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Double covers of Cartan modular curves

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We present a strategy to obtain explicit equations for the modular double covers associated respectively to both a split and a non-split Cartan subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$ with p prime. Then we apply it successfully to the level 13 case.

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1. Introduction

Non-cuspidal rational points on modular curves parametrize elliptic curves over $\mathbb{Q}$ with particular properties regarding the associated Galois representation modulo some positive integer N. Having equations for such modular curves helps to explicitly determine elliptic curves with a given Galois representation modulo N. When $N = p$ is a prime number, modular curves associated to maximal subgroups of $\mathrm{GL}_2(\mathbb{F}_p)$ with surjective determinant, such as normalizers of Cartan subgroups, play an important role in the framework of Serre's uniformity problem, which concerns the determination of elliptic curves with surjective Galois representation modulo p.

Recent work has been done in determining equations of modular curves of Cartan type. An affine plane model for $X_0(169) \cong X_{\mathrm{s}}(13)$ is computed in [Ken80][Ken81]. An equation for $X^+_{\mathrm{ns}}(13)$ and $X^+_{\mathrm{s}}(13)$ is computed in [Bar14], while models of $X^+_{\mathrm{ns}}(p)$ for $p = 17, 19$ can be found in [MS17]. An equation for $X_{\mathrm{ns}}(11)$ and the double cover $X_{\mathrm{ns}}(11) \to X^+_{\mathrm{ns}}(11)$ is computed in [DFGS14]. Furthermore, it is used in [FNS17] to solve the generalized Fermat equation with exponents $2, 3, 11$ and in [Zyw15] to classify the possible images of Galois representations modulo 11 associated to elliptic curves.

In this paper we present a strategy to compute birational models of modular curves $X_{\mathrm{s}}(p)$ and $X_{\mathrm{ns}}(p)$ obtained as double covers of their explicitly given quotients $X^+_{\mathrm{s}}(p)$ and $X^+_{\mathrm{ns}}(p)$, associated respectively to the normalizer of a split and a non-split Cartan subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$. As an application, we compute singular models in $\mathbb{A}^3$ for the genus 8 curves $X_{\mathrm{s}}(13)$ and $X_{\mathrm{ns}}(13)$. For completeness and further check we also compute smooth equations of the canonical model of such curves in $\mathbb{P}^7$ using classical methods and give explicit maps between all the known models of $X_{\mathrm{s}}(13)$ and $X_{\mathrm{ns}}(13)$.

In particular, in Section 3 we recall the method to compute equations of the canonical model, in Section 4 we describe the new strategy in general, and in Section 5 we apply the two methods to the level 13 case, obtaining the equations. Furthermore, in Section 6 we also compute a birational map between the two models found.

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2. Notation and basic facts

Let $\mathcal{H} = \{\tau \in \mathbb{C}, \mathrm{Im}(\tau) > 0\}$ be the complex upper half-plane and let $\mathcal{H}^* = \mathcal{H} \cup \{\infty\} \cup \mathbb{Q}$, both endowed with the action of $\mathrm{SL}_2(\mathbb{R})$ given by fractional linear transformation. Given a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, we can consider the modular curve $X(\Gamma)$ associated to Γ which is obtained by providing the orbit space $\Gamma\backslash\mathcal{H}^*$ with the structure of a compact Riemann surface. When $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ we denote the associated modular curve of genus 0 by $X(1)$. The complex points of $X(1)$ parametrize elliptic curves over $\mathbb{C}$ up to isomorphism. There is a morphism from every modular curve $X(\Gamma)$ to the modular curve $X(1)$ which is given by the group inclusion $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$. This morphism is called the j-map of $X(\Gamma)$.

Let N be a positive integer and let H be a subgroup of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. We can associate to H the congruence subgroup $\Gamma_H \overset{\mathrm{def}}{=} \{x \in \mathrm{SL}_2(\mathbb{Z})$ such that $x \pmod N \in H\}$ and the modular curve $X_H \overset{\mathrm{def}}{=} X(\Gamma_H)$, which admits the structure of projective algebraic curve. When the determinant homomorphism $\det : H \to (\mathbb{Z}/N\mathbb{Z})^\times$ is surjective, X_H can be defined over $\mathbb{Q}$. Furthermore, if the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ belongs to H, then, for any number field K, the K-rational points on X_H parametrize the elliptic curves defined over K such that H contains the image of the associated Galois representation modulo N, given by the action of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ on N-torsion points.

If we take H to be a Borel subgroup of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, we obtain the classical modular curve $X_0(N)$. This curve has an automorphism w_N, the Atkin–Lehner involution, induced by the action of the matrix $\begin{pmatrix} 0 & -\frac{1}{\sqrt{N}} \\ \sqrt{N} & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ on $\mathcal{H}$. This automorphism allows us to define the quotient curve $X_0^+(N) \overset{\mathrm{def}}{=} X_0(N)/\langle w_N \rangle$.

Let now $N = p$ be a prime number. If we take H to be a split or non-split Cartan subgroup of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ (see [Ser72, p. 278, Section 2.1]) we denote the associated modular curve by $X_{\mathrm{s}}(p)$ and $X_{\mathrm{ns}}(p)$, respectively. Any Cartan subgroup of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ has index 2 in its normalizer. Thus we obtain involutions w_{s} and w_{ns} and double covers $X_{\mathrm{s}}(p) \to X_{\mathrm{s}}^+(p)$ and $X_{\mathrm{ns}}(p) \to X_{\mathrm{ns}}^+(p)$, where $X_{\mathrm{s}}^+(p)$ and $X_{\mathrm{ns}}^+(p)$ are the modular curves associated to the normalizer of a split and a non-split Cartan subgroup of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$, respectively. Moreover we have $X_{\mathrm{s}}^+(p) = X_{\mathrm{s}}(p)/\langle w_{\mathrm{s}} \rangle$ and $X_{\mathrm{ns}}^+(p) = X_{\mathrm{ns}}(p)/\langle w_{\mathrm{ns}} \rangle$. Since the congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ associated to a split Cartan subgroup of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ is conjugate to the congruence subgroup associated to a Borel subgroup of $\mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z})$, we get the isomorphism $X_{\mathrm{s}}(p) \cong X_0(p^2)$. Furthermore, the involution w_s of $X_{\mathrm{s}}(p)$ corresponds to the Atkin–Lehner involution of $X_0(p^2)$, so that also $X_{\mathrm{s}}^+(p)$ is isomorphic to $X_0^+(p^2)$.

3. Explicit equations of modular curves using the canonical embedding

In this section we briefly recall how to get explicit equations for the canonical model of modular curves when we know enough Fourier coefficients of a basis of the vector space of modular forms corresponding to the space of differentials of the curve (see [Mer17] for more details).

Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, let $\mathcal{S}_2(\Gamma)$ be the $\mathbb{C}$-vector space of the cusp forms of weight 2 with respect to Γ. We know that $\mathcal{S}_2(\Gamma)$ is isomorphic to the $\mathbb{C}$-vector space of holomorphic differentials $\Omega^1(X(\Gamma))$ via the map $f(\tau) \mapsto f(\tau)d\tau$ (see [DS05, p. 81, Theorem 3.3.1]). Using this isomorphism when the genus g of $X(\Gamma)$ is greater than 2, we get the following realization of the canonical map

$$\varphi\colon X(\Gamma) \to \mathbb{P}^{g-1}(\mathbb{C})$$
$$\Gamma\tau \mapsto (f_1(\tau) : \ldots : f_g(\tau)),$$

where $\tau \in \mathcal{H}^*$ and $\mathcal{B} = \{f_1, \ldots, f_g\}$ is a $\mathbb{C}$-basis of $\mathcal{S}_2(\Gamma)$. The Enriques–Petri Theorem (see [GH78, Chapter 4, Section 3, p. 535] or [SD73]), states that the canonical model of a complete non-singular non-hyperelliptic curve is entirely cut out by quadrics and cubics.

Though the Enriques–Petri Theorem is proved only over algebraically closed fields, when $X(\Gamma)$ can be defined over $\mathbb{Q}$, we can try to look for quadratic and cubic equations over $\mathbb{Q}$ for the image of φ. Then we can check if the zero locus Z of such equations, which contains by construction the image of φ, is an algebraic curve with the same genus as $X(\Gamma)$. If this is the case, an application of the Riemann–Hurwitz formula tells us that the morphism $\varphi\colon X(\Gamma) \to Z$ is an isomorphism.

We know that $X_0(N)$ is not hyperelliptic when $N > 71$ ([Ogg74, p. 451, Theorem 2]) while $X_0^+(p^r)$, with p prime and r a positive integer, is not hyperelliptic when its genus is bigger than 2 ([Has97, p. 370, Theorem B]). Furthermore, $X_{\mathrm{ns}}(p)$ is not hyperelliptic when $p \geqslant 11$ and $X_{\mathrm{ns}}^+(p)$ is not hyperelliptic when $p \geqslant 13$ ([Dos16, p. 76, Theorem 1.1]). Hence, in all these cases, we need to look only for equations of degree $d = 2, 3$ for the image of φ. This is done in the following way.

3.1. Algorithm description

Consider the basis $\mathcal{B} = \{f_1, \ldots, f_g\}$ of $\mathcal{S}_2(\Gamma)$. We can think of the f_i as power series in $\mathbb{C}[[q]]$ through their Fourier q-expansion. Let's fix the degree d of the equations and suppose we know the first m coefficients of such power series, with $m > d(2g-2)$. This condition on m guarantees that if we have a polynomial F with rational coefficients and g unknowns such that $F(f_1, \ldots, f_g) \equiv 0 \pmod{q^{m+1}}$, then $F(f_1, \ldots, f_g) = 0$ (see [BGJGP05, Section 2.1, Lemma 2.2, p. 1329]). Now we evaluate all the monomials of degree d in the basis $\mathcal{B}$, obtaining elements of $\mathbb{C}[[q]]$ of which we know the first m coefficients. In this way we get m vectors generating a subspace S of $\mathbb{C}^k$ where k is the number of monomials of degree d with fixed coefficient. A basis of the space $S^\perp$, the orthogonal space to S in $\mathbb{C}^k$, gives the coefficients of the desired equations for the image of φ. These coefficients belong to a number field that depends on the choice of the basis $\mathcal{B}$ and it is not necessarily $\mathbb{Q}$.

If the Fourier coefficients of the elements of $\mathcal{B}$ belong to $\mathbb{Q}$, then the coefficients of the equations belong to $\mathbb{Q}$ too. This, for example, happens for the curves $X_0(p), X_{\mathrm{s}}(p)$ and $X_{\mathrm{s}}^+(p)$. But in general we cannot obtain Fourier coefficients in $\mathbb{Q}$ if the modular curve $X(\Gamma)$ does not have a rational cusp. However, if the modular curve can be defined over $\mathbb{Q}$, one can expect to obtain rational equations also in the case there is not any rational cusp. This, for example, can be done for the curves $X_{\mathrm{ns}}(p)$ and $X_{\mathrm{ns}}^+(p)$. We briefly explain the process in Section 5.2 with particular emphasis on the case $p = 13$, while more details can be found in [MS17].

Once we have equations defined over $\mathbb{Q}$ we can assume that their coefficients belong to $\mathbb{Z}$. In this case we are interested in reducing the size of the coefficients of these equations and minimizing the number of primes ℓ such that the model has bad reduction modulo ℓ. There is a number field K which contains all the coefficients of all the elements of $\mathcal{B}$ ([DS05, p. 234, Theorem 6.5.1]). We can assume that the Fourier coefficients of the basis $\mathcal{B}$ are algebraic integers, hence their coordinates with respect to a suitable chosen basis of K over $\mathbb{Q}$ belong to $\mathbb{Z}$. To reduce the size of the coefficients of the equations we can apply the LLL algorithm, first to the Fourier coefficients of $\mathcal{B}$ and then to the $\mathbb{Z}$-basis of the space $S^{\perp}$. We know that if the elements of $\mathcal{B}$ are linearly dependent modulo a prime number ℓ, then the canonical model of the curve that we find is singular modulo ℓ. In [Mer17, Algorithm 2.1] there is a description of how to modify $\mathcal{B}$ such that the elements of $\mathcal{B}$ are linearly independent for each prime ℓ.

4. Equations of modular double covers

Let $\pi\colon X \to Y$ be a double cover of modular curves of the type $X_{\mathrm{s}}(p) \to X_{\mathrm{s}}^{+}(p)$ or $X_{\mathrm{ns}}(p) \to X_{\mathrm{ns}}^{+}(p)$ where p is a prime number. We have $j_X = j_Y \circ \pi$ where j_X and j_Y are the j-maps to $X(1) \cong \mathbb{P}^1$ of respectively X and Y. In this section we describe a general strategy for determining equations for X and π starting from existing equations of Y and j_Y. This is basically the same strategy used in [DFGS14] to find equations for the modular double cover $X_{\mathrm{ns}}(11) \to X_{\mathrm{ns}}^{+}(11)$ and we describe here how it can be applied in general. In section 5 we apply it successfully to the double covers $X_{\mathrm{s}}(13) \to X_{\mathrm{s}}^{+}(13)$ and $X_{\mathrm{ns}}(13) \to X_{\mathrm{ns}}^{+}(13)$.

Let K be the function field over $\mathbb{Q}$ of Y and let $p_1 = 0, \ldots, p_k = 0 \in \mathbb{Q}[x_1, \ldots, x_h]$ be affine equations for Y. We have $K \cong \mathbb{Q}(x_1, \ldots, x_h)/(p_1, \ldots, p_k)$. Let now L be the function field of X over $\mathbb{Q}$, and let $K \subset L$ be the field inclusion given by the morphism π. We would like to express L in the form

$$L = K(\sqrt{q})$$

where q is a square-free polynomial in K, so that equations for X, possibly singular, are $p_1 = 0, \ldots, p_k = 0, t^2 - q = 0$ in the variables $x_1, \ldots, x_h, t$. We start by observing the following facts.

Proposition 4.1. *Let $L = K(\sqrt{q})$ for some non-square element q in K. Then the points of Y over which the morphism π ramifies are exactly the zeros and poles of odd order for q.*

Proof. This is a consequence of Kummer Theory ([Sti09, p. 122, Proposition 3.7.3(b)]). □

Proposition 4.2. *Let q be a function in K such that $L = K(\sqrt{q})$ and let f be another function of K whose zeros and poles of odd order are the points of Y over which the morphism π ramifies. If the rational 2-torsion of the Jacobian of Y is trivial, then $L = K(\sqrt{\lambda f})$ for some constant $\lambda \in \mathbb{Q}$.*

Proof. Proposition 4.1 implies that all the zeros and poles of odd order for both q and f are the points of Y over which the morphism π ramifies. This means that the function q/f has zeros and poles of even order, i.e. $\mathrm{div}(q/f) = 2D$ for some degree 0 divisor D of Y. Since q and f have rational coefficients, the triviality of the rational 2-torsion in the Jacobian implies that $\mathrm{div}(q/f) = \mathrm{div}(h^2)$ for some $h \in K$. Thus $q/f = \lambda h^2$ for some constant $\lambda \in \mathbb{Q}$, so that $\sqrt{q}$ and $\sqrt{\lambda f}$ generate the same field over K. □

Therefore, assuming the hypothesis of Proposition 4.2, if we find a function f on Y whose zeros and poles of odd order are the points over which π ramifies, then we have determined the function field of X to be of the form $L = K(\sqrt{\lambda f})$ for some constant $\lambda \in \mathbb{Q}$. Note that in this situation, f and λ can be both multiplied by a square (of respectively a rational function or a rational constant), and the field L would still be generated by $\sqrt{\lambda f}$ over K. We can then suppose f to be a polynomial function.

4.1. Determining the ramification points

The points over which the morphism π ramifies could be determined if we have equations for the modular j-map $j_Y : Y \to X(1) \cong \mathbb{P}^1$. In fact π can ramify only over points in which the function j_Y has a pole, a zero or it is equal to 1728 (see [DS05, Chapter 2, Section 2.3]). In other words, π can ramify over a cusp or over those elliptic points of Y whose preimage by π does not contain elliptic points of X. Furthermore, in both our cases, the number of cusps in X is exactly the double of the number of cusps in Y (see [Ogg74, p. 454, Proposition 3], [Bar10, Proposition 7.10]).

Let p be a prime and r a positive integer. A study of the elliptic points of $X_{\mathrm{ns}}(p^r)$ and $X^+_{\mathrm{ns}}(p^r)$ can be found in [Bar10], while the analogous study for $X_0(N)$ is classical for any positive integer N ([DS05, p. 92, Section 3.7]) and therefore for $X_{\mathrm{s}}(p^r) \cong X_0(p^{2r})$. In [Ogg74, Section 2] the author computes the number of ramification points of the map $X_0(N) \to X_0^+(N)$ (and therefore of $X_{\mathrm{s}}(p^r) \to X_{\mathrm{s}}^+(p^r)$), but he does not determine the order of the elliptic points associated.

Remark 4.3. A direct computation with cosets representatives shows that the number of elliptic points e_2 and e_3 for the curve $X_{\mathrm{s}}(p^r)$ is

$$e_2 = \begin{cases} 2 & \text{if } p \equiv 1 \pmod 4 \\ 0 & \text{otherwise} \end{cases} \qquad e_3 = \begin{cases} 2 & \text{if } p \equiv 1 \pmod 3 \\ 0 & \text{otherwise} \end{cases}$$

while for the curve $X_{\mathrm{s}}^+(p^r)$ we have

$$e_2 = \begin{cases} 1 + \frac{p^{r-1}}{2}(p-1) & \text{if } p \equiv 1 \pmod 4 \\ \frac{p^{r-1}}{2}(p+1) & \text{if } p \equiv 3 \pmod 4 \\ 2^{r-1} & \text{if } p = 2 \end{cases} \qquad e_3 = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 3 \\ 0 & \text{otherwise} \end{cases}$$

4.2. Determining f

Once we have determined the coordinates of the points $P_1, \ldots, P_r$ of Y over which π ramifies, one can try to find a rational function f on Y, whose zeros and poles of odd order are exactly $P_1, \ldots, P_r$. A procedure for doing this, which does not necessarily give a solution, but it has been successful for $X_{\mathrm{ns}}^+(11)$, $X_{\mathrm{ns}}^+(13)$ and $X_{\mathrm{s}}^+(13)$ is the following.

Let D be the divisor on Y given by $P_1+\cdots+P_r$. This divisor is defined over $\mathbb{Q}$ because π is defined over $\mathbb{Q}$. Furthermore r must be even because of the Riemann–Hurwitz formula. Let $Q_1, \ldots, Q_k$ be the expected rational points on Y, which are only the CM-points of class number one, when $Y = X_{\mathrm{ns}}^+(p)$ or the CM-points of class number one plus the rational cusp, when $Y = X_{\mathrm{s}}^+(p)$. Then we compute Riemann–Roch spaces of the type

$$B(n_1, \ldots, n_k) \overset{\mathrm{def}}{=} H^0\left(-D + \sum_{i=1}^{k} n_i Q_i\right)$$

where n_i is even for every i and $\sum_{i=1}^{k} n_i = r$. Since the divisor $-D + \sum_{i=1}^{k} n_i Q_i$ has degree 0 then $B(n_1, \ldots, n_k)$ has dimension 0 or 1. If we find such a space of dimension 1 and we can give a basis of rational functions of it, then we have found our function f.

4.3. Verifying the triviality of the rational 2-torsion in the Jacobian

Let A be an abelian variety over $\mathbb{Q}$ and let $\ell \neq 2$ be a prime of good reduction for A. If A has a nontrivial rational point of 2-torsion, then the number of points of A over $\mathbb{F}_\ell$ must be even because $\ell \neq 2$ and the reduction modulo ℓ is an isomorphism on the 2-torsion points of A. Hence, in our case, to prove the triviality of the rational 2-torsion in the Jacobian $\mathrm{Jac}(Y)$ of Y, it is enough to find a prime number $\ell \neq 2, p$ such that the quantity $\#\mathrm{Jac}(Y)(\mathbb{F}_\ell)$ is odd.

Computing $\#\mathrm{Jac}(Y)(\mathbb{F}_\ell)$ can be done recalling that the Jacobian of Y is isogenous to some factor of the Jacobian of $X_0^+(p^2)$, the whole Jacobian in the split case, and the new part in the non-split case. Therefore we need to compute the number of $\mathbb{F}_\ell$-rational points of such factor which can be done using the Eichler–Shimura relations that relate the characteristic polynomial of the ℓ-th Frobenius endomorphism acting on the Jacobian of $X_0(p^2)$ with the characteristic polynomial of the Hecke operator T_ℓ (for an example of this, see the proof of [Dos16, p. 76, Theorem 1.1]).

4.4. Determining λ

Once we have equations of X up to multiplication by a constant $\lambda \in Q$, we can determine λ by analyzing the field of definitions of special values of the function f.

Let $Q \in Y$ be a rational CM-point, therefore of class number one. Then the two points in X over Q are defined over the CM-field of the elliptic curve associated to Q

(see [Ser97, pp. 194–195]). This allows us to determine the constant λ because $\sqrt{\lambda f(Q)}$ must generate the CM-field.

5. Results for level 13

We recall that $X_{\mathrm{s}}^{+}(13)$ and $X_{\mathrm{ns}}^{+}(13)$ are both curves of genus 3. They are isomorphic, and an equation for both of them is the following quartic in $\mathbb{P}^2$

$$p(X,Y,Z) \stackrel{\text{def}}{=} (-Y-Z)X^3 + (2Y^2+ZY)X^2 + (-Y^3+ZY^2-2Z^2Y+Z^3)X + \\ +(2Z^2Y^2-3Z^3Y) = 0 \quad (5.1)$$

Starting from this equation it is possible to obtain formulas for the different j-maps of both $X_{\mathrm{s}}^{+}(13)$ and $X_{\mathrm{ns}}^{+}(13)$ (see [Bar14] for details and explicit formulas) which we will call respectively j_{s} and j_{ns}.

5.1. Singular equations of $X_{\mathrm{s}}(13)$ and $X_{\mathrm{ns}}(13)$ in $\mathbb{A}^3$

Here we apply the strategy described in Section 4 to obtain equations of $X_{\mathrm{s}}(13)$ and $X_{\mathrm{ns}}(13)$ starting from the known same equation (5.1) of $X_{\mathrm{s}}^{+}(13)$ and $X_{\mathrm{ns}}^{+}(13)$.

We begin by looking for the coordinates (in the model given by equation (5.1)) of the points of $X_{\mathrm{s}}^{+}(13)$ and $X_{\mathrm{ns}}^{+}(13)$ over which the two modular double covers $\pi_{\mathrm{s}}\colon X_{\mathrm{s}}(13) \to X_{\mathrm{s}}^{+}(13)$ and $\pi_{\mathrm{ns}}\colon X_{\mathrm{ns}}(13) \to X_{\mathrm{ns}}^{+}(13)$ ramify. Remark 4.3 and [Bar10, p. 2768, Proposition 7.10] tell us that both these double covers ramify over six elliptic points of order 2. This implies that such points are among the simple zeros of the function $j-1728$. In [Bar14, Appendix A], we can find explicit affine formulas for j_{s} and j_{ns} which are both in the form $j(x,y) = h(x,y)/k(x,y)$, where h and k are polynomials with integer coefficients, $x \stackrel{\text{def}}{=} X/Z$ and $y \stackrel{\text{def}}{=} Y/Z$.

Then to find the simple zeros of $j(x,y)-1728$ we compute the resultant with respect to x of the polynomial $p(x,y,1)$, defining the affine equation of the curve, and the polynomial $h(x,y)-1728k(x,y)$. This resultant has the form:

$$(x-1)(43x^6-194x^5-115x^4+692x^3+85x^2-498x+243)\varphi(x)$$

for the split case and

$$(2888x^6+12500x^5+13443x^4+24786x^3+134781x^2+230254x+120131)\psi(x)$$

for the non-split case, where φ and ψ are a product of irreducible polynomials with multiplicity higher than one. Using MAGMA ([BCFS]) we can check that both the polynomials of degree 6, made explicit above, define functions on $X_{\mathrm{s}}^{+}(13) \cong X_{\mathrm{ns}}^{+}(13)$ with 18 simple zeros, which are divided in two Galois orbits of cardinality 12 and 6. We call $P_1,\ldots,P_6$ the simple zeros in the Galois orbit of cardinality 6. Since the modular double covers π_{s} and π_{ns} are defined over $\mathbb{Q}$, the set of points that ramify in one of them must be composed of entire Galois orbits. This implies that such points are exactly the points $P_1, \ldots, P_6$.

The Jacobian of $X_s^+(13) \cong X_{ns}^+(13)$ doesn't have any rational torsion ([BPS16, p. 60, Example 12.9.3.]). Hence, Propositions 4.1 and 4.2 imply that we can determine the function fields of $X_s(13)$ and $X_{ns}(13)$ up to a constant, by adding to the function field of $X_s^+(13) \cong X_{ns}^+(13)$ the square root of a function f whose zeros and poles of odd order are the points $P_1, \ldots, P_6$. Following the procedure described in Section 4.2, and using MAGMA, we compute a basis of each Riemann–Roch space of type

$$H^0\left(-(P_1 + \cdots + P_6) + \sum_{i=1}^{7} n_i Q_i\right)$$

with $n_1, \ldots, n_7$ even integers such that $-12 \leqslant n_1, \ldots, n_7 \leqslant 12$ and $\sum_{i=1}^{7} n_i = 6$, and where $Q_1, \ldots, Q_7$ are the 7 rational points of $X_s^+(13) \cong X_{ns}^+(13)$ associated to the rational CM-points of class number one, or to the rational cusp in the case of $X_s^+(13)$ (see [Bar14, p. 275, Table 1.1]).

With these conditions on the coefficients, the divisors $-(P_1+\cdots+P_6)+\sum_{i=1}^{7} n_i Q_i$ have degree 0, hence the associated Riemann–Roch spaces have dimension 0 or 1. There are indeed a few of these spaces which are non-trivial. Among them, we choose one that is associated to a divisor of the lower degree for which the associated space is non-trivial. For the split case we choose the space generated by

$$\begin{aligned} f_s(x,y) =& \frac{6y^5 + 18y^4 - 17y^3 - 37y^2 - 5y + 3}{y^4} x^2 + \\ &+ \frac{-5y^5 - y^4 + 18y^3 - 17y^2 - y + 6}{y^3} x + \\ &+ \frac{10y^5 - 6y^4 - 38y^3 + 28y^2 + 14y - 3}{y^4}, \end{aligned}$$

which is a function of degree 10, and we choose for the non-split case the space generated by

$$\begin{aligned} f_{ns}(x,y) =& \frac{-11y^8 - 20y^7 - 41y^6 - 85y^5 - 260y^4 - 586y^3 - 635y^2 - 312y - 56}{y^4} x^2 + \\ &+ \frac{22y^9 + 29y^8 + 51y^7 + 109y^6 + 372y^5 + 866y^4 + 1124y^3 + 840y^2 + 340y + 56}{y^4} x + \\ &+ \frac{-11y^9 + 2y^8 - y^7 + 8y^6 - 70y^5 - 68y^4 - 453y^3 - 1016y^2 - 740y - 168}{y^3}, \end{aligned}$$

which is a function of degree 12. We can get rid of denominators by multiplying f_s and f_ns by y^4 which is a square in the function field of $X_\mathrm{s}^+(13) \cong X_\mathrm{ns}^+(13)$. We obtain the polynomial functions

$$\begin{aligned}
q_\mathrm{s}(x,y) =& (6y^5+18y^4-17y^3-37y^2-5y+3)x^2+\\
&+(-5y^6-y^5+18y^4-17y^3-y^2+6y)x+\\
&+10y^5-6y^4-38y^3+28y^2+14y-3\\
q_\mathrm{ns}(x,y) =& (-11y^8-20y^7-41y^6-85y^5-260y^4-586y^3-635y^2-312y-56)x^2+\\
&+(22y^9+29y^8+51y^7+109y^6+372y^5+866y^4+1124y^3+840y^2+340y+56)x+\\
&-11y^{10}+2y^9-y^8+8y^7-70y^6-68y^5-453y^4-1016y^3-740y^2-168y.
\end{aligned}$$

Now we can determine the constant λ of Section 4.4. Note that

$$\begin{aligned}
q_\mathrm{s}(0,0) &= -3 & q_\mathrm{s}(0,3/2) &= -3^3\cdot 2^{-4}\\
q_\mathrm{ns}(-1,0) &= -7\cdot 2^4 & q_\mathrm{ns}(0,3/2) &= -163\cdot 2^{-10}\cdot 3^{10}
\end{aligned}$$

which is consistent with [Bar14, p. 275, Table 1.1]. We have thus obtained singular models in $\mathbb{A}^3$ for $X_\mathrm{s}(13)$ and $X_\mathrm{ns}(13)$ given respectively by the equations

$$\begin{cases} p(x,y,1)=0\\ t^2-q_\mathrm{s}(x,y)=0\end{cases} \quad \text{and} \quad \begin{cases} p(x,y,1)=0\\ t^2-q_\mathrm{ns}(x,y)=0\end{cases}$$

For both curves, the double cover over $X_\mathrm{s}^+(13) \cong X_\mathrm{ns}^+(13)$ has equation $(x,y,t)\mapsto(x,y)$.

5.2. Smooth equations of $X_\mathrm{s}(13)$ and $X_\mathrm{ns}(13)$ in $\mathbb{P}^7$

To find equations describing $X_\mathrm{s}(13)$ it is enough to take a basis for $\mathcal{S}_2(\Gamma_0(169))$ using the software MAGMA ([BCFS]) or William Stein's tables ([Ste12]). Then operate as shown in the proof of Corollary 6.5.6 in [DS05, p. 238] to find a basis with rational integer Fourier coefficients. Finally, we apply the method explained in Section 3. The equations obtained are shown in the Appendix.

The equations for the map $\pi_\mathrm{s}\colon X_\mathrm{s}(13)\to X_\mathrm{s}^+(13)$, using this model for $X_\mathrm{s}(13)$ are obtained in the following way. Let $\mathcal{B}=\{f_1,\dots,f_g\}$ be a basis of eigenforms for $\mathcal{S}_2(\Gamma_0(p^2))$ and we assume that the first g^+ elements of $\mathcal{B}$ are invariant with respect of the action of w_{p^2}. Hence, $\mathcal{B}^+=\{f_1,\dots,f_{g^+}\}$ is a basis for $\Omega^1(X_\mathrm{s}^+(p))$. So, the canonical embedding gives a canonical model $\mathcal{C}$ in $\mathbb{P}^{g-1}$ for $X_\mathrm{s}(p)$ using $\mathcal{B}$ and a canonical model $\mathcal{C}^+$ in $\mathbb{P}^{g^+-1}$ for $X_\mathrm{s}^+(p)$ using $\mathcal{B}^+$, and we have the morphism

$$\begin{aligned}
\pi\colon \mathcal{C} &\longrightarrow \mathcal{C}^+,\\
(x_1:\ldots:x_{g^+}:x_{g^++1}:\ldots:x_g) &\longmapsto (x_1:\ldots:x_{g^+}).
\end{aligned}$$

Let t and t^+ be the compositions of all the invertible projective linear transformations of $\mathbb{P}^{g-1}$ and $\mathbb{P}^{g^+-1}$ respectively, that we use in algorithm 3.1 to obtain better models of $X_{\mathrm{s}}(p)$ and $X_{\mathrm{s}}^+(p)$. These transformations include the elimination of bad primes as well as the application of LLL algorithm to the Fourier coefficients of modular forms and to the coefficients of the equations. We have the following commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ t\ } & \mathcal{C}_1 \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ \mathcal{C}^+ & \xrightarrow[\ t^+\ ]{} & \mathcal{C}_1^+ \end{array}$$

where π_1 is just the composition $t^{-1} \circ \pi \circ t^+$. In the case $p = 13$ we have that $\mathcal{C}_1^+$ is the model given by equation (5.1), $\mathcal{C}_1$ is the model for $X_{\mathrm{s}}(13)$ obtained in Section 3 and written in the Appendix, while the map π_1 is π_{s}.

Finding equations describing $X_{\mathrm{ns}}(13)$ is more difficult because we don't have the Fourier coefficients of a basis for $\mathcal{S}_2(\Gamma_{\mathrm{ns}}(13))$. To find these Fourier coefficients we use a basis of $\mathcal{S}_2(\Gamma_0(169))^{\mathrm{new}}$ and some representation theory of $G := \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$. We use the newforms because the jacobian of $X_{\mathrm{ns}}(p)$ is isogenous over $\mathbb{Q}$ to the new part of the jacobian of $X_0(p^2)$ (see [Che98], [dSE00]).

The irreducible complex representations of the finite group G are divided into three kinds: representations of dimension $p - 1$, representations of dimension p and representations of dimension $p + 1$. The representations of dimension $p - 1$ are also called cuspidal representations and this kind of representation is parametrized by characters $\theta\colon \mathbb{F}_{p^2}^* \to \mathbb{C}^*$. The other two kinds of representations are called principal series representations and are parametrized by characters μ of the upper triangular matrices subgroup of G. We have a representation of dimension p if μ is the quadratic character, and we have a representation of dimension $p + 1$ otherwise.

Let V_f be the $\mathbb{C}[G]$-span of an element f of a basis of eigenforms for $\mathcal{S}_2(\Gamma_0(p^2))^{\mathrm{new}}$. We know that V_f is a complex irreducible representation that is a principal series representation if f is a twist of an eigenform h of $\mathcal{S}_2(\Gamma_1(p))$. This means that the Fourier coefficient a_n of f is equal, for each n, to $\chi(n)b_n$, where χ is a character of $\mathbb{F}_p$ and b_n is the n-th Fourier coefficient of h. The dimension of V_f is p if the form h is in $\mathcal{S}_2(\Gamma_0(p))$ and is $p + 1$ otherwise. If f is not a twist of a lower level form, then V_f is a cuspidal representation. One can find elements invariant under the action of a non-split Cartan subgroup of G using the related trace $\sum_g gf$, where the sum is taken over all elements g in the non-split Cartan subgroup considered. To get equations over $\mathbb{Q}$ we need to multiply this trace by a constant ε_f defined as follows:

$$\varepsilon_f := \begin{cases} \frac{\tau(\chi)\tau(\chi^2)}{a_p} & \text{if } f \text{ is a twist of an eigenform } h \text{ of } \mathcal{S}_2(\Gamma_1(p)) \text{ by a character } \chi, \\ 1 & \text{otherwise.} \end{cases}$$

Here a_p is the p-th Fourier coefficient of h and $\tau(\chi)$ is the Gauss sum $\sum_{k=1}^{p-1}\chi(k)\zeta_p^k$, with $\zeta_p = e^{\frac{2\pi i}{p}}$, and the same for $\tau(\chi^2)$.

Last step is to take the Galois trace for each conjugate of f. We mean the following. The Fourier coefficients of f are in a number field K_f that we identify with $\mathbb{Q}[x]/(p(x))$ where p is a monic irreducible polynomial of degree s. Let $\alpha_1, \dots, \alpha_s$ be the complex roots of p, which correspond to the embeddings of K_f in $\mathbb{C}$. Let f_i be the element in the conjugacy class of f corresponding to the embedding of K_f associated to α_i and suppose $f = f_1$. As we said we take the non-split Cartan trace $\hat{f} := \varepsilon_f \sum_g gf$. This form has Fourier coefficients in $K_f(\zeta_p)^+$. Now we take the traces $\tilde{f}_k := \mathrm{Tr}(\alpha_1^{k-1}\hat{f}) = \sum_{i=1}^{s}\alpha_i^{k-1}\hat{f}_i$, for $k = 1, \dots, s$, where Tr is the usual trace map from $K_f(\zeta_p)^+$ to $\mathbb{Q}(\zeta_p)^+$. The forms $\tilde{f}_1, \dots, \tilde{f}_s$ have Fourier coefficients in $\mathbb{Q}(\zeta_p)^+$ and collecting them for each conjugacy class of $\mathcal{S}_2(\Gamma_0(p^2))^{\mathrm{new}}$, we get a basis that gives equations over $\mathbb{Q}$ for the corresponding curve, for more details see [MS17] or, for the cuspidal case, see [Bar14].

We have that $\mathcal{S}_2(\Gamma_0(169))^{\mathrm{new}}$ has dimension 8 and let $\mathcal{B} = \{f_1, \dots, f_8\}$ be a basis of eigenforms of $\mathcal{S}_2(\Gamma_0(169))^{\mathrm{new}}$. Three of the forms in $\mathcal{B}$ are conjugate with respect to the Galois action and form a basis for the w_N-invariant forms in $\mathcal{S}_2(\Gamma_0(169))^{\mathrm{new}}$; they are not twist of some lower level form, so the irreducible representations associated are all cuspidal. Two Galois conjugate forms in $\mathcal{B}$ are twists of a form of $\mathcal{S}_2(\Gamma_1(13))^{\mathrm{new}}$, which is a complex vector space of dimension 2. The last three forms in $\mathcal{B}$ are conjugate with respect to the Galois action and they are not twists of some lower level form, so the associated irreducible representations are all cuspidal.

The equations obtained are written in the Appendix together with the equations for the map $\pi_{\mathrm{ns}}\colon X_{\mathrm{ns}}(13) \to X_{\mathrm{ns}}^+(13)$ which are obtained analogously to the split case.

6. Maps from the canonical models to other models

To compute maps from the canonical model $\mathcal{C}$ to a different model $\mathcal{C}'$, we use the reverse-mapping correspondence between curves and function fields. What we do is finding an injective field homomorphism ι from the function field $\mathcal{F}'$ of $\mathcal{C}'$ to the function field $\mathcal{F}$ of the canonical model $\mathcal{C}$. To achieve this, we need a way to go from rational functions on the canonical model to their Laurent q-expansion and vice versa. One direction is easy. Indeed, we know that the x_i in the equations of the canonical models in Section 5.2 correspond to elements in a specific basis of cusp forms, that we found beforehand. On the other hand recognizing Laurent q-series as rational functions in the x_i requires more work.

Let's place ourselves in the affine chart of $\mathcal{C}$ where $x_8 \neq 0$. The function field of $\mathcal{C}$ is generated by the functions $\frac{x_1}{x_8}, \dots, \frac{x_7}{x_8}$ which are all well defined in the affine chart we chose. Let h_i be the Laurent q-expansion of $\frac{x_i}{x_8}$ for $i = 1, \dots, 7$. Let f be an element of $\mathcal{F}'$ and suppose we know the Laurent q-expansion of $\iota(f) \in \mathcal{F}$. We want to write $\iota(f)$ in the form

$$\iota(f) = \frac{p(h_1, \dots, h_7)}{q(h_1, \dots, h_7)},$$

where p and q are suitable polynomials. We write the previous equality as

$$p(h_1, \dots, h_7) - \iota(f) q(h_1, \dots, h_7) = 0,$$

where the left hand side above can be seen as a linear combination of Laurent q-series, assuming we know the degree of the polynomials p and q.

Therefore, if we know the first m Laurent coefficients of $\iota(f), h_1, \dots, h_7$, with $m > d(2g-2) = 14d$ and d is the maximum of the degrees of p and q, it is easy to compute the coefficients of p and q in the same way explained in the algorithm 3.1 description for the coefficients of the polynomial F, i.e. we have m vectors generating a subspace S and we want a basis of $S^{\perp}$. If we don't know the degree of p and q, we make computations just trying sufficiently large degree of p and q until we find some non-trivial relations among Laurent coefficients.

6.1. Map to Kenku's affine plane model of $X_0(169)$

In [Ken80] and [Ken81], Kenku gives an explicit plane affine model of $X_0(169)$ which is naturally isomorphic to $X_{\mathrm{s}}(13)$. Let X and Y be the coordinates in the affine model of $X_0(169)$ described in Kenku's paper. They correspond to Laurent q-series obtained by

$$X(\tau) = \frac{13\eta^2(169\tau)}{\eta^2(\tau)}, \qquad Y(\tau) = \frac{\eta^2(\tau)}{\eta^2(13\tau)},$$

where η is the classical Dedekind eta function $\eta(\tau) = q^{\frac{1}{24}} \prod_n (1-q^n)$ and $q = e^{2\pi i \tau}$. We consider the affine model for $X_{\mathrm{s}}(13)$ as the affine chart where $x_8 \neq 0$ in the projective model described by equations in Section 5.2. Then we have the field isomorphism

$$\begin{aligned} \iota\colon \mathbb{Q}(X_0(169)) &\longrightarrow \mathbb{Q}(X_{\mathrm{s}}(13)) \\ X &\longmapsto U \\ Y &\longmapsto V, \end{aligned}$$

where

$$\begin{aligned} \text{numerator}(U) &= 117x_1^2 - 13x_1x_2 + 13x_1x_3 + 13x_4x_6 + 13x_4x_7 + \\ &\quad + 26x_4x_8 - 13x_5^2 + 13x_6x_8 + 13x_7^2, \\ \text{denominator}(U) &= 238x_3^2 + 215x_3x_4 + 215x_3x_5 + 429x_3x_6 - 419x_3x_7 + 36x_3x_8 - 89x_4^2 + \\ &\quad + 185x_4x_5 + 130x_4x_6 - 505x_4x_7 - 313x_4x_8 + 305x_5^2 + 217x_5x_6 + \\ &\quad + 145x_5x_7 - 46x_5x_8 + 28x_6^2 - 7x_6x_7 + 352x_6x_8 + 351x_7^2 - 3x_7x_8 - 2x_8^2; \\ \text{numerator}(V) &= 4637022x_1^2 + 4624659x_4x_6 - 5060016x_4x_7 + 14784393x_4x_8 + \\ &\quad - 6782997x_5^2 - 19275477x_5x_6 - 8559018x_5x_7 + 1545960x_5x_8 + \end{aligned}$$

$$
\begin{aligned}
& - 28694289x_6^2 - 8134854x_6x_7 - 4473261x_6x_8 + 6858072x_7^2 + \\
& + 2366208x_7x_8 - 3989778x_8^2, \\
\text{denominator}(V) = & -209376188x_3^2 - 196485388x_3x_4 - 183091120x_3x_5 - 421799299x_3x_6 + \\
& + 371436944x_3x_7 - 136573881x_3x_8 + 89731271x_4^2 - 151182225x_4x_5 + \\
& - 140218639x_4x_6 + 488527387x_4x_7 + 280939604x_4x_8 - 277852129x_5^2 + \\
& - 207933217x_5x_6 - 146929317x_5x_7 + 17764144x_5x_8 - 15033364x_6^2 + \\
& + 3885141x_6x_7 - 323708963x_6x_8 - 329322311x_7^2 - 3989778x_7x_8.
\end{aligned}
$$

6.2. Desingularization maps to the affine models of section 5.1

The function fields defined by the equations found in Section 5.2 and the function fields defined by the equations found in Section 5.1 are both isomorphic to the function field of the associated modular curve, which is $X_{\mathrm{s}}(13)$ or $X_{\mathrm{ns}}(13)$. Here we give an explicit isomorphism between the function fields defined by the two models, in both the split and the non-split case.

Let $\mathcal{C}$ be the smooth projective model defined in Section 5.2 and let $\mathcal{C}'$ be the singular affine model defined in Section 5.1. We have the following situation

$$
\begin{array}{ccc}
\mathcal{C} & \overset{\varphi}{\dashrightarrow} & \mathcal{C}' \\
{\scriptstyle \pi} \searrow & & \swarrow {\scriptstyle \pi'} \\
\mathcal{C}_{\mathrm{p}}^{+} & \hookleftarrow & \mathcal{C}_{\mathrm{a}}^{+}
\end{array}
$$

where $\mathcal{C}_{\mathrm{p}}^{+}$ is the curve defined by equation (5.1), $\mathcal{C}_{\mathrm{a}}^{+}$ is the affine chart in which $Z \neq 0$, the map π is π_{s} or π_{ns} depending on whether we are dealing with the split or the non-split case, π' is the double cover given by $(x, y, t) \mapsto (x, y)$ and φ is a birational map that makes the diagram commute on some affine chart of $\mathcal{C}$. The isomorphism from the function field of $\mathcal{C}'$ generated by x, y, t to the function field of $\mathcal{C}$ is given in the form

$$
\begin{aligned}
\varphi^*\colon \mathbb{Q}(\mathcal{C}') & \xrightarrow{\cong} \mathbb{Q}(\mathcal{C}) \\
x & \longmapsto X/Z \\
y & \longmapsto Y/Z \\
t & \longmapsto \tilde{t}
\end{aligned}
$$

where X, Y, Z are the one defined in Section 5.2 in the equations of π_{s} and π_{ns}, and $\tilde{t} \overset{\mathrm{def}}{=} s \cdot (Y/Z)^2$, with s being a square root of $\pi_{\mathrm{s}}^* f_{\mathrm{s}}$ or $\pi_{\mathrm{ns}}^* f_{\mathrm{ns}}$ depending on whether we are dealing with the split or the non-split case. To determine s we take a square root of the Laurent q-expansion of $\pi_{\mathrm{s}}^* f_{\mathrm{s}}$ or $\pi_{\mathrm{ns}}^* f_{\mathrm{ns}}$ and then we recognize it as a rational function in the x_i, as explained in the beginning of Section 6. In the split case we get

$$s = \frac{4x_1 - x_2 - x_3 + x_4 - 3x_5 - x_6 - 2x_7 + x_8}{-x_2 + x_3 + x_4}$$

and for the non-split case we get

$$\begin{aligned}
\text{numerator}(s) &= 78953974807x_1^2 + 26x_1x_2 - 25x_1x_3 - x_1x_4 + 2x_3^2 + \\
&+ 238115162692x_4x_7 + 209337250703x_4x_8 - 582346348536x_5^2 + \\
&+ 727177285412x_5x_6 + 78542213920x_5x_7 - 563548816331x_5x_8 + \\
&+ 65380280758x_6^2 - 244488381626x_6x_7 + 82647686352x_6x_8 + \\
&+ 136959277010x_7^2 + 250609762421x_7x_8 - 257891423548x_8^2, \\
\text{denominator}(s) &= 33279035581x_3^2 - 20440236060x_3x_4 + 161001990516x_3x_5 + \\
&+ 177481085270x_3x_6 - 284601313488x_3x_7 - 214125958084x_3x_8 + \\
&- 116902000189x_4^2 + 103367036819x_4x_5 + 124067876928x_4x_6 + \\
&- 155405328616x_4x_7 - 193679032128x_4x_8 - 57688123584x_5^2 + \\
&- 123976858837x_5x_6 - 194732784800x_5x_7 - 165341806053x_5x_8 + \\
&- 126114432327x_6^2 + 524882113804x_6x_7 + 271440452599x_6x_8 + \\
&- 236487356215x_7^2 - 365606104840x_7x_8 - 113208254802x_8^2.
\end{aligned}$$

Appendix A. Equations for the canonical models of $X_{\mathrm{s}}(13)$ and $X_{\mathrm{ns}}(13)$

The curve $X_{\mathrm{s}}(13)$ of $g = 8$ can be explicitly given by the following 15 equations in $\mathbb{P}^7$.

$$x_1x_2 - x_1x_3 - x_2^2 - x_2x_4 + x_2x_5 + x_3x_6 - x_3x_7 - x_4x_6 - x_4x_7 - x_4x_8 = 0$$

$$\begin{aligned}
&- x_1^2 + 2x_1x_2 + x_1x_4 - x_1x_5 + x_1x_6 + x_3^2 + x_3x_5 + x_3x_6 - x_3x_7 - x_4^2 + x_4x_5 - x_4x_7 + \\
&\quad - x_5x_8 + x_6^2 + x_6x_8 + x_7^2 = 0
\end{aligned}$$

$$\begin{aligned}
&x_1^2 + x_1x_3 - x_1x_4 + x_1x_5 + x_1x_7 + x_1x_8 - x_3x_4 + x_3x_7 + x_3x_8 + x_4^2 - 2x_4x_5 - x_5x_6 + \\
&\quad + x_5x_7 + x_5x_8 - x_6^2 = 0
\end{aligned}$$

$$\begin{aligned}
&- x_1x_6 - 2x_1x_8 + x_2^2 + x_2x_4 - x_2x_5 - x_3x_6 - x_3x_8 - x_4x_6 + x_4x_8 + x_5x_6 - x_5x_7 + \\
&\quad - x_5x_8 + x_6^2 = 0
\end{aligned}$$

$$- x_1x_2 + x_1x_3 - x_1x_4 - 2x_1x_6 + x_2^2 - x_3^2 - x_3x_5 - 2x_4^2 + x_4x_6 + x_5^2 + x_5x_6 = 0$$

$$x_1x_2 - x_1x_4 + x_2^2 + 2x_2x_4 - x_2x_5 + x_3x_6 + x_3x_7 - x_3x_8 + x_4x_6 + x_5x_6 - x_5x_7 = 0$$

$$\begin{aligned}
&- x_1x_2 + x_1x_6 - x_2^2 - x_2x_5 - x_3x_4 + x_3x_5 - x_3x_6 + x_3x_8 + x_4x_6 + x_4x_8 - x_5x_6 + \\
&\quad + x_5x_7 + x_5x_8 - x_6^2 + x_6x_8 + x_7^2 = 0
\end{aligned}$$

$$\begin{aligned}
&x_1x_2 - x_1x_3 + x_2x_3 + x_2x_4 + x_2x_5 + x_3x_7 - x_3x_8 + x_4^2 - x_4x_6 + x_4x_8 - x_5^2 - x_5x_7 + \\
&\quad - x_6x_8 - x_7^2 = 0
\end{aligned}$$

$$- x_1x_3 - x_1x_4 - x_1x_6 - 2x_2x_4 + x_2x_6 + x_3x_8 - x_4x_7 - x_5x_6 = 0$$
$$x_1x_2 - x_1x_8 - x_2x_5 - x_2x_7 + x_2x_8 - x_3x_7 - x_4x_5 - x_4x_7 + x_5^2 - x_5x_8 - x_6x_7 + x_6x_8 + \\ + x_7^2 = 0$$
$$2x_1x_3 + x_1x_4 + x_1x_7 - x_1x_8 + x_2^2 + x_2x_5 + x_2x_7 + x_3^2 + x_3x_6 - x_3x_7 + x_3x_8 - x_4x_5 + \\ - x_4x_6 - x_4x_7 + x_5x_6 + x_5x_7 + x_6^2 + x_6x_8 + x_7^2 = 0$$
$$x_1x_3 - x_1x_4 - x_1x_5 - x_1x_6 - x_2x_4 + x_2x_6 - x_2x_8 - 2x_3x_4 - x_3x_6 - x_3x_7 + x_3x_8 - x_4^2 + \\ - x_4x_5 + x_6x_7 = 0$$
$$x_1x_5 - x_1x_6 - x_1x_7 + x_2x_4 + x_2x_5 - x_2x_6 - x_2x_8 + x_3x_4 - x_3x_8 - x_4x_6 - x_4x_7 + \\ - x_4x_8 + x_5^2 + x_5x_6 + x_6x_7 - x_6x_8 - x_7^2 = 0$$
$$- x_1^2 - x_1x_2 - x_1x_4 + x_1x_6 + x_2^2 + x_2x_4 - x_2x_8 + x_3x_4 - 2x_3x_5 + x_3x_7 + x_4^2 + x_4x_6 + \\ - x_4x_8 - x_5^2 + x_6x_7 - x_6x_8 - x_7^2 = 0$$
$$x_1^2 - x_1x_2 - x_1x_3 + x_1x_7 + 2x_1x_8 + x_2x_3 - x_2x_7 + x_3^2 + x_3x_4 + x_3x_5 + x_4^2 + 2x_4x_5 + \\ - x_5x_6 - x_5x_8 = 0.$$

This curve has only two rational points: the two cusps ([Ken80, p. 241, Theorem 1], [Ken81]). Using the previous equations, these rational points have the following coordinates.

Rational points
$(-2 : -1 : -4 : 3 : 6 : -3 : 1 : 4)$
$(0 : 0 : 0 : 0 : 0 : 0 : 0 : 1)$

The map $\pi_s \colon X_s(13) \to X_s^+(13)$, using the previous model for $X_s(13)$ and the model (5.1) for $X_s^+(13)$, is

$$\begin{cases} X = -x_1 + x_2 + 2x_4 + x_5 - x_6 + x_7 - x_8 \\ Y = -x_2 - x_3 + x_4 + x_5 - x_6 - x_8 \\ Z = -x_1 - x_2 - 2x_4 + x_5 + x_6. \end{cases}$$

The curve $X_{ns}(13)$ of $g = 8$ can be explicitly given by the following 15 equations in $\mathbb{P}^7$.

$$x_1^2 - x_1x_3 - x_1x_4 - x_1x_7 + x_1x_8 + x_2x_4 + x_2x_5 + 2x_3x_4 - 2x_3x_5 - x_3x_8 + 2x_4x_5 + \\ + x_4x_7 + x_5x_8 - x_7^2 + x_7x_8 = 0,$$
$$- x_1x_3 + 2x_1x_5 + x_1x_8 - 2x_3x_4 - x_3x_5 + x_3x_6 - x_3x_7 - x_4x_5 - x_4x_6 + x_4x_7 + \\ + x_4x_8 - x_5^2 + x_5x_6 - 3x_5x_8 - x_6x_7 - 3x_6x_8 + x_7^2 - x_8^2 = 0,$$
$$- x_1x_3 + 2x_1x_4 + x_1x_5 - 2x_1x_6 + 4x_1x_8 + x_2x_4 + x_2x_5 - x_3x_4 + x_3x_6 - x_3x_7 - x_4^2 + \\ + x_4x_5 - 2x_4x_8 + 2x_5x_7 + x_5x_8 - 2x_6x_8 + x_7x_8 - x_8^2 = 0,$$

$$x_1x_3 + x_1x_4 + x_1x_5 - 3x_1x_6 + x_1x_7 + 2x_1x_8 - x_2x_3 - x_2x_4 + x_2x_5 + x_2x_6 - x_3^2 + $$
$$- x_3x_4 - x_3x_5 + x_3x_6 - x_3x_8 - 2x_4x_5 - x_4x_8 + x_5x_6 + x_5x_7 + 2x_6^2 - 2x_6x_7 + x_7^2 + $$
$$+ x_7x_8 - x_8^2 = 0,$$
$$x_1x_2 - x_1x_3 + x_1x_5 + x_1x_6 - x_1x_7 + x_1x_8 + x_2^2 + x_2x_3 - x_2x_4 - x_2x_5 - x_2x_6 + x_3^2 + $$
$$- x_3x_4 - x_3x_5 - x_3x_6 + x_3x_8 - x_4^2 + x_4x_5 + 2x_4x_6 + x_4x_7 - 2x_4x_8 - x_5^2 + 2x_5x_6 + $$
$$+ x_5x_7 - 2x_5x_8 + x_6x_7 - x_6x_8 + x_7x_8 - x_8^2 = 0,$$
$$2x_1x_2 + x_1x_3 - x_1x_4 + x_1x_6 - x_1x_7 - x_1x_8 + x_2x_3 - 2x_2x_4 - x_2x_5 - x_2x_6 + x_2x_7 + $$
$$+ x_3^2 - 2x_3x_4 - x_3x_6 - x_3x_7 + x_4x_5 + x_4x_6 + x_4x_7 + 2x_4x_8 + x_5x_6 - 2x_5x_8 + $$
$$+ x_6x_7 - x_6x_8 = 0,$$
$$- x_1^2 + x_1x_2 + 2x_1x_3 + x_1x_5 - x_1x_6 - x_1x_7 + 2x_1x_8 - x_2^2 - x_2x_3 + x_2x_6 + x_2x_7 + $$
$$+ x_2x_8 - x_3x_4 - x_3x_5 - x_3x_8 - x_4^2 + x_4x_6 + x_5^2 - x_5x_6 - x_6x_7 - x_6x_8 = 0,$$
$$- x_1^2 - x_1x_2 + x_1x_5 + 2x_1x_6 + x_1x_7 + x_1x_8 + x_2x_3 - x_2x_4 - x_2x_5 + x_2x_7 + $$
$$+ x_2x_8 - x_3x_5 + x_3x_6 - x_3x_7 + x_4x_5 - x_4x_6 + x_4x_7 - x_5^2 + x_5x_6 + $$
$$+ x_5x_7 - x_5x_8 - 2x_6x_8 + x_7x_8 - x_8^2 = 0,$$
$$- 2x_1x_2 + 2x_1x_3 - x_1x_4 - x_1x_5 + x_1x_7 - x_1x_8 - x_2x_4 + 2x_2x_5 + 2x_2x_6 + x_2x_8 - x_3^2 + $$
$$+ x_3x_4 + x_3x_5 + x_3x_6 + x_3x_7 - x_3x_8 + x_4^2 + x_4x_5 + x_4x_7 - x_5^2 - 2x_5x_6 - x_5x_7 + $$
$$+ x_5x_8 - x_6x_7 + x_6x_8 - x_7^2 + x_7x_8 = 0,$$
$$- 2x_1x_3 - x_1x_4 + x_1x_5 - x_1x_7 + 2x_1x_8 + x_2^2 + x_2x_3 - x_2x_4 - x_2x_7 + x_3x_4 + x_3x_5 + $$
$$+ 2x_3x_6 - 2x_3x_7 + 2x_3x_8 - x_4^2 + 2x_4x_5 + 2x_4x_7 - x_4x_8 - x_5^2 + 2x_5x_7 - x_5x_8 + $$
$$+ 2x_6x_7 - 2x_6x_8 + 2x_7x_8 - 2x_8^2 = 0,$$
$$- x_1x_2 + 2x_1x_4 - x_1x_6 + x_1x_7 + x_1x_8 - x_2^2 + 2x_2x_4 + x_2x_5 - x_2x_6 + 2x_2x_7 + $$
$$+ 2x_2x_8 - x_4x_6 - x_4x_7 - x_4x_8 + x_5x_6 + x_5x_7 + x_5x_8 = 0,$$
$$x_1x_3 + 2x_1x_4 - x_1x_5 - x_1x_6 + x_1x_7 + x_1x_8 - x_2^2 - x_2x_3 - x_2x_4 + x_2x_5 + x_2x_6 + $$
$$+ x_2x_7 - 2x_2x_8 - x_3^2 + 2x_3x_5 + x_3x_6 + x_3x_7 - x_3x_8 + x_4x_5 - x_4x_6 - x_4x_7 - x_4x_8 + $$
$$- x_5x_6 + x_5x_7 + 2x_5x_8 - x_6x_7 + x_6x_8 - x_7^2 + x_7x_8 = 0,$$
$$- x_1^2 + x_1x_2 + 2x_1x_3 - x_1x_4 + x_1x_6 - x_1x_7 - x_2x_3 - 2x_2x_4 - 2x_2x_5 - x_2x_7 - x_2x_8 + $$
$$- x_3^2 - x_3x_5 + x_3x_7 - x_3x_8 + x_4^2 + x_4x_5 + 2x_4x_7 + x_4x_8 + x_5x_6 + x_5x_7 - x_5x_8 + $$
$$+ 2x_6x_8 + 2x_7^2 + 2x_7x_8 - 2x_8^2 = 0,$$
$$x_1^2 + 2x_1x_2 - x_1x_3 - x_1x_4 + x_1x_6 - x_1x_8 - x_2^2 + 2x_2x_3 - 2x_2x_5 + x_2x_7 + 3x_3^2 - x_3x_4 + $$
$$- 2x_3x_6 - x_3x_7 - x_4^2 + 3x_4x_6 + 2x_5^2 + x_5x_6 + x_5x_7 - 2x_6^2 - x_6x_7 + x_6x_8 - x_7^2 + $$
$$- 2x_7x_8 + 2x_8^2 = 0,$$

$$2x_1^2 - 2x_1x_2 + x_1x_4 + 3x_1x_5 - 2x_1x_6 - 2x_1x_7 - 2x_1x_8 + x_2^2 - x_2x_3 - 3x_2x_5 - x_2x_7 + \\ - 3x_3x_4 + x_3x_6 + x_3x_8 + x_4^2 + 3x_4x_5 - 2x_4x_6 + x_4x_7 + x_4x_8 + 2x_5^2 - 4x_5x_6 + \\ - 2x_5x_8 + 2x_6x_7 + x_7^2 - 2x_7x_8 + x_8^2 = 0.$$

We know that this curve doesn't have rational points. The map $\pi_{\mathrm{ns}}\colon X_{\mathrm{ns}}(13) \to X_{\mathrm{ns}}^+(13)$, using the previous model for $X_{\mathrm{ns}}(13)$ and the model (5.1) for $X_{\mathrm{ns}}^+(13)$, is

$$\begin{cases} X = -3x_1 + 2x_2 \\ Y = -3x_1 + x_2 + 2x_4 - 2x_5 \\ Z = x_1 + x_2 + x_4 - x_5. \end{cases}$$

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