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On the existence of L^2 -valued thermodynamic entropy solutions for a hyperbolic system with boundary conditions

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ABSTRACT

We prove existence of L^2 -weak solutions of a quasilinear wave equation with boundary conditions. This describes the isothermal evolution of a one dimensional non-linear elastic material, attached to a fixed point on one side and subject to a force (tension) applied to the other side. The L^2 -valued solutions appear naturally when studying the hydrodynamic limit from a microscopic dynamics of a chain of anharmonic springs connected to a thermal bath. The proof of the existence is done using a vanishing viscosity approximation with extra Neumann boundary conditions added. In this setting we obtain a uniform a priori estimate in L^2 , allowing us to use L^2 Young measures, together with the classical tools of compensated compactness. We then prove that the viscous solutions converge to weak solutions of the quasilinear wave equation strongly in L^p , for any $p \in [1, 2)$, that satisfy, in a weak sense, the boundary conditions. Furthermore, these solutions satisfy, beside the local Lax entropy condition, the Clausius inequality: the change of the free energy is bounded by the work done by the boundary tension. In this sense they are the correct thermodynamic solutions, and we conjecture their uniqueness.

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1. Introduction

The problem of existence of weak solutions for hyperbolic systems of conservation law in a bounded domain has been studied for solutions that are of bounded variation or in L^∞ [1]. In the scalar case some works extend to L^∞ solutions, obtained from viscous approximations [2]. But viscous approximations require extra boundary conditions that are usually taken of Dirichlet type.

We present here an approach based on viscosity approximations, where the extra boundary conditions are of Neumann type, to reflect the conservative nature of the viscous approximation. We consider here the quasilinear wave equation

$$\begin{cases} r_t - p_x = 0 \\ p_t - \tau(r)_x = 0 \end{cases}, \quad (t, x) \in \mathbb{R}_+ \times [0, 1] \quad (1.1)$$

where $\tau(r)$ is a strictly increasing regular function of r such that $0 < c_1 \leq \tau'(r) \leq c_2$, for some constant c_1, c_2 . In Section 2, we will require some more technical assumption

for τ . We add to the system the following boundary conditions:

$$p(t, 0) = 0, \quad \tau(r(t, 1)) = \bar{\tau}(t) \tag{1.2}$$

and initial data

$$r(0, x) = r_0(x), \quad p(0, x) = p_0(x). \tag{1.3}$$

The boundary tension $\bar{\tau} : \mathbb{R}_+ \rightarrow \mathbb{R}$ is smooth and bounded with bounded derivative.

Eq. (1.1) describes the isothermal evolution of an elastic material in Lagrangian coordinates. The *material* point $x \in [0, 1]$ has a *volume strain* $r(t, x)$ at time t (that can also have negative values), and momentum (velocity) $p(t, x)$. The Eulerian position of the material point x , with respect to the position of the particle 0, is given by $q(t, x) = \int_0^x r(t, y)dy$, so that we can identify the position of the material point $x = 1$ as the total extension of the material:

$$L(t) = q(t, 1) = \int_0^1 r(t, y)dy. \tag{1.4}$$

Let $T < \infty$ be given and arbitrary, and define $Q_T := [0, T] \times [0, 1]$. We shall construct weak solutions $\bar{u}(t, y) = (\bar{r}(t, y), \bar{p}(t, y))$, $(t, y) \in Q_T$, to the quasilinear wave equation such that $\bar{u}(t, \cdot) \in L^2(0, 1)$ for all $t \leq T$ and satisfy the initial and boundary conditions in the following weak sense:

$$\int_0^1 \varphi(t, x)\bar{r}(t, x)dx - \int_0^1 \varphi(0, x)r_0(x)dx = \int_0^t \int_0^1 (\varphi_s \bar{r} - \varphi_x \bar{p})dxds \tag{1.5}$$

$$\int_0^1 \psi(t, x)\bar{p}(t, x)dx - \int_0^1 \psi(0, x)p_0(x)dx = \int_0^t \int_0^1 (\psi_s \bar{p} - \psi_x \tau(\bar{r}))dxds + \int_0^t \psi(s, 1)\bar{\tau}(s)ds \tag{1.6}$$

for all functions $\varphi, \psi \in C^1(Q_T)$ such that $\varphi(t, 1) = \psi(t, 0) = 0$ for all $t \geq 0$.

Define the *free energy* of the system, associated to a profile $u(x) = (r(x), p(x)) \in L^2(0, 1)$, as

$$\mathcal{F}(u) := \int_0^1 \left(\frac{p^2(x)}{2} + F(r(x)) \right) dx \tag{1.7}$$

where $F(r)$ is a primitive of $\tau(r)$ ($F'(r) = \tau(r)$), such that $\frac{c_1}{2}r^2 \leq F(r) \leq \frac{c_2}{2}r^2$ for any $r \in \mathbb{R}$. This is possible thanks to the bounds we required on τ' .

The solution \bar{u} of (1.5) and (1.6) that we obtain has the following properties:

- $\bar{u} \in L^\infty(0, T; L^2(0, 1))$
- $\bar{u}(0, x) = u_0(x)$ for a.e. x ;
- For any $\phi \in C^1([0, 1])$, the application

$$t \mapsto \int_0^1 \phi(x)\bar{u}(t, x)dx \tag{1.8}$$

is Lipschitz continuous over $[0, T]$;

- \bar{u} satisfies *Clausius inequality*:

$$\mathcal{F}(\bar{u}(t)) - \mathcal{F}(u_0) \leq W(t), \quad \forall t \in [0, T] \tag{1.9}$$

where $u_0 = (r_0, p_0)$ and

$$W(t) := - \int_0^t \bar{\tau}'(s) \int_0^1 \bar{r}(s, x) dx ds + \bar{\tau}(t) \int_0^1 \bar{r}(t, x) dx - \bar{\tau}(0) \int_0^1 r_0(x) dx \quad (1.10)$$

is the work done by the external tension up to time t . In this sense, we call our solution a *thermodynamic entropy solution*. For general discussion of the connection of such thermodynamic solutions to the usual definition of entropic solutions, see [3,4].

Remark. If $\bar{r}(t, x)$ is differentiable with respect to time, we may perform an integration by parts and obtain

$$W(t) = \int_0^t \bar{\tau}(s) dL(s). \quad (1.11)$$

This recovers the usual mechanical definition of the work.

The construction of the solution is obtained from the following viscosity approximation

$$\begin{cases} r_t^\delta - p_x^\delta = \delta r_{xx}^\delta \\ p_t^\delta - \tau(r^\delta)_x = \delta p_{xx}^\delta \end{cases}, \quad (t, x) \in \mathbb{R}_+ \times [0, 1] \quad (1.12)$$

with boundary conditions

$$p^\delta(t, 0) = 0, \quad \tau(r^\delta(t, 1)) = \bar{\tau}(t), \quad p_x^\delta(t, 1) = 0, \quad r_x^\delta(t, 0) = 0 \quad (1.13)$$

and initial data

$$r^\delta(0, x) = r_0^\delta(x), \quad p^\delta(0, x) = p_0^\delta(x) \quad (1.14)$$

such that r_0^δ and p_0^δ are compatible with the boundary conditions, regular enough (see (3.4) and (3.5)) and converge to r_0 and p_0 , respectively, as $\delta \rightarrow 0$.

Note that in the viscous approximation we have added two Neumann boundary conditions that reflect the *conservative* nature of the viscous perturbation. Under these conditions we have

$$\int_0^1 |u^\delta(t, x)|^2 dx + \delta \int_0^t \int_0^1 |u_x^\delta(s, x)|^2 dx ds \leq C, \quad \forall t \geq 0 \quad (1.15)$$

where C is independent of t and δ . It is thus clear that $\{u^\delta\}_{\delta>0}$ and $\{\sqrt{\delta}u_x^\delta\}_{\delta>0}$ are uniformly bounded in $L^2(Q_T)$. Then we rely on the existence of a family of bounded Lax entropy-entropy fluxes as in [5–7], that allows us to apply the compensated compactness in the L^2 version. The conditions assumed on $\tau(r)$ are in fact those required to apply [7] results. Under a slight different set of conditions, another L^p extension of the compensated compactness argument can be found in [8].

1.1. Physical motivations

The problem arises naturally considering hydrodynamic limit for a non-linear chain of anharmonic oscillators in contact with a heat bath at a given temperature [9,10]. This microscopic dynamics models an isothermal transformation with two locally conserved quantities that evolve, on the macroscopic scale, following (1.1).

Consider $N + 1$ particles on the real line and, for $i = 0, \dots, N$, call q_i and p_i the positions and the momenta of the i th particle, respectively. Particles i and $i - 1$ interact via a nonlinear potential $V(q_i - q_{i-1})$. Particle $i = 0$ is at position $q_0 = 0$ and does not move, i.e. $p_0(t) = 0$. There is a time dependent force (tension) $\bar{\tau}(t)$ acting on the last particle. Then, defining $r_i := q_i - q_{i-1}$ we have a system with Hamiltonian

$$H_N(t) = \sum_{i=1}^N \left(\frac{p_i^2}{2} + V(r_i) \right) - \bar{\tau}(t) \sum_{i=1}^N r_i. \tag{1.16}$$

The interaction with a heat bath at temperature β^{-1} is modeled by a stochastic perturbation of the dynamics that acts as a *microscopic stochastic viscosity*. Defining the discrete gradient and laplacian as

$$\nabla a_i = a_{i+1} - a_i, \quad \Delta a_i = a_{i+1} + a_{i-1} - 2a_i,$$

the evolution equations are then given by the following system of stochastic differential equations:

$$\left\{ \begin{array}{l} dr_1 = p_1 dt + \delta \nabla V'(r_1) dt - \sqrt{2\beta^{-1}\delta} d\tilde{w}_1 \\ dr_i = \nabla p_{i-1} dt + \delta \Delta V'(r_i) dt - \sqrt{2\beta^{-1}\delta} \nabla d\tilde{w}_{i-1}, \quad 2 \leq i \leq N - 1 \\ dr_N = \nabla p_{N-1} dt + \delta(\bar{\tau}(t) + V'(r_{N-1}) - 2V'(r_N)) - \sqrt{2\beta^{-1}\delta} \nabla d\tilde{w}_{N-1}, \\ dp_1 = \nabla V'(r_1) dt + \delta(p_2 - 2p_1) dt - \sqrt{2\beta^{-1}\delta} \nabla dw_1, \\ dp_j = \nabla V'(r_j) dt + \delta \Delta p_j dt - \sqrt{2\beta^{-1}\delta} \nabla dw_{j-1}, \quad 2 \leq j \leq N - 1 \\ dp_N = (\bar{\tau}(t) - V'(r_N)) dt - \delta \nabla p_{N-1} dt + \sqrt{2\beta^{-1}\delta} dw_{N-1} \end{array} \right. \tag{1.17}$$

Here $\beta^{-1} > 0$ is the temperature of the heat bath, and $\{w_i\}_{i=1}^{N-1}, \{\tilde{w}_i\}_{i=1}^{N-1}$ are families of independent Brownian motions. The parameter δ is the intensity of the action of the heat bath, and is chosen depending on N such that $\delta \sim o(N)$. When $\delta = 0$, Eq. (1.17) is just the Newton deterministic equations for the Hamiltonian (1.16). Notice the correspondence of the boundary conditions in (1.17) with the one chosen in (1.12).

One of the effects of the action of the stochastic heat bath is to fix, in a large time scale, the variance of the velocities (i.e. the temperature) at β^{-1} , and establish a local equilibrium, where space-time averages of $V'(r_i)$ around a macroscopic particle number $[Nx]$ at a macroscopic time Nt converges to the equilibrium tension $\tau(r(t, x), \beta)$ at temperature β^{-1} and volume stretch $r(t, x)$. Since β is fixed by the heat bath and do not evolve in time, we drop it from the notation in the sequel.

The hydrodynamic limit consists in proving that, for any continuous function $G(x)$ on $[0, 1]$,

$$\frac{1}{N} \sum_{i=1}^N G\left(\frac{i}{N}\right) \begin{pmatrix} r_i(Nt) \\ p_i(Nt) \end{pmatrix} \xrightarrow{N \rightarrow \infty} \int_0^1 G(x) \begin{pmatrix} r(t, x) \\ p(t, x) \end{pmatrix} dx, \tag{1.18}$$

in probability, with $(r(t, x), p(t, x))$ satisfying (1.5), (1.6). Of course a complete proof would require the uniqueness of such L^2 valued solutions that satisfy (1.9): this remains an open

problem. The results contained in [10] states that the limit distribution of the empirical distribution defined on the RHS of (1.18), concentrates on the possible solutions of (1.5) and (1.6) that satisfy (1.9). Since we have no uniqueness result, we cannot assure that the solutions constructed in the present paper coincide with those obtained with the hydrodynamic limit from (1.17). One can however conjecture that this is the case.

This stochastic model was already considered by Fritz [11] in the infinite volume without boundary conditions, and in [9], but without the characterization of the boundary conditions.

In the hydrodynamic limit only L^2 bounds are available and we are constrained to consider L^2 valued solutions. Since these solutions do not have definite values on the boundary, boundary conditions have only a dynamical meaning in the sense of an evolution in L^2 given by (1.5), (1.6).

2. Hyperbolic system and the existence of weak solutions

For $r, p : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$, consider the hyperbolic system

$$\begin{cases} r_t - p_x = 0 & p(t, 0) = 0 & r(t, 1) = \tau^{-1}(\bar{\tau}(t)) \\ p_t - \tau(r)_x = 0 & p(0, x) = p_0(x) & r(0, x) = r_0(x) \end{cases} \tag{2.1}$$

The nonlinearity $\tau \in C^3(\mathbb{R})$ is chosen to have the following properties.

- (τ -i) $c_1 \leq \tau'(r) \leq c_2$ for some $c_1, c_2 > 0$ and all $r \in \mathbb{R}$;
- (τ -ii) $\tau''(r) \neq 0$ for all $r \in \mathbb{R}$;
- (τ -iii) $\tau''(r), \tau'''(r) \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

We also assume that $\bar{\tau} : \mathbb{R}_+ \rightarrow \mathbb{R}$ is smooth. Moreover, there is a time T_* such that $\bar{\tau}'(t) = 0$ for all $t \geq T_*$. The initial data $r_0, p_0 \in L^2(0, 1)$ are compatible with the boundary conditions.

Remark. Conditions (τ -i) and (τ -ii) ensure that the system is strictly hyperbolic and genuinely nonlinear, respectively. Condition (τ -iii) is used later on to ensure some boundedness properties of the Lax entropies.

Theorem 2.1. *System (2.1) admits a weak solution $\bar{u} = (\bar{r}, \bar{p})$ in the sense of (1.5) and (1.6), such that $\bar{u} \in L^\infty(0, T; L^2(0, 1))$, $\bar{u}(0, x) = u_0(x)$ for a.e. x ; and it satisfies the Clausius inequality:*

$$\mathcal{F}(\bar{u}(t)) - \mathcal{F}(u_0) \leq W(t), \quad \forall t \in [0, T] \tag{2.2}$$

with $W(t)$ as in (1.10). Furthermore \bar{u} satisfies the local Lax entropy condition in the sense specified in Section 6.

3. Viscous approximation and energy estimates

We consider the following parabolic approximation of the hyperbolic system (2.1)

$$\begin{cases} r_t^\delta - p_x^\delta = \delta r_{xx}^\delta \\ p_t^\delta - \tau(r^\delta)_x = \delta p_{xx}^\delta \end{cases}, \quad (t, x) \in \mathbb{R}_+ \times [0, 1] \tag{3.1}$$

for $\delta > 0$, with the boundary conditions:

$$p^\delta(t, 0) = 0, \quad r^\delta(t, 1) = \tau^{-1}(\bar{\tau}(t)), \quad p_x^\delta(t, 1) = 0, \quad r_x^\delta(t, 0) = 0, \tag{3.2}$$

and initial data:

$$p^\delta(0, x) = p_0^\delta(x), \quad r^\delta(0, x) = r_0^\delta(x). \tag{3.3}$$

The initial data $r_0^\delta, p_0^\delta \in C^\infty([0, 1])$ are mollifications of r_0 and p_0 compatible with the boundary conditions:

$$p_0^\delta(0) = 0 \quad r_0^\delta(1) = \tau^{-1}(\bar{\tau}(0)), \quad \partial_x p_0^\delta(1) = 0 \quad \partial_x r_0^\delta(0) = 0. \tag{3.4}$$

Moreover, there is C independent of δ such that

$$\|r_0^\delta\|_{L^2} + \|p_0^\delta\|_{L^2} + \|\sqrt{\delta}\partial_x r_0^\delta\|_{L^2} + \|\sqrt{\delta}\partial_x p_0^\delta\|_{L^2} \leq C \tag{3.5}$$

and $(r_0^\delta, p_0^\delta) \rightarrow (r_0, p_0)$ strongly in $L^2(0, 1)$.

As shown in [12] in a more general setting, this system admits a global classical solution (r^δ, p^δ) , with

$$r^\delta, p^\delta \in C^1(\mathbb{R}_+; C^0([0, 1])) \cap C^0(\mathbb{R}_+; C^2([0, 1])).$$

Remark. (i) We added two extra Neumann conditions, namely $p_x^\delta(t, 1) = r_x^\delta(t, 0) = 0$. These conditions reflect the conservative nature of the viscous perturbation, and are required in order to obtain the correct production of free energy.

(ii) One could introduce a *nonlinear* viscosity term: $\delta\tau(r^\delta)_{xx}$ in place of δr_{xx}^δ . This is a term which comes naturally from a microscopic derivation of system (3.1), as described in the introduction (see also [13]). Nevertheless, this does not drastically change the problem, thus we shall consider only the linear viscosity δr_{xx}^δ .

Theorem 3.1 (Energy estimate). *There there is a constant $C > 0$ independent of t and δ such that*

$$\int_0^1 |u^\delta(t, x)|^2 dx + \delta \int_0^t \int_0^1 |u_x^\delta(s, x)|^2 dx ds \leq C \tag{3.6}$$

for all $t \geq 0$ and $\delta > 0$.

Proof. Let F be a primitive of τ such that $\frac{\epsilon_1}{2}r^2 \leq F(r) \leq \frac{\epsilon_2}{2}r^2$. By a direct calculation we have

$$\int_0^1 \left(\frac{(p^\delta)^2}{2} + F(r^\delta) \right) dx \Big|_{t=0}^{t=T} + \int_0^T \int_0^1 \left(\delta(r_x^\delta)^2 + \delta(p_x^\delta)^2 \right) dx dt = \int_0^T \bar{\tau}(t) \int_0^1 r_t^\delta dx dt \tag{3.7}$$

$$= \left(\bar{\tau}(t) \int_0^1 r^\delta dx \right) \Big|_{t=0}^{t=T} - \int_0^T \bar{\tau}'(t) \int_0^1 r^\delta dx dt. \tag{3.8}$$

Write, for some $\epsilon > 0$ to be chosen later,

$$\bar{\tau}(T) \int_0^1 r^\delta(T, x) dx \leq |\bar{\tau}(T)| \left(\frac{1}{2\epsilon} + \frac{\epsilon}{2} \int_0^1 (r^\delta)^2(T, x) dx \right) \leq \frac{C_{\bar{\tau}}}{2\epsilon} + \frac{C_{\bar{\tau}}\epsilon}{2} \int_0^1 (r^\delta)^2(T, x) dx \tag{3.9}$$

where $C_{\bar{\tau}} = \sup_{t \geq 0} (|\bar{\tau}(t)| + |\bar{\tau}'(t)|)$ depends on $\bar{\tau}$ only.

Using $F(r) \geq \frac{c_1}{2} r^2$ we obtain

$$\begin{aligned} \left(\frac{c_1}{2} - \frac{C_{\bar{\tau}}\varepsilon}{2}\right) \int_0^1 (r^\delta)^2(T, x) dx + \frac{1}{2} \int_0^1 (p^\delta)^2(T, x) dx + \int_0^T \int_0^1 \left(\delta(r_x^\delta)^2 + \delta(p_x^\delta)^2\right) dx dt \\ \leq \frac{C_{\bar{\tau}}}{2\varepsilon} + C_0 - \int_0^T \bar{\tau}'(t) \int_0^1 r^\delta(t, x) dx dt. \end{aligned} \tag{3.10}$$

Recall that there is $T_* > 0$ such that $\bar{\tau}'(t) = 0$ for $t \geq T_*$. Then, for $T < T_*$, we write

$$\begin{aligned} \left(\frac{c_1}{2} - \frac{C_{\bar{\tau}}\varepsilon}{2}\right) \int_0^1 (r^\delta)^2(T, x) dx + \frac{1}{2} \int_0^1 (p^\delta)^2(T, x) dx + \int_0^T \int_0^1 \left(\delta(r_x^\delta)^2 + \delta(p_x^\delta)^2\right) dx dt \\ \leq \frac{C_{\bar{\tau}}}{2\varepsilon} + C_0 + \frac{C_{\bar{\tau}}^2}{2} T + \frac{1}{2} \int_0^T \int_0^1 (r^\delta)^2(t, x) dx \end{aligned} \tag{3.11}$$

$$\begin{aligned} \leq \frac{C_{\bar{\tau}}}{2\varepsilon} + C_0 + \frac{C_{\bar{\tau}}^2}{2} T + \frac{1}{2} \int_0^T \int_0^1 \left((r^\delta)^2(t, x) + (p^\delta)^2(t, x)\right) dx dt \\ + \frac{1}{2} \int_0^T \int_0^t \int_0^1 \left(\delta(r_x^\delta)^2 + \delta(p_x^\delta)^2\right) dx ds dt \end{aligned} \tag{3.12}$$

where C_0 depends on the initial data only. Choosing $\varepsilon = c_1/(2C_{\bar{\tau}})$ gives

$$\frac{c_1}{4} J(T) \leq C_0 + \frac{C_{\bar{\tau}}^2}{c_1} + \frac{C_{\bar{\tau}}^2}{2} T + \frac{1}{2} \int_0^T J(t) dt, \tag{3.13}$$

where

$$J(t) = \int_0^1 \left((r^\delta)^2(t, x) + (p^\delta)^2(t, x)\right) dx + \int_0^t \int_0^1 \left(\delta(r_x^\delta)^2 + \delta(p_x^\delta)^2\right) dx ds. \tag{3.14}$$

We apply Gronwall's inequality. This, together with $T < T_*$, gives

$$\begin{aligned} J(T) &\leq \frac{4c_1 C_0 + 2C_{\bar{\tau}}^2(2 + c_1 T)}{c_1^2} \exp\left(\frac{2T}{c_1}\right) \\ &\leq \frac{4c_1 C_0 + 2C_{\bar{\tau}}^2(2 + c_1 T_*)}{c_1^2} \exp\left(\frac{2T_*}{c_1}\right) := C_0(c_1, \bar{\tau}), \end{aligned} \tag{3.15}$$

for all $T \in [0, T_*)$, where $C_0(c_1, \bar{\tau})$ is independent of T and δ

On the other hand, if $T \geq T_*$, we have

$$\begin{aligned} \left(\frac{c_1}{2} - \frac{C_{\bar{\tau}}\varepsilon}{2}\right) \int_0^1 (r^\delta)^2(T, x) dx + \frac{1}{2} \int_0^1 (p^\delta)^2(T, x) dx + \int_0^T \int_0^1 \left(\delta(r_x^\delta)^2 + \delta(p_x^\delta)^2\right) dx dt \\ \leq \frac{C_{\bar{\tau}}}{2\varepsilon} + C_0 - \int_0^{T_*} \bar{\tau}'(t) \int_0^1 r^\delta(t, x) dx dt \end{aligned} \tag{3.16}$$

and the integral at the right-hand side is uniformly bounded in T , δ and δ , since

$$\begin{aligned}
 - \int_0^{T_*} \bar{\tau}'(t) \int_0^1 r^\delta(t, x) dx dt &\leq C_{\bar{\tau}} \int_0^{T_*} \int_0^1 |r^\delta(t, x)| dx dt \\
 &\leq C_{\bar{\tau}} T_* \left(\frac{1}{T_*} \int_0^{T_*} \int_0^1 (r^\delta)^2(t, x) dx dt \right)^{1/2} \\
 &\leq C_{\bar{\tau}} \sqrt{T_*} \left(\int_0^{T_*} J(t) dt \right)^{1/2} \\
 &\leq C_{\bar{\tau}} T_* \sqrt{C_0(c_1, \bar{\tau})}
 \end{aligned} \tag{3.17}$$

□

From (3.7) we also immediately obtain the following

Corollary 3.2 (Viscous Clausius inequality).

$$\mathcal{F}(u^\delta(t)) - \mathcal{F}(u_0^\delta) \leq - \int_0^t \bar{\tau}'(s) \int_0^1 r^\delta(s, x) dx + \bar{\tau}(t) \int_0^1 r^\delta(t, x) dx - \bar{\tau}(0) \int_0^1 r_0^\delta(x) dx. \tag{3.18}$$

4. L^2 young measures and compensated compactness

Throughout this section, for any fixed $T > 0$ let $u^\delta(t, x) := (r^\delta(t, x), p^\delta(t, x))$ be a strong solution of (3.1) on Q_T . By Theorem 3.1 and after a time integration over $[0, T]$ we obtain

$$\|u^\delta\|_{L^2(Q_T)} \leq C \tag{4.1}$$

for some C independent of δ . Thus we can extract from $\{u^\delta\}_{\delta > 0}$ a subsequence that is weakly convergent in $L^2(Q_T)$. Namely, up to a subsequence, there exists $\bar{u} = (\bar{r}, \bar{p}) \in L^2(Q_T)$ such that

$$\lim_{\delta \rightarrow 0} \int_{Q_T} u^\delta \varphi = \int_{Q_T} \bar{u} \varphi, \quad \forall \varphi \in L^2(Q_T). \tag{4.2}$$

All the limits $\delta \rightarrow 0$ taken below are intended along a chosen subsequence.

In this section, we want to show that for any $\phi \in L^2(Q_T)$ we have

$$\lim_{\delta \rightarrow 0} \int_{Q_T} \phi(t, x) \tau(r^\delta(t, x)) dx dt = \int_{Q_T} \phi(t, x) \tau(\bar{r}(t, x)) dx dt. \tag{4.3}$$

This is done using a L^2 version of the compensated compactness, which is usually performed in L^∞ .

From the solution $u^\delta(t, x)$, we define the following Young measure on $Q_T \times \mathbb{R}^2$:

$$\nu_{t,x}^\delta := \delta_{u^\delta(t,x)}, \tag{4.4}$$

which is a Dirac mass centered at u^δ , i.e.

$$\int_{Q_T} J(t, x) f(u^\delta(t, x)) dx dt = \int_{Q_T} \int_{\mathbb{R}^2} J(t, x) f(\xi) d\nu_{t,x}^\delta(\xi) dx dt$$

for all measurable $J : Q_T \rightarrow \mathbb{R}$ and $f : \mathbb{R}^2 \rightarrow \mathbb{R}$.

Since we have L^2 bounds on u^δ , we refer at $\nu_{t,x}^\delta$ as a L^2 -Young measure [14]. In particular we have, from (4.1)

$$\int_{Q_T} \int_{\mathbb{R}^2} |\xi|^2 d\nu_{t,x}^\delta(\xi) dxdt \leq C. \tag{4.5}$$

We call \mathcal{Y} the set of Young measures on $Q_T \times \mathbb{R}^2$ and we make it a metric space by endowing it with the Prohorov’s metric. By proposition 4.1 of [15], the set

$$K_C := \left\{ \nu \in \mathcal{Y} : \int_{Q_T} \int_{\mathbb{R}^2} |\xi|^2 d\nu_{t,x}(\xi) dxdt \leq C \right\} \tag{4.6}$$

is compact in \mathcal{Y} . Then, by the fundamental theorem for Young measures ([14], section 2), there exists $\bar{\nu}_{t,x} \in \mathcal{Y}$ so that, up to a subsequence,

$$\lim_{\delta \rightarrow 0} \int_{Q_T} \int_{\mathbb{R}^2} J(t,x) f(\xi) d\nu_{t,x}^\delta(\xi) dxdt = \int_{Q_T} \int_{\mathbb{R}^2} J(t,x) f(\xi) d\bar{\nu}_{t,x}(\xi) dxdt \tag{4.7}$$

for all continuous and bounded $J : Q_T \rightarrow \mathbb{R}$ and $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. We shall simply write $\nu^\delta \rightarrow \bar{\nu}$ in place of (4.7). By a simple adaptation of proposition 4.2 of [15], (4.7) can be extended to a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $f(\xi)/|\xi|^2 \rightarrow 0$ as $|\xi| \rightarrow +\infty$.

In order to obtain (4.3), we need to prove that the limit Young measure $\bar{\nu}$ is a Dirac mass: $\bar{\nu}_{t,x} = \delta_{\bar{u}(t,x)}$, for some $\bar{u} \in L^2(Q_T)$ and for almost every $(t,x) \in Q_T$. This is done using the classical argument by Tartar and Murat.

Definition 4.1. A Lax entropy-entropy flux pair for system (2.1) is a couple of differentiable functions $(\eta, q) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$\begin{cases} \eta_r + q_p = 0 \\ \tau'(r)\eta_p + q_r = 0 \end{cases} \tag{4.8}$$

We show that Tartar’s equation holds for any two suitable entropy pairs (η, q) and (η', q') to be specified below and almost all $(t,x) \in Q_T$:

$$\langle \eta q' - \eta' q, \bar{\nu}_{t,x} \rangle = \langle \eta, \bar{\nu}_{t,x} \rangle \langle q', \bar{\nu}_{t,x} \rangle - \langle \eta', \bar{\nu}_{t,x} \rangle \langle q, \bar{\nu}_{t,x} \rangle, \tag{4.9}$$

where

$$\langle f, \bar{\nu}_{t,x} \rangle := \int_{\mathbb{R}^2} f(\xi) d\bar{\nu}_{t,x}(\xi) \tag{4.10}$$

for any measurable f . We employ the following argument due to Shearer [7].

Accordingly to Lemma 2 in [7], there exists a family of half-plain supported entropy-entropy fluxes (η, q) such that η and q are bounded together with their first and second derivatives. These are explicitly given as follows. We define $z(r) := \int_0^r \sqrt{\tau'(\rho)} d\rho$ and we define the Riemann coordinates $w_1 = p + z, w_2 = p - z$. We also pass from the dependent variables η, q to H, Q as follows:

$$\eta = \frac{1}{2}(\tau')^{-1/4}(H + Q) \tag{4.11}$$

$$q = \frac{1}{2}(\tau')^{+1/4}(H - Q) \tag{4.12}$$

so that (4.8) becomes

$$\begin{cases} H_{w_1} = aQ \\ H_{w_2} = -aQ \end{cases} \tag{4.13}$$

where

$$a(w_1 - w_2) = \frac{\tau''\left(r\left(\frac{w_1-w_2}{2}\right)\right)}{8\left(\tau'\left(r\left(\frac{w_1-w_2}{2}\right)\right)\right)^{3/2}}. \tag{4.14}$$

Then we fix $\bar{w}_1, \bar{w}_2 \in \mathbb{R}$ and we solve (4.13) with Goursat data given on the lines $w_1 = \bar{w}_1$ and $w_2 = \bar{w}_2$:

$$\begin{aligned} H(\bar{w}_1, w_2) &= g(w_2) \\ Q(w_1, \bar{w}_2) &= 0, \end{aligned} \tag{4.15}$$

where g is continuous and compactly supported. Then one can explicitly solve (4.13) and get

$$\begin{aligned} H(w_1, w_2) &= g(w_2) + \sum_{n=1}^{\infty} (\mathcal{A}^n g)(w_1, w_2) \\ Q(w_1, w_2) &= - \int_{\bar{w}_2}^{w_2} a(w_1 - v)H(w_1, v)dv, \end{aligned} \tag{4.16}$$

where the operator \mathcal{A} acts on functions $f \in L^1_{loc}(\mathbb{R}^2)$ as follows:

$$(\mathcal{A}f)(w_1, w_2) = - \int_{\bar{w}_1}^{w_1} \int_{\bar{w}_2}^{w_2} a(v - w_2)a(v - u)f(v, u)dudv. \tag{4.17}$$

Finally, going back to η and q and using our assumptions on τ it is possible to show ([7], Lemma 2) that η and q are bounded, together with their first and second derivatives.

Now we have a suitable family of entropy-entropy flux pair, we use Tartar-Murat Lemma in order to derive Tartar’s equation (4.9). We evaluate (η, q) along the approximate solutions u^δ and compute the entropy production:

$$\eta(u^\delta)_t + q(u^\delta)_x = \delta \left(\eta_r r_x^\delta + \eta_p p_x^\delta \right)_x - \delta \left(\eta_{rr} (r_x^\delta)^2 + \eta_{pp} (p_x^\delta)^2 + 2\eta_{rp} r_x^\delta p_x^\delta \right) \tag{4.18}$$

Since η_r and η_p are bounded and $\sqrt{\delta}r_x^\delta, \sqrt{\delta}p_x^\delta$ are bounded in $L^2(Q_T)$, we have

$$\lim_{\delta \rightarrow 0} \delta \left(\eta_r r_x^\delta + \eta_p p_x^\delta \right)_x = 0 \quad \text{in } H^{-1}(Q_T), \tag{4.19}$$

while

$$\left\| \delta \left(\eta_{rr} (r_x^\delta)^2 + \eta_{pp} (p_x^\delta)^2 + 2\eta_{rp} r_x^\delta p_x^\delta \right) \right\|_{L^1(Q_T)} \leq C \tag{4.20}$$

uniformly with respect to δ . Thus we have obtained an equality of the form

$$\eta(u^\delta)_t + q(u^\delta)_x = \chi^\delta + \psi^\delta,$$

where $\{\chi^\delta\}_{\delta>0}$ lies in a compact set of $H^{-1}(Q_T)$ and $\{\psi^\delta\}_{\delta>0}$ is bounded in $L^1(Q_T)$. Moreover, since η and q are bounded, $\{\eta(u^\delta)_t + q(u^\delta)_x\}_{\delta>0}$ is bounded in $W^{-1,p}(Q_T)$ for some $p > 2$.

Therefore, we can apply Tartar-Murat and the div-curl lemma (cf [16], Theorem 16.2.1 and Lemma 16.2.2) and obtain Tartar’s equation (4.9).

The final step is to use Tartar’s equation to prove that the support of the limit Young measure $\bar{\nu}_{t,x}$ is a point. This is done in lemmas 4 to 7 of [7] and leads to the following

Proposition 4.2. *There exists a $\bar{u} \in L^2(Q_T)$ such that $\bar{\nu}_{t,x} = \delta_{\bar{u}(t,x)}$ for almost all $(t, x) \in Q_T$. Moreover, $u^\delta \rightarrow \bar{u}$ strongly in $L^p(Q_T)$ for any $p \in [1, 2)$.*

4.1. Regularity

Proposition 4.3. *For the function \bar{u} obtained in section 4,*

$$\bar{u} \in L^\infty(0, T; L^2(0, 1)).$$

Proof. Since $u^\delta \rightarrow \bar{u}$ in L^p strong for $p < 2$, we can extract a subsequence $\{u^{\delta_k}\}_{k \in \mathbb{N}}$ that converges pointwise to \bar{u} for almost all t and x . In particular, for almost all t , the sequence $u^{\delta_k}(t, x)$ converges for almost all x . Therefore, by Fatou lemma and **Theorem 3.1**,

$$\int_0^1 |\bar{u}(t, x)|^2 dx \leq \liminf_{k \rightarrow \infty} \int_0^1 |u^{\delta_k}(t, x)|^2 dx \leq C \tag{4.21}$$

for almost all $t \in [0, T]$. □

The proof of next lemma is standard and therefore omitted.

Lemma 4.4. *Let $a(t) := \tau^{-1}(\bar{\tau}(t))$. Then, the solutions (r^δ, p^δ) of the viscous system (3.1) can be written as follows:*

$$\begin{aligned} r^\delta(t, x) &= a(t) + \int_0^1 G_r^\delta(x, x', t)(r_0^\delta(x') - a(0)) dx' \\ &\quad + \int_0^t \int_0^1 G_r^\delta(x, x', t - t')(\partial_{x'} p^\delta(t', x') - a(t')) dx' dt' \end{aligned} \tag{4.22}$$

$$p^\delta(t, x) = \int_0^1 G_p^\delta(x, x', t) p_0^\delta(x') dx' + \int_0^t \int_0^1 G_p^\delta(x, x', t - t') \partial_{x'} \tau(r^\delta(t', x')) dx' dt' \tag{4.23}$$

where the G_r^δ and G_p^δ are Green functions of the heat operator $\partial_t - \delta \partial_{xx}$ with homogeneous boundary conditions:

$$G_r^\delta(1, x', t) = \partial_x G_r^\delta(0, x', t) = 0 \tag{4.24}$$

$$G_p^\delta(0, x', t) = \partial_x G_p^\delta(1, x', t) = 0 \tag{4.25}$$

for all $x' \in [0, 1], t \geq 0$ and $\delta > 0$.

The Green’s functions $G_r^\delta(x, x', t)$ and $G_p^\delta(x, x', t)$ are symmetric under the exchange of x and x' . Moreover we have the following identities

$$\partial_x G_p^\delta(x, x', t) = -\partial_{x'} G_r^\delta(x, x', t), \tag{4.26}$$

$$\partial_x G_r^\delta(x, x', t) = -\partial_{x'} G_p^\delta(x, x', t). \tag{4.27}$$

Remark. The functions G_r^δ and G_p^δ have the following explicit forms:

$$G_p^\delta(x, x', t) = \frac{1}{2} \sum_{n \text{ odd}} e^{-t\delta\lambda_n} \sin(\sqrt{\lambda_n}x) \sin(\sqrt{\lambda_n}x') \tag{4.28}$$

$$G_r^\delta(x, x', t) = \frac{1}{2} \sum_{n \text{ odd}} e^{-t\delta\lambda_n} \cos(\sqrt{\lambda_n}x) \cos(\sqrt{\lambda_n}x'), \tag{4.29}$$

with $\lambda_n = \frac{n^2\pi^2}{4}$.

Proposition 4.5. For any $\phi \in C^1([0, 1])$, the application

$$t \mapsto I_\phi(t) := \int_0^1 \phi(x)\bar{u}(t, x)dx \tag{4.30}$$

is Lipschitz continuous. Consequently $\bar{u}(t, \cdot) \in L^2(0, 1)$ for all $t \geq 0$.

Proof. We prove the statement for \bar{p} , as the proof for \bar{r} is similar. Furthermore, we prove the proposition only between 0 and t , as in the general case, say between t_1 and t , it is enough to replace the initial term $p_0^\delta(x)$ with $p^\delta(t_1, x)$. We let

$$I_\phi^\delta(t) := \int_0^1 \phi(x)p^\delta(t, x)dx \tag{4.31}$$

and evaluate

$$\begin{aligned} I_\phi(t) - I_\phi(0) &= \int_0^1 \int_0^1 \phi(x)G_p^\delta(x, x', t)p_0^\delta(x')dx'dx - \int_0^1 \phi(x)p_0^\delta(x)dx \\ &\quad + \int_0^t \int_0^1 \int_0^1 \phi(x)G_p^\delta(x, x', t-t')\partial_{x'}\tau(r^\delta(t', x'))dx dx' dt' \\ &= \int_0^1 \int_0^1 \phi(x) \left[G_p^\delta(x, x', t) - \delta(x, x') \right] p_0^\delta(x')dx'dx \\ &\quad + \int_0^t \int_0^1 \int_0^1 \phi(x)\partial_x G_r^\delta(x, x', t-t')\tau(r^\delta(t', x'))dx dx' dt' + \int_0^t \int_0^1 \phi(x)G_p^\delta(x, 1, t-t')\bar{\tau}(t')dx dt', \end{aligned} \tag{4.33}$$

where we have used the symmetry of G_p^δ as well as the property $\partial_{x'}G_p^\delta = -\partial_x G_r^\delta$. The boundary term is estimated as

$$\begin{aligned} \left| \int_0^t \int_0^1 \phi(x)G_p^\delta(x, 1, t-t')\bar{\tau}(t')dx dt' \right| &\leq \|\bar{\tau}\|_\infty \int_0^t \left| \int_0^1 \phi(x)G_p^\delta(x, 1, t-t')dx \right| dt' \\ &\leq t\|\bar{\tau}\|_\infty \|\phi\|_{L^2}. \end{aligned} \tag{4.34}$$

We estimate the term involving τ as

$$\begin{aligned}
 & \left| \int_0^t \int_0^1 \int_0^1 \phi(x) \partial_x G_r^\delta(x, x', t - t') \tau(r^\delta(t', x')) dx dx' dt' \right| \\
 &= \left| \frac{1}{2} \sum_{n \text{ odd}} \int_0^1 \sqrt{\lambda_n} \sin(\sqrt{\lambda_n} x) \phi(x) dx \int_0^t dt' e^{-(t-t')\lambda_n \delta} \int_0^1 \cos(\sqrt{\lambda_n} x') \tau(r^\delta(t', x')) dx' \right| \\
 &\leq \|\phi'\|_{L^2} \left[\frac{1}{2} \sum_{n \text{ odd}} \left(\int_0^t dt' e^{-(t-t')\lambda_n \delta} \int_0^1 \cos(\sqrt{\lambda_n} x') \tau(r^\delta(t', x')) dx' \right)^2 \right]^{1/2} \\
 &\leq \|\phi'\|_{L^2} \left[\frac{1}{2} \sum_{n \text{ odd}} \left(\frac{1 - e^{-2t\lambda_n \delta}}{2\lambda_n \delta} \right) \int_0^t dt' \left(\int_0^1 \cos(\sqrt{\lambda_n} x') \tau(r^\delta(t', x')) dx' \right)^2 \right]^{1/2} \\
 &\leq \|\phi'\|_{L^2} \left[\frac{t}{2} \int_0^t dt' \sum_{n \text{ odd}} \left(\int_0^1 \cos(\sqrt{\lambda_n} x') \tau(r^\delta(t', x')) dx' \right)^2 \right]^{1/2} \\
 &= \|\phi'\|_{L^2} \left[t \int_0^t dt' \int_0^1 \tau(r^\delta(t', x'))^2 dx' \right]^{1/2}
 \end{aligned} \tag{4.35}$$

$$\leq t \|\phi'\|_{L^2} \left\| \int_0^1 \tau(r^\delta(\cdot, x'))^2 dx' \right\|_{L^\infty(0, T)}^{1/2} \leq Ct \|\phi'\|_{L^2} \tag{4.36}$$

where C is independent of t and δ .

In order to estimate the first term of (4.33) we write

$$\begin{aligned}
 & \int_0^1 \int_0^1 \phi(x) \left[G_p^\delta(x, x', t) - \delta(x, x') \right] p_0^\delta(x') dx' dx \\
 &= \frac{1}{2} \sum_{n \text{ odd}} \int_0^1 \sin(\sqrt{\lambda_n} x) \phi(x) dx (e^{-t\lambda_n \delta} - 1) \int_0^1 \sin(\sqrt{\lambda_n} x') p_0^\delta(x') dx' \\
 &\leq t \delta \frac{1}{2} \sum_{n \text{ odd}} \lambda_n \left| \int_0^1 \sin(\sqrt{\lambda_n} x) \phi(x) dx \right| \left| \int_0^1 \sin(\sqrt{\lambda_n} x') p_0^\delta(x') dx' \right| \\
 &\leq t \|\phi'\|_{L^2} \|\delta \partial_x p_0^\delta\|_{L^2} \leq t \|\phi'\|_{L^2} C
 \end{aligned} \tag{4.37}$$

where we have used the assumption that $\{\sqrt{\delta} \partial_x p_0^\delta\}_{\delta > 0}$ is bounded in $L^2(0, 1)$. We have also assumed, without loss of generality, $\delta \leq 1$.

Putting everything together, we have obtained

$$\left| I_\phi^\delta(t) - I_\phi^\delta(0) \right| \leq tC (\|\phi'\|_{L^2} + \|\phi\|_{L^2}) \tag{4.38}$$

for some constant C independent of t and δ . This leads to the conclusion after passing to the limit $\delta \rightarrow 0$. □

5. Proof of Theorem 2.1 and Clausius inequality

All is left to prove is that the function \bar{u} obtained in the previous section is to a weak solution of the hyperbolic system (2.1), in the sense of Section 2. Let $\psi \in C^1(Q_T)$ with $\psi(t, 0) = 0$ for all $t \in [0, T]$. Then, for any $t \in [0, T]$ we have

$$\begin{aligned} 0 &= \int_0^t \int_0^1 \left(\psi p_s^\delta - \psi \tau (r^\delta)_x - \delta \psi p_{xx}^\delta \right) dx ds \\ &= \int_0^1 \psi(t, x) p^\delta(t, x) dx - \int_0^1 \psi(0, x) p_0^\delta(x) dx + \\ &\quad - \int_0^t \int_0^1 \left(\psi_s p^\delta - \psi_x \tau (r^\delta) - \delta \psi_x p_x^\delta \right) dx ds - \int_0^\infty \psi(t, 1) \bar{\tau}(t) dt. \end{aligned} \tag{5.1}$$

where we have used the initial-boundary conditions $\tau(r^\delta(t, 1)) = \bar{\tau}(t)$ and $p_x^\delta(t, 1) = 0, p^\delta(0, x) = p_0^\delta(x)$ as well as $\psi(t, 0) = 0$. Since p_0^δ converges to p_0 in $L^2(0, 1)$, we have

$$\lim_{\delta \rightarrow 0} \int_0^1 \psi(0, x) p_0^\delta(x) dx = \int_0^1 \psi(0, x) p_0(x) dx. \tag{5.2}$$

Furthermore, Theorem 3.1 implies $\sqrt{\delta} p_x^\delta \in L^2(Q_T)$, consequently $\int_0^t \int_0^1 \delta \psi_x p_x^\delta dx ds$ vanishes as $\delta \rightarrow 0$. Moreover, (4.2) implies, along the subsequence that defines $\bar{u} = (\bar{r}, \bar{p})$,

$$\lim_{\delta \rightarrow 0} \int_0^t \int_0^1 \psi_x p^\delta dx ds = \int_0^t \int_0^1 \psi_x \bar{p} dx dt, \tag{5.3}$$

while by (4.3) we have that

$$\lim_{\delta \rightarrow 0} \int_0^t \int_0^1 \psi_x \tau (r^\delta) dx ds = \int_0^t \int_0^1 \psi_x \tau (\bar{r}) dx ds,$$

so that (1.6) is satisfied. The (1.5) is linear and it follows similarly.

Proposition 5.1. *The solution \bar{u} satisfies Clausius inequality*

$$\mathcal{F}(\bar{u}(t)) - \mathcal{F}(u_0) \leq W(t) \tag{5.4}$$

for all $t \in [0, T]$, where

$$W(t) = - \int_0^t \bar{\tau}'(s) \int_0^1 \bar{r}(s, x) dx + \bar{\tau}(t) \int_0^1 \bar{r}(t, x) dx - \bar{\tau}(0) \int_0^1 r_0(x) dx. \tag{5.5}$$

Proof. By Proposition 4.3, Corollary 3.1, and Lemma 4.5, we have, for all $t \in [0, T]$,

$$\int_0^1 \left(\frac{\bar{p}^2(t, x)}{2} + F(\bar{r}(t, x)) \right) dx \leq \liminf_{k \rightarrow \infty} \int_0^1 \left(\frac{(p^{\delta_k})^2(t, x)}{2} + F(r^{\delta_k}(t, x)) \right) dx \tag{5.6}$$

$$\leq \lim_{k \rightarrow \infty} \left(\mathcal{F}(u_0^{\delta_k}) - \int_0^t \bar{\tau}'(s) \int_0^1 r^{\delta_k}(s, x) dx ds + \bar{\tau}(t) \int_0^1 r^{\delta_k}(t, x) dx - \bar{\tau}(0) \int_0^1 r_0^{\delta_k}(x) dx \right) \tag{5.7}$$

$$= \mathcal{F}(u_0) - \int_0^t \bar{\tau}'(s) \int_0^1 \bar{r}(s, x) dx ds + \bar{\tau}(t) \int_0^1 \bar{r}(t, x) dx - \bar{\tau}(0) \int_0^1 r_0(x) dx, \tag{5.8}$$

where we have used the fact that u_0^δ converges to u_0 in L^2 strongly in order to conclude that $\mathcal{F}(u_0^\delta) \rightarrow \mathcal{F}(u_0)$. Moreover, all the integrals are well defined, since the application

$$t \mapsto \int_0^1 \bar{r}(t, x) dx$$

is continuous. □

Thanks to the Clausius inequality, the solutions we have constructed are natural candidates for being the thermodynamic entropy solution of Eq. (3.1) and one can conjecture that such limit is unique.

6. Lax entropy condition

From (4.18), if η is a convex Lax entropy, we have that

$$\eta(u^\delta)_t + q(u^\delta)_x \geq \delta \left(\eta_r r_x^\delta + \eta_p p_x^\delta \right)_x. \quad (6.1)$$

If η grows at most quadratically, the right hand side of (6.1) vanished in $H^{-1}(Q_T)$, and by (4.19), for the limit we have that $\eta(u)_t + q(u)_x \geq 0$ as a distribution in $H^{-1}(Q_T)$. This is the usual *local* characterization of weak entropy solutions, that is independent of the boundary conditions and does not give informations of the *behavior* at the boundary. Our solutions obtained from viscosity approximation satisfy such local entropy condition.

Our point is that this local characterization should be implemented by the global entropy production for the entropy given by the free energy $\eta(u) = \mathcal{F}(u)$, i.e. the Clausius inequality.

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