EXTINCTION IN A FINITE TIME FOR PARABOLIC EQUATIONS OF FAST DIFFUSION TYPE ON MANIFOLDS

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Abstract. We prove extinction in a finite time for a singular parabolic equation on a Riemannian manifold, under suitable assumptions on the Riemannian metric and on the inhomogeneous coefficient appearing in the equation. The result relies on a suitable embedding theorem, of which we present a new proof.

1. Introduction

We look at the Cauchy problem for the nonlinear parabolic equation

$$\rho(x)\frac{\partial u}{\partial t} = \operatorname{div}(u^{m-1}|\nabla u|^{p-2}\nabla u)\,, \qquad (x,t) \in S_T = M \times (0,T)\,, \quad (1.1)$$

$$u(x,0) = u_0(x)\,, \qquad x \in M\,. \quad (1.2)$$

We assume that

$$p + m - 3 < 0\,, \qquad N > p > 1\,, \quad (1.3)$$

that is we consider the fast diffusion case. Here M is a noncompact Riemannian manifold of topological dimension N, whose measure is denoted here by μ. We denote by $d(x)$ for $x \in M$ the distance from a fixed point $x_0 \in M$, and by $V(R)$ the volume of the geodesic ball $B_R(x_0)$, $R > 0$.

Assume that the following isoperimetrical inequality holds true for all measurable subsets $U \subset M$ with a Lipschitz continuous boundary ∂U

$$|\partial U|_{N-1} \geq g(\mu(U))\,, \quad (1.4)$$

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where $g(s)$ is an increasing function for $s > 0$. In addition we assume that

$$\omega(s) := \frac{s^{\frac{N-1}{N}}}{g(s)}, \qquad s > 0, \tag{1.5}$$

is non decreasing.
In what follows we denote with a slight abuse of notation $\rho(x) = \rho(d(x))$, where we assume $\rho(s)$ to be a continuous decreasing function for $s \geq 0$, $\rho(0) = 1$. We use the function $\rho^*(s)$, $s > 0$, defined as the decreasing rearrangement of $\rho(d(x))$.
We also need the following assumption, to prove a kind of Hardy inequality:

$$\int_0^s y^{-p} g(y)^p \,\mathrm{d}y \leq c s^{-p+1} g(s)^p, \qquad s > 0, \tag{1.6}$$

for a given constant $c > 0$.

Remark 1.1. In the Euclidean case $g(s) = s^{(N-1)/N}$ it is easily seen that (1.6) is equivalent to $p < N$.
Taking for example M as the one of the manifolds with cylindrical end of [2], whose metric is (out of a compact set) $\mathrm{d}t^2 + t^{2k}\,\mathrm{d}M_0$, where $\mathrm{d}M_0$ is the metric of a compact manifold M_0, $0 < k < 1$, one could see that assumption (1.6) amounts essentially to the non-parabolicity of M in the sense of [2], i.e., to $k > (p-1)/(N-1)$. In this case, $g(s) = \gamma \min\{s^{\frac{N-1}{N}}, s^\alpha\}$, $\alpha = k(N-1)/(k(N-1)+1)$, and such a condition is equivalent to the restriction $\alpha > (p-1)/p$. □

In this note we prove two results. First we prove the following embedding result. Below we set $p^* = Np/(N-p)$ and $\beta = N(p+m-3)+p$.

Theorem 1.2. *Assume* $1 < p < N$, (1.4), (1.5), (1.6). *Then for all* $u \in W^{1,p}(M)$ *we have*

$$\left(\int_M |u|^{p^*} \omega(V(d(x)))^{-p^*} \,\mathrm{d}\mu \right)^{\frac{N-p}{N}} \leq C \int_M |\nabla u|^p \,\mathrm{d}\mu, \tag{1.7}$$

for a suitable constant $C > 0$ *independent of* u.

The result above was proved in [3], in a different framework. Our proof uses a different technique, relying on a symmetrization approach and on the use of Hardy inequality, which seems to us to be sharp and very straightforward.
Next we apply the weighted inequality in (1.7) to the proof of our extinction result. Let us note however that the embedding may be also employed to prove for example sup bounds and blow up estimates;

this will be pursued elsewhere. We refer to [5] for a definition of weak solution to our problem (the definition given there carries over straightforwardly to our setting).

Theorem 1.3. *Assume that u is a nonnegative weak solution to* (1.1)–(1.2)*, where we assume* (1.4)*,* (1.5)*,* (1.3)*,* (1.6) *and*

$$\int_M \Big\{\rho(x)\omega^\delta\Big(V(d(x))\Big)\Big\}^{\frac{p^*}{p^*-\delta}} \,\mathrm{d}\mu < \infty\,. \tag{1.8}$$

Here $\delta > p$ may be any value in (p, p^*) if $\beta \le 0$, and any value $p < \delta < p/(p+m-2) < p^*$ if $\beta > 0$.
Then u becomes identically zero in $\boldsymbol{R}^N$ in a finite time.

Similar extinction results are well known in the literature, when $\rho = 1$ and the equation is singular, see [1]. In this note we consider the case of the inhomogeneous fast diffusion equation, for which we quote [6]. Our extinction result reduces to the one there for the special cases of Euclidean metric, that is when ω is constant, and of $\rho(s) = (1+s)^{-\ell}$, $\beta > 0$: i.e., we have extinction if $\ell > \ell^* = \beta/(p+m-2)$.

2. Proof of the Weighted Sobolev Inequality

We have by Hardy-Littlewood inequality

$$I := \left(\int_M |u|^{p^*}\omega(V(d(x)))^{-p^*}\,\mathrm{d}\mu\right)^{\frac{N-p}{N}} \le \left(\int_0^\infty (u^*(s))^{p^*}\Big(\omega(V(d(x)))^{-p^*}\Big)^*\,\mathrm{d}s\right)^{\frac{N-p}{N}}\,. \tag{2.1}$$

By definition

$$\int_0^\infty (u^*(s))^{p^*}\Big(\omega(V(d(x)))^{-p^*}\Big)^*\,\mathrm{d}s = \int_0^\infty (u^*(s))^{p^*}(\omega(s))^{-p^*}\,\mathrm{d}s\,.$$

Next we have by the properties of Lorentz spaces (see [4])

$$\left(\int_0^\infty (u^*(s))^{p^*}(\omega(s))^{-p^*}\,\mathrm{d}s\right)^{\frac{N-p}{N}} \le c\int_0^\infty (u^*(s))^p s^{\frac{p}{p^*}-1}(\omega(s))^{-p}\,\mathrm{d}s = c\int_0^\infty (u^*(s))^p s^{-p}(g(s))^p\,\mathrm{d}s\,. \tag{2.2}$$

Next we prove the Hardy inequality

$$\int_0^\infty (u^*(s))^p s^{-p}(g(s))^p \,\mathrm{d}s \le c\int_0^\infty (-u^*_s(s))^p (g(s))^p \,\mathrm{d}s\,. \tag{2.3}$$

We have

$$\frac{\partial}{\partial s}\left[(u^*(s))^p\left(\int_0^s y^{-p}(g(y))^p\,\mathrm{d}y\right)\right] = p(u^*(s))^{p-1}u^*_s(s)\int_0^s y^{-p}(g(y))^p\,\mathrm{d}y + (u^*(s))^p s^{-p}(g(s))^p\,.$$

Integrating this equality between 0 and ∞ we have

$$\int_0^\infty (u^*(s))^p s^{-p}(g(s))^p\,\mathrm{d}s = p\int_0^\infty\left((u^*(s))^{p-1}(-u^*_s(s))\int_0^s y^{-p}(g(y))^p\,\mathrm{d}y\right)\mathrm{d}s\,. \tag{2.4}$$

By the Holder inequality we obtain

$$p\int_0^\infty\left((u^*(s))^{p-1}(-u^*_s(s))\int_0^s y^{-p}(g(y))^p\,\mathrm{d}y\right)\mathrm{d}s \le \left(\int_0^\infty (u^*(s))^p s^{-p}(g(s))^p\,\mathrm{d}s\right)^{\frac{p-1}{p}}$$
$$\times\left(\int_0^\infty(-u^*_s(s))^p\left(\int_0^s y^{-p}(g(y))^p\,\mathrm{d}y\right)^p[s^{-p}(g(s))^p]^{-(p-1)}\,\mathrm{d}s\right)^{\frac{1}{p}}\,. \tag{2.5}$$

Applying (1.6) to the right-hand side of (2.5) and combining it with (2.4) we arrive at (2.3). Finally from Polia-Szego principle we obtain

$$I \le c\int_0^\infty(-u^*_s(s))^p(g(s))^p\,\mathrm{d}s \le c\int_M |\nabla u|^p\,\mathrm{d}\mu\,, \tag{2.6}$$

where the last inequality is proved below. Following Talenti's approach we estimate by Hölder inequality

$$\frac{1}{h}\int_{\{t<|u|\le t+h\}}|\nabla u|\,\mathrm{d}\mu \le \left(\frac{1}{h}\int_{\{t<|u|\le t+h\}}|\nabla u|^p\,\mathrm{d}\mu\right)^{\frac{1}{p}}\left(\frac{1}{h}|\{t<|u|\le t+h\}|\right)^{\frac{p-1}{p}}\,.$$

On letting $h \to 0$, we obtain

$$P(t) := |\{|u| = t\}|_{N-1} = \left(-\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\{t<|u|\}} |\nabla u|^p \,\mathrm{d}\mu \right)^{\frac{1}{p}} \Big(-\frac{\mathrm{d}}{\mathrm{d}t} v(t) \Big)^{\frac{p-1}{p}}, \tag{2.7}$$

where $v(t) := |\{t < |u|\}|$. Next by the isoperimetrical inequality (1.4) we have

$$P(t) \geq g(v(t)) \,. \tag{2.8}$$

When we set $s = v(t)$ we get $t = u^*(s)$, and

$$\frac{\mathrm{d}v}{\mathrm{d}t}(t) = u^*_s(s) \,.$$

Thus (2.7), (2.8) imply

$$g(s)^p(-u^*_s(s))^p \leq -\frac{\mathrm{d}}{\mathrm{d}s} \int\limits_{\{u^*(s)<|u|\}} |\nabla u|^p \,\mathrm{d}\mu \,.$$

Finally, on integrating the last inequality over $(0, \infty)$, we arrive at the desired result.
The theorem is proved.

3. Extinction in finite time. Proof of Theorem 1.3.

On multiplying both the sides of the equation (1.1) by u^θ, $\theta > 0$ such that $p + m + \theta > 2$ and integrating py parts we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_M \rho v^\delta \,\mathrm{d}\mu = -\gamma \int\limits_M |\nabla v|^p \,\mathrm{d}\mu \,, \tag{3.1}$$

where, owing to (1.3),

$$v = u^{\frac{p+m+\theta-2}{p}} \,, \qquad \delta = \delta(\theta) = \frac{(1+\theta)p}{p+m+\theta-2} > p \,.$$

We select θ so that the value of δ is the one given in (1.8); the requirement $p^* > \delta$ translates into $\theta > \theta_0 := (3 - p - m)N/p - 1$; note that $\theta_0 > 2 - p - m$ under our assumptions. If $\theta_0 \geq 0$ then no other requirement is needed on δ other than $\delta < p^*$; otherwise one must impose $\delta < \delta(0+) = p/(p + m - 2)$ which belongs to (p, p^*), when $\theta_0 < 0$.
Applying Holder inequality and the embedding in (1.7) to v we get

$$\int\limits_M \rho v^\delta \,\mathrm{d}\mu \leq C \left(\int\limits_M |\nabla v|^p \,\mathrm{d}\mu \right)^{\frac{\delta}{p}} \left(\int\limits_M (\rho(x)\omega^\delta \Big(V(d(x))\Big)\Big)^{\frac{p^*}{p^*-\delta}} \,\mathrm{d}\mu \right)^{\frac{p^*-\delta}{p^*}} . \tag{3.2}$$

On setting

$$E(t) := \int_M \rho(x) v^\delta(x,t)\,\mathrm{d}\mu\,,$$

we have that, when we appeal to (1.8),

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) \le -\gamma E(t)^{\frac{p}{\delta}}\,.$$

Since $\frac{p}{\delta} < 1$ the last inequality leads to extinction in finite time: $E(t) \to 0$ as $t \to \bar{t}-$ for some $\bar{t} < +\infty$.

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