CESÀRO AVERAGES FOR GOLDBACH REPRESENTATIONS WITH SUMMANDS IN ARITHMETIC PROGRESSIONS

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Abstract. Let $\Lambda(n)$ be the von Mangoldt-function, let $n \geq 2$ be an integer and let

$$R_G(n; q, a, b) := \sum_{\substack{m_1+m_2=n \\ m_1 \equiv a \bmod q \\ m_2 \equiv b \bmod q}} \Lambda(m_1)\Lambda(m_2)$$

be the counting function for the Goldbach numbers with summands in arithmetic progression modulo a common integer q. We prove an asymptotic formula for the weighted average, with Cesàro weight of order $k > 1$, with $k \in \mathbb{R}$, of this function. Our result is uniform in a suitable range for q.

Keywords: Goldbach representations, Cesàro averages, primes in arithmetic progressions
MSC[2010]: Primary 11P32, Secondary 44A10

1. Introduction

In recent years a great deal of papers treated various questions related to the average number of representations of an integer as a sum of prime numbers, or powers of primes. Early instances of such problems can be traced back at least to the work of Walfisz [24], and Chandrasekharan & Narasimhan [8]. A more detailed introduction to these topics can be found in Languasco's survey paper [11]. About a decade or so, Languasco & Zaccagnini used the above techniques in the papers [13] and [14], in order to gain a deeper understanding of asymptotic formulae in two important problems of additive number theory, namely the Hardy-Littlewood and the Goldbach problem respectively. This brought about a large amount of work, culminating in Brüdern, Kaczorowski & Perelli's paper [3], where they improve on the results in [14] using a radically new approach in the spirit of the classical proof of the Prime Number Theorem.

Among recent papers dealing with such averages we also mention the following: Unweighted averages have been considered by Languasco & Zaccagnini [12], and by Bhowmik, Halupczok, Matsumoto & Suzuki [2]. Weighted averages appear in a fairly large number of papers, as Cantarini [4], [5], [6], [7]; Languasco & Zaccagnini [15], [16], [17]; Goldston & Yang [10].

Here we deal with averages with a Cesàro weight, with the constraint that the summands in the Goldbach representations lie in fixed arithmetic progressions modulo a common integer q. The main advantage in using weighted averages instead of the classical averages (see [2]) is that the Cesàro weights makes functions of the type $f(x) := \sum_{n \leq x} a_n (x-n)^k$, where $k \in \mathbb{R}_0^+$, "smoother" as k increases. Since the main formula is based on an inverse integral transform, the regularity of the function $f(x)$ is strictly connected to the "quality" of the asymptotic formula. In our specific case, we are able to obtain an asymptotic formula that holds uniformly for a suitable range of q which clearly highlights the role of the exceptional zero. Furthermore, some heuristic argument suggests that our formula holds for all $k > 0$ and may be a reason for further research in the future.

Date: March 22, 2021.

Let q be a positive integer. For integers a and b coprime to q we define

$$R_G(n;q,a,b) := \sum_{\substack{m_1+m_2=n \\ m_1\equiv a \bmod q \\ m_2\equiv b \bmod q}} \Lambda(m_1)\Lambda(m_2),$$

where Λ is the usual von Mangoldt-function. We could consider a more general definition with the summands lying in progressions with different q_1 and q_2, but then the conditions $m_1 \equiv a \bmod q_1$ and $m_2 \equiv b \bmod q_2$ would become $m_1 \bmod q \in \mathcal{R}_1$ and $m_2 \bmod q \in \mathcal{R}_2$, where q is the least common multiple of q_1 and q_2 and $\mathcal{R}_1$ and $\mathcal{R}_2$ are suitable sets of residue classes $\bmod\, q$. Averages of $R_G(n;q,a,b)$ have been treated by many authors in the past years, see the introduction of [2] and, e.g., Ruppel [21] and Suzuki [22], as it is known that almost all even integers satisfying some congruence condition can be written as the sum of two primes in congruence classes in analogy with the Goldbach problem.

For $z = x + iy$ with $x \in \mathbb{R}^+$ and $y \in \mathbb{R}$ we also set

$$\widetilde{S}_{a,q}(z) := \sum_{\substack{m\geq 1 \\ m\equiv a \bmod q}} \Lambda(m)\, \mathrm{e}^{-mz}.$$

It is clear that R_G is the generating function of $\widetilde{S}_{a,q}(z)\widetilde{S}_{b,q}(z)$, that is

$$\widetilde{S}_{a,q}(z)\widetilde{S}_{b,q}(z) = \sum_{m\geq 1} R_G(m;q,a,b)\, \mathrm{e}^{-mz}.$$

Our goal is to study averages of the quantities R_G, so we introduce a real parameter $k \geq 0$ and define

$$\Sigma_k(N;q,a,b) := \sum_{n\leq N} R_G(n;q,a,b)\frac{(N-n)^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_{(x)} \mathrm{e}^{Nz}\widetilde{S}_{a,q}(z)\widetilde{S}_{b,q}(z)\,\frac{\mathrm{d}z}{z^{k+1}}, \tag{1}$$

where Γ is the Euler Gamma-function. We will express Σ_k as a sum of a main term, a secondary term and other smaller terms, depending explicitly on the zeros of the relevant Dirichlet L-functions. The inversion of infinite integral and series in (1) can be justified as in [14] for $k > 0$. See §§5–6.

This method was used in most of the papers quoted above to deal with weights with $k > 1$, while Brüdern, Kaczorowski & Perelli [3] used their new approach to deal with the range $k > 0$ in the case of the Goldbach problem. The latter approach carries over, with modifications, for fixed q. Our goal in this paper, however, is to obtain a uniform version with an explicit dependence on the possible exceptional zero. Therefore, we believe that using a simpler method in a smaller range for k has independent interest.

For simplicity, we denote by $\mathcal{Z}(\chi)$ the set of non-trivial zeros of the Dirichlet L-function $L(s,\chi)$. We can now define the terms in our decomposition:

$$M_k^{(1)}(N;q) = \frac{N^{k+2}}{\phi(q)^2\Gamma(k+3)}, \tag{2}$$

$$M_k^{(2)}(N;q,a) = -\frac{1}{\phi(q)^2}\sum_{\chi \bmod q} \overline{\chi}(a) \sum_{\rho\in\mathcal{Z}(\chi^*)} \frac{\Gamma(\rho)N^{k+1+\rho}}{\Gamma(k+2+\rho)}, \tag{3}$$

$$M_k^{(3)}(N;q,a,b) = \frac{1}{\phi(q)^2}\sum_{\chi_1,\chi_2 \bmod q} \overline{\chi}_1(a)\overline{\chi}_2(b) \sum_{\substack{\rho_1\in\mathcal{Z}(\chi_1^*) \\ \rho_2\in\mathcal{Z}(\chi_2^*)}} \frac{\Gamma(\rho_1)\Gamma(\rho_2)N^{k+\rho_1+\rho_2}}{\Gamma(k+1+\rho_1+\rho_2)}, \tag{4}$$

where here and throughout the paper χ^* denotes the primitive character that induces χ. It is also convenient to write

$$M_k(N;q,a,b) = M_k^{(1)}(N;q) + M_k^{(2)}(N;q,a) + M_k^{(2)}(N;q,b) + M_k^{(3)}(N;q,a,b). \qquad (5)$$

Throughout the paper, implicit constants may depend only on k. Our main result is the following theorem.

Theorem 1. *For $k > 1$ and as $N \to +\infty$ we have*

$$\Sigma_k(N;q,a,b) = M_k(N;q,a,b) + O_k((g(q) + \log N)N^{k+1}),$$

uniformly for $q \leq (\log N)^A$ and every a, b with $(a,q) = (b,q) = 1$, for any fixed $A > 0$ and where

$$g(q) = \begin{cases} \log^2(q) & \text{if there is no exceptional zero modulo } q, \\ q^{1/2}\log^2(q) & \text{if there is an exceptional zero modulo } q. \end{cases} \qquad (6)$$

We notice that we essentially detect the second main term in Bhowmik et al. [2] for $k = 0$, although we prove our Theorem 1 only for $k > 1$. The case $q = 1$ was treated in [14], and in fact we closely follow its proof. The weak uniformity bound in our Theorem and the value of g are both due to the possible exceptional (or Siegel) zero of a Dirichlet L-function attached to a primitive real character modulo a "small" q. The turning point of the proof of Theorem 1 is Lemma 1, which we prove in §3. In fact, a weaker version would suffice in establishing the bounds which are necessary for the exchange of single and double series with the line integral, because in this case we may assume that q is fixed.

As usual in this kind of problems, it is technically convenient to work with the von Mangoldt-function $\Lambda(n)$ instead of functions that run only over primes, because we can use the important tool of the explicit formula. This trick does not have an important impact in the study of asymptotic formulae. Let

$$r_G(n;q,a,b;\alpha,\beta) := \sum_{\substack{p_1^\alpha + p_2^\beta = n \\ p_1^\alpha \equiv a \bmod q \\ p_2^\beta \equiv b \bmod q}} \log(p_1)\log(p_2)$$

and

$$\sigma_k(N;q,a,b;\alpha,\beta) := \sum_{n\leq N} r_G(n;q,a,b;\alpha,\beta)\frac{(N-n)^k}{\Gamma(k+1)}.$$

We easily see that the difference $\Sigma_k(N;q,a,b) - \sigma_k(N;q,a,b;1,1)$ is essentially dominated by $\sigma_k(N;q,a,b;1,2) + \sigma_k(N;q,a,b;2,1)$. These terms can be evaluated using the technique in [16] with $\ell_1 = 1$, $\ell_2 = 2$. Of course, if, say, a is a quadratic non-residue modulo q, then $\sigma_k(N;q,a,b;2,1) = 0$, and similarly for b. Anyway, the total contribution of prime powers is of order $O_k\left(N^{k+3/2}\right)$ at most, with implicit constant which is uniform in q since $\sigma_k(N;q,a,b;\alpha,\beta) \leq \sigma_k(N;1,0,0;\alpha,\beta)$.

Acknowledgements. We thank the referee for several suggestions.

2. Outline of the proof

The four dominant terms in the statement, that is, in (5), arise multiplying formally the leading terms for $\widetilde{S}_{a,q}$ and $\widetilde{S}_{b,q}$ provided by Lemma 1: see (9). We have to show that we may exchange the summations with integration on the vertical line $\Re(z) = x$, and also that the error term is small. We need the identity

$$\frac{1}{2\pi i}\int_{(a)} u^{-s}e^u \, du = \frac{1}{\Gamma(s)} \qquad (7)$$

for $a > 0$ and its variant

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}Du}}{(a+\mathrm{i}u)^s}\,\mathrm{d}u = \begin{cases} D^{s-1}\mathrm{e}^{-aD}/\Gamma(s) & \text{if } D > 0,\\ 0 & \text{if } D < 0,\end{cases}$$

in order to prove that (1) holds. Here $\Re(a) > 0$ and $\Re(s) > 0$. For the proof, see de Azevedo Pribitkin [1]. For brevity, we write

$$\widetilde{S}_{a,q}(z) = \frac{1}{\phi(q)}(\mathcal{M}(z;q,a) + \mathcal{E}(z;q,a)), \tag{8}$$

where

$$\mathcal{M}(z;q,a) = \frac{1}{z} - \sum_{\chi \bmod q} \overline{\chi}(a) \sum_{\rho\in\mathcal{Z}(\chi^*)} z^{-\rho}\Gamma(\rho). \tag{9}$$

The bound for $\mathcal{E}(z;q,a)$ is provided by Lemma 1. We now substitute into (1) and find that

$$\Sigma_k(N;q,a,b) = \frac{1}{2\pi\mathrm{i}\phi^2(q)}\Big(\int_{(x)} \mathrm{e}^{Nz}\mathcal{M}(z;q,a)\mathcal{M}(z;q,b)\,\frac{\mathrm{d}z}{z^{k+1}} \tag{10}$$

$$+ \int_{(x)} \mathrm{e}^{Nz}(\mathcal{M}(z;q,a)\mathcal{E}(z;q,b) + \mathcal{E}(z;q,a)\mathcal{M}(z;q,b))\,\frac{\mathrm{d}z}{z^{k+1}} \tag{11}$$

$$+ \int_{(x)} \mathrm{e}^{Nz}\mathcal{E}(z;q,a)\mathcal{E}(z;q,b)\,\frac{\mathrm{d}z}{z^{k+1}}\Big). \tag{12}$$

Expanding the first term in (10), exchanging summation with integration and using identity (7), we recover the terms in (2), (3) and (4), that is, the main term $M_k(N;q,a,b)$ defined in (5). The proofs that the exchanges are legitimate are in §§5–6, while the proofs that single and double sums over zeros in (3) and (4) respectively converge follow closely the argument in [14] and [16], with minor modifications in the choices of the regions. We do not include details.

Now we deal with the error terms. We remark that

$$|z|^{-1} \ll \begin{cases} x^{-1} & \text{if } |y| \le x,\\ |y|^{-1} & \text{if } |y| > x.\end{cases} \tag{13}$$

We will eventually choose $x = N^{-1}$. Hence, by Lemma 1 or simply by the Brun-Titchmarsh inequality we have

$$\widetilde{S}_{a,q}(z) \ll \frac{1}{\phi(q)x},$$

provided that $q \le (\log N)^A$ for any fixed $A > 0$. Hence

$$|\mathcal{M}(z;q,a)| \le \phi(q)\widetilde{S}(x;q,a) + |\mathcal{E}(z;q,a)| \ll x^{-1} + |\mathcal{E}(z;q,a)|.$$

Therefore we have

$$\Big|\int_{(x)} \mathrm{e}^{Nz}\mathcal{M}(z;q,a)\mathcal{E}(z;q,b)\,\frac{\mathrm{d}z}{z^{k+1}}\Big| \ll \mathrm{e}^{Nx}\int_{(x)} |\mathcal{M}(z;q,a)\mathcal{E}(z;q,b)|\,\frac{\mathrm{d}y}{|z|^{k+1}}$$

$$\ll \mathrm{e}^{Nx}\int_{(x)} (x^{-1} + |\mathcal{E}(z;q,a)|)\,|\mathcal{E}(z;q,b)|\,\frac{\mathrm{d}y}{|z|^{k+1}}.$$

For brevity we set

$$\mathcal{L}(x;q,a) := \int_{(x)} |\mathcal{E}(z;q,a)|^2\,\frac{\mathrm{d}y}{|z|^{k+1}}. \tag{14}$$

By the Cauchy-Schwarz inequality, the total contribution of the terms in (11) and (12) is

$$\ll \mathrm{e}^{Nx}x^{-1}\int_{(x)} (|\mathcal{E}(z;q,a)| + |\mathcal{E}(z;q,b)|)\,\frac{\mathrm{d}y}{|z|^{k+1}} + \mathrm{e}^{Nx}\int_{(x)} |\mathcal{E}(z;q,a)|\cdot|\mathcal{E}(z;q,b)|\,\frac{\mathrm{d}y}{|z|^{k+1}}$$

$$\ll \mathrm{e}^{Nx} x^{-1} \sum_{h\in\{a,b\}} \Big(\int_{(x)} \frac{\mathrm{d}y}{|z|^{k+1}} \mathcal{L}(x;q,h) \Big)^{1/2} + \mathrm{e}^{Nx} \Big(\mathcal{L}(x;q,a) \mathcal{L}(x;q,b) \Big)^{1/2}$$
$$\ll \mathrm{e}^{Nx} \max_{(h,q)=1} \Big(x^{-1-k/2} \mathcal{L}(x;q,h)^{1/2} + \mathcal{L}(x;q,h) \Big),$$

by (13). We choose $x = 1/N$ and use Lemma 2 to bound (14). We will complete the proof of Theorem 1 in §4.

3. Lemmas

For a Dirichlet character χ mod q let

$$\widetilde{S}(z;\chi) := \sum_{m\geq 1} \chi(m)\Lambda(m)\mathrm{e}^{-mz},$$

so that, by orthogonality,

$$\widetilde{S}_{a,q}(z) = \frac{1}{\phi(q)} \sum_{\chi \bmod q} \overline{\chi}(a) \widetilde{S}(z;\chi). \tag{15}$$

We now express $\widetilde{S}(\chi)$ by means of $\widetilde{S}(\chi^*)$, where χ^* mod q^* is the primitive character that induces χ. We have

$$\Big|\widetilde{S}(z;\chi) - \widetilde{S}(z;\chi^*)\Big| \leq \sum_{\substack{m\geq 1\\(m,q)>1}} \Lambda(m)\mathrm{e}^{-mx} = \sum_{p|q} \log(p) \sum_{\nu\geq 1} \mathrm{e}^{-p^\nu x} \leq \mathrm{e}^{-x}\log(q). \tag{16}$$

We recall some properties of the Dirichlet L-functions that we need in the proof of Lemma 1. First, let χ be a primitive odd character modulo $q > 1$ and let

$$b(\chi) = \frac{L'}{L}(0,\chi) = -\frac{1}{2}\log\frac{q}{\pi} - \frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{1}{2}\Big) + B(\chi),$$

where $b(\chi)$ is defined in §19 of Davenport [9] and $B(\chi)$ appears in the Weierstrass product for $L(s,\chi)$. If χ is even, we let

$$b(\chi) = \lim_{s\to 0}\Big(\frac{L'}{L}(s,\chi) - \frac{1}{s}\Big) = -\frac{1}{2}\log\frac{q}{\pi} + B(\chi) - \lim_{s\to 0}\Big(\frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big) + \frac{1}{s}\Big).$$

By the argument on pages 118–119 of Davenport [9] we have

$$b(\chi) = O\left(\log(q)\right) - \sum_{|\gamma|<1} \frac{1}{\rho}.$$

The Riemann-von Mangoldt formula for the number of zeros implies that the number of summands on the right is $\ll \log(q)$. Each of the summands is $\ll \log(q)$, unless χ is a real character such that the relative L-function has an exceptional zero $\widetilde{\beta} \in [1 - c/\log(q), 1]$. Hence

$$b(\chi) = O\left(\log^2(q)\right) - \frac{1}{\widetilde{\beta}} - \frac{1}{1-\widetilde{\beta}} = O\left(\log^2(q)\right) - \frac{1}{1-\widetilde{\beta}}.$$

The terms containing the exceptional zero are to be omitted if it does not exist. We now recall the upper bound $(1-\widetilde{\beta})^{-1} \ll q^{1/2}\log^2(q)$ which is (12) on page 96 of Davenport. Hence we have $b(\chi) = O\left(g(q)\right)$. Furthermore, for $\sigma = \Re(w) \in [-1,2]$, by (4) in §16 of [9] we have

$$\frac{L'}{L}(w,\chi) = \sum_{\substack{\rho\\|t-\gamma|<1}} \frac{1}{w-\rho} + O\left(\log(q(|t|+2))\right).$$

For w on the line $\Re(s) = -\frac{1}{2}$ the summands are uniformly bounded, and their number is $\ll \log(q(|t|+2))$ by the Riemann-von Mangoldt formula for the L-functions. Hence

$$\frac{L'}{L}\Big(-\frac{1}{2}+\mathrm{i}t,\chi\Big) \ll \log\big(q(|t|+2)\big). \tag{17}$$

Lemma 1. *Let $\mathcal{M}(z;q,a)$ and $\mathcal{E}(z;q,a)$ be defined by* (8) *and* (9). *Then*

$$\mathcal{E}(z;q,a) \ll g(q) + 1 + \log(|z|) + |z|^{1/2}(\log(q)+1)\cdot\begin{cases}1 & \text{if } |y|\le x,\\ 1+\log^2(|y|/x) & \text{if } |y|>x.\end{cases} \tag{18}$$

The implicit constant is absolute.

Proof. Assume for the time being that χ is a primitive character, and let $\delta(\chi)=1$ if χ is principal and $\delta(\chi)=0$ otherwise. Following the proof in §4 of Linnik [18], or Lemma 4.1 of [14], we have

$$\widetilde{S}(z;\chi) = -\frac{1}{2\pi\mathrm{i}}\int_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty} z^{-w}\Gamma(w)\frac{L'}{L}(w,\chi)\,\mathrm{d}w = \frac{\delta(\chi)}{z} - \sum_{\rho\in\mathcal{Z}(\chi)} z^{-\rho}\Gamma(\rho) + Q(z;\chi) + R(z;\chi),$$

say, where

$$R(z;\chi) := -\frac{1}{2\pi\mathrm{i}}\int_{-1/2-\mathrm{i}\infty}^{-1/2+\mathrm{i}\infty} z^{-w}\Gamma(w)\frac{L'}{L}(w,\chi)\,\mathrm{d}w$$

and

$$Q(z;\chi) := \begin{cases}-(L'/L)(0,\chi) & \text{if } \chi \text{ is an odd character}\\ -\gamma + b(\chi) + \log(z) & \text{if } \chi \text{ is an even character,}\end{cases} \tag{19}$$

taking into account the double pole of the integrand at $s=0$. Hence, recalling (15) and (16), we have

$$\begin{aligned}
\widetilde{S}_{a,q}(z) &= \frac{1}{\phi(q)}\sum_{\chi \bmod q}\overline{\chi}(a)\widetilde{S}(z;\chi^*) + O\left(\frac{1}{\phi(q)}\sum_{\chi \bmod q}\left|\widetilde{S}(z;\chi) - \widetilde{S}(z;\chi^*)\right|\right)\\
&= \frac{1}{\phi(q)z} - \frac{1}{\phi(q)}\sum_{\chi \bmod q}\overline{\chi}(a)\sum_{\rho\in\mathcal{Z}(\chi^*)} z^{-\rho}\Gamma(\rho)\\
&\qquad + O\left(\frac{1}{\phi(q)}\sum_{\chi \bmod q}(|R(z;\chi^*)| + |Q(z;\chi^*)|) + \mathrm{e}^{-x}\log(q)\right)\\
&= \frac{1}{\phi(q)}\mathcal{M}(z;q,a) + O\left(\frac{1}{\phi(q)}\sum_{\chi \bmod q}(|R(z;\chi^*)| + |Q(z;\chi^*)|) + \mathrm{e}^{-x}\log(q)\right).
\end{aligned}$$

In order to treat $R(z;\chi)$ we need the bound (17) and $\Gamma(w) \ll |t|^{-1}\mathrm{e}^{-\pi|t|/2}$, (see Titchmarsh [23] §4.42) which is valid for $|t|\to+\infty$. We split the line $\sigma=-1/2$ into the set $L_c = \{w = -1/2+\mathrm{i}t\colon |t|>c\}$, where $c>0$ is a suitable large absolute constant, and its complement. Arguing as in the proof of Lemma 4.1 of [14], and examining various cases we find

$$\int_{L_c} z^{-w}\Gamma(w)\frac{L'}{L}(w,\chi)\,\mathrm{d}w \ll |z|^{1/2}\cdot\begin{cases}\log(q)+1 & \text{if } |y|\le x,\\ \log^2(|y|/x) + 1 + \log(q)(\log(|y|/x)+1) & \text{if } |y|>x.\end{cases}$$

Finally, the integration over the complement of L_c yields a contribution $O\left(1+\log(q)\right)$.

We now turn to the estimation of $|Q(z;\chi)|$. If χ is an odd character, then we have

$$\frac{L'}{L}(0,\chi) \ll \log(q),$$

by formula 10.35 of §10 of Montgomery & Vaughan [19]. If χ is even, we argue as above in (6), and by (19) we have

$$Q(z;\chi) \ll g(q) + |\log(z)|.$$

This implies that $\mathcal{E}$ satisfies the bound in (18). □

Lemma 2. *For $k > 1$ we have*

$$\mathcal{L}(N^{-1};q,a) := \int_{(1/N)} |\mathcal{E}(z;q,a)|^2 \frac{\mathrm{d}y}{|z|^{k+1}} \ll_k N^k \left(g\left(q\right) + \log\left(N\right)\right)^2.$$

Proof. We have

$$\begin{aligned}\mathcal{L}(N^{-1};q,a) &\ll N^{k+1} \int_{-1/N}^{1/N} |\mathcal{E}\left(z;q,a\right)|^2 \mathrm{d}y + \int_{1/N}^{+\infty} \frac{|\mathcal{E}\left(z;q,a\right)|^2}{y^{k+1}} \mathrm{d}y \\ &:= N^{k+1} I_1 + I_2,\end{aligned}$$

say. By Lemma 1 we easily get

$$\begin{aligned}I_1 &\ll \int_{-1/N}^{1/N} \left(g\left(q\right) + \log\left(N\right) + N^{-1/2}\left(\log\left(q\right)+1\right)\right)^2 \mathrm{d}y \\ &\ll N^{-1}\left(g\left(q\right) + \log\left(N\right)\right)^2.\end{aligned}$$

Now we analyse I_2. Using again Lemma 1 we obtain

$$\begin{aligned}I_2 &\ll \int_{1/N}^{+\infty} \frac{\left(g\left(q\right) + 1 + |\log\left(y\right)| + y^{1/2}\left(\log\left(q\right)+1\right)\left(1+\log^2\left(Ny\right)\right)\right)^2}{y^{k+1}} \mathrm{d}y \\ &\ll \int_{1/N}^{+\infty} \frac{\left(g\left(q\right) + 1 + |\log\left(y\right)|\right)^2}{y^{k+1}} \mathrm{d}y + \left(\log\left(q\right)+1\right)^2 \int_{1/N}^{+\infty} \frac{\left(1+\log^2\left(Ny\right)\right)^2}{y^k} \mathrm{d}y.\end{aligned}$$

The first summand can be easily estimated:

$$\begin{aligned}\int_{1/N}^{+\infty} \frac{\left(g\left(q\right) + 1 + |\log\left(y\right)|\right)^2}{y^{k+1}} \mathrm{d}y &\ll_k N^k \left(g\left(q\right)+1\right)^2 + \int_{1/N}^{+\infty} \frac{\log^2\left(y\right)}{y^{k+1}} \mathrm{d}y \\ &\ll_k N^k \left(\left(g\left(q\right)+1\right)^2 + \log^2\left(N\right)\right).\end{aligned}$$

We use the change of variables $v = Ny$ for the integral of the last summand and we get

$$\begin{aligned}\int_{1/N}^{+\infty} \frac{\left(1+\log^2\left(Ny\right)\right)^2}{y^k} \mathrm{d}y &\ll_k \left(N^{k-1} + \int_{1/N}^{+\infty} \frac{\log^4\left(Ny\right)}{y^k} \mathrm{d}y\right) \\ &\ll_k N^{k-1}\left(1 + \int_1^{+\infty} \frac{\log^4\left(v\right)}{v^k} \mathrm{d}v\right) \ll_k N^{k-1}\end{aligned}$$

since $k > 1$, and Lemma 2 follows. □

4. Completion of the proof of Theorem 1

In order to complete the proof it is enough to see that by Lemma 2 we have

$$\begin{aligned}&\max_{(h,q)=1} \left(N^{1+k/2} \mathcal{L}(N^{-1};q,h)^{1/2} + \mathcal{L}(N^{-1};q,h)\right) \\ &\ll N^{k+1}(g(q) + \log(N)) + N^k (g(q) + \log(N))^2 \\ &\ll N^{k+1}(g(q) + \log(N)),\end{aligned}$$

where the last estimation follows from the bound for q.

5. Interchange of the series over zeros with the line integral

It remains to prove that all the interchanges of the series and the integrals are legitimate.

We now state without proof two technical lemmas that can be proved by means of small variations on the arguments in Lemmas 2 and 3 in [14]. In both cases, we assume that χ mod q is a primitive character and we let $\rho_\chi = \beta_\chi + i\gamma_\chi$ run over the non-trivial zeros of the associated Dirichlet L-function. Furthermore, we let $\alpha > 1$ be a parameter.

Lemma 3. *We have*

$$\sum_{\gamma_\chi > 0} \gamma_\chi^{\beta_\chi - 1/2} \int_1^{+\infty} \log^c(u) \exp\Big(-\gamma_\chi \arctan\Big(\frac{1}{u}\Big)\Big) \frac{\mathrm{d}u}{u^{\alpha+\beta_\chi}} \ll_{\alpha,c} \sum_{\gamma_\chi > 0} \gamma_\chi^{1/2-\alpha}$$

for all $c \ge 0$. The series on the left converges for $\alpha > 3/2$ and diverges otherwise.

Lemma 4. *Let $z = x + iy$ with $x \in (0,1)$ and $y \in \mathbb{R}$. We have*

$$\sum_{\rho_\chi} |\gamma_\chi|^{\beta_\chi - 1/2} \int_{\mathbb{Y}} \log^c\Big(\frac{|y|}{x}\Big) \exp\Big(\gamma_\chi \arctan\Big(\frac{y}{x}\Big) - \frac{\pi}{2}|\gamma_\chi|\Big) \frac{\mathrm{d}y}{|z|^{\alpha+\beta_\chi}} \ll_{\alpha,c} x^{-\alpha} \sum_{\gamma_\chi > 0} \gamma_\chi^{\beta_\chi - 1/2} \exp\Big(-\frac{\pi}{4}\gamma_\chi\Big),$$

for all $c \ge 0$, where $\mathbb{Y} = \mathbb{Y}_1 \cup \mathbb{Y}_2$, $\mathbb{Y}_1 = \{y \in \mathbb{R} \colon y\gamma_\chi \le 0\}$ and $\mathbb{Y}_2 = \{y \in [-x, x] \colon y\gamma_\chi > 0\}$.

Let $z = x + iy$ and $w = u + iv$, where $u \ge 0$. We recall that

$$|z^w| = |z|^u \exp\Big(v \arctan\Big(\frac{y}{x}\Big)\Big) \tag{20}$$

and

$$|\Gamma(w)| \le (2\pi)^{1/2} |w|^{u-1/2} e^{-\pi|v|/2} \exp\Big(\frac{1}{6|w|}\Big). \tag{21}$$

The latter form of the Stirling formula can be found in Olver et al. [20], Lemma 5.6(ii) equation 5.6.9. We want to establish the absolute convergence of

$$\sum_{\chi \bmod q} \sum_{\rho \in \mathcal{Z}(\chi^*)} |\Gamma(\rho)| \int_{(1/N)} |e^{Nz}| \cdot |z^{-k-2-\rho}| \cdot |\mathrm{d}z|.$$

Using (20) and (21) we find

$$|\Gamma(\rho)| \int_{(1/N)} |z^{-k-2-\rho}| \cdot |\mathrm{d}z| \ll |\gamma|^{\beta-1/2} \exp\Big(\frac{1}{6|\gamma|}\Big) \int_{\mathbb{R}} \exp\Big(\gamma \arctan(Ny) - \frac{\pi}{2}|\gamma|\Big) \frac{\mathrm{d}y}{|z|^{k+2+\beta}}.$$

We sum over $\rho \in \mathcal{Z}(\chi^*)$ and we can assume by symmetry that $\gamma > 0$. Recalling Lemma 4, we have

$$\sum_{\gamma > 0} |\Gamma(\rho)| \int_{\mathbb{Y}} |e^{Nz}| \cdot |z^{-k-2-\rho}| \cdot |\mathrm{d}z| \ll_k N^{k+2} \sum_{\gamma > 0} \gamma^{\beta-1/2} \exp\Big(-\frac{\pi}{4}\gamma\Big).$$

We still have to deal with the case $\gamma > 0$ and $y > 1/N$. Using the identity $\arctan(y) + \arctan(1/y) = \pi/2$, we have

$$\begin{aligned}
&\sum_{\gamma > 0} |\Gamma(\rho)| \int_{1/N}^{+\infty} |e^{Nz}| \cdot |z^{-k-2-\rho}| \cdot |\mathrm{d}z| \\
&\qquad \ll \sum_{\gamma > 0} \gamma^{\beta-1/2} \int_{1/N}^{+\infty} \exp\Big(\gamma \arctan(Ny) - \frac{\pi}{2}\gamma\Big) \frac{\mathrm{d}y}{y^{k+2+\beta}} \\
&\qquad \ll \sum_{\gamma > 0} \gamma^{\beta-1/2} \int_{1/N}^{+\infty} \exp\big(-\gamma \arctan(1/(Ny))\big) \frac{\mathrm{d}y}{y^{k+2+\beta}}
\end{aligned}$$

$$= \sum_{\gamma>0} \gamma^{\beta-1/2} N^{k+1+\beta} \int_1^{+\infty} \exp\Big(-\gamma \arctan(1/u)\Big) \frac{\mathrm{d}u}{u^{k+2+\beta}}.$$

Using Lemma 3 we see that this is

$$\ll_k N^{k+2} \sum_{\gamma>0} \gamma^{-k-3/2},$$

which converges for $k > -1/2$, by standard zero-density estimates.

6. Interchange of the double series over zeros with the line integral

We start examining

$$\sum_{\chi_1 \bmod q} \sum_{\rho_1 \in \mathcal{Z}(\chi_1^*)} |\Gamma(\rho_1)| \int_{(1/N)} |\mathrm{e}^{Nz}| \cdot |z^{-k-1-\rho_1}| \cdot \Big| \sum_{\chi_2 \bmod q} \overline{\chi}_2(b) \sum_{\rho_2 \in \mathcal{Z}(\chi_2^*)} \Gamma(\rho_2) z^{-\rho_2} \Big| \cdot |\mathrm{d}z|.$$

Lemma 1 implies that

$$\Big| \sum_{\chi_2 \bmod q} \overline{\chi}_2(b) \sum_{\rho_2 \in \mathcal{Z}(\chi_2^*)} \Gamma(\rho_2) z^{-\rho_2} \Big| \le \phi(q) |\widetilde{S}_{a,q}(z)| + \frac{1}{|z|} + \sum_{\chi_2 \bmod q} \left(|R(z;\chi_2^*)| + |Q(z;\chi_2^*)|\right)$$

$$\ll_q \frac{1}{|z|} + N + \log\left(|z|\right) + |z|^{1/2} \cdot \begin{cases} 1 & |y| \le 1/N \\ 1 + \log^2\left(N\,|y|\right) & |y| > 1/N, \end{cases}$$

hence we have to study

$$\sum_{\chi_1 \bmod q} \sum_{\rho_1 \in \mathcal{Z}(\chi_1^*)} |\Gamma(\rho_1)| (A_1 + A_2 + A_3)$$

where

$$A_1 = \int_{-1/N}^{1/N} N^{k+1+\beta_1} \exp(\gamma_1 \arctan\left(Ny\right)) \Big(N + \log(N) + N^{-1/2}\Big) \mathrm{d}y,$$

$$A_2 = \int_{1/N}^{+\infty} y^{-k-1-\beta_1} \exp(\gamma_1 \arctan\left(Ny\right)) \Big(\frac{1}{y} + N + |\log(y)| + y^{1/2}(1 + \log^2(Ny))\Big) \mathrm{d}y,$$

$$A_3 = \int_{-\infty}^{-1/N} |y|^{-k-1-\beta_1} \exp(\gamma_1 \arctan\left(Ny\right)) \Big(\frac{1}{|y|} + N + |\log(|y|)| + |y|^{1/2}(1 + \log^2(N|y|))\Big) \mathrm{d}y.$$

By symmetry, we can assume that $\gamma_1 > 0$. Clearly, by (21), we have

$$\sum_{\chi_1 \bmod q} \sum_{\rho_1 \in \mathcal{Z}(\chi_1^*)} |\Gamma(\rho_1)| A_1 \ll_q N^{k+2} \sum_{\chi_1 \bmod q} \sum_{\gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \exp\left(\frac{-\pi \gamma_1}{4}\right)$$

and the series is obviously convergent, by standard density estimates.

Now, we consider A_2: we have, again by (21), that

$$\sum_{\chi_1 \bmod q} \sum_{\rho_1 \in \mathcal{Z}(\chi_1^*)} |\Gamma(\rho_1)| A_2 \ll_q \sum_{\chi_1 \bmod q} \sum_{\gamma_1 > 0} \gamma_1^{\beta_1 - 1/2}$$

$$\times \int_{1/N}^{+\infty} y^{-k-1-\beta_1} \exp\left(\gamma_1 \left(\arctan\left(Ny\right) - \frac{\pi}{2}\right)\right) \left(\frac{1}{y} + N + \left|\log\left(y\right)\right| + y^{1/2}\left(1 + \log^2\left(Ny\right)\right)\right) \mathrm{d}y.$$

Then, taking $v = Ny$, we obtain

$$\sum_{\chi_1 \bmod q} \sum_{\rho_1 \in \mathcal{Z}(\chi_1^*)} |\Gamma(\rho_1)| A_2 \ll_q N^k \sum_{\chi_1 \bmod q} \sum_{\gamma_1 > 0} N^{\beta_1} \gamma_1^{\beta_1 - 1/2}$$

$$\times \int_1^{+\infty} v^{-k-1-\beta_1} \exp\left(-\gamma_1 \arctan\left(\frac{1}{v}\right)\right)\left(\frac{N}{v} + N + \log(vN) + v^{1/2}N^{-1/2}\left(1+\log^2(v)\right)\right) \mathrm{d}v$$

and now, by Lemma 3, it is easy to see that the integral and the series are convergent for $k > 1$. If $y \in (-\infty, -1/N)$ the result is trivial since $\exp\left(\gamma_1\left(\arctan(Ny) - \frac{\pi}{2}\right)\right) \leq \exp\left(-\gamma_1 \frac{\pi}{2}\right)$ and the condition $k > 1$ suffices to ensure the convergence of the integral.

Finally, if we consider

$$\sum_{\chi_1 \bmod q} \sum_{\chi_2 \bmod q} \sum_{\rho_1 \in Z(\chi_1^\star)} |\Gamma(\rho_1)| \sum_{\rho_2 \in Z(\chi_2^\star)} |\Gamma(\rho_2)| \int_{(1/N)} \left|e^{Nz}\right| \left|z^{-k-1-\rho_1-\rho_2}\right| |\mathrm{d}z|,$$

we can simply argue as in [14], equation (21), getting the convergence for $k > 1$.

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