i$. The second and third ones take into account, respectively, the design cost (it is technological more difficult and more expensive to manufacture beams whose magnetization varies more pronouncedly) and the technological cost of applying magnetic fields $\vec{\boldsymbol{h}}$ of larger intensities. From the mathematical viewpoint, they together guarantee *the coerciveness* of $\mathcal{C}$.

Our precise mathematical formulation of the problem thus involves three ingredients:

(i) the *admissible space*

$$\mathcal{H} = \{(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) : \vec{\boldsymbol{h}} \in \mathbb{R}^{2n}, \alpha \in H^1_{0L}(I), \boldsymbol{\vartheta} \in H^1_{0L}(I)^n\} = \mathbb{R}^{2n} \times H^1_{0L}(I) \times H^1_{0L}(I)^n, \tag{1.11}$$

where

$$H^1_{0L}(I) := \{v \in H^1(I) : v(0) = 0\}; \tag{1.12}$$

(ii) the *cost functional* $\mathcal{C} : \mathcal{H} \to \mathbb{R}$, defined by (1.10) for all $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) \in \mathcal{H}$;

(iii) the *admissible set*

$$\mathcal{A} = \left\{ (\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) \in \mathcal{H} : \ \vartheta_i \text{ solves } (P_{\vartheta_i}) \text{ for every } i = 1, \ldots, n \right\}. \tag{1.13}$$

With these three ingredients, we may formulate the following *Optimal Control-Design Problem*:

$$\text{minimize } \mathcal{C}(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) \text{ among all } (\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) \in \mathcal{A}. \tag{1.14}$$

1.5. **Our results.** Using the direct method of the Calculus of Variations, we prove in Section 3 the existence of a minimizer:

Proposition 1.2. *For any $\overline{\boldsymbol{\vartheta}} \in L^2(I)^n$, the cost functional $\mathcal{C}$ has a minimizer in the admissible set $\mathcal{A}$. Furthermore, any minimizer is such that*

$$\left(\max_{i=1,\ldots,n} \{ |\vec{h}_i| \} \right)^2 \leq \frac{\overline{\Theta}^2}{\gamma}, \qquad \textit{where} \quad \overline{\Theta}^2 = \sum_{i=1}^n \int_0^1 \overline{\vartheta}_i^2. \tag{1.15}$$

An important consequence of (1.15) is that, for $\overline{\Theta}$ sufficiently small and/or γ sufficiently large, each of the applied magnetic field $\vec{h}_i$ satisfies (1.7); thus, if $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$ is a minimizer with $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_n)$, then each state ϑ_i is the unique solution of its state system (P_{ϑ_i}): this means that the mechanical equilibria identified by the minimization of $\mathcal{C}$ are stable. In other words, for $\overline{\Theta}$ sufficiently small and/or γ sufficiently large each configuration ϑ_i corresponds to a stable minimizer of the magnetoelastic energy if the corresponding $\vec{\boldsymbol{h}}$ and α are taken as fixed.

However, the previous result neither implies uniqueness of the *triplet* $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$, nor provides a constructive scheme for its numerical approximation. Focusing on these two aspects, we investigate the Lagrange-multiplier reformulation of (1.14). This reformulation amounts to finding a critical point of the *Lagrangian*

$$\mathcal{L}(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}, \boldsymbol{\lambda}) := \mathcal{C}(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) - \sum_{i=1}^n \int_0^1 \lambda_i \left(-\vartheta_i'' - \vec{h}_i \cdot D\vec{m}(\alpha + \vartheta_i) \right),$$

where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ is the *Lagrange multiplier*. The differentiation of $\mathcal{L}$ yields, formally, the following system:

$$\begin{cases} (P_{\vartheta_i}) : & -\vartheta_i'' - \vec{h}_i \cdot D\vec{m}(\alpha + \vartheta_i) = 0, & \vartheta_i(0) = \vartheta_i'(1) = 0 \\ (P_{\lambda_i}) : & -\lambda_i'' - \lambda_i \vec{h}_i \cdot D^2\vec{m}(\alpha + \vartheta_i) = \vartheta_i - \bar{\vartheta}_i, & \lambda_i(0) = \lambda_i'(1) = 0 \\ (P_\alpha) : & -\varepsilon\alpha'' + \displaystyle\sum_{i=1}^n \lambda_i \vec{h}_i \cdot D^2\vec{m}(\alpha + \vartheta_i) = 0, & \alpha(0) = \alpha'(1) = 0 \\ (P_{\vec{h}_i}) : & \vec{h}_i = -\dfrac{1}{\gamma} \displaystyle\int_0^1 \lambda_i D\vec{m}(\alpha + \vartheta_i) & \end{cases} \tag{1.16}$$

for every $i = 1, \ldots, n$, where

$$D^2\vec{m}(v) = (-\cos v, -\sin v), \qquad v \in \mathbb{R}, \tag{1.17}$$

is the second derivative of $\vec{m}$. According to the standard theory of constrained minimization through Lagrange multipliers in Banach spaces, whose main results we summarize in the Appendix, a mimimizer $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$ of the cost functional in the admissible set corresponds to a stationary point $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}, \boldsymbol{\lambda})$ for some Lagrange multiplier $\boldsymbol{\lambda}$ only if that point is *regular*, in the sense that the *constraint mapping* $G : \mathcal{H} \to (H^1_{0L}(I)^n)'$ (the dual of $H^1_{0L}(I)^n$), defined by

$$\langle G(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}), \boldsymbol{u}\rangle = \sum_{i=1}^n \left\{ \int_0^1 \vartheta_i' u_i' - \int_0^1 \vec{h}_i \cdot D\vec{m}(\alpha + \vartheta_i) u_i \right\} \quad \text{for all } \boldsymbol{u} \in H^1_{0L}(I)^n, \tag{1.18}$$

is *Fréchet differentiable* at $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$ and its differential DG is *surjective*. We apply this theory in Section 3, where we study the Fréchet differentiability of the cost function $\mathcal{C}$ and of the constraint mapping G, as well as the surjectivity of the Fréchet differential of the latter. We show (see the proof of Proposition 5.2) that a miminizer $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}, \boldsymbol{\lambda}) \in \mathcal{A}$ is a regular point of the admissible set $\mathcal{A}$ if

$$\max_{i=1,\ldots,n} \{|\vec{h}_i|\} < c_p^{-2}. \tag{1.19}$$

Thanks to this fact, we deduce the following result:

Proposition 1.3. *Let* $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$ *be a minimizer of* $\mathcal{C}$ *in* $\mathcal{A}$ *such that* (1.19) *holds. Then there exists a Lagrange multiplier* $\boldsymbol{\lambda} \in H^1_{0L}(I)^n$ *such that* $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}, \boldsymbol{\lambda})$ *is a solution of system* (1.16)*. Furthermore,* $\alpha'(0) = 0$.

Remark 1.4. As an immediate consequence of Proposition 1.2, the bound (1.15), and Proposition 1.3, we obtain that if $\gamma > \overline{\Theta}^2 c_p^4$ then there exists a solution $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}, \boldsymbol{\lambda}) \in \mathcal{H} \times H^1_{0L}(I)^n$ to system (1.16) such that $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$ is a minimizer of $\mathcal{C}$ in $\mathcal{A}$.

The existence of a Lagrange multiplier justifies the approach proposed in [8] to numerically approximate the minimizer of $\mathcal{C}$, which is based on (1.16). In Section 6 we prove by a contraction argument that, at least for γ sufficiently large, System (1.16) has a unique solution (see Proposition 6.2). As a by-product, we have:

Theorem 1.5. *Let* $\overline{\boldsymbol{\vartheta}} \in C([0, 1])^n$*,* $\varepsilon > 0$*, and let* $K > 0$ *such that*

$$K < c_p^{-2}. \tag{1.20}$$

Then exists $\gamma_* = \gamma_*(\overline{\boldsymbol{\vartheta}}, \varepsilon, K)$ *such that for every* $\gamma > \gamma_*$ *there exists a unique solution of system* (1.16) *within the following set:*

$$(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}, \boldsymbol{\lambda}) \in \mathcal{H} \times H^1_{0L}(I)^n \quad \textit{such that} \quad \max_{i=1,\ldots,n} \{|\vec{h}_i|\} \le K < c_p^{-2}. \tag{1.21}$$

Proposition 1.2, Proposition 1.3 and Theorem 1.5 combine into the main result of the present paper:

Corollary 1.6. *Let* $\overline{\boldsymbol{\vartheta}} \in C([0, 1])^n$*,* $\varepsilon > 0$*, and let* $K > 0$ *such that* (1.20) *holds. Then there exists* $\gamma_{**} = \gamma_{**}(\overline{\boldsymbol{\vartheta}}, \varepsilon, K)$ *such that for any* $\gamma > \gamma_{**}$ *the minimizer in Proposition 1.2 is unique. Furthermore, it coincides with the unique solution to* (1.16)*, whence it is smooth and such that* $\alpha'(0) = 0$.

Proof. Let $(\vec{\boldsymbol{h}}^{(j)}, \alpha^{(j)}, \boldsymbol{\vartheta}^{(j)})$, $j = 1, 2$ be two minimizers. Let $\gamma > \overline{\Theta}^2/K^2$. By (1.15) in Proposition 1.2, both minimizers satisfy

$$\left(\max_{i=1,\ldots,n} |\vec{h}_i^{(j)}| \right)^2 \le \frac{\overline{\Theta}^2}{\gamma} < K^2 < c_p^{-4}, \qquad j = 1, 2. \tag{1.22}$$

In particular, (1.19) holds for both. Hence, by Proposition 1.3, there exist $\boldsymbol{\lambda}^{(j)} \in H^1_{0L}(I)^n$ such that $(\vec{\boldsymbol{h}}^{(j)}, \alpha^{(j)}, \boldsymbol{\vartheta}^{(j)}, \boldsymbol{\lambda}^{(j)}) \in \mathcal{H} \times H^1_{0L}(I)^n$ are solutions to system (1.16). Assume in addition that $\gamma > \gamma_*(\overline{\boldsymbol{\vartheta}}, \varepsilon, K)$. Then, by (1.22) and Theorem 1.5, the two quadruplets, whence the two minimizers, coincide: therefore the proof is complete by choosing $\gamma_{**} = \max\{\overline{\Theta}^2/K^2, \gamma_*(\overline{\boldsymbol{\vartheta}}, \varepsilon, K)\}$. □

Corollary 1.6 states that for $\gamma > \gamma_{**}$ the minimum is unique and may be numerically approximated by solving the Euler-Lagrange system (1.16) (hence, not necessarily by a direct approach, although the latter is used to prove the existence of the minumum). In fact, it is through the uniqueness of the solution of the Euler-Lagrange system that we are able to assert the uniqueness of the minimum. Also, note that Corollary 1.6 holds for any target $\overline{\boldsymbol{\vartheta}}$, even very large ones (in this respect, see Section 7).

Remark 1.7 (Interpretation of the constant K)**.** The lower bound γ_{**} depends on the target shapes $\overline{\boldsymbol{\vartheta}} = (\overline{\vartheta}_i)$, on the penalization ε, and on a constant K which we are free to choose consistent with the restriction $K < c_p^{-2}$. Corollary 1.6 guarantees that none of the control fields $\vec{h}_i$ will exceed K in intensity if $\gamma > \gamma_{**}$. Thus, the constant K may be chosen to be the maximum intensity (in renormalized units) that available devices can apply, if this intensity does not exceed c_p^{-2}.

Further remarks and open questions are presented in the concluding Section 7.

2. Notation and preliminaries

In this section we introduce some notation as a complement to that already defined in the Introduction, and we collect preliminary results that will be needed in our subsequent developments. Other standard results are contained in the Appendix.

Given a vector $\vec{v} = (v_x, v_y) \in \mathbb{R}^2$, we let $|\vec{v}| = \sqrt{v_x^2 + v_y^2}$ be its Euclidean norm and, for $\vec{w}$ another vector, we let $\vec{v}\cdot\vec{w} = v_x w_x + v_y w_y$ be the scalar product between $\vec{v}$ and $\vec{w}$. Given a list of vectors $\vec{\boldsymbol{v}} = (\vec{v}_1, \ldots, \vec{v}_n)$, with $\vec{v}_i \in \mathbb{R}^2$ for $i = 1, \ldots, n$, we let $|\vec{\boldsymbol{v}}| := |\vec{v}_1| + \cdots + |\vec{v}_n|$. For $f : I \to \mathbb{R}$ a measurable function, we use the abbreviation $\|f\|_p \equiv \|f\|_{L^p(I)}$ for all exponents $p \geq 1$. We use similar abbreviations for measurable vector-valued functions. We recall that, by the Sobolev embedding theorem, $H^1_{0L}(I) \subset C([0,1])$, where $C([0,1])$ is the space of the continuous functions on $[0,1]$. We record for later use the inequality

$$\|v\|_\infty^2 \leq \int_0^1 (v')^2 \quad \text{for all } v \in H^1_{0L}(I) \tag{2.1}$$

which is sharp, as can be seen by taking $v(x) = x$.

We denote by $c_p = 2/\pi$ the *best constant* in the *Poincaré-type inequality*:

$$\int_0^1 v^2 \leq c_p^2 \int_0^1 (v')^2 \quad \text{for all } v \in H^1_{0L}(I). \tag{2.2}$$

It follows from (2.2) and from the definition (1.12) that

$$\|v\|^2 := \int_0^1 (v')^2 \tag{2.3}$$

is equivalent to the Sobolev norm on $H^1_{0L}(I)$. Accordingly, we henceforth shall use the norm (2.3) to endow $H^1_{0L}(I)$ with a Hilbert-space structure. For $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) \in \mathcal{H}$ we define $\|(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})\|_{\mathcal{H}} = |\vec{\boldsymbol{h}}| + \|\alpha\| + \|\boldsymbol{\vartheta}\|$. It follows that $\mathcal{H}$ is an Hilbert space.

For $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$, $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ Banach spaces we denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$, and we let $\|\cdot\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})}$ be the operator norm. Moreover, we write $\langle\cdot,\cdot\rangle_{\mathcal{X}',\mathcal{X}}$ to denote the pairing between a Banach space $\mathcal{X}$ and its dual: in fact, we will omit the indexing whenever the space $\mathcal{X}$ is clear from the context.

If not otherwise specified, we will denote by C a generic constant whose value may possibly change within the same chain of inequalities, and by $C(\cdot)$ constants whose value only depend on the parameters and variables listed within parentheses.

Finally, we observe that the function $\vec{m} : \mathbb{R} \to \mathbb{R}^2$ defined in (1.2) is bounded, infinitely differentiable and its N-th derivative, defined consistently with (1.6), is

$$D^N \vec{m}(v) = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}^N \vec{m}(v), \qquad \text{for all } N \in \mathbb{N}, \tag{2.4}$$

namely, $D^N \vec{m}(v)$ is the vector obtained by rotating $\vec{m}(v)$ in the counter-clockwise direction by the amount $N\pi/2$. Thus,

$$\left| D^N \vec{m}(v) \right| = 1, \quad \text{for all } v \in \mathbb{R} \text{ and } N \in \mathbb{N}. \tag{2.5}$$

Hence,

$$|D^N \vec{m}(v_1) - D^N \vec{m}(v_2)| \leq \int_{v_1}^{v_2} |D^{N+1} \vec{m}(v)| dv = |v_1 - v_2| \quad \text{for all } v_1, v_2 \in \mathbb{R}. \tag{2.6}$$

As a consequence of this observation, we record three bounds which will be used several times.

Lemma 2.1. *Let* $(\vec{h}, \alpha, \vartheta, \lambda), (\tilde{\vec{h}}, \tilde{\alpha}, \tilde{\vartheta}, \tilde{\lambda}) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ *and* $N \in \mathbb{N}$. *Then*

$$\left| \lambda D^N \vec{m}(\alpha + \vartheta) - \tilde{\lambda} D^N \vec{m}(\tilde{\alpha} + \tilde{\vartheta}) \right| \leq |\lambda - \tilde{\lambda}| + |\tilde{\lambda}| \left(|\alpha - \tilde{\alpha}| + |\vartheta - \tilde{\vartheta}| \right), \tag{2.7a}$$

$$\left| \vec{h} \cdot D^N \vec{m}(\alpha + \vartheta) - \tilde{\vec{h}} \cdot D^N \vec{m}(\tilde{\alpha} + \tilde{\vartheta}) \right| \leq |\vec{h} - \tilde{\vec{h}}| + |\tilde{\vec{h}}| \left(|\alpha - \tilde{\alpha}| + |\vartheta - \tilde{\vartheta}| \right), \tag{2.7b}$$

$$\begin{aligned} \left| \lambda \vec{h} \cdot D^N \vec{m}(\alpha + \vartheta) - \tilde{\lambda} \tilde{\vec{h}} \cdot D^N \vec{m}(\tilde{\alpha} + \tilde{\vartheta}) \right| \leq\ & |\vec{h}||\lambda - \tilde{\lambda}| + |\tilde{\lambda}||\vec{h} - \tilde{\vec{h}}| \\ & + |\tilde{\vec{h}}||\tilde{\lambda}| \left(|\alpha - \tilde{\alpha}| + |\vartheta - \tilde{\vartheta}| \right). \end{aligned} \tag{2.7c}$$

3. Existence of a minimizer

In this section we address the existence of a minimizer to the optimal control-design problem (1.14).

Proof of Proposition 1.2. We recall that the admissible set is defined in (1.13). We begin by noting that $(\mathbf{0}, 0, \vec{\mathbf{0}}) \in \mathcal{A}$, hence $\mathcal{A}$ is not empty. Next, we let

$$m = \inf_{\mathcal{A}} C(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}),$$

and we consider a minimizing sequence, i.e. a sequence $\{(\vec{\boldsymbol{h}}_k, \alpha_k, \boldsymbol{\vartheta}_k)\} \subset \mathcal{A}$ with $\vec{\boldsymbol{h}}_k = (\vec{h}_{k1}, \ldots, \vec{h}_{kn})$, $\vec{h}_{ki} = (h_{kix}, h_{kiy})$, and $\boldsymbol{\vartheta}_k = (\vartheta_{k1}, \ldots, \vartheta_{kn})$, such that $\mathcal{C}(\vec{\boldsymbol{h}}_k, \alpha_k, \boldsymbol{\vartheta}_k) \to m$ as $k \to +\infty$. In particular, by the definition of $\mathcal{C}$, a constant C exists such that

$$\sum_{i=1}^n \int_0^1 |\vartheta_{ki}|^2 + \int_0^1 |\alpha_k'|^2 + \sum_{i=1}^n |\vec{h}_{ki}|^2 \leq C \tag{3.1}$$

for all $k \in \mathbb{N}$. Moreover ϑ_{ki} satisfies

$$\int_0^1 \left(\vartheta_{ki}' v' - \vec{h}_{ki} \cdot D\vec{m}(\alpha_k + \vartheta_{ki}) v \right) = 0, \qquad \forall v \in H^1_{0L}(I) \tag{3.2}$$

for all $k \in \mathbb{N}$ and $i = 1, \ldots, n$. Choosing ϑ_{ki} as test function in (3.2) and recalling (3.1), we obtain

$$\|\vartheta_{ki}\| \leq 2C \quad \text{for all } k \in \mathbb{N} \text{ and every } i = 1, \ldots, n.$$

Hence, by a standard compactness argument, $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) \in \mathcal{H}$ exists such that, by passing to a subsequence (not relabeled),

$$\begin{aligned} &\vec{h}_{ki} \to \vec{h}_i \text{ in } \mathbb{R}^2, \text{ for every } i = 1, \ldots, n, \\ &\alpha_k \to \alpha \text{ weakly in } H^1_{0L}(I) \text{ and uniformly in } C([0,1]), \\ &\vartheta_{ki} \to \vartheta_i \text{ weakly in } H^1_{0L}(I) \text{ and uniformly in } C([0,1]), \text{ for every } i = 1, \ldots, n. \end{aligned} \tag{3.3}$$

Letting k tend to infinity in (3.2) and using the convergence statement (3.3), we conclude that ϑ_i is a weak solution of (P_{ϑ_i}) for every $i = 1, \ldots, n$, so $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) \in \mathcal{A}$. Moreover, by lower semi-continuity,

$$\mathcal{C}(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) \leq \liminf_{k\to\infty} \mathcal{C}(\vec{\boldsymbol{h}}_k, \alpha_k, \boldsymbol{\vartheta}_k) = m.$$

This implies that $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$ is a minimizer of $\mathcal{C}$. In addition, since $(\mathbf{0}, 0, \vec{\mathbf{0}}) \in \mathcal{A}$, for any minimizer we have

$$\frac{\gamma}{2} \sum_{i=1}^n |\vec{h}_i|^2 \leq \mathcal{C}(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) \leq \mathcal{C}(\mathbf{0}, 0, \vec{\mathbf{0}}) = \frac{1}{2} \sum_{i=1}^n |\overline{\vartheta}_i|^2,$$

which implies (1.15). □

4. The basic equation

The next sections will be devoted to the analysis of the Lagrange-multiplier system (1.16). Its equations, (P_{ϑ_i}), (P_{λ_i}), and (P_α), share the following structure:

$$\begin{cases} -v'' + f(s, v) = 0 & \text{in } (0,1), \\ v(0) = 0, \\ v'(1) = 0. \end{cases} \tag{4.1}$$

In particular, with reference to (1.16), we have that

$$\text{(4.1) is equivalent to:} \begin{cases} (P_{\vartheta_i}) \text{ for } f(s,v) = -\vec{h}_i \cdot D\vec{m}(\alpha(s) + v); \\ (P_{\lambda_i}) \text{ for } f(s,v) = -v\vec{h}_i \cdot D^2\vec{m}(\alpha(s) + \vartheta_i(s)) + \overline{\vartheta}_i(s) - \vartheta_i(s); \\ (P_\alpha) \text{ for } f(s,v) = \dfrac{1}{\varepsilon} \displaystyle\sum_{i=1}^n \lambda_i(s) \vec{h}_i \cdot D^2\vec{m}(v + \vartheta_i(s)). \end{cases} \tag{4.2}$$

Definition 4.1. *Let $f \in L^1(I \times \mathbb{R})$. A function v belonging to $H^1_{0L}(I)$ is a (weak) solution to problem* (4.1) *if*

$$\int_0^1 v'w' + \int_0^1 f(s,v)w = 0, \qquad \textit{for all } w \in H^1_{0L}(I). \tag{4.3}$$

In the following Lemma we provide (to the extent we need) uniqueness, existence, and boundedness results for solutions of (4.1).

Lemma 4.2. *Let $f \in L^\infty(I \times \mathbb{R})$, let $L \in (0, c_p^{-2})$ be such that*

$$|f(s, v_1) - f(s, v_2)| \leq L|v_1 - v_2| \quad \textit{for a.e. } s \in I \textit{ and for all } v_1, v_2 \in \mathbb{R}, \tag{4.4}$$

and let c_p be defined by (2.2)*. Then there exists a unique solution v to problem* (4.1) *in the sense of Definition 4.1, and this solution satisfies the bounds*

$$\|v\| \leq \frac{c_p}{1 - Lc_p^2}\|f(s,0)\|_\infty, \quad \|v\|_\infty \leq \|f\|_\infty. \tag{4.5}$$

Proof. Let $f_0(s) = f(s,0)$. For $g(s,v) = \int_0^v f(s,t)dt$, we let:

$$\mathscr{F}(w) = \frac{1}{2}\int_0^1 w'^2 + \int_0^1 g(s,w), \qquad w \in H^1_{0L}(I).$$

This position defines a Gâteaux-differentiable, weakly-lower semicontinuous functional $\mathscr{F} : H^1_{0L}(I) \to \mathbb{R}$. Since

$$|g(s,v)| \leq \int_0^v |f(s,t)|dt \overset{(4.4)}{\leq} |f_0(s)||v| + \frac{1}{2}L|v|^2,$$

we have

$$\mathscr{F}(w) \geq \frac{1}{2}\int_0^1 w'^2 - \int_0^1 |f_0 w| - \frac{1}{2}L\int_0^1 w^2 \geq \frac{1}{2}(1 - Lc_p^2)\|w\|^2 - \|f_0\|_2\|w\|_2,$$

whence, by (2.2),

$$\mathscr{F}(w) \geq \frac{1}{2}(1 - Lc_p^2)\|w\|^2 - c_p\|f_0\|_\infty\|w\|.$$

This inequality implies that $\mathscr{F}$ is *coercive*, thanks to the hypothesis $L < c_p^{-2}$. The coercivity and the lower semicontinuity of $\mathscr{F}$ imply, by a standard argument, that $\mathscr{F}$ has a minimizer v in $H^1_{0L}(I)$ (see [13]). Since $\mathscr{F}$ is Gâteaux differentiable, v is also a weak solution of Problem (4.1).

In order to prove uniqueness, let v_1 and v_2 be two weak solutions of (4.1). According to Definition 4.1, $v_1 - v_2$ is a legal test function for the weak formulation of (4.1). We use this test in (4.3). On taking the difference between the resulting equations we obtain:

$$\int_0^1 [(v_1 - v_2)']^2 + \int_0^1 (f(\cdot, v_1) - f(\cdot, v_2))(v_1 - v_2) = 0.$$

It follows from the assumption on f and from the Poincaré inequality (2.2) that

$$\int_0^1 (f(\cdot, v_1) - f(\cdot, v_2))(v_1 - v_2) \leq L\int_0^1 |v_1 - v_2|^2 \leq Lc_p^2\|v_1 - v_2\|^2,$$

whence

$$(1 - Lc_p^2)\|v_1 - v_2\|^2 \leq 0,$$

and thence $v_1 = v_2$, given that $Lc_p^2 < 1$.

Taking v as test function in (4.3) we obtain that

$$\begin{aligned}\|v\|^2 = \int_0^1 v'^2 \;&=\; -\int_0^1 f(s,v)v \leq \int_0^1 |f_0(s)||v| + L\int_0^1 |v|^2 \\ &\leq\; \|f_0\|_\infty\left(\int_0^1 v^2\right)^{1/2} + L\int_0^1 |v|^2 \overset{(2.2)}{\leq} c_p\|f_0\|_\infty\|v\| + Lc_p^2\|v\|^2,\end{aligned}$$

whence the first bound in (4.5). Finally, the second bound in (4.5) is immediate from the representation formula

$$v(s) = \int_0^s \int_{s'}^1 f(s'', v(s''))ds''\, ds'.$$

□

Remark 4.3 (Regularity and boundary values of the solution to Problem (4.1))**.** It follows from standard arguments (see e.g. [5, Proposition A]) that if $v \in H^1_{0L}(I)$ is a weak solution to problem (4.1), then

$$v \in H^2(I) = \left\{ w \in L^2(I) : w', w'' \in L^2(I) \right\}.$$

The Sobolev embedding theorem (see for instance Sec. 2.1 of [6]) implies that $v \in C^1([0,1])$, and that the boundary conditions are satisfied pointwise. Indeed, since $v \in H^1_{0L}(I)$, we have that $v(0) = 0$. Moreover, multiplying by an arbitrary function $w \in H^1_{0L}(I)$ and integrating in I equation (4.1) we obtain

$$\int_0^1 -v''w + \int_0^1 f(s,v)w = 0 \qquad \text{for all } w \in H^1_{0L}(I). \tag{4.6}$$

Integrating by parts the first term of the l.h.s. of (4.6) we have

$$v'(1)w(1) = \int_0^1 (v'w)' = \int_0^1 v'w' + \int_0^1 f(s,v)w \overset{(4.3)}{=} 0 \qquad \text{for all } w \in H^1_{0L}(I).$$

This implies that $v'(1) = 0$.

As a by-product of the previous discussion, we obtain:

Corollary 4.4. *Let $\vec{h} \in \mathbb{R}^2$ and $\alpha \in C([0,1])$. Then the energy functional $\mathcal{E}$ defined by* (1.5) *has a minimizer $\vartheta \in H^1_{0L}$. Furthermore, $\vartheta \in C^1([0,1])$ solves (P_ϑ) and ϑ is unique if* (1.7) *holds.*

Proof. Existence of a minimizer of $\mathcal{E}$ in $H^1_{0L}(I)$ can be proved with the same arguments used in Section 3. By standard variational considerations, ϑ is a solution to (4.3) with $f(s,\vartheta) = -\vec{h} \cdot D\vec{m}(\alpha(s) + \vartheta)$: hence, by Lemma 4.2, it is unique if $|\vec{h}| < c_p^{-2}$. The remaining statement follows from Remark 4.3. □

5. The Lagrange multiplier formulation

We recall the definition (1.18) of the constraint mapping:

$$\langle G(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}), \boldsymbol{u} \rangle = \sum_{i=1}^{n} \left\{ \int_0^1 \vartheta_i' u_i' - \int_0^1 \vec{h}_i \cdot D\vec{m}(\alpha + \vartheta_i) u_i \right\}. \tag{5.1}$$

Since $|\langle G(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}), \boldsymbol{u} \rangle| \le C(\vec{\boldsymbol{h}}, \boldsymbol{\vartheta}) \|\boldsymbol{u}\|$ for every $\boldsymbol{u} \in H^1_{0L}(I)^n$, $G(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$ is a linear bounded functional. Thus (5.1) defines a map $G : \mathcal{H} \to (H^1_{0L}(I)^n)'$. Thanks to the equivalence

$$(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) \in \mathcal{A} \quad \Leftrightarrow \quad G(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) = 0,$$

we can write

$$\mathcal{A} \overset{(1.13)}{=} \left\{ (\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) \in \mathcal{H} : G(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}) = 0 \right\}. \tag{5.2}$$

Proposition A.5 in the Appendix of this paper provides sufficient conditions for the existence of the Lagrange multiplier $\boldsymbol{\lambda}$. We are going to use this proposition as a tool to characterize the minimizers of $\mathcal{C}$ in $\mathcal{A}$. To this aim, we need to assess the regularity of the functional $\mathcal{C}$ and of the operator G; the next statement concerns their Fréchet differentiability, which we shall obtain as a consequence of Proposition A.4 and the following lemma.

Lemma 5.1. *The operators $\mathcal{C}:\mathcal{H}\to\mathbb{R}$ and $G:\mathcal{H}\to(H^1_{0L}(I)^n)'$ are $\mathcal{C}$ and G are C^1, with $D\mathcal{C}:\mathcal{H}\to\mathcal{H}'$ and $DG:\mathcal{H}\to\mathcal{L}(\mathcal{H},(H^1_{0L}(I)^n)')$ being represented by*

$$D\mathcal{C}(\vec{\boldsymbol{h}},\alpha,\boldsymbol{\vartheta})(\vec{\boldsymbol{k}},\beta,\boldsymbol{\iota})=\gamma\sum_{i=1}^n\vec{h}_i\cdot\vec{k}_i+\varepsilon\int_0^1\alpha'\beta'+\sum_{i=1}^n\int_0^1(\vartheta_i-\bar{\vartheta}_i)\iota_i,\tag{5.3}$$

respectively

$$\begin{aligned}\langle DG(\vec{\boldsymbol{h}},\alpha,\boldsymbol{\vartheta})(\vec{\boldsymbol{k}},\beta,\boldsymbol{\iota}),\boldsymbol{u}\rangle &= \sum_{i=1}^n\int_0^1-\vec{k}_i\cdot D\vec{m}(\alpha+\vartheta_i)u_i\\ &+\sum_{i=1}^n\int_0^1\Big\{\iota_i'u_i'-\vec{h}_i\cdot\big(D^2\vec{m}(\alpha+\vartheta_i)\iota_i+D^2\vec{m}(\alpha+\vartheta_i)\beta\big)u_i\Big\},\end{aligned}\tag{5.4}$$

for every $(\vec{\boldsymbol{h}},\alpha,\boldsymbol{\vartheta}),(\vec{\boldsymbol{k}},\beta,\boldsymbol{\iota})\in\mathcal{H}$ and $\boldsymbol{u}\in H^1_{0L}(I)^n$.

Proof. Fix $\boldsymbol{\varphi}:=(\vec{\boldsymbol{h}},\alpha,\boldsymbol{\vartheta})\in\mathcal{H}$. We consider a sequence $\{\boldsymbol{\varphi}_k\}:=\{(\vec{\boldsymbol{h}}_k,\alpha_k,\boldsymbol{\vartheta}_k)\}$, with $\vec{\boldsymbol{h}}_k=(\vec{h}_{k1},\dots,\vec{h}_{kn})$, $\vec{h}_{ki}=(h_{kix},h_{kiy})$, and $\boldsymbol{\vartheta}_k=(\vartheta_{k1},\dots,\vartheta_{kn})$ such that $\|\boldsymbol{\varphi}_k-\boldsymbol{\varphi}\|_{\mathcal{H}}\to 0$ as $k\to+\infty$. In particular, C exists such that

$$|\vec{\boldsymbol{h}}_k|\le C.\tag{5.5}$$

First we focus on $\mathcal{C}$. We trivially have $\mathcal{C}(\vec{\boldsymbol{h}}_k,\alpha_k,\boldsymbol{\vartheta}_k)\to\mathcal{C}(\vec{\boldsymbol{h}},\alpha,\boldsymbol{\vartheta})$ in $\mathbb{R}$, hence $\mathcal{C}$ is continuous. The Gâteaux derivative $\mathcal{C}'(\boldsymbol{\varphi})$ can be computed explicitly via Definition A.1, and it coincides with the right-hand side of (5.3). In order to show that $\mathcal{C}$ is C^1, we write (using Cauchy-Schwarz inequality)

$$\begin{aligned}&|\left(\mathcal{C}'(\boldsymbol{\varphi})-\mathcal{C}'(\boldsymbol{\varphi}_k)\right)(\vec{\boldsymbol{k}},\beta,\boldsymbol{\iota})|\\ &\overset{(5.3)}{\le}\left\{\gamma\sum_{i=1}^n|\vec{h}_i-\vec{h}_{ki}|+\varepsilon\|\alpha-\alpha_k\|+\sum_{i=1}^n\|\vartheta_i-\vartheta_{ki}\|_2\right\}\|(\vec{\boldsymbol{k}},\beta,\boldsymbol{\iota})\|_{\mathcal{H}},\end{aligned}$$

hence

$$\|\mathcal{C}'(\boldsymbol{\varphi})-\mathcal{C}'(\boldsymbol{\varphi}_k)\|_{\mathcal{H}'}\le C\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_k\|_{\mathcal{H}}.$$

Thus the Gâteaux derivative $\mathcal{C}'$ of $\mathcal{C}$ is (Lipschitz) continuous with respect to the operator norm and, by applying Proposition A.4, we conclude that $\mathcal{C}$ is Fréchet differentiable, that its differential is $D\mathcal{C}=\mathcal{C}'$ and that therefore $\mathcal{C}$ is C^1.

Now we focus our attention on G, by first proving that G is continuous. To this aim, we fix $\boldsymbol{u}=(u_1,\dots,u_n)\in H^1_{0L}(I)^n$ and we compute:

$$\begin{aligned}\langle G(\boldsymbol{\varphi})-G(\boldsymbol{\varphi}_k),\boldsymbol{u}\rangle &\overset{(5.1)}{=}\sum_{i=1}^n\int_0^1(\vartheta_i-\vartheta_{ki})'u_i'-\sum_{i=1}^n\int_0^1\Big(\vec{h}_i\cdot D\vec{m}(\alpha+\vartheta_i)-\vec{h}_{ki}\cdot D\vec{m}(\alpha_k+\vartheta_{ki})\Big)u_i\\ &\overset{(2.7b)}{\le}\|\boldsymbol{\vartheta}-\boldsymbol{\vartheta}_k\|\|\boldsymbol{u}\|+\sum_{i=1}^n\int_0^1\left(|\vec{h}_i-\vec{h}_{ki}|+|\vec{h}_{ki}|\left(|\alpha-\alpha_k|+|\vartheta_i-\vartheta_{ki}|\right)\right)|u_i|.\end{aligned}$$

Then, by making use of Hölder and Poincaré inequalities, we deduce the inequality

$$|\langle G(\boldsymbol{\varphi})-G(\boldsymbol{\varphi}_k),\boldsymbol{u}\rangle|\le C\Big((1+|\vec{\boldsymbol{h}}_k|)\|\boldsymbol{\vartheta}-\boldsymbol{\vartheta}_k\|+|\vec{\boldsymbol{h}}-\vec{\boldsymbol{h}}_k|+|\vec{\boldsymbol{h}}_k|\|\alpha-\alpha_k\|\Big)\|\boldsymbol{u}\|,$$

whence

$$\begin{aligned}\|G(\boldsymbol{\varphi})-G(\boldsymbol{\varphi}_k)\|_{(H^1_{0L}(I)^n)'} &\le C\Big((1+|\vec{\boldsymbol{h}}_k|)\|\boldsymbol{\vartheta}-\boldsymbol{\vartheta}_k\|+|\vec{\boldsymbol{h}}-\vec{\boldsymbol{h}}_k|+|\vec{\boldsymbol{h}}_k|\|\alpha-\alpha_k\|\Big)\\ &\overset{(5.5)}{\le} C\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_k\|_{\mathcal{H}}\to 0,\end{aligned}$$

hence G is (Lipschitz) continuous. The Gâteaux derivative $G'(\boldsymbol{\varphi}) : \mathcal{H} \to (H^1_{0L}(I)^n)'$ can be computed explicitly via its definition, and it coincides with the right-hand side of (5.4). Hence we deduce that

$$\begin{aligned}
&\langle (G'(\boldsymbol{\varphi}) - G'(\boldsymbol{\varphi}_k))\,(\vec{\boldsymbol{k}}, \beta, \boldsymbol{\iota}), \boldsymbol{u}\rangle \\
&= \; -\sum_{i=1}^n \int_0^1 \vec{k}_i \cdot (D\vec{m}(\alpha + \vartheta_i) - D\vec{m}(\alpha_k + \vartheta_{ki}))u_i \\
&\quad -\sum_{i=1}^n \int_0^1 \Big(\vec{h}_i \cdot D^2\vec{m}(\alpha + \vartheta_i) - \vec{h}_{ki} \cdot D^2\vec{m}(\alpha_k + \vartheta_{ki})\Big)\,(\beta + \iota_i)u_i \\
&\overset{(2.7b)}{\leq} \; \sum_{i=1}^n \int_0^1 |\vec{k}_i|(|\alpha - \alpha_k| + |\vartheta_i - \vartheta_{ki}|)|u_i| \\
&\quad +\sum_{i=1}^n \int_0^1 \Big(|\vec{h}_i - \vec{h}_{ki}| + |\vec{h}_{ki}|(|\alpha - \alpha_k| + |\vartheta_i - \vartheta_{ki}|)\Big)\,|\beta + \iota_i||u_i| \\
&\overset{(5.5)}{\leq} \; C\left(\Big(|\vec{\boldsymbol{h}} - \vec{\boldsymbol{h}}_k| + \|\alpha - \alpha_k\| + \|\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_k\|\Big)\,(|\vec{\boldsymbol{k}}| + \|\beta\| + \|\boldsymbol{\iota}\|)\right)\|\boldsymbol{u}\|,
\end{aligned}$$

where in the last inequality we have also used Hölder and Poincaré inequalities. It follows that

$$\begin{aligned}
\|(G'(\boldsymbol{\varphi}) - G'(\boldsymbol{\varphi}_k))(\vec{\boldsymbol{k}}, \beta, \boldsymbol{\iota})\|_{(H^1_{0L}(I)^n)'} &= \sup_{\|\boldsymbol{u}\|_{H^1_{0L}(I)^n}=1} |\langle (G'(\boldsymbol{\varphi}) - G'(\boldsymbol{\varphi}_k))(\vec{\boldsymbol{k}}, \beta, \boldsymbol{\iota}), \boldsymbol{u}\rangle| \\
&\leq C\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_k\|_{\mathcal{H}}\|(\vec{\boldsymbol{k}}, \beta, \boldsymbol{\iota})\|_{\mathcal{H}},
\end{aligned}$$

hence that

$$\|G'(\boldsymbol{\varphi}) - G'(\boldsymbol{\varphi}_k)\|_{\mathcal{L}(\mathcal{H},(H^1_{0L}(I)^n)')} = \sup_{\|(\vec{\boldsymbol{k}},\beta,\boldsymbol{\iota})\|_{\mathcal{H}}=1} \|(G'(\boldsymbol{\varphi}) - G'(\boldsymbol{\varphi}_k))(\vec{\boldsymbol{k}}, \beta, \boldsymbol{\iota})\|_{(H^1_{0L}(I)^n)'} \leq C\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_k\|_{\mathcal{H}}.$$

This implies, applying Proposition A.4, that $DG = G'$ and that G is C^1. □

We give a necessary condition for minimizers of $\mathcal{C}$ in $\mathcal{A}$.

Proposition 5.2. *Let* $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$ *be a minimizer of* $\mathcal{C}$ *in* $\mathcal{A}$ *such that* $\max_{i=1,\dots,n} |\vec{h}_i| < c_p^{-2}$. *There exists a Lagrange multiplier* $\boldsymbol{\lambda} \in H^1_{0L}(I)^n$ *such that* $(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta}, \boldsymbol{\lambda})$ *satisfies*

$$D\mathcal{C}(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})(\cdot) = \langle DG(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})(\cdot), \boldsymbol{\lambda}\rangle \qquad \text{in } \mathcal{H}'. \tag{5.6}$$

Proof. In view of Lemma 5.1 and Proposition A.5, we only need to prove that $DG(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$ is surjective, that is, for every $T \in (H^1_{0L}(I)^n)'$ there exists $\boldsymbol{\psi} \in \mathcal{H}$ such that

$$\langle DG(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})(\boldsymbol{\psi}), \boldsymbol{u}\rangle = \langle T, \boldsymbol{u}\rangle \quad \text{for all } \boldsymbol{u} = (u_1, \dots, u_n) \in H^1_{0L}(I). \tag{5.7}$$

It turns out that it suffices to choose $\boldsymbol{\psi} = (\boldsymbol{v}, 0, \vec{\boldsymbol{0}})$ with $\boldsymbol{v} \in H^1_{0L}(I)^n$. In this case, the l.h.s. of (5.7) defines a bilinear form, $a : H^1_{0L}(I)^n \times H^1_{0L}(I)^n \to \mathbb{R}$:

$$a(\boldsymbol{v}, \boldsymbol{u}) := \langle DG(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})(\boldsymbol{v}, 0, \vec{\boldsymbol{0}}), \boldsymbol{u}\rangle \overset{(5.4)}{=} \sum_{i=1}^n \int_0^1 v_i'u_i' - \vec{h}_i \cdot D^2\vec{m}(\alpha + \vartheta_i)v_iu_i. \tag{5.8}$$

If we prove that there exists $\boldsymbol{v} \in H^1_{0L}(I)^n$ such that

$$a(\boldsymbol{v}, \boldsymbol{u}) = \langle T, \boldsymbol{u}\rangle \quad \text{for any } \boldsymbol{u} \in H^1_{0L}(I)^n, \tag{5.9}$$

then (5.7) holds and $DG(\vec{\boldsymbol{h}}, \alpha, \boldsymbol{\vartheta})$ is surjective. By Lemma 5.1 we have that

$$|a(\boldsymbol{v}, \boldsymbol{u})| \leq C\|\boldsymbol{v}\|\|\boldsymbol{u}\|,$$

so a is a continuous form. Moreover,

$$a(\boldsymbol{u},\boldsymbol{u}) = \sum_{i=1}^n \int_0^1 \left\{ (u_i')^2 - \vec{h}_i \cdot D^2\vec{m}(\alpha+\vartheta_i)u_i^2 \right\}$$
$$\overset{(2.5)}{\geq} \sum_{i=1}^n \int_0^1 \left\{ (u_i')^2 - |\vec{h}_i| u_i^2 \right\} \overset{(2.2)}{\geq} \left(1 - c_p^2 \max_{i=1,\dots,n} \{|\vec{h}_i|\} \right) \|\boldsymbol{u}\|^2.$$

Hence, by (1.19), we have that a is coercive. Then, by the Lax-Milgram theorem, there exists $\boldsymbol{v} \in H^1_{0L}(I)^n$ such that (5.9) is satisfied. □

Now, starting from Proposition 5.2, we prove Proposition 1.3, which makes (5.6) explicit.

Proof of Proposition 1.3. It follows from (5.3) and (5.4) that (5.6) evaluated in $(\vec{\boldsymbol{k}},\beta,\boldsymbol{\iota}) \in \mathcal{H}$ is equivalent to

$$\gamma \sum_{i=1}^n \vec{h}_i \cdot \vec{k}_i + \varepsilon \int_0^1 \alpha'\beta' + \sum_{i=1}^n \int_0^1 (\vartheta_i - \bar{\vartheta}_i)\iota_i = \sum_{i=1}^n \int_0^1 -\vec{k}_i \cdot \lambda_i D\vec{m}(\alpha+\vartheta_i)$$
$$+ \sum_{i=1}^n \int_0^1 -\lambda_i \vec{h}_i \cdot D^2\vec{m}(\alpha+\vartheta_i)\beta + \sum_{i=1}^n \int_0^1 \left\{ \lambda_i'\iota_i' - \lambda_i \vec{h}_i \cdot D^2\vec{m}(\alpha+\vartheta_i)\iota_i \right\}. \tag{5.10}$$

Since DC and DG are linear w.r.t. $(\vec{\boldsymbol{k}},\beta,\boldsymbol{\iota})$, expanding (5.10) with respect to each component, we obtain that it is equivalent to

$$\begin{cases} \int_0^1 \left\{ \lambda_i'\iota_i' - \lambda_i \vec{h}_i \cdot D^2\vec{m}(\alpha+\vartheta_i)\iota_i \right\} = \int_0^1 (\vartheta_i - \bar{\vartheta}_i)\iota_i, \\ \varepsilon \int_0^1 \alpha'\beta' + \sum_{i=1}^n \int_0^1 \lambda_i \vec{h}_i \cdot D^2\vec{m}(\alpha+\vartheta_i)\beta = 0, & \forall i = 1,\dots,n. \\ \gamma \vec{k}_i \cdot \vec{h}_i = -\vec{k}_i \cdot \int_0^1 \lambda_i D\vec{m}(\alpha+\vartheta_i), \end{cases} \tag{5.11}$$

It follows from Remark 4.3 and adding to (5.11) the constraint $(\vec{\boldsymbol{h}},\alpha,\boldsymbol{\vartheta}) \in \mathcal{A}$ that $(\vec{\boldsymbol{h}},\alpha,\boldsymbol{\vartheta},\boldsymbol{\lambda})$ is a solution to (1.16).

In order to deduce $\alpha'(0) = 0$, we recall that $\vartheta_i, \alpha, \lambda_i \in C^1([0,1])$ (see Remark 4.3). Therefore, using (P_α), $\alpha \in C^2(I)$ and

$$\alpha'(1) - \alpha'(0) = \int_0^1 \alpha'' \overset{(P_\alpha)}{=} \frac{1}{\varepsilon} \sum_{i=1}^n \int_0^1 \lambda_i \vec{h}_i \cdot D^2\vec{m}(\alpha+\vartheta_i)$$
$$\overset{(2.4)}{=} \frac{1}{\varepsilon} \sum_{i=1}^n \int_0^1 \vec{h}_i \cdot \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \lambda_i D\vec{m}(\alpha+\vartheta_i)$$
$$\overset{(P_{\vec{h}_i})}{=} -\frac{\gamma}{\varepsilon} \sum_{i=1}^n \vec{h}_i \cdot \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \vec{h}_i = 0.$$

Hence $\alpha'(1) = \alpha'(0) \overset{(P_\alpha)}{=} 0$. □

6. A constructive scheme; uniqueness of solutions to the Lagrange multiplier formulation

In this section we introduce a constructive scheme to obtain solutions of the Euler-Lagrange formulation (1.16). We will prove its contractivity and, as a by-product, uniqueness of solutions to (1.16) (Theorem 1.5). The scheme consists of two steps and works as follows.

Step 1. In the first step, we fix $\alpha \in C([0,1])$. We introduce the set

$$D := \{\vec{\boldsymbol{h}} \in \mathbb{R}^{2n} : \max_{1\le i\le n} |\vec{h}_i| \le K\}, \quad \text{with } K < c_p^{-2} \text{ (cf. (1.20))}. \tag{6.1}$$

We will show that the chain

$$\vec{\boldsymbol{h}} \overset{(P_{\vartheta_i})}{\mapsto} \boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_n) \overset{(P_{\lambda_i})}{\mapsto} \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \overset{(P_{\vec{h}_i})}{\mapsto} \vec{\boldsymbol{T}}^{(\alpha)}(\vec{\boldsymbol{h}}) \in \mathbb{R}^{2n} \tag{6.2}$$

defines a map $\vec{\boldsymbol{T}}^{(\alpha)} : D \to D$. We then show that $\vec{\boldsymbol{T}}^{(\alpha)}$ is a contraction for γ sufficiently large. Then, by Proposition A.6, there exists a unique fixed point of $\vec{\boldsymbol{T}}^{(\alpha)}$ in D, $\vec{\boldsymbol{h}}(\alpha)$:

$$\vec{\boldsymbol{h}}(\alpha) = \vec{\boldsymbol{T}}^{(\alpha)}(\vec{\boldsymbol{h}}(\alpha)).$$

Step 2. In view of Step 1, we define $A : C([0,1]) \to C([0,1])$ as the unique function such that

$$\begin{cases} -A(\alpha)'' = -\dfrac{1}{\varepsilon}\displaystyle\sum_{i=1}^{n} \lambda_i(\alpha)\vec{h}_i(\alpha) \cdot D^2(\alpha + \vartheta_i(\alpha)) & \text{in } (0,1), \\ A(\alpha)|_0 = A(\alpha)'|_1 = 0, \end{cases} \tag{6.3}$$

where $\vartheta_i(\alpha), \lambda_i(\alpha)$ are the unique solutions to (P_{ϑ_i}), resp. (P_{λ_i}), with $\vec{\boldsymbol{h}} = \vec{\boldsymbol{h}}(\alpha)$. We will prove that A is a contraction, hence it has a unique fixed point, for γ sufficiently large.

Thanks to (6.3) and to (6.2), a quadruplet $(\vec{\boldsymbol{h}}(\alpha), \alpha, \boldsymbol{\vartheta}(\alpha), \boldsymbol{\lambda}(\alpha))$ is a solution to System (1.16) if and only if α is a fixed point of A. In particular, this implies the uniqueness result in Theorem 1.5.

Remark 6.1. There are three key features of System (1.16) that allow us to show that the maps $\vec{\boldsymbol{T}}^{(\alpha)}$ and A are contractions. Namely:

- the boundary-value problems in (1.16) share the same structure, that of (4.1);
- all lowest-order terms on the left-hand sides of the differential equations in (1.16) are *proportional* to the norm $|\vec{h}_i|$ of the applied fields, which in turn are controlled (for a minimizer) by the target shapes $\overline{\boldsymbol{\vartheta}}$ and by the regularization constant γ through the bound (1.15);
- the equation for $\vec{h}_i$ in (1.16) contain on the right-hand side the pre-factor γ^{-1}. Accordingly, as long as γ is large, we can control the applied fieds and hence also the solutions of (1.16).

We now prove the assertions formulated above.

Proposition 6.2. *Let D as in* (6.1). *Then:*

(*i*) *For any $\alpha \in C([0,1])$ there exists γ_1 (depending on $\bar{\vartheta}$ and K) such that for any $\gamma > \gamma_1$ the map $\vec{\boldsymbol{T}}^{(\alpha)} : D \to \mathbb{R}^{2n}$ defined in* (6.2) *has a unique fixed point in D, $\vec{\boldsymbol{h}}(\alpha)$; in particular,*

$$\max_{1\le i\le n} |\vec{h}_i(\alpha)| \le K < c_p^{-2};$$

(ii) *there exists $\gamma_* > \gamma_1$ (depending on $\overline{\vartheta}$, K, and ε) such that the map $A : C([0,1]) \to C([0,1])$ defined in (6.3) has a unique fixed point, $\alpha = A(\alpha)$. Furthermore, $\alpha \in C^2([0,1])$ and $\alpha(0) = \alpha'(1) = 0$;*

(iii) *consequently, Theorem 1.5 holds.*

Proof. We divide the proof into steps.

(A). There exists γ_0 such that $\vec{\boldsymbol{T}}^{(\alpha)}$ maps D in itself. Let $\vec{\boldsymbol{h}} \in D$. Thanks to the first equivalence in (4.2), (2.7b), and (1.20), we can apply Lemma 4.2 with $L = K$: for every $i = 1, \ldots, n$ there exists a unique solution $\vartheta_i \in H^1_{0L}(I)$ of (P_{ϑ_i}), which satisfies

$$\|\vartheta_i\|_\infty \overset{(4.5)_2}{\leq} K, \quad i = 1, \ldots, n. \tag{6.4}$$

By the same argument, the second equivalence in (4.2) and (2.7c) allow to apply Lemma 4.2 with $L = K$ and $f_0 = \overline{\vartheta}_i - \vartheta_i$: for every $i = 1, \ldots, n$ there exists a unique solution $\lambda_i \in H^1_{0L}(I)$ of (P_{λ_i}), such that

$$\begin{aligned} \|\lambda_i\|_\infty &\overset{(4.5)_1}{\leq} \frac{c_p}{1 - Kc_p^2}\|\vartheta_i - \overline{\vartheta}_i\|_\infty \leq \frac{c_p}{1 - Kc_p^2}\left(\|\vartheta_i\|_\infty + \|\overline{\vartheta}_i\|_\infty\right) \\ &\overset{(6.4)}{\leq} \frac{c_p}{1 - Kc_p^2}\left(K + \|\overline{\vartheta}_i\|_\infty\right) = C, \end{aligned} \tag{6.5}$$

where from now on C denotes a generic constant depending on $\overline{\boldsymbol{\vartheta}}$ and K, but independent of γ and ε. Therefore

$$|\vec{T}_i^{(\alpha)}(\vec{\boldsymbol{h}})| \overset{(6.2),(1.16)_4}{=} \frac{1}{\gamma}\left|\int_0^1 \lambda_i D\vec{m}(\alpha + \vartheta_i)\right| \overset{(2.5)}{\leq} \frac{2}{\gamma}\|\lambda_i\|_\infty \overset{(6.5)}{\leq} \frac{C}{\gamma}, \quad i = 1, \ldots, n. \tag{6.6}$$

This implies that, for $\gamma > \gamma_0$ large enough, the operator $\vec{\boldsymbol{T}}^{(\alpha)}$ maps D in itself.

For reasons which will be clarified later, we postpone the proof of (i), and for the moment we assume it to be true.

(B). Proof of (ii) assuming (i). Assume $\gamma > \gamma_1$, with γ_1 as given in (i). For α and $\tilde{\alpha}$ in $C([0,1])$, let $\vartheta_i = \vartheta_i(\alpha)$ and $\tilde{\vartheta}_i = \tilde{\vartheta}_i(\alpha)$, resp. $\lambda_i = \lambda_i(\alpha)$ and $\tilde{\lambda}_i = \tilde{\lambda}_i(\alpha)$, be the unique solutions to (P_{ϑ_i}), resp. (P_{λ_i}), with $\vec{\boldsymbol{h}} = \vec{\boldsymbol{h}}(\alpha)$ and $\tilde{\vec{\boldsymbol{h}}} = \vec{\boldsymbol{h}}(\tilde{\alpha})$ as defined in (i). It follows from (6.4) and (6.5) that

$$\|\vartheta_i\|_\infty + \|\lambda_i\|_\infty \leq C \quad \text{and} \quad \|\tilde{\vartheta}_i\|_\infty + \|\tilde{\lambda}_i\|_\infty \leq C, \quad i = 1, \ldots, n. \tag{6.7}$$

Let $A(\alpha)$ and $A(\tilde{\alpha})$ be defined by (6.3), and note that (6.3) is equivalent to

$$A(\alpha)(s) := -\frac{1}{\varepsilon}\sum_{i=1}^n \int_0^s \int_{s'}^1 \lambda_i(s'')\vec{h}_i \cdot D^2(\alpha(s'') + \vartheta_i(s''))ds''ds', \qquad \forall s \in [0,1]\,; \tag{6.8}$$

in particular, $A(\alpha) \in C^2([0,1])$. Therefore

$$
\begin{aligned}
&\|A(\alpha) - A(\tilde\alpha)\|_\infty \\
&\overset{(6.8)}{\le} \quad \frac{1}{\varepsilon}\sum_{i=1}^{n} \|\lambda_i \vec h_i \cdot D^2\vec m(\alpha+\vartheta_i) - \tilde\lambda_i \vec{\tilde h}_i \cdot D^2 \vec m(\tilde\alpha + \tilde\vartheta_i)\|_\infty \\
&\overset{(2.7c)}{\le} \quad \frac{1}{\varepsilon}\sum_{i=1}^{n} \Big(|\vec h_i| \|\lambda_i - \tilde\lambda_i\|_\infty + \|\tilde\lambda_i\|_\infty |\vec h_i - \vec{\tilde h}_i| + |\vec{\tilde h}_i| \|\tilde\lambda_i\|_\infty \Big(\|\alpha - \tilde\alpha\|_\infty + \|\vartheta_i - \tilde\vartheta_i\|_\infty \Big)\Big) \\
&\overset{(6.1),(6.7)}{\le} \quad \frac{C}{\varepsilon}\sum_{i=1}^{n} \Big(|\vec h_i - \vec{\tilde h}_i| + \|\alpha - \tilde\alpha\|_\infty + \|\vartheta_i - \tilde\vartheta_i\|_\infty + \|\lambda_i - \tilde\lambda_i\|_\infty \Big). \qquad (6.9)
\end{aligned}
$$

Now we will estimate the right hand side of (6.9). Taking $\lambda_i - \tilde\lambda_i$ as test function in the weak formulations for λ_i and $\tilde\lambda_i$ (cf. (4.2) and (4.3)) and subtracting the resulting equations, we obtain

$$
\begin{aligned}
\int_0^1 |(\lambda_i - \tilde\lambda_i)'|^2 \quad &= \quad \int_0^1 \Big(\lambda_i \vec h \cdot D^2 \vec m(\alpha + \vartheta_i) - \tilde\lambda_i \vec{\tilde h}_i \cdot D^2\vec m(\tilde\alpha + \tilde\vartheta_i)\Big)(\lambda_i - \tilde\lambda_i) \\
&\quad + \int_0^1 (\vartheta_i - \tilde\vartheta_i)(\lambda_i - \tilde\lambda_i) \\
&\overset{(2.7c)}{\le} \Big(\|\tilde\lambda_i\|_\infty |\vec{\tilde h}_i| + 1\Big) \int_0^1 |\vartheta_i - \tilde\vartheta_i||\lambda_i - \tilde\lambda_i| \\
&\quad + \|\tilde\lambda_i\|_\infty |\vec h_i - \vec{\tilde h}_i| \int_0^1 |\lambda_i - \tilde\lambda_i| + \|\tilde\lambda_i\|_\infty |\vec{\tilde h}_i| \int_0^1 |\alpha - \tilde\alpha||\lambda_i - \tilde\lambda_i| \\
&\quad + |\vec h_i| \int_0^1 |\lambda_i - \tilde\lambda_i|^2. \qquad (6.10)
\end{aligned}
$$

We estimate the last summand on the r.h.s. of (6.10) using the Poincaré inequality and the definition of D:

$$
|\vec h_i| \int_0^1 |\lambda_i - \tilde\lambda_i|^2 \overset{(2.2),(6.1)}{\le} K c_p^2 \int_0^1 |(\lambda_i - \tilde\lambda_i)'|^2.
$$

Absorbing this summand on the left-hand side of (6.10), we obtain:

$$
\begin{aligned}
\underbrace{(1 - K c_p^2)}_{>0 \text{ by } (1.20)} \int_0^1 |(\lambda_i - \tilde\lambda_i)'|^2 &\le \Big(\int_0^1 |\lambda_i - \tilde\lambda_i| \Big) \times \\
&\times \Big(\Big(\|\tilde\lambda_i\|_\infty |\vec{\tilde h}_i| + 1 \Big) \|\vartheta_i - \tilde\vartheta_i\|_\infty + \|\tilde\lambda_i\|_\infty |\vec h_i - \vec{\tilde h}_i| + \|\tilde\lambda_i\|_\infty |\vec{\tilde h}_i| \|\alpha - \tilde\alpha\|_\infty \Big) \\
&\overset{(6.1),(6.7)}{\le} C \Big(\int_0^1 |\lambda_i - \tilde\lambda_i| \Big) \Big(|\vec h_i - \vec{\tilde h}_i| + \|\vartheta_i - \tilde\vartheta_i\|_\infty + \|\alpha - \tilde\alpha\|_\infty \Big).
\end{aligned}
$$

Using Hölder and Poincaré inequalities, we deduce

$$
\|\lambda_i - \tilde\lambda_i\|_\infty \overset{(2.1)}{\le} \|\lambda_i - \tilde\lambda_i\| \le C \Big(|\vec h_i - \vec{\tilde h}_i| + \|\vartheta_i - \tilde\vartheta_i\|_\infty + \|\alpha - \tilde\alpha\|_\infty \Big). \qquad (6.11)
$$

In order to estimate $\|\vartheta_i - \tilde\vartheta_i\|_\infty$, we follow the same line of argument. We choose $\vartheta_i - \tilde\vartheta_i$ as test function in the weak formulations for ϑ_i and for $\tilde\vartheta_i$ (cf. (4.3)): subtracting the resulting equations, we have

$$\int_0^1 |(\vartheta_i - \tilde{\vartheta}_i)'|^2 \quad = \quad \int_0^1 \big(\vec{h}_i \cdot D\vec{m}(\alpha+\vartheta_i) - \vec{\tilde{h}}_i \cdot D\vec{m}(\tilde{\alpha}+\tilde{\vartheta}_i)\big)(\vartheta_i - \tilde{\vartheta}_i)$$

$$\overset{(2.7b)}{\leq} \quad \Big(|\vec{h}_i - \vec{\tilde{h}}_i| + |\vec{h}_i|\|\alpha - \tilde{\alpha}\|_\infty\Big)\int_0^1 |\vartheta_i - \tilde{\vartheta}_i| + |\vec{h}_i|\int_0^1 |\vartheta_i - \tilde{\vartheta}_i|^2.$$

As above, the second summand may be absorbed on the left-hand side via Poincaré inequality and the assumption that $K < c_p^{-2}$, whereas the first one can be treated by Hölder and Poincaré inequality (the specific constant being irrelevant in this case). Altogether, we obtain

$$\|\vartheta_i - \tilde{\vartheta}_i\|_\infty \overset{(2.1)}{\leq} \|\vartheta_i - \tilde{\vartheta}_i\| \leq C\left(|\vec{h}_i - \vec{\tilde{h}}_i| + \|\alpha - \tilde{\alpha}\|_\infty\right). \tag{6.12}$$

Now we estimate $|\vec{h}_i - \vec{\tilde{h}}_i|$. By the definition of $T_i^{(\alpha)}$ we deduce that

$$|\vec{h}_i - \vec{\tilde{h}}_i| \quad \leq \quad \frac{1}{\gamma}\|\lambda_i D\vec{m}(\alpha+\vartheta_i) - \tilde{\lambda}_i D\vec{m}(\tilde{\alpha}+\tilde{\vartheta}_i)\|_\infty$$

$$\overset{(2.7a),(6.7)}{\leq} \quad \frac{C}{\gamma}\left(\|\alpha - \tilde{\alpha}\|_\infty + \|\vartheta_i - \tilde{\vartheta}_i\|_\infty + \|\lambda_i - \tilde{\lambda}_i\|_\infty\right). \tag{6.13}$$

Inserting (6.11) and (6.12) in (6.13), we deduce that there exists C_2 such that

$$|\vec{h}_i - \vec{\tilde{h}}_i| \leq \frac{C_2}{\gamma}\left(\|\alpha - \tilde{\alpha}\|_\infty + |\vec{h}_i - \vec{\tilde{h}}_i|\right). \tag{6.14}$$

Taking $\gamma_2 = C_2$, we have $C_2/\gamma < 1$ for $\gamma > \gamma_2$, so that

$$|\vec{h}_i - \vec{\tilde{h}}_i| \leq \frac{C_2}{\gamma - C_2}\|\alpha - \tilde{\alpha}\|_\infty \tag{6.15}$$

for $\gamma > \gamma_2$. Using (6.15) in (6.12), we obtain

$$\|\vartheta_i - \tilde{\vartheta}_i\|_\infty \leq \frac{C}{\gamma - C_2}\|\alpha - \tilde{\alpha}\|_\infty. \tag{6.16}$$

In turn, using (6.15) and (6.16) in (6.11), we obtain

$$\|\lambda_i - \tilde{\lambda}_i\|_\infty \leq \frac{C}{\gamma - C_2}\|\alpha - \tilde{\alpha}\|_\infty. \tag{6.17}$$

Finally, inserting (6.15), (6.16), and (6.17) into (6.9), we deduce that there exist C_3 such that

$$\|A(\alpha) - A(\tilde{\alpha})\|_\infty \leq \frac{1}{\varepsilon}\frac{C_3}{\gamma - C_2}\|\alpha - \tilde{\alpha}\|_\infty. \tag{6.18}$$

We now set $\gamma_3 = C_2 + C_3/\varepsilon > \gamma_2$, so that the prefactor in (6.18) is smaller than 1 for every $\gamma > \max\{\gamma_1, \gamma_3\} =: \gamma_*$. By Proposition A.6, for $\gamma > \gamma_*$ there exists a unique fixed point $\alpha \in C([0,1])$ of A.

We finally return to the proof of (i), which we postponed since its proof is simpler than that of (ii), in that we may use the same estimates as in (B) with $\alpha = \tilde{\alpha}$.

(C). Proof of (i). We prove that $\boldsymbol{\vec{T}}^{(\alpha)}$ is a contraction. Let $\gamma > \gamma_0$, as given in (A). Given $\boldsymbol{\vec{h}}, \boldsymbol{\vec{\tilde{h}}} \in D$, we define ϑ_i and $\tilde{\vartheta}_i$, resp. λ_i and $\tilde{\lambda}_i$, as the corresponding unique solutions of (P_{ϑ_i}), resp. (P_{λ_i}). Then, the same arguments of (B) may be applied with $\alpha = \tilde{\alpha}$, yielding

$$\|\lambda_i - \tilde{\lambda}_i\|_\infty \leq C\left(|\vec{h}_i - \vec{\tilde{h}}_i| + \|\vartheta_i - \tilde{\vartheta}_i\|_\infty\right) \tag{6.19}$$

(cf. (6.11)) and

$$\|\vartheta_i - \tilde\vartheta_i\|_\infty \overset{(2.1)}{\le} \|\vartheta_i - \tilde\vartheta_i\| \le C|\vec h_i - \tilde{\vec h}_i| \tag{6.20}$$

(cf. (6.12)). Therefore

$$\begin{aligned}|\vec T_i^{(\alpha)}(\vec{\boldsymbol h}) - \vec T_i^{(\alpha)}(\tilde{\vec{\boldsymbol h}})| &\overset{(6.2),(1.16)_4}{=} \frac{1}{\gamma}\left|\int_0^1 \Big(\lambda_i D\vec m(\alpha+\vartheta_i) - \tilde\lambda_i D\vec m(\alpha+\tilde\vartheta_i)\Big)\right| \\ &\overset{(2.7a),(6.5)}{\le} \frac{C}{\gamma}\Big\{\|\lambda_i - \tilde\lambda_i\|_\infty + \|\vartheta_i - \tilde\vartheta_i\|_\infty\Big\} \overset{(6.19),(6.20)}{\le} \frac{C_1}{\gamma}|\vec h_i - \tilde{\vec h}_i|.\end{aligned}$$

Choosing $\gamma_1 = C_1$, we conclude that $\vec{\boldsymbol T}^{(\alpha)}$ is a contraction for every $\gamma > \gamma_1$.

(D). Proof of (iii). Theorem 1.5 is an immediate consequence of (i) and (ii). Indeed, let $\alpha = A(\alpha)$ be the fixed point of A identified in (ii), and let $\vec{\boldsymbol h}(\alpha) = \vec{\boldsymbol T}^{(\alpha)}(\vec{\boldsymbol h}(\alpha))$ be the fixed point identified in (i). Then, by construction, the quadruplet $(\vec{\boldsymbol h}(\alpha), \alpha, \boldsymbol\vartheta(\alpha), \boldsymbol\lambda(\alpha))$ is a solution to system (1.16) in the class (1.21). Viceversa, if two solutions of (1.16) exist in that class, then they are both fixed points of A, hence they coincide.

□

Remark 6.3. Under the provision that $K < 1$, a bound similar to (6.19) might be obtained from the representation formula

$$\lambda_i - \tilde\lambda_i = \int_0^s \int_{s'}^1 \Big(\lambda_i \vec h_i \cdot D^2\vec m(\alpha+\vartheta_i) - \tilde\lambda_i \tilde{\vec h}_i \cdot D^2\vec m(\alpha+\tilde\vartheta_i)\Big)\, ds'' ds'.$$

Indeed, from

$$\|\lambda_i - \tilde\lambda_i\|_\infty \overset{(2.7c)}{\le} \|\tilde\lambda_i\|_\infty |\tilde{\vec h}_i| \|\vartheta_i - \tilde\vartheta_i\|_\infty + \|\tilde\lambda_i\|_\infty |\vec h_i - \tilde{\vec h}_i| + |\vec h_i| \|\lambda_i - \tilde\lambda_i\|_\infty,$$

we obtain, for $K < 1$,

$$\begin{aligned}\|\lambda_i - \tilde\lambda_i\|_\infty &\overset{(6.5),(6.1)}{\le} \frac{1}{1-K}\Big(KC(\overline{\boldsymbol\vartheta}, K)\|\vartheta_i - \tilde\vartheta_i\|_\infty + C(\overline{\boldsymbol\vartheta}, K)|\vec h_i - \tilde{\vec h}_i|\Big) \\ &\le C(\overline{\boldsymbol\vartheta}, K)\big(\|\vartheta_i - \tilde\vartheta_i\|_\infty + |\vec h_i - \tilde{\vec h}_i|\big).\end{aligned}$$

However, the requirement $K < 1$ is stricter than our assumption (1.20) because $c_p < 1$.

Remark 6.4. The constant γ_* in Theorem 1.5 increases monotonically with K and blows up as K tends to c_p^{-2}. Since $\gamma_{**} = \max(\overline{\Theta}^2/K^2, \gamma_*)$, we have that γ_{**} blows up as K tends to 0 and as K tends to c_p^{-2}. The blow up for K small is obvious: as K tends to 0 the maximum allowed applied field tends to 0 in intensity, and to limit the applied field we need γ large. The blow up for K approaching c_p^{-2} is essentially technical, and follows from the estimates in the proof of Theorem 1.5 becoming degenerate as K tends to c_p^{-2}. Ultimately, this is because we rely only on the first estimate in (4.5) in Lemma 4.2, which becomes degenerate when the Lipschitz constant L (identified with the intensity of the magnetic field) approaches c_p^{-2}.

Let us conclude the Section with a digression on the case in which α is fixed. Part (i) of Proposition 6.2 may be rephrased as follows:

Proposition 6.5. *Let $\overline{\boldsymbol\vartheta} \in C([0,1])^n$, $\varepsilon > 0$, and let $K > 0$ such that (1.20) holds. Then there exists $\gamma_* = \gamma_*(\overline{\boldsymbol\vartheta}, K)$ such that for every $\gamma > \gamma_*$ and any $\alpha \in C([0,1])$ there exists a* *unique solution of the system*

$$\begin{cases} -\vartheta_i'' - \vec{h}_i \cdot D\vec{m}(\alpha+\vartheta_i) = 0, & \vartheta_i(0) = \vartheta_i'(1) = 0 \\ -\lambda_i'' - \lambda_i \vec{h}_i \cdot D^2\vec{m}(\alpha+\vartheta_i) = \vartheta_i - \bar{\vartheta}_i, & \lambda_i(0) = \lambda_i'(1) = 0 \quad \forall i = 1\dots,n \\ \vec{h}_i = -\dfrac{1}{\gamma}\displaystyle\int_0^1 \lambda_i D\vec{m}(\alpha+\vartheta_i) \end{cases} \tag{6.21}$$

within the following set:

$$(\vec{\boldsymbol{h}}, \boldsymbol{\vartheta}, \boldsymbol{\lambda}) \in D \times H^1_{0L}(I)^n \times H^1_{0L}(I)^n. \tag{6.22}$$

Remark 6.6. For fixed α, the solution in Proposition 6.5 is the unique stationary point of the functional

$$\tilde{\mathcal{C}}(\vec{\boldsymbol{h}}, \boldsymbol{\vartheta}) = \frac{1}{2}\sum_{i=1}^n \int_0^1 |\vartheta_i - \bar{\vartheta}_i|^2 + \frac{\gamma}{2}\sum_{i=1}^n |\vec{h}_i|^2, \tag{6.23}$$

in the admissible set

$$(\vec{\boldsymbol{h}}, \boldsymbol{\vartheta}) \in \tilde{\mathcal{A}} := \left\{ (\vec{\boldsymbol{h}}, \boldsymbol{\vartheta}) \in D \times H^1_{0L}(I)^n : \ \vartheta_i \text{ solves } (P_{\vartheta_i}) \text{ for every } i = 1, \dots, n \right\}.$$

Therefore, arguing as we did for the full problem, for γ sufficiently large it follows from Proposition 6.5 that:

- $\tilde{\mathcal{C}}$ has a unique minimizer;
- looking for the minimum $(\vec{\boldsymbol{h}}, \boldsymbol{\vartheta})$ of $\tilde{\mathcal{C}}$ is equivalent to looking for the fixed point of $\vec{\boldsymbol{T}}^{(\alpha)}$.

7. Concluding remarks and open problems

We have considered a beam clamped at one side, modeled as a planar elastica. The beam has a permanent magnetization (the design α), hence it deforms under the action of spatially-constant magnetic fields $\vec{h}_i$, $i = 1, \dots, n$ (the controls). Given a list of n prescribed target shapes $\bar{\vartheta}_i$ ($i = 1, \dots, n$), we have looked for optimal design and controls in order for the corresponding shapes ϑ_i ($i = 1, \dots, n$) of the beam to get as close as possible to the corresponding targets. Choosing the cost functional as in (1.10) has lead us to the formulation of an optimal design-control problem (cf. (1.14)), whose minimization has been studied by both direct and indirect methods. Loosely speaking, we have shown that:

- minimizers $(\alpha, \vec{\boldsymbol{h}})$ exist (Proposition 1.2);
- provided the intensity of $\vec{\boldsymbol{h}}$ is sufficiently small (cf. (1.19)), minimizers solve the Lagrange multiplier formulation (1.16) (Proposition 1.3);
- if the parameter γ penalizing the cost of the fields' intensity is sufficiently high, the minimizer is unique, satisfies (1.19), and is the unique solution to the Lagrange multiplier formulation (1.16) (Theorem 1.5 and Corollary 1.6).

In what follows, we briefly discuss a numerical scheme which naturally emerges from the proof of Theorem 1.5, as well as a different choice of the cost functional, using residuals. We also point out open question related to uniqueness and to the refinement of estimates, as well as two possible generalizations of our choice of the cost.

• **The numerical scheme.** The proof of Theorem 1.5 suggests an alternative to the numerical scheme proposed in [8]. The new scheme is based on two nested loops. In the inner loop, α is fixed and $\vec{\boldsymbol{h}}$, $\boldsymbol{\lambda}$, and $\boldsymbol{\vartheta}$ are computed by a fixed point iteration scheme which uses, in the order, equations (P_{ϑ_i}), (P_{λ_i}) and $(P_{\vec{h}_i})$; in the outer loop, α is updated by using the equation (P_α) with $\vec{\boldsymbol{h}}$, $\boldsymbol{\lambda}$, and $\boldsymbol{\vartheta}$ obtained from the inner loop. Each loop terminates when the update of each variable results in an increment below a certain tolerance *tol*. The algorithm is described in the pseudocode aside. Note that, in this algorithm, steps to be performed for $i = 1, \dots n$ do not need to be carried out sequentially, but can also be done in parallel, since they are independent on each other.

Initialisation:
  $\alpha \longleftarrow$ initial guess $\alpha^{(0)}$;
  $\vec{h}_i \longleftarrow$ initial guess $\vec{h}_i^{(0)}$, $i = 1, \dots n$;
  $\lambda_i \longleftarrow$ initial guess $\lambda_i^{(0)}$, $i = 1, \dots n$;
  $tol \longleftarrow$ tolerance;
repeat
  repeat
    $\vartheta_i \longleftarrow$ solve (P_{ϑ_i}), $i = 1, \dots, n$;
    $\lambda_i \longleftarrow$ solve (P_{λ_i}), $i = 1, \dots, n$;
    $\vec{h}_i^{old} \longleftarrow \vec{h}_i$;
    $\vec{h}_i \longleftarrow$ solve $(P_{\vec{h}_i})$, $i = 1, \dots, n$;
  until $\max_{i=1}^n |\vec{h}_i^{old} - \vec{h}_i| \le tol$;
  $\alpha^{old} \longleftarrow \alpha$;
  $\alpha \longleftarrow$ solve (P_α);
until $\|\alpha^{old} - \alpha\|_\infty \le tol$;

• **Using residuals to assess shape attainment.** Shape programming has been addressed in [17] under slightly more general conditions than those considered in this paper. In particular, [17] allows the magnetization intensity to be non-constant and the magnetic field to be non-uniform, and assigns a different weight to each shape. The unknown fields are represented through their Fourier expansion truncated at order k. Within our framework (constant magnetic intensity, uniform applied field, and same weight for all shapes), the approach proposed in [17] would lead to the minimization of the following functional:

$$\widetilde{E}(\alpha, \vec{\boldsymbol{h}}) = \sum_{i=1}^{n} \int_0^1 \left| -\overline{\vartheta}_i'' - \vec{h}_i \cdot \vec{m}(\alpha + \overline{\vartheta}_i) \right|^2 . \tag{7.1}$$

The integrands in (7.1) represent *residuals*, in the sense that they vanish if the targets $\overline{\vartheta}_i$ are themselves solution to the state equation (P_{ϑ_i}). Such minimization would be carried out in the space of designs α whose first k Fourier coefficients are in a bounded set and control fields $\vec{\boldsymbol{h}}$ whose magnitude does not exceed a constant K. However, from the purely theoretical point of view, it would be useful to have estimates of the attainment error

$$E(\alpha, \vec{\boldsymbol{h}}) = \frac{1}{2} \sum_{i=1}^{n} \int_0^1 |\overline{\vartheta}_i - \Theta_\alpha(\vec{h}_i)|^2$$

(cf. (1.9)) for solutions of both the optimization problem considered in [17] and the problem considered in this paper. In this respect, a first problem to be solved would be obtaining a bound of $E(\alpha, \vec{\boldsymbol{h}})$ in terms of $\widetilde{E}(\alpha, \vec{\boldsymbol{h}})$, where $(\alpha, \vec{\boldsymbol{h}})$ is a minimizer of (7.1).

• **The condition** $|\vec{h}| < c_p^{-2}$**.** Within our approach, even if $n = 1$ and $\alpha, \vec{h}$ are fixed, we can guarantee uniqueness of solutions to Problem (P_ϑ) only if $|\vec{h}| < c_p^{-2}$ (cf.(1.7)), a condition which then drives our analysis of the full problem. We have reasons to believe that this restriction is not optimal. For example, the bifurcation diagrams in [7] provide numerical evidence, at least for the clamped elastica, that uniqueness of equilibria holds for loads two or three times higher than those for which uniqueness is guaranteed by convexity of the energy or fixed-point arguments applied to the Euler-Lagrange system. We expect that this is due to a gap, which seems nontrivial to be captured, between loss of convexity and emergence of local critical points. The same considerations arise when comparing the theory and the numerical simulations in [9], a paper which investigates equilibria of a cantilever undergoing a uniformly-distributed vertical dead load. Improving the conditions on $\vec{h}$ which guarantee uniqueness seems to be an interesting open problem that deserves attention.

• **$\boldsymbol{\vartheta}$ versus $\overline{\boldsymbol{\vartheta}}$.** As we mentioned in the Introduction, Corollary 1.6 holds for any target $\overline{\boldsymbol{\vartheta}}$ (even very large ones). However, the minimizing state $\boldsymbol{\vartheta}$ will anyway be such that $\|\vartheta_i\|_\infty \le K$ for all $i = 1, \dots, n$ (see (6.4) in the proof of Theorem 1.5), meaning that the minimizing states may turn out not to be "close" to the targets when the latter ones are large. Also with an eye to the evolutive case, in which the n targets would reperesent discrete snapshots of a continuous movement, this limitation points towards a refinement of the estimates of $\boldsymbol{\vartheta}$ in terms of $\overline{\boldsymbol{\vartheta}}$.

• **Variable intensity of the magnetization.** Further developments of the present work may include a variable intensity of the magnetization. In this case, if we let $\mu(s)M_0$ be the magnetization density in the undeformed configuration, then the energy functional (2.5) would be replaced by

$$\widetilde{\mathcal{E}}(\vartheta) = \int_0^1 \frac{1}{2}(\vartheta')^2 - \mu\vec{h}\cdot\vec{m}(\vartheta+\alpha).$$

Such modification would also require a regularization to limit the oscillations of μ, as well as a penalization of negative values. Instead of choosing μ and α as design variables for the magnetization, one might choose the vector $\vec{\mu} = \mu\vec{m}(\alpha)$. In terms of this vector, the energy would take the form

$$\widetilde{\mathcal{E}}(\vartheta) = \int_0^1 \frac{1}{2}(\vartheta')^2 - \vec{h}\cdot\mathbf{R}(\vartheta)\vec{\mu},$$

where $\mathbf{R}(v) = \begin{pmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{pmatrix}$ is the counterclockwise rotation of the angle v. Such extension should be accompanied by a penalization of the oscillation of the vector field $\vec{\mu}$.

• **Non-quadratic costs.** A non trivial generalization of the present work we may consider more general costs, of the form

$$\sum_{i=1}^{n} \|\overline{\vartheta}_i - \vartheta\|_{L^p} + e(\|\alpha'\|_{L^p}) + g\Big(\big(\sum_{i=1}^{n} |\vec{h}_i|^p\big)^{1/p}\Big) \tag{7.2}$$

where $p \in [2, +\infty]$ and e and g are convex functions (for instance, the indicator functions of convex sets). Such more general situation would likely require techniques completely different from those used in this paper, especially in the case of e and g nonsmooth.

Appendix

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}}), (\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be Banach spaces, $\mathcal{U}$ an open subset of $\mathcal{X}$. We shall also consider a generic map $F : \mathcal{U} \to \mathcal{Y}$.

Definition A.1. *F has Gâteaux derivative $F'(x_0)$ at the point $x_0 \in \mathcal{U}$ if there exist*

$$F'(x_0)(v) := \lim_{t\to 0} \frac{F(x_0+tv) - F(x_0)}{t}, \qquad \forall v \in \mathcal{X}.$$

Definition A.2. *F is called Fréchet differentiable at $x_0 \in \mathcal{U}$ if there exists $DF(x_0) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ such that*

$$\lim_{\|h\|_{\mathcal{X}} \to 0} \frac{\|F(x_0 + h) - F(x) - DF(x_0)(h)\|_{\mathcal{Y}}}{\|h\|_{\mathcal{X}}} = 0.$$

Moreover we give the notion of continuous differentiable operator. Let T belong to $\mathcal{L}(\mathcal{X}, \mathcal{Y})$. We recall that the operator norm is defined by

$$\|T\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} := \sup_{\{0 \neq x \in \mathcal{X}\}} \frac{\|T(x)\|_{\mathcal{Y}}}{\|x\|_{\mathcal{X}}} = \sup_{\|x\|_{\mathcal{X}}=1} \|T(x)\|_{\mathcal{Y}}.$$

Definition A.3. *We say that F is C^1 if $DF(x)$ exists for every $x \in \mathcal{U}$ and $DF : \mathcal{U} \to \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a continuous operator.*

We recall the following proposition linking Gâteaux derivability and Fréchet differentiability.

Proposition A.4. *If F admits Gâteaux derivative $F'(x)$ in an open neighborhood $\mathcal{V} \subset \mathcal{U}$ of x_0 and $F' : \mathcal{V} \to \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is continuous at x_0, then F is Fréchet differentiable at x_0 and $DF(x_0) = F'(x_0)$. Moreover if $F' : \mathcal{U} \to \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a continuous operator, then $DF = F'$ and F is C^1.*

Proof. See [25], p. 274. □

We denote by $\mathcal{X}'$ the dual space of $\mathcal{X}$ and by $\langle \cdot, \cdot \rangle : \mathcal{X}' \times \mathcal{X} \to \mathbb{R}$ the duality pairing defined as $\langle S, t \rangle = S(t)$, for every $t \in \mathcal{X}, S \in \mathcal{X}'$.

Proposition A.5 (Existence of a Lagrange multiplier: [25], p. 270)**.** *Let $f : \mathcal{U} \subset \mathcal{X} \to \mathbb{R}$ and $G : \mathcal{U} \subset \mathcal{X} \to \mathcal{Y}$ be C^1 on an open neighborhood $\mathcal{U}$ of $\tilde{x}$. Suppose that $\tilde{x}$ is an extremum of f on the set $\{x \in \mathcal{U} : G(x) = 0\}$ and that*

$$DG(\tilde{x}) : \mathcal{X} \to \mathcal{Y} \qquad \text{is a surjective linear operator.}$$

Then there exists a Lagrange multiplier $\lambda \in \mathcal{Y}'$ such that

$$Df(\tilde{x}) - \langle \lambda, DG(\tilde{x}) \rangle = 0.$$

Proposition A.6 (Contraction Theorem [24, p.18])**.** *Let $T : \mathcal{X} \to \mathcal{X}$. If $L \in (0, 1)$ exists such that*

$$\|T(x) - T(y)\|_{\mathcal{X}} \leq L\|x - y\|_{\mathcal{X}}, \qquad \forall x, y \in \mathcal{X},$$

then T admits a unique fixed point $x^* \in \mathcal{X}$ (i.e. $T(x^*) = x^*$).

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