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Diagonal sections of copulas, multivariate conditional hazard rates and distributions of order statistics for minimally stable lifetimes

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Abstract: As a motivating problem, we aim to study some special aspects of the marginal distributions of the order statistics for exchangeable and (more generally) for *minimally stable* non-negative random variables $T_1, \dots, T_r$. In any case, we assume that $T_1, \dots, T_r$ are identically distributed, with a common survival function $\overline{G}$ and their survival copula is denoted by K. The diagonal sections of K, along with $\overline{G}$, are possible tools to describe the information needed to recover the laws of order statistics.
When attention is restricted to the absolutely continuous case, such a joint distribution can be described in terms of the associated multivariate conditional hazard rate (m.c.h.r.) functions. We then study the distributions of the order statistics of $T_1, \dots, T_r$ also in terms of the system of the m.c.h.r. functions. We compare and, in a sense, we combine the two different approaches in order to obtain different detailed formulas and to analyze some probabilistic aspects for the distributions of interest. This study also leads us to compare the two cases of exchangeable and minimally stable variables both in terms of copulas and of m.c.h.r. functions. The paper concludes with the analysis of two remarkable special cases of stochastic dependence, namely Archimedean copulas and load sharing models. This analysis will allow us to provide some illustrative examples, and some discussion about peculiar aspects of our results.

Keywords: Minimally stable random vectors, diagonal sections of survival copulas, diagonal dependence, t-exchangeability, absolute continuity, Archimedean copulas, load-sharing models

MSC: 60E05, 62H05, 62G30, 60K10, 62N05

1 Introduction

Concerning the basic role of the concept of copula and of the Sklar's theorem in the analysis of stochastic dependence, a main issue is the study of the distributions of the order statistics $X_{1:r}, \dots, X_{r:r}$ for a set of interdependent random variables $X_1, \dots, X_r$. On the one hand, the condition of exchangeability is specially relevant (see in particular Galambos [11]) in such a study. On the other hand, the marginal distributions of $X_{1:r}, \dots, X_{r:r}$ are strictly related to the diagonal sections of copulas (see, e.g., Jaworski [12], Durante and Sempi [9]). For these reasons, in the theory of order statistics, the study of diagonal sections of copulas has been mainly concentrated on the case of exchangeable random variables.

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Really, in such a study, the assumption of exchangeability can at any rate be replaced by the more general condition that, for $d = 2, \dots, r - 1$, all the diagonal sections of the d-dimensional marginal copulas do coincide. Such a condition has been attracting more and more interest in the recent literature, where it has been however designated by means of different terminologies. In fact, such a condition can actually manifest under different mathematical forms, as we will discuss in details. For our purposes it is specially convenient to look at it as the condition that $X_1, \dots, X_r$ are minimally stable (see Definition 3 below).

In this note we concentrate attention on the case of non-negative, minimally stable, random variables which we denote by $T_1, \dots, T_r$.

Generally, concerning with non-negative random variables, stochastic dependence can also be conveniently described in terms of stochastic intensities of related counting processes. See in particular Arjas [1], Bremaud [3], Arjas and Norros [2]. Such a description, in particular, can be based on the knowledge of the so-called *multivariate conditional hazard rates (m.c.h.r.)* functions, when attention is restricted to the absolutely continuous case (see in particular the papers by Shaked and Shanthikumar [26–28]). In such a case the family of those functions gives rise to a method to describe a joint distribution, which is alternative to the one based on copulas and marginal distributions or on the joint density function.

From an analytical view-point the two methods are actually equivalent: on the one hand the family of the m.c.h.r. functions can be obtained in terms of the joint density function, on the other hand the joint density can be recovered when the m.c.h.r. functions are known. As a matter of fact, however, the corresponding formulas are not easily handleable in general cases. The two methods, furthermore, are respectively apt to explain completely different aspects of stochastic dependence.

In this paper we aim to establish a bridge between the two different approaches. Maintaining the attention focused on the minimally stable case, then, we are primarily interested in the relations tying the system of the diagonal sections with the system formed by the m.c.h.r. functions. Such relations will allow us to detect, both in terms of copulas and in terms of the m.c.h.r. functions, which are the minimal sets of functions able to convey sufficient information to recover the family of the marginal distributions of the order statistics $T_{1:r}, \dots, T_{r:r}$.

In such a framework, interesting questions also concern with understanding the real difference between the cases when $T_1, \dots, T_r$ are exchangeable and when they are minimally stable. On this purpose, the differences between the two properties will be detailed both using the language of copulas and the language of the m.c.h.r. functions. Still by using and combining the two approaches, we will also face the problem of constructing examples of random variables $T_1, \dots, T_r$ which are minimally stable but not exchangeable.

More in details, the plan of this paper goes as follows.

In Section 2 we introduce some needed notation and then we review basic facts about distributions of order statistics, about diagonal sections of copulas, and about the relations tying these two families of objects. We also show in details the equivalence among different forms under which one can represent the condition that $T_1, \dots, T_r$ are minimally stable. Some relevant remarks are given and an example is presented concerning the construction of random variables which are minimally stable but not exchangeable.

In Section 3 we will first recall, in general, the definition and some basic aspects of the family of the multivariate conditional hazard rate functions. We will then show special features of the cases where the lifetimes $T_1, \dots, T_r$ are exchangeable or minimally stable. In this frame, the results of Section 2 will emerge as natural tools to obtain, in Section 4, the relations existing among diagonal sections of copulas, the distributions of order statistics, and a special subclass $\mathcal{C}$ of multivariate conditional hazard rates (see (37) and (39)), corresponding to the (unconditional) one-dimensional hazard rates of $\min(T_1, \dots, T_\ell)$, $\ell = 1, \dots, r$. See in particular Propositions 18 and 19.

In order to demonstrate some special aspects of the results presented in the Sections 3 and 4, Section 5 will be devoted to a detailed discussion of the remarkable cases of Archimedean copulas and of minimally stable time homogeneous load-sharing models. Some more general examples will be presented in the Appendix.

Often, along the paper, the term *lifetime* will be used as a short-hand for "non-negative random variable".

Notation: For any natural number n, we set $[n] := \{1, 2, \dots, n\}$.

For any subset $J \subseteq [n]$, we denote by $|J|$ the cardinality of J, and as usual we denote by J^c the complementary set of J, i.e., the set of indices $[n] \setminus J$. Furthermore, for any $k \leq |J|$ we denote by

$$\Pi_k(J) = \left\{(j_1, \dots, j_k) : j_\ell \in J,\ \forall\, \ell = 1, .., k,\ j_\ell \neq j_h,\ \forall \ell \neq h\right\},$$

the set of k-permutations of J. When $k = |J|$ we drop the index k and write simply $\Pi(J)$. The symbol

$$(n)_k := n(n-1)\cdots(n-(k-1))) = |\Pi_k([n])|$$

denotes the number of k-permutations in $\Pi_k([n])$.

For any subset $A = \{j_1, \dots, j_\ell\} \subset [n]$ we denote by $\mathbf{e}_A$ the vector whose i-th component is equal to 1 if $i \in A$, and is equal to 0, otherwise.

2 Diagonal sections and distributions of order statistics

Let $T_1, \dots, T_r$ denote r non-negative random variables, defined on a same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with joint survival function

$$\overline{F}(t_1, \dots, t_r) := \mathbb{P}\left(T_1 > t_1, \dots, T_r > t_r\right),$$

and survival copula $K : [0, 1]^r \to [0, 1]$.

All over the paper we generally assume the following conditions, unless specified otherwise,

(H1) the random variables $T_1, \dots, T_r$ are identically distributed with common one-dimensional marginal survival function $\overline{G}$, i.e.,

$$\overline{G}(t) := \mathbb{P}\left(T_j > t\right), \quad \text{for } j = 1, \dots, r, \text{ and for } t > 0.$$

(H2) $\overline{G}(t)$ is continuous, strictly positive, and strictly decreasing on $(0, \infty)$.

(H3) the random variables $T_1, \dots, T_r$ are no-ties, i.e., $\mathbb{P}(T_i = T_j) = 0$, for $i \neq j$.

Since $T_1, \dots, T_r$ are non-negative, condition **(H2)** implies that $\overline{G}(0) = 1$ and that $\overline{G}(t)$ is invertible. Though **(H2)** is not strictly necessary (as, for example, in Proposition 8), we assume it for simplicity's sake (for example we use **(H2)** in Proposition 5, Remark 6 and Corollary 9, within this section and somewhere else, within the other sections). Condition **(H3)** allows the order statistics $T_{1:r}, \dots, T_{r:r}$ of $(T_1, \dots, T_r)$ to be defined without ambiguity. We denote by

$$\overline{G}_{1:r}(t) := \mathbb{P}\left(T_{1:r} > t\right), \dots, \overline{G}_{r:r}(t) := \mathbb{P}\left(T_{r:r} > t\right) \tag{1}$$

the corresponding marginal survival functions.

Note that the order statistics $T_{1:r}, \dots, T_{r:r}$ may be considered as the jump times of the counting process

$$N(t) := \sum_{i=1}^{r} \mathbf{1}_{\{T_i \leq t\}}$$

i.e., the process such that $N(t) = k$ for $t \in [T_{k:r}, T_{k+1:r})$, where we have set $T_{0:r} = 0$ and $T_{r+1:r} = \infty$.

Before continuing we recall the following definition.

Definition 1. *For a r-dimensional copula C the* **diagonal section** *is the function*

$$\delta^C : [0, 1] \to [0, 1]; \quad u \mapsto \delta^C(u) = C(u, u, \dots, u)$$

Furthermore, for any $A \subset [r]$, by δ^C_A we denote the diagonal section of the marginal copula, corresponding to the A-components, i.e., the function

$$\delta^C_A : [0, 1] \to [0, 1]; \quad u \mapsto \delta^C_A(u) := C(u\mathbf{e}_A + \mathbf{e}_{A^c}).$$

In particular we refer to the functions

$$\delta^C_{[\ell]}(u) = C(\overbrace{u, \dots, u}^{\ell \text{ times}}, \overbrace{1, \dots, 1}^{r-\ell \text{ times}}), \quad 2 \le \ell \le r$$

as the **diagonal sections associated to** C.

For the functions $\delta^C_{[\ell]}$ we will also use the shorter notation δ^C_ℓ, namely

$$\delta^C_\ell(u) := \delta^C_{[\ell]}(u), \quad 2 \le \ell \le r,$$

and the shorter term ***diagonal sections*** of C. Such terminology turns out to be convenient for the ensuing arguments. It is clear that, for $A \subsetneq [r]$, with $|A| = \ell$ and $A \ne [\ell]$, the two functions δ^C_A and δ^C_ℓ are generally different.
It is also clear that $\delta^C_\ell(u)$ is an increasing function and that $\delta^C_2(u) \ge \delta^C_3(u) \ge \dots \ge \delta^C_r(u)$. Conditions, for a function $\delta : [0, 1] \to [0, 1]$ to be the diagonal section of a copula, are given, in particular, in Jaworski [12], and Durante and Sempi [9].

In what follows, when dealing with the diagonal sections associated to the survival copula K of $T_1, \dots, T_r$ we drop the superscript, i.e., we set

$$\delta_\ell(u) := \delta^K_\ell(u), \quad 2 \le \ell \le r.$$

Assume for the moment that the joint survival function $\overline{F}(t_1, \dots, t_r)$ is exchangeable, namely

$$\overline{F}(t_1, \dots, t_r) = K\left(\overline{G}(t_1), \dots, \overline{G}(t_r)\right), \quad \text{for } t_1, \dots, t_r > 0,$$

with K permutation-invariant.

As well-known, a direct relationship can be established between δ_r and the probability law of the minimal order statistics $T_{1:r}$, in fact one immediately obtains, for $t > 0$,

$$\overline{G}_{1:r}(t) = \mathbb{P}(T_1 > t, \dots, T_r > t) = \delta_r(\overline{G}(t)). \tag{2}$$

By taking into account exchangeability of $T_1, \dots, T_r$ one can similarly write

$$\begin{aligned}
&\mathbb{P}\left(T_{j_1} > t, \dots, T_{j_d} > t\right) = \mathbb{P}(T_1 > t, \dots, T_d > t), \\
&= \overline{F}(t, \dots, t, 0, \dots, 0) = K\left(\overline{G}(t), \dots, \overline{G}(t), 1, \dots, 1\right) = \delta_d(\overline{G}(t)),
\end{aligned} \tag{3}$$

for $d = 1, 2, \dots, r-1$ and for any subset of indices $J = \{j_1, .., j_d\} \subset [r]$ of cardinality d. Whence one can write

$$\overline{G}_{\ell:r}(t) = \sum_{h=r-\ell+1}^{r} (-1)^{h-r-1+\ell} \binom{r}{h} \binom{h-1}{r-\ell} \delta_h(\overline{G}(t)), \quad \ell = 1, \dots, r. \tag{4}$$

In fact, by using (3), the latter formula is readily obtained from the formula expressing the survival functions of the order statistics of exchangeable variables in terms of the survival functions of the minima within subsets of the same variables (see in particular David and Nagaraja, p. 46 [5], Jaworski and Rychlik [13], Rychlik [22]).

As we will see in Proposition 7, formula (4) for $\overline{G}_{\ell:r}(t)$ is still valid when the joint distribution of $T_1, \dots, T_r$ satisfy the specific symmetry conditions recalled in Definitions 2 and 3, below. Such conditions are actually weaker than exchangeability, and turn out to be equivalent each other (see Proposition 5 below).

Definition 2. *We will say that the random variables $T_1, \dots, T_r$ are* t**-Exchangeable** *if for every $t \ge 0$, the binary random variables $X_i(t) = \mathbf{1}_{\{T_i > t\}}$, $i = 1, \dots, r$, are exchangeable, or equivalently the events $\{T_i > t\}$, $i = 1, \dots, r$, are exchangeable.*

We will briefly refer to the previous property as t-EX.

Definition 3. *The random variables $T_1, \dots, T_r$ are said* **minimally stable***, when, for any $\ell = 1, \dots, r$ and for any subset $A = \{j_1, \dots, j_\ell\} \subseteq [r]$*

$$\mathbb{P}\left(T_{j_1} > t, \dots, T_{j_\ell} > t\right) = \mathbb{P}\left(T_1 > t, \dots, T_\ell > t\right), \quad \forall t > 0, \tag{5}$$

namely $\mathbb{P}\left(T_{j_1} > t, \dots, T_{j_\ell} > t\right) = \overline{F}(\underbrace{t, \dots, t}_{\ell\ times}, \underbrace{0, \dots, 0}_{r-\ell\ times}), \forall t > 0.$

Finally we recall the strictly related concept of diagonal dependent copulas (see Navarro and Fernandez-Sanchez [17]). Such a concept can be obtained as a special case of the one of k-diagonal dependence, for $k \le r$, as introduced by Okolewski in [21].

Definition 4. *Let C be an r-dimensional copula C. The copula C is said to be a* **k-diagonal dependent copula***, with $k \le r$, if for any subsets $A, B \subset [r]$, with $|A| = |B| \le k$*

$$\delta_A^C(u) = \delta_B^C(u), \quad \forall\, u \in [0, 1]. \tag{6}$$

When $k = r$, the copula C is said **diagonal dependent**.

As in Navarro and Fernandez-Sanchez [17] we briefly refer to the property of diagonal dependence as DD.

The following result can be obtained by taking into account basic and well known properties of exchangeable binary random variables originally obtained by de Finetti (see [6]). See also Navarro et al. [18].

Proposition 5. *Under the conditions* **(H1)–(H3)** *the following properties are equivalent*
(i) *The random variables $T_1, \dots, T_r$, are t-Exchangeable;*
(ii) *For all $H, H' \subseteq \{1, 2, \dots, r\}$, with $|H| = |H'|$*

$$\mathbb{P}\big(T_j > t, \forall j \in H, T_i \le t, \forall i \notin H\big) = \mathbb{P}\big(T_j > t, \forall j \in H', T_i \le t, \forall i \notin H'\big). \tag{7}$$

(iii) *The random variables $T_1, \dots, T_r$ are minimally stable;*
(iv) *The random variables $T_1, \dots, T_r$ are identically distributed and their survival copula K is diagonal dependent.*

Proof. Properties ***(i)*** and ***(ii)*** are clearly equivalent: indeed

$$\mathbb{P}\big(T_j > t, \forall j \in H, T_i \le t, \forall i \notin H\big) = \mathbb{P}\big(X_j(t) = 1, \forall j \in H, X_i(t) = 0, \forall i \notin H\big).$$

Similarly properties ***(iii)*** and ***(iv)*** are equivalent: indeed if $T_1, \dots, T_r$ are minimally stable, then by taking $\ell = 1$ in (5), they are identically distributed, and therefore for all $A \subset [r]$ with $|A| = \ell$

$$\mathbb{P}(T_i > t,\ \forall i \in A) = K(\overline{G}(t)\mathbf{e}_A + \mathbf{e}_{A^c}) = \delta_\ell(\overline{G}(t)), \quad \forall t \ge 0,$$

and $\overline{G}(t)$ is invertible, in view of the regularity condition **(H2)**.
Finally ***(iv)*** is equivalent to ***(ii)***, in view of the inclusion-exclusion formula.

□

Remark 6. *The previous result (Proposition 5) holds true also without the regularity assumption* **(H2)** *on $\overline{G}$, but in the general case an extension of the notion of Diagonal Dependence is needed (see Navarro et al. [18]).*

The interest for the properties ***(i)*** and ***(iv)*** had independently emerged in the two papers Marichal et al. [15] and Navarro and Fernandez-Sanchez [17] with reference to the field of systems' reliability. Still in the same framework, furthermore, the study of conditions for the equivalence between ***(i)*** and ***(iv)*** has been developed in Navarro et al. [18]. See Remark 14 below for details about the connections with system reliability.

We are now in a position to establish the following result.

Proposition 7. *Assume* **(H1)–(H3)** *and any condition among* ***(i)–(iv)****. Then the equations* (4) *hold.*

Actually the validity of (4) hinges on the Eq. (3), which only requires the DD property of the survival copula and the identical distribution of T_i, $i = 1, \dots, r$.

From now on we make the following further assumption

(H4) the random variables $T_1, \dots, T_r$ are minimally stable.

Namely we assume the condition ***(iii)*** of Proposition 5 and, at a time, we highlight that such a result just ensures the validity of the equivalence among all the conditions ***(i)–(iv)***, under our standing hypotheses **(H1)–(H3)**.

Under the assumptions **(H1)–(H4)** we thus proceed to establish detailed results concerning the relations between the following families of functions

$$\mathcal{A} := \{\overline{G}; \delta_2, \dots, \delta_r\}, \quad \mathcal{B} := \{\overline{G}_{1:r}, \dots, \overline{G}_{r:r}\}. \tag{8}$$

Since the marginal survival functions $\overline{G}_{1:r}, \dots, \overline{G}_{r:r}$ are determined by the knowledge of the joint distribution of $T_1, \dots, T_r$ then, in principle, the family $\mathcal{B}$ should depend on the survival copula K and the common survival marginal $\overline{G}(t)$. Actually the full knowledge of K is not necessary, since the knowledge of the associated diagonal sections is sufficient as shown by the formula (4). More precisely the families $\mathcal{A}$ and $\mathcal{B}$ convey the same amount of information concerning the joint distribution of $T_1, \dots, T_r$, as we point out in details in what follows and summarize in Proposition 10.

To this end we start by recalling that when $\overline{G}_{1:r}, \dots, \overline{G}_{r:r}$ are known, we can easily recover the common marginal survival function $\overline{G}(t)$. Indeed, the random variables $T_1,\dots,T_r$ are identically distributed and therefore

$$\overline{G}(t) = \frac{1}{r} \sum_{k=1}^{r} \overline{G}_{k:r}(t), \tag{9}$$

as immediately follows by observing that $\sum_{h=1}^{r} \mathbf{1}_{\{T_h > t\}} = \sum_{k=1}^{r} \mathbf{1}_{\{T_{k:r} > t\}}$.

Furthermore the same formula (4) would permit to recover, step-by-step, the functions $\delta_2, \dots, \delta_r$. Here we follow a different path and the detailed formula is given in the Corollary 9 of the following result.

Proposition 8. *Under the conditions* **(H1)–(H4)***, for every $d \in [r]$, and $J \subseteq [r]$, with $|J| = d$*

$$\mathbb{P}\big(T_j > t,\ \forall j \in J\big) = \sum_{h=d}^{r} \frac{(h)_d}{(r)_d} \left(\overline{G}_{r-h+1:r}(t) - \overline{G}_{r-h:r}(t)\right) \tag{10}$$

$$= \frac{d}{(r)_d} \sum_{k=1}^{r-d+1} (r-k)_{d-1}\, \overline{G}_{k:r}(t), \quad t > 0, \tag{11}$$

where by convention $\overline{G}_{0:r}(t) = 0$, for $t > 0$.

Also for what concerns the proof of Proposition 8, similarly to what we mentioned for Proposition 5, one could apply well-known and simple results (see, e.g., de Finetti [6]) about exchangeable binary random variables. For the ease of the reader we give a self-contained, and detailed, proof at the end of this section. Here we only point out that the most important ingredient of the proof amounts to the validity of the following identity for any subset $J \subset [r]$:

$$\mathbb{P}\big(T_j > t,\ \forall j \in J\big) = \sum_{K:\, K \subseteq J^c} \mathbb{P}\big(T_j > t,\ \forall j \in J \cup K,\ T_i \le t,\ \forall i \notin J \cup K\big) \tag{12}$$

or equivalently

$$\mathbb{P}\big(T_j > t,\ \forall j \in J\big) = \sum_{H:\, H \supseteq J} \mathbb{P}\big(T_j > t,\ \forall j \in H,\ T_i \le t,\ \forall i \notin H\big). \tag{13}$$

From Proposition 8 we get the expression of δ_d in terms of the marginal survival functions $\overline{G}_{k:r}(t)$, $k = 1, \dots, r$.

Corollary 9. *Under the conditions* **(H1)–(H4)**, *for every* $d \in \{1, 2, \dots, r\}$, *the following equalities hold*

$$\delta_d(u) = \sum_{h=d}^{r} \frac{(h)_d}{(r)_d} \left(\overline{G}_{r-h+1:r}\left(\overline{G}^{-1}(u)\right) - \overline{G}_{r-h:r}\left(\overline{G}^{-1}(u)\right) \right) \tag{14}$$

$$= \frac{d}{(r)_d} \sum_{h=1}^{r-d+1} (r-h)_{d-1}\, \overline{G}_{h:r}\left(\overline{G}^{-1}(u)\right), \quad u \in [0, 1]. \tag{15}$$

Indeed, this result is a consequence of equations (10), (11), and (3), since, as already observed the condition appearing in (3) also holds for minimally stable variables.

The above results, concerning the relations tying the families $\mathcal{A}$ and $\mathcal{B}$, will be now summarized by means of the following proposition.

Proposition 10. *Under the conditions* **(H1)–(H4)**, *the family of the survival functions* $\mathcal{B} = \left\{\overline{G}_{1:r}, \dots, \overline{G}_{r:r}\right\}$ *is determined by the family* $\mathcal{A} = \left\{\overline{G}, \delta_2, \dots, \delta_r\right\}$ *by means of formula* (4). *Viceversa the family* $\mathcal{A}$ *is determined by the family* $\mathcal{B}$ *by means of formula* (14) *(or* (15)*) and formula* (9).

Remark 11. *For non-exchangeable, but minimally stable lifetimes* $T_1, \dots, T_r$ *there still exist exchangeable lifetimes* $\widetilde{T}_1, \dots, \widetilde{T}_r$, *such that* $\mathbb{P}(\widetilde{T}_j > t) = \mathbb{P}(T_j > t) = \overline{G}(t)$, *and the diagonal sections* $\widetilde{\delta}_\ell(u)$ *associated to their survival copula* $\widetilde{K}$ *coincide with the diagonal sections* $\delta_\ell(u)$ *associated to the survival copula* K. *Indeed* $\widetilde{K}$ *may be constructed by symmetrizing* K*:*

$$\widetilde{K}(u_1, \dots, u_r) = \frac{1}{r!} \sum_{\sigma \in \Pi([r])} K(u_{\sigma_1}, \dots, u_{\sigma_r}).$$

The above construction can be of help in obtaining the explicit form of $\mathbb{P}(T_j > t, \forall j \in A)$ *in some special cases (see in particular Subsection 5.2).*

Remark 12. *The problem of constructing examples of vectors which are not exchangeable, but still minimally stable, naturally arises. Since minimally stable variables* $T_1, \dots, T_r$ *are identically distributed, constructing such examples is equivalent to constructing diagonal-dependent copulas, which are not exchangeable. In the following Example 13 we present a simple path to such a construction. Other examples can be found in Navarro, Fernandez-Sanchez [17], and Navarro et al. [18]. See also Example 31 in the Appendix.*

Example 13. *First of all we notice that, when* $r = 2$, *then any pair* (T_1, T_2) *of identical distributed random variables is minimally stable, but in general is not exchangeable. Similarly, and trivially, any 2-dimensional copula* C *is minimally stable. Indeed* $\delta_2^C(u) = C(u, u)$, *and* $\delta_1^C(u) = C(u, 1) = C(1, u) = u$. *Starting from the copula* C *one may define two 3-dimensional copulas as follows*

$$C_{(1,2,3)}(u_1, u_2, u_3) := \frac{1}{3}\left[C(u_1, u_2)u_3 + C(u_2, u_3)u_1 + C(u_3, u_1)u_2\right]$$

$$C_{(3,2,1)}(u_1, u_2, u_3) := \frac{1}{3}\left[C(u_3, u_2)u_1 + C(u_2, u_1)u_3 + C(u_1, u_3)u_2\right]$$

respectively obtained as the symmetric mixture over the cyclic permutations of $(1, 2, 3)$ *and the cyclic permutations of* $(3, 2, 1)$. *Notice that when* C *is non-exchangeable, then* $C_{(1,2,3)}$ *and* $C_{(3,2,1)}$ *are non-exchangeable: indeed if* $u, v \in (0, 1)$ *are such that* $C(u, v) \neq C(v, u)$ *then, for example,*

$$C_{(1,2,3)}(u, v, 1) := \frac{1}{3}\left[C(u, v) + C(v, 1)u + C(1, u)v\right] = \frac{1}{3}C(u, v) + \frac{2}{3}uv,$$

which is clearly different from

$$C_{(1,2,3)}(v, u, 1) := \frac{1}{3}\left[C(v, u) + C(u, 1)v + C(1, v)u\right] = \frac{1}{3}C(v, u) + \frac{2}{3}uv,$$

thus $C_{(1,2,3)}$ is non-exchangeable, though the 2-dimensional marginal distributions are all equal, namely

$$C_{(1,2,3)}(u,v,1) = C_{(1,2,3)}(v,1,u) = C_{(1,2,3)}(1,u,v) = \frac{1}{3}\,C(v,u) + \frac{2}{3}\,uv.$$

By iterating the above construction, one can also obtain a DD, non-exchangeable, copula which is n-dimensional. Details can be found in the Appendix (see Example 30).

Remark 14. *As already mentioned, the topics developed in the papers Marichal et al. [15], Navarro and Fernandez-Sanchez [17], and Navarro et al. [18] are motivated by questions arising in the field of systems' reliability. More precisely these papers deal with the so-called signature representation for the survival function $R_S^{(\varphi)}(t)$ of the lifetime T_S of a binary coherent system S, made with r binary components, with structure function φ, and for which the random variables $T_1, \dots, T_r$ have the meaning of the components' lifetimes. The signature of S is a probability distribution $\mathbf{s}^{(\varphi)} := (s_1^{(\varphi)}, \dots, s_r^{(\varphi)})$ over $[r]$ which is a combinatorial invariant associated to φ (see in particular Samaniego [24]). The signature representation means that the equation*

$$R_S^{(\varphi)}(t) = \sum_{h=1}^{r} s_h^{(\varphi)} \overline{G}_{h:r}(t) \tag{16}$$

holds for any $t > 0$.
Under our standing hypotheses **(H1)—(H4)**, *the properties* ***(i)*** *and* ***(iv)*** *are equivalent and also imply the signature representation* (16) *for the survival function $R_S^{(\varphi)}(t)$. When the functions $\overline{G}_{1:r}, \dots, \overline{G}_{r:r}$ are known, one can then recover from* (16) *the function $R_S^{(\varphi)}(t)$, relatively to any structure φ for which the signature $\mathbf{s}^{(\varphi)}$ is known. At the same time the family of all the functions $R_S^{(\varphi)}(t)$ in particular contains the family $\mathcal{B} = \{\overline{G}_{1:r}, \dots, \overline{G}_{r:r}\}$. In fact, the survival functions $\overline{G}_{k:r}$ can be seen as the reliability functions of the coherent systems of the type k-out-of-r, for $k = 1, \dots, r$.*

We end this section with the afore announced proof of Proposition 8.

Proof of Proposition 8. We start by recalling that $N(t) = \sum_{i=1}^{r} \mathbf{1}_{\{T_i \le t\}}$ and observing that

$$\mathbb{P}(N(t) = r-h) = \sum_{J:|J|=h} \mathbb{P}(T_j > t,\ \forall j \in J,\ \ T_i \le t,\ \forall\, i \notin J)$$

so that, thanks to Eq. (7)

$$\mathbb{P}(N(t) = r-h) = \binom{r}{h} \mathbb{P}(T_j > t,\ \forall j \in \{1,2,..,h\},\ \ T_i \le t,\ \ \forall i \in \{h+1,\dots,r\}),$$

or equivalently, for any $H \subset [r]$, with $|H| = h$

$$\mathbb{P}\big(T_j > t,\ \forall j \in H,\ \ T_i \le t,\ \forall i \notin H\big) = \frac{1}{\binom{r}{h}}\,\mathbb{P}(N(t) = r-h). \tag{17}$$

On the other hand, we observe that

$$\begin{aligned} &\mathbb{P}(N(t) = r-h) = \mathbb{P}(T_{r-h:r} \le t < T_{r-h+1:r}) \\ =&\mathbb{P}(T_{r-h+1:r} > t) - \mathbb{P}(T_{r-h:r} > t) = \overline{G}_{r-h+1:r}(t) - \overline{G}_{r-h:r}(t). \end{aligned} \tag{18}$$

Then the thesis follows immediately: indeed, for every $J \subset \{1,2,\dots,r\}$ with $|J| = d$, Eq.s (17) and (18), together with (12), imply (with the convention that $\binom{0}{0} = 1$)

$$\mathbb{P}\big(T_j > t,\ \forall j \in J\big) = \sum_{h=d}^{r} \binom{r-d}{h-d} \frac{1}{\binom{r}{h}} \ \big(\overline{G}_{r-h+1:r}(t) - \overline{G}_{r-h:r}(t)\big)\,,$$

and formula (10) follows by observing that

$$\frac{\binom{r-d}{h-d}}{\binom{r}{h}} = \frac{(h)_d}{(r)_d}.$$

Finally, from (10), taking into account the convention that $\overline{G}_{0:r}(t) = 0$, one obtains

$$\begin{aligned}(r)_d\, \mathbb{P}\big(T_j > t,\ \forall j \in J\big) &= \sum_{h=d}^{r} (h)_d \overline{G}_{r-(h-1):r}(t) - \sum_{h=d}^{r-1} (h)_d \overline{G}_{r-h:r}(t) \\ &= \sum_{k=d-1}^{r-1} (k+1)_d \overline{G}_{r-k:r}(t) - \sum_{h=d}^{r-1} (h)_d \overline{G}_{r-h:r}(t) \\ &= d!\, \overline{G}_{r-(d-1):r}(t) + \sum_{k=d}^{r-1} \big[(k+1)_d - (k)_d\big]\, \overline{G}_{r-k:r}(t).\end{aligned}$$

Therefore, by observing that

$$(k+1)_d - (k)_d = (k)_{d-1}\,[k+1-(k-(d-1))] = (k)_{d-1}\, d,$$

one gets

$$\mathbb{P}\big(T_j > t,\ \forall j \in J\big) = \frac{d}{(r)_d} \sum_{k=d-1}^{r-1} (k)_{d-1} \overline{G}_{r-k:r}(t),$$

Then formula (11) follows by setting $h = r - k$ in the last sum. □

3 The use of multivariate conditional hazard rates

In this section attention will be restricted to random vectors of lifetimes with absolutely continuous joint distributions, so that their probabilistic properties can be alternatively described in terms of the multivariate conditional hazard rate (m.c.h.r.) functions. In a first part of this section we recall the definition of the m.c.h.r. functions associated to generic random variables $\{V_j, j \in [n]\}$, reviewing related properties and providing some references. Furthermore, for $A \subset [n]$, we focus attention on the joint distributions of $\{V_j,\ j \in A\}$. In particular we analyze the probability distributions of their minima by means of the m.c.h.r. functions associated to $\{V_j,\ j \in A\}$. In the second part, coming back to our lifetimes $T_1, \dots, T_r$, and adding absolute continuity condition to our standing hypotheses, we characterize both the exchangeability and the minimal stability conditions by means of the m.c.h.r. functions associated to $\{T_j, j \in [r]\}$.

3.1 Multivariate conditional hazard rates and distribution of minima

In this subsection we briefly recall some definitions and basic properties of multivariate conditional hazard rate functions for n non-negative random variables $V_1, \dots, V_n$ with an absolutely continuous joint distribution whose joint density function is denoted by $f_{\mathbf{V}}$. For simplicity's sake we will assume moreover that there exists a version of the joint density which is strictly positive on $\mathbb{R}_+^n$, i.e., $f_{\mathbf{V}}(v_1, .., v_n) > 0$, when $v_i > 0$, for all $i = 1, 2, \dots, n$.

For $k = 1, \dots, n-1$, and for any k-permutation $\mathbf{j} = (j_1, \dots, j_k) \in \Pi_k([n])$, the symbol $\mathbf{V_j}$ denotes the vector of lifetimes $(V_{j_1}, \cdots, V_{j_k})$; for any subset $J \subseteq [n]$ we denote

$$V_{1:J} := \min_{j \in J} V_j; \tag{19}$$

furthermore, if $\mathbf{j} \in \Pi(J)$, for $0 < v_1 < \cdots < v_k \le v$ the symbol

$$\mathbf{V_j} = \mathbf{v}; \quad V_{1:J^c} > v \tag{20}$$

briefly denotes the observation

$$V_{j_1} = v_1, \cdots, V_{j_k} = v_k, \quad \min_{j \in J^c} V_j > v.$$

The observation $\mathbf{V_j} = \mathbf{v};\ V_{1:J^c} > v$ in (20) is often called a "dynamic history".

For the given $k \ge 1$, $J \subset [n]$, with $|J| = k$, $\mathbf{j} = (j_1, \ldots, j_k) \in \Pi(J)$, $v > 0$, $0 < v_1 < \cdots < v_k \le v$, $j \notin J$, the multivariate conditional hazard rates (m.c.h.r.) function $v \mapsto \lambda_{j|\mathbf{j}}(v|v_1, \cdots, v_k)$ is defined by the limit (where it exists)

$$\lambda_{j|j_1,\ldots,j_k}(v|v_1, \cdots, v_k) := \lim_{\Delta \to 0^+} \frac{\mathbb{P}\big(V_j \le v + \Delta | \mathbf{V_j} = \mathbf{v}; V_{1:J^c} > v\big)}{\Delta}, \quad a.e. \tag{21}$$

Note that the above m.c.h.r. functions $\lambda_{j|j_1,\ldots,j_k}$ depend also on the version of the conditional probability.

Furthermore, for any $j \in [n]$, the m.c.h.r. function $\lambda_{j|\emptyset}(v) : [0, \infty) \to [0, \infty)$ is defined by the limit (where it exists)

$$\lambda_{j|\emptyset}(v) := \lim_{\Delta \to 0^+} \frac{\mathbb{P}\big(V_j \le v + \Delta | V_{1:n} > v\big)}{\Delta}, \quad a.e. \tag{22}$$

In the sequel we will use the convention

$$\lambda_{j|j_1,\ldots,j_k}(v|v_1, \cdots, v_k) = \lambda_{j|\emptyset}(v), \quad \text{when } k = 0. \tag{23}$$

The above limits make sense in view of the assumption of absolute continuity of the joint distribution of $V_1, \ldots, V_n$ and the m.c.h.r. functions can be seen as direct extensions of the common concept of hazard rate function of a univariate non-negative random variable.

For the random vector $\mathbf{V} \equiv (V_1, \ldots, V_n)$, the system of the m.c.h.r. functions in (21) and (22) can be computed in terms of the joint density $f_{\mathbf{V}}$. It is remarkable the circumstance that the function f_V can be obtained from the knowledge of the set of all the m.c.h.r. functions in terms of a formula, that we are going to recall next. Preliminarily, we first notice an obvious difference between f_V and the functions $\lambda_{j|j_1,\ldots j_k}(v|v_1, \ldots, v_k)$: while the arguments $v_1, \ldots, v_n$ of f_V are generally not ordered, the arguments $v_1, \ldots, v_k$ of the functions $\lambda_{j|j_1,\ldots j_k}(v|v_1, \ldots, v_k)$ are necessarily listed in increasing order by definition. Furthermore, for given non-ordered values $v_1, \ldots, v_n$, we denote by $v_{1:n}, \ldots, v_{n:n}$ the same values rearranged in increasing order. Then the following formula holds: for $(v_1, \ldots, v_n)$, let $\mathbf{j} = (j_1, \ldots, j_n)$ a permutation in $\Pi([n])$ such that $v_{1:n} = v_{j_1} \le v_{2:n} = v_{j_2} \le \cdots \le v_{n:n} = v_{j_n}$,

$$\begin{aligned} f_{\mathbf{V}}(v_1, \ldots, v_n) = \Pi_{k=1}^{n} \lambda_{j_k|j_1,\ldots,j_{k-1}}(v_{j_k}|v_{j_1}, \ldots, v_{j_{k-1}}) \cdot \\ \cdot\, e^{-\int_{v_{j_{k-1}}}^{v_{j_k}} \Lambda_{j_1,\ldots,j_{k-1}}(u|v_{j_1},\ldots,v_{j_{k-1}})\, du}, \end{aligned} \tag{24}$$

where we have set $v_{j_0} = 0$,

$$\Lambda_{j_1,\ldots,j_{k-1}}(u|v_{j_1}, \ldots, v_{j_{k-1}}) = \sum_{j \notin \{j_1,\ldots,j_{k-1}\}} \lambda_{j|j_1,\ldots,j_{k-1}}(u|v_{j_1}, \ldots, v_{j_{k-1}}), \tag{25}$$

and we have used the convention that, when $k = 1$,

$$\begin{aligned} \lambda_{j|j_1,\ldots,j_{k-1}}(u|v_{j_1}, \ldots, v_{j_{k-1}}) &= \lambda_{j|\emptyset}(u), \\ \Lambda_{j_1,\ldots,j_{k-1}}(u|v_{j_1}, \ldots, v_{j_{k-1}})\Big|_{k=1} &= \Lambda_{\emptyset}(u) = \sum_{j \in [n]} \lambda_{j|\emptyset}(u). \end{aligned} \tag{26}$$

For proofs, details, and for general aspects of the m.c.h.r. functions see Shaked, Shantikumar [26, 27]. See also the reviews contained within the more recent papers Shaked, Shantikumar [28], Spizzichino [30].

For any subset of indices $A = \{h_1, \ldots, h_{|A|}\} \subset [n]$, one can also consider joint density of the random vector $(V_{h_1}, \ldots V_{h_{|A|}})$. Such a density may be defined by means of a different family of m.c.h.r. functions, related to the choice of the set A. Namely for any $k \le |A|$ and for any k-permutation $(j_1, \dots, j_k) \in \Pi_k(A)$, $j \in A \setminus \{j_1, \dots, j_k\}$, $0 < v_1 < \cdots < v_k \le v$, we can consider the m.c.h.r. functions defined as follows

$$\lambda^A_{j|j_1,\dots,j_k}(v|v_1, \cdots, v_k) := \lim_{\Delta \to 0^+} \frac{\mathbb{P}\big(V_j \le v + \Delta | \mathbf{V_j} = \mathbf{v}; V_{1:A\setminus\{j_1,\dots,j_k\}} > v\big)}{\Delta}, \quad a.e. \tag{27}$$

and, for $j \in A$, $v > 0$

$$\lambda^A_{j|\emptyset}(v) := \lim_{\Delta \to 0^+} \frac{\mathbb{P}\big(V_j \le v + \Delta | V_{1:A} > v\big)}{\Delta}, \quad a.e. \tag{28}$$

In view of the characterization of minimal stability, we are interested in the distributions of minima over different subsets of $[r]$. It is therefore relevant to highlight that, in particular, the functions $\lambda_{j|\emptyset}(v)$, for $j \in [n]$, are strictly related to the marginal law of the minimal order statistic $V_{1:n} \equiv V_{1:[n]} = \min_{j=1,\dots,n} V_j$. In this respect the following identity holds:

$$\mathbb{P}(V_{1:n} > v) = \exp\Big\{ - \int_0^v \sum_{j=1}^n \lambda_{j|\emptyset}(s) ds \Big\} = \exp\Big\{ - \int_0^v \Lambda_\emptyset(s) ds \Big\}. \tag{29}$$

(See, e.g., De Santis et al. [7], where a more detailed description of the probabilistic behavior of $V_{i:n}$ in terms of $\lambda_{j|\emptyset}(v)$, for $j \in [n]$, is pointed out). Similarly, we can also consider the survival function of $V_{1:A}$, the minimal order statistic among the variables V_j, with $j \in A$, for $A \subset [r]$. With the notation introduced so far, one can write

$$\mathbb{P}(V_{1:A} > v) = \exp\Big\{ - \int_0^v \sum_{j \in A} \lambda^A_{j|\emptyset}(s) ds \Big\} = \exp\left\{ - \int_0^v \Lambda^A_\emptyset(s)\, ds \right\}, \tag{30}$$

where

$$\Lambda^A_\emptyset(t) := \sum_{j \in A} \lambda^A_{j|\emptyset}(t). \tag{31}$$

Notice that therefore the functions $\Lambda_\emptyset(t)$ and $\Lambda^A_\emptyset(t)$ can be respectively interpreted as the "usual" hazard rate functions for the random variables $V_{1:n}$ and $V_{1:A}$. This observation will be a key point for the discussion in Section 4.

3.2 The m.c.h.r. functions and characterizations of Exchangeability and of Minimal Stability

Here we come back to our lifetimes $T_1, \dots, T_r$. We maintain the condition **(H1)**, whereas the conditions **(H2)** and **(H3)** are replaced by the following stronger condition:

(H5) The joint distribution of $T_1, \dots, T_r$ is absolutely continuous, with the joint density such that

$$f_{\mathbf{T}}(t_1, .., t_r) > 0, \quad a.e. \text{ in } \mathbb{R}^r_+$$

We start with the exchangeable case, noticing that such a case leads to a remarkable simplification of notation, technical results, and conceptual aspects concerning the m.c.h.r. functions.

First notice that the symmetry conditions among the different random variables, as requested by exchangeability, imply a specially simple form for the m.c.h.r. functions. More precisely, the functions $\lambda_{j|i_1,\dots,i_k}(t|t_{i_1}, \dots, t_{i_k})$ cannot depend on the index $j \notin \{i_1, \dots, i_k\}$. Furthermore all the k-permutations $(i_1, \dots, i_k)$ are to be considered as similar one another and thus the dependence of a m.c.h.r. function w.r.t. to $(i_1, \dots, i_k)$ is encoded in the number k. For the present exchangeable case, for any $t > 0$, $0 < t_1 < \cdots < t_k \le t$, we then introduce the symbols $\mu(t|k; t_1, \dots, t_k)$ and $\mu(t|0)$ with the following meaning: for any $\mathbf{j} = (j_1, \dots, j_k) \in \Pi(I)$

$$\lambda_{j|j_1,\dots,j_k}(t|t_1, \dots, t_k) = \mu(t|k; t_1, \dots, t_k), \qquad \lambda_{j|\emptyset}(t) = \mu(t|0). \tag{32}$$

Thus, for any $(t_1, \dots, t_r) \in \mathbb{R}_+^r$, and denoting by $t_{1:r}, \dots, t_{r:r}$ the values $t_1, \dots, t_r$ rearranged in an increasing order, by Eq. (24), $f_{\mathbf{T}}$ takes the form

$$f_{\mathbf{T}}(t_1, \dots, t_r) = \Pi_{k=0}^{r-1} \mu(t_{k+1:r} | k; t_{1:r}, \dots, t_{k:r}) e^{-(r-k) \int_{t_{k:r}}^{t_{k+1:r}} \mu(s|k;t_{1:r},\dots,t_{k:r})\, ds}, \tag{33}$$

where we used the further convention that $t_{0:k} = 0$.

On the other hand, when the form (32) is assumed for the family of the m.c.h.r. functions, the consequent formula (33) shows that $f_{\mathbf{T}}$ actually depends on the arguments $t_1, \dots, t_r$ only through the ordered values $t_{1:r}, \dots, t_{r:r}$ and thus it is necessarily exchangeable. In conclusion, the following characterization of exchangeability holds (see also Spizzichino [29], chap.2).

Proposition 15. *Non-negative random variables $T_1, \dots, T_r$ with a strictly positive joint density are exchangeable if and only if the corresponding m.c.h.r. functions are of the form* (32).

As far as minimal stability is concerned, one can find natural conditions involving the hazard rates functions $\Lambda_\emptyset^A(t)$ of the minima $T_{1:A}$, for $A \subset [r]$. See in particular Lemma 17 in the next Section 4, where this topic is dealt with in some details. Here we point out that the functions $\Lambda_\emptyset^A(t)$ can be recovered once the m.c.h.r. functions $\lambda_{j|\emptyset}^A(t)$, associated to the random variables $\{T_j,\ j \in A\}$, are known (see (31)). However in general, when the distribution of $T_1, \dots, T_r$ is specified in terms of the associated m.c.h.r. functions $\lambda_{j|j_1,\dots,j_k}(t|t_1, \dots, t_k)$, it is not easy to recover the m.c.h.r. functions $\lambda_{j|\emptyset}^A(t)$ associated to $\{T_j,\ j \in A\}$. Therefore it is useful to find conditions for minimal stability expressed directly in terms of the m.c.h.r. functions $\lambda_{j|j_1,\dots,j_k}(t|t_1, \dots, t_k)$. On this purpose, taking into account the equivalence between condition (7) and minimal stability (i.e., between conditions ***(ii)*** and ***(iii)*** of Proposition 5), it is relevant to express

$$\mathbb{P}\big(T_j > t, \forall j \in A,\ T_i \le t, \forall i \in [r] \setminus A\big)$$

in terms of the m.c.h.r. functions. To this end for any d-permutation $\mathbf{j} = (j_1, \dots, j_d)$, we set

$$\begin{aligned} \Psi(t; [r], \mathbf{j}) :=& \mathbb{P}\big(T_{j_1} < T_{j_2} < \cdots < T_{j_d} \le t,\ T_i > t \,\forall i \notin \{j_1, \dots, j_d\}\big) \\ =& \int_0^t ds_d \int_0^{s_d} ds_{d-1} \cdots \int_0^{s_2} ds_1\, e^{-\int_{s_d}^t \Lambda_{j_1,\dots,j_d}(\tau|s_1,\dots,s_d) d\tau} \cdot \\ & \cdot \prod_{\ell=1}^d \lambda_{j_\ell|j_1,\dots,j_{\ell-1}}(s_\ell|s_1, \dots, s_{\ell-1}) e^{-\int_{s_{\ell-1}}^{s_\ell} \Lambda_{j_1,\dots,j_{\ell-1}}(\tau|s_1,\dots,s_{\ell-1}) d\tau}. \end{aligned} \tag{34}$$

Then, for any subset $A \subset [r]$, we can write

$$\mathbb{P}\big(T_j > t, \forall j \in A,\ T_i \le t, \forall i \in [r] \setminus A\big) = \sum_{\mathbf{j} \in \Pi([r] \setminus A)} \Psi(t; [r], \mathbf{j}). \tag{35}$$

The above Eq. (35), together with Eq. (34), and Proposition 5 can be used to get the following characterization of minimal stability.

Proposition 16. *Non-negative random variables $T_1, \dots, T_r$ with a strictly positive joint density are minimally stable if and only if the corresponding m.c.h.r. satisfy the condition that whenever $A, B \subset [r]$, with $|A| = |B| \le r-1$, then*

$$\sum_{\mathbf{j} \in \Pi(A)} \Psi(t; [r], \mathbf{j}) = \sum_{\mathbf{j}' \in \Pi(B)} \Psi(t; [r], \mathbf{j}'), \quad t > 0, \tag{36}$$

where we have used the notation (34).

Observe that the exchangeability condition implies the identity $\Psi(t; [r], \mathbf{j}) = \Psi(t; [r], \mathbf{j}')$ for any pair of d-permutations $\mathbf{j}, \mathbf{j}'$, so that condition (36) is trivially satisfied.

The characterization of Proposition 16, besides its conceptual meaning, reveals to be effective in some special cases. In particular we will use it when dealing with a subclass of Load Sharing models (see Subsection 5.2).

4 Relations among diagonal sections of DD copulas, distributions of order statistics, and hazard rates of minima

Concerning with the joint distribution of minimally stable lifetimes $T_1, \dots, T_r$, it has been pointed out in Proposition 10 that the two systems of functions $\mathcal{A} = \{\overline{G}; \delta_2, \dots, \delta_r\}$ and $\mathcal{B} = \{\overline{G}_{1:r}, \dots, \overline{G}_{r:r}\}$ convey the same information about the joint distribution of $T_1,\dots,T_r$ and that they can be then recovered one from the other. Like in Subsection 3.2 we assume **(H1)** and **(H5)**, where the joint distribution of $T_1,\dots,T_r$ can be described in terms of the corresponding m.c.h.r. functions. In terms of those functions, we aim to single out characteristics of the joint distribution, whose knowledge may be equivalent, under the condition **(H4)**, to that of the systems of functions $\mathcal{A}$ and $\mathcal{B}$ defined in (8). It will emerge that the information contained in the systems $\mathcal{A}$ and $\mathcal{B}$ is equivalent to the knowledge embedded in the systems of functions defined by

$$\mathcal{C} =: \{\Lambda_\emptyset^{[1]}, \dots, \Lambda_\emptyset^{[r]}\}. \tag{37}$$

Such equivalence is demonstrated by the relations tying $\mathcal{C}$ with $\mathcal{A}$ and $\mathcal{B}$. Such relations will be detailed below by means of the following Propositions 18 and 19. More precisely, in Proposition 18 we express each of the families $\mathcal{A}$ and $\mathcal{B}$ in terms of $\mathcal{C}$, whereas in Proposition 19 the family $\mathcal{C}$ is expressed in terms of $\mathcal{A}$ and in terms of $\mathcal{B}$.

We start by giving a characterization of minimal stability in terms of the hazard rates of minima. As already observed (see Eq.s (28) and (30)) the m.c.h.r. functions $\lambda_{j|\emptyset}^A(t)$, $j \in A$, are related to the law of the minimum on an arbitrary set $A \subset [r]$, indeed

$$\mathbb{P}(T_{1:A} > t) = \exp\left\{ -\int_0^t \Lambda_\emptyset^A(s)\, ds \right\}, \quad t > 0, \tag{38}$$

where $\Lambda_\emptyset^A(t) = \sum_{j\in A} \lambda_{j|\emptyset}^A(t)$ is the one-dimensional failure rate of $T_{1:A}$. Concerning with this notation, observe that the failure rate $\Lambda_\emptyset^{[r]}(t)$ coincides with $\Lambda_\emptyset(t) = \sum_{j=1}^r \lambda_{j|\emptyset}(t)$, so that $\overline{G}_{1:r}(t) = \mathbb{P}(T_{1:r} > t) = \exp\left\{ -\int_0^t \Lambda_\emptyset(s)\, ds\right\}$, (see also Eq. (29)). Equation (38) leads immediately to the following simple characterization of minimal stability.

Lemma 17. *Assume condition* **(H5)** *and assume that the hazard rates $\Lambda_\emptyset^A(t)$ of the minima $T_{1:A}$ are known, for every non-empty subset $A \subset [r]$. Then each of the following conditions is necessary and sufficient for the minimal stability condition* **(H4)***:*

$$\forall A \subset [r], \quad \Lambda_\emptyset^A(t) = \Lambda_\emptyset^{[d]}(t) \quad a.e.,\ t > 0 \quad \text{where } d = |A|, \tag{39}$$

and

$$\forall A \subset [r], \quad \mathbb{P}(T_{1:A} > t) = \exp\left\{ -\int_0^t \Lambda_\emptyset^{[d]}(s)\, ds \right\}, \quad t > 0, \quad \text{where } d = |A|. \tag{40}$$

Proof. Conditions (39) and (40) are clearly equivalent to each other, and if (40) holds then **(H4)** is immediate. Viceversa, by the assumption of minimal stability we may restrict attention on the subset of variables $T_1, \dots, T_d$, which are minimally stable, as well. Indeed, since $\mathbb{P}(T_{1:A} > t) = \mathbb{P}(T_{1:B} > t)$ for any $A, B \subseteq [r]$ such that $|A| = |B|$, then (40) follows, and therefore also (39) holds. □

Clearly when the m.c.h.r. functions $\lambda_{j|\emptyset}^A(t)$ are known, for every non-empty subset $A \subset [r]$, then (39) is equivalent to

$$\forall A \subset [r], \quad \sum_{j=1}^d \lambda_{j|\emptyset}^A(t) = \sum_{j=1}^d \lambda_{j|\emptyset}^{[d]}(t), \quad a.e.,\ t > 0, \quad \text{where } d = |A|, \tag{41}$$

Proposition 18. *Assume the minimal stability condition* **(H4)***, and the joint absolute continuity condition* **(H5)***. Assume furthermore that the family $\mathcal{C}$ of the hazard rate functions $\Lambda_\emptyset^{[d]}(t)$ is known. Then*

(i) *the family $\mathcal{A}$ is given by:*

$$\overline{G}(t) = \exp\left\{ -\int_0^t \Lambda_\emptyset^{[1]}(s)\,ds \right\}, \quad \textit{for any } t > 0, \tag{42}$$

and for $d = 2, \dots, r$ and for any $u \in [0, 1]$,

$$\delta_d(u) = \exp\left\{ -\int_0^{\overline{G}^{-1}(u)} \Lambda_\emptyset^{[d]}(s)\,ds \right\}; \tag{43}$$

(ii) *the family $\mathcal{B}$ is given by:*
for $\ell = 1, \dots, r$ and for any $t > 0$,

$$\overline{G}_{\ell:r}(t) = \sum_{h=r-\ell+1}^{r} (-1)^{h-r-1+\ell} \binom{r}{h} \binom{h-1}{r-\ell} \exp\left\{ -\int_0^t \Lambda_\emptyset^{[h]}(s)\,ds \right\}. \tag{44}$$

*Proof of **(i)**.* Due to minimal stability the random variables T_i share the marginal survival function with T_1, i.e.,

$$\exp\left\{ -\int_0^t \lambda_{i|\emptyset}(s)\,ds \right\} = \exp\left\{ -\int_0^t \lambda_{1|\emptyset}(s)\,ds \right\}, \quad t > 0.$$

Therefore $\lambda_{i|\emptyset}(t) = \lambda_{1|\emptyset}(t) \equiv \Lambda_\emptyset^{[1]}(t)$, for any $t > 0$, and (42) follows.

As already observed, we may concentrate attention on the random variables $T_1, \dots, T_d$. Therefore to prove (43), on the one hand one has

$$\mathbb{P}(\min_{i=1,\dots,d} T_i > t) = \exp\left\{ -\int_0^t \Lambda_\emptyset^{[d]}(s)\,ds \right\}.$$

On the other hand, taking into account (3) we can also write

$$\mathbb{P}(\min_{i=1,\dots,d} T_i > t) = \delta_d(\overline{G}(t)).$$

Thus Eq. (43) is immediately achieved by comparing the preceding two formulas and recalling that condition **(H5)** implies condition **(H2)**, which in its turn implies that $\overline{G}$ is invertible.
*Proof of **(ii)**.* Taking into account Proposition 7, Eq. (44) is immediately achieved by combining Eq. (43) with Eq. (4). □

Proposition 19. *Assume the minimal stability condition* **(H4)**, *and the joint absolute continuity condition* **(H5)**.
(i) *If the family $\mathcal{A} = \{\overline{G}, \delta_2, \dots, \delta_r\}$ is known, then, setting $\delta_1(u) = u$,*

$$\Lambda_\emptyset^{[\ell]}(t) = -\frac{d}{dt} \log\left[\delta_\ell(\overline{G}(t))\right], \quad a.e.\ t > 0, \quad \ell = 1, 2, \dots, r. \tag{45}$$

(ii) *If the family $\mathcal{B} = \{\overline{G}_{1:r}, \dots, \overline{G}_{r:r}\}$ is known, then, respectively denoting by $g_{1:r}, \dots, g_{r:r}$ the probability density of the order statistics $T_{1:r}, \dots, T_{r:r}$,*

$$\Lambda_\emptyset^{[\ell]}(t) = \frac{\sum_{h=\ell}^{r} (h)_\ell \left(g_{r-h+1:r}(t) - g_{r-h:r}(t) \right)}{\sum_{h=\ell}^{r} (h)_\ell \left(\overline{G}_{r-h+1:r}(t) - \overline{G}_{r-h:r}(t) \right)}$$
$$= \frac{\sum_{k=1}^{r-(\ell-1)} (r-k)_{\ell-1}\, g_{k:r}(t)}{\sum_{k=1}^{r-(\ell-1)} (r-k)_{\ell-1}\, \overline{G}_{k:r}(t)}, \qquad a.e.\ t > 0, \quad \ell = 1, 2, \dots, r. \tag{46}$$

*Proof of **(i)**.* For $\ell = 1$ Eq. (45) follows immediately by Eq. (42). For $\ell = 2, \dots, r$, Eq. (45) is immediately achieved by inverting the Eq. (43).

*Proof of **(ii)***. Eq. (46) is obtained by resorting to Eq. (45) and Proposition 8, together with the circumstance that $\mathbb{P}\big(T_{1:A} > t\big) = \delta_d\big(\overline{G}(t)\big)$. □

These results can be further specialized to the case of exchangeable times $T_1, \dots, T_r$. For any $d = 1, \dots, r$, the exchangeable random lifetimes $T_1, \dots, T_d$, are characterized by the m.c.h.r. functions $\mu^{[d]}(t|k; t_1, .., t_k)$ for $k = 0, \dots, d-1$, $0 < t_1 < \cdots < t_k \leq t$, and therefore (see (32) and Proposition 15 for $r = d$)

$$\Lambda_{\emptyset}^{[d]}(t) = d\mu^{[d]}(t|0). \tag{47}$$

In view of this remark, the set of functions $\mathcal{C}$ is equivalent to the set of functions

$$\mathcal{C}' := \{\mu^{[1]}(t|0), \dots, \mu^{[r]}(t|0)\}, \tag{48}$$

which is therefore equivalent also to the systems of functions $\mathcal{A}$ and $\mathcal{B}$.

More precisely, when $T_1, \dots, T_r$ are exchangeable and satisfy **(H5)**, then, with the above notation, Eq.s (42), (43) and (44) can be rewritten as

$$\overline{G}(t) = \exp\left\{-\int_0^t \mu^{[1]}(s|0)\, ds\right\}, \quad t > 0, \tag{49}$$

$$\delta_d(u) = \exp\left\{-d \int_0^{\overline{G}^{-1}(u)} \mu^{[d]}(s|0)\, ds\right\}, \quad u \in [0,1],\ d = 2, \dots, r, \tag{50}$$

and

$$\overline{G}_{\ell:r}(t) = \sum_{h=r-\ell+1}^{r} (-1)^{h-r-1+\ell} \binom{r}{h} \binom{h-1}{r-\ell} \exp\left\{-h \int_0^t \mu^{[h]}(s|0)\, ds\right\}. \tag{51}$$

Similarly Eq.s (45) and (46) take the form

$$\mu^{[\ell]}(t|0) = -\frac{1}{\ell}\frac{d}{dt}\log\left[\delta_\ell(\overline{G}(t))\right], \quad a.e.\ t > 0, \quad \ell = 1, 2, \dots, r, \tag{52}$$

$$\begin{aligned}\mu^{[\ell]}(t|0) &= \frac{1}{\ell}\,\frac{\sum_{h=\ell}^{r}(h)_\ell\left(g_{r-h+1:r}(t) - g_{r-h:r}(t)\right)}{\sum_{h=\ell}^{r}(h)_\ell\left(\overline{G}_{r-h+1:r}(t) - \overline{G}_{r-h:r}(t)\right)} \\ &= \frac{1}{\ell}\,\frac{\sum_{k=1}^{r-(\ell-1)}(r-k)_{\ell-1}\, g_{k:r}(t)}{\sum_{k=1}^{r-(\ell-1)}(r-k)_{\ell-1}\, \overline{G}_{k:r}(t)}, \quad a.e.\ t > 0, \quad \ell = 1, \dots, r. \end{aligned}\tag{53}$$

5 Special cases

The arguments developed in the previous sections will now be illustrated by considering in Subsections 5.1 and 5.2 the two remarkable classes of models respectively defined by Archimedean copulas and by multivariate conditional hazard rate functions satisfying the load-sharing condition. These choices in a sense correspond to the simplest possible forms admitted in the two types of descriptions of a joint distribution for lifetimes, respectively.

In particular, the analysis of these classes will allow us to obtain some examples of application for some of the results derived so far, by showing the special form taken by related formulas. Subsection 5.1 is tailored to illustrate basic aspects of the exchangeable case. On the other hand the arguments in Subsection 5.2 permit to pave the way for a better understanding of the differences between exchangeability and minimal stability, and to present some heuristic ideas at the basis of the construction of minimally stable, but non-exchangeable, multivariate models.

5.1 Archimedean Copulas

Let us consider the case when the survival copula K of $T_1, \dots, T_r$ is Archimedean with generator ψ. For simplicity's sake we assume that K is a strict Archimedean copula, i.e., the generator ψ is a strictly decreasing, continuous and convex function, such that $\psi(1) = 0$ and $\psi(0^+) = \infty$, so that $K = C_\psi$, with

$$C_\psi(u_1, \dots, u_r) := \psi^{-1}\big(\psi(u_1) + \cdots + \psi(u_r)\big).$$

It is important to recall that the function C_ψ is an r-dimensional copula if and only if the inverse function ψ^{-1} is r-monotonic (see Theorem 6.3.6 in Schweizer and Sklar [25], Nelsen [20], see also Mc Neil and Nešlehová [16]). By Definition 1 it is immediately seen that the diagonal sections associated to $K = C_\psi$ assume the form

$$\delta_\ell(u) = \psi^{-1}(\ell\psi(u)), \quad 2 \le \ell \le r.$$

Furthermore, the survival copula K being symmetric, the model $T_1, \dots, T_r$ is exchangeable when the lifetimes share the same common marginal survival function $\overline{G}$, i.e., under condition **(H1)**. Therefore, once ψ and $\overline{G}$ are given, then the family $\mathcal{A}$ in (8) coincides with the family $\{\overline{G}(t), \psi^{-1}(2\psi(u)), \dots, \psi^{-1}(r\psi(u))\}$. From Eq. (4) (see also Proposition 10) we know how the family $\mathcal{B}$ is generally obtained from $\mathcal{A}$. In the present case, one can more precisely write

$$\overline{G}_{\ell:r(t)} = \sum_{h=r-\ell+1}^{r} (-1)^{h-r-1+\ell} \binom{r}{h} \binom{h-1}{r-\ell} \psi^{-1}(h\psi(\overline{G}(t))), \quad 1 \le \ell \le r.$$

Moreover, under the further regularity condition **(H5)**, by Eq. (52) one can get the m.c.h.r. functions

$$\begin{aligned} \mu^{[\ell]}(t|0) &= -\frac{1}{\ell}\frac{d}{dt}\log\left[\psi^{-1}\big(\ell\psi(\overline{G}(t))\big)\right], \quad a.e. \\ &= \frac{1}{\psi'\Big(\psi^{-1}\big(\ell\psi(\overline{G}(t))\big)\Big)}\,\psi'\big(\overline{G}(t)\big)\,g(t), \quad a.e. \end{aligned} \tag{54}$$

Conversely, when the family $\mathcal{B}$ of the survival distribution functions $\overline{G}_{k:r}$ is given, then we can recover the family $\mathcal{A}$. In this respect we stress that, even if the explicit expression of the generator ψ is not known, still by Corollary 9 (see also Proposition 10), from $\mathcal{B}$ we immediately get the diagonal sections $\delta_d(u)$, $d = 2, \dots, r$. Then the following question naturally arises:

Does the knowledge of $\mathcal{B}$ allow us to identify the generator ψ?

In other words we wonder whether the identity $\delta_d(u) = \psi^{-1}(d\psi(u))$ for any $d = 2, \dots, r$ is sufficient to identify ψ. We briefly discuss about this problem in Remark 20 below. To this end it is useful to write down explicitly the case $d = r$:

$$\psi^{-1}(r\psi(u)) = \delta_r(u) = \overline{G}_{1:r}\big(\overline{G}^{-1}(u)\big),$$

and the case $d = r - 1$:

$$\psi^{-1}((r-1)\psi(u)) = \delta_{r-1}(u) = \frac{1}{r}\overline{G}_{2:r}\big(\overline{G}^{-1}(u)\big) + \left(1 - \frac{1}{r}\right)\delta_r(u).$$

Remark 20. *It is interesting to point out (see Jaworski [12]) that when $r > 2$ the generator ψ is uniquely determined (up to a multiplicative constant) by the pair δ_r and δ_{r-1}, though the proof of the latter claim is not* *constructive. On the other hand in general one cannot uniquely determine ψ only from δ_r, when $r > 2$, since there may exist infinite generators with the same diagonal section δ_r (see again [12]). However, when the diagonal δ_r satisfy the condition $\delta_r'(1^-) = r$, then Erderly et al. [10] show that ψ is uniquely determined (up to a multiplicative constant) by δ_r and, furthermore, ψ can be approximated thanks to the following formula:*

$$\psi(u) \propto \lim_{m\to\infty} r^m \left(1 - \delta_r^{-m}(u)\right),$$

where δ_r^{-m} is the composition of δ_r^{-1} with itself m times.

A further particular case within Archimedean models is the Schur-constant case, i.e., when

$$\mathbb{P}(T_1 > t_1, \ldots, T_r > t_r) = \overline{G}(t_1 + \cdots + t_r), \tag{55}$$

corresponding to the choice $\overline{G} = \psi^{-1}$ (see, e.g., [29]). In this case both the diagonal sections δ_h and the survival functions $\overline{G}_{\ell:r}$ are determined by the marginal survival function $\overline{G}(t)$:

$$\delta_h(\overline{G}(t)) = \psi^{-1}(h\psi(\overline{G}(t))) = \overline{G}(ht) \tag{56}$$

$$\overline{G}_{\ell:r}(t) = \sum_{h=r-\ell+1}^{r} (-1)^{h-r-1+\ell} \binom{r}{h} \binom{h-1}{r-\ell} \overline{G}(ht), \quad 1 \le \ell \le r.$$

Furthermore, in view of the particularly simple form (56) of the diagonal sections, Eq. (52) can be rewritten as

$$\mu^{[\ell]}(t|0) \;=\; -\frac{1}{\ell}\frac{d}{dt}\log\left[\overline{G}(\ell t)\right] = \frac{g(\ell t)}{\overline{G}(\ell t)}, \quad a.e. \tag{57}$$

Note that for Schur-constant models one could get the above results also directly, taking into account that (55) implies

$$\mathbb{P}(T_1 > t, \ldots, T_\ell > t) = \mathbb{P}(T_1 > t, \ldots, T_\ell > t, T_{\ell+1} > 0, \ldots, T_r > 0) = \overline{G}(\ell\, t).$$

Example 21. *In order to illustrate how to compute the m.c.h.r. functions for Archimedean models, in this example we consider two special cases within the class of Archimedean models sharing the same generator*

$$\psi(u) = (u^{-\alpha} - 1)^{\frac{1}{\beta}}, \quad \alpha > 0,\ \beta \ge 1.$$

Note that the inverse function $\psi^{-1}(t) = \frac{1}{(t^\beta+1)^\alpha}$ is completely monotonic, so that C_ψ is a copula for any $r \ge 2$. The first case is the Archimedean model with $\overline{G}(t) = e^{-t}$. Then, for any $A \subset [r]$ with $|A| = \ell$, one has

$$\begin{aligned}\mathbb{P}(T_{1:A} > t) &= \delta_\ell(\overline{G}(t)) = \psi^{-1}(\ell\psi(\overline{G}(t)))\\ &= \frac{1}{\left(\left(\ell(e^{t\alpha}-1)^{\frac{1}{\beta}}\right)^\beta + 1\right)^\alpha} = \frac{1}{\left(\ell^\beta e^{t\alpha} - \ell^\beta + 1\right)^\alpha}.\end{aligned}$$

Therefore by (54) *and taking into account that*

$$\psi'(u) = -\frac{\alpha}{\beta}\, u^{-(\alpha+1)}\, \left(u^{-\alpha} - 1\right)^{\frac{1}{\beta}-1},$$

we get

$$\mu^{[\ell]}(t|0) = \frac{e^{\alpha t}\left(e^{\alpha t} - 1\right)^{\frac{1}{\beta}-1}}{\left(\ell^\beta e^{t\alpha} - \ell^\beta + 1\right)^{\alpha^2+\alpha}\left(\left(\ell^\beta e^{t\alpha} - \ell^\beta + 1\right)^{\alpha^2} - 1\right)^{\frac{1}{\beta}-1}}.$$

As a second case, we consider the Schur-constant model with the same generator ψ, which corresponds to the choice

$$\overline{G}(t) = \psi^{-1}(t) = \frac{1}{(t^\beta + 1)^\alpha}, \quad g(t) = \alpha\,\beta\frac{t^{\beta-1}}{(t^\beta+1)^{\alpha+1}}.$$

Then by Eq.s (56) *and* (57) *we get*

$$\mathbb{P}(T_{1:A} > t) = \frac{1}{\left((\ell t)^\beta + 1\right)^\alpha}, \quad \mu^{[\ell]}(t|0) = \alpha\,\beta\frac{(\ell t)^{\beta-1}}{(\ell t)^\beta + 1}.$$

5.2 Time homogeneous load-sharing models

Load sharing models are characterized by the condition that the m.c.h.r. functions depend on current time and on the set of failed components at the current time, but do not depend on the failure times. Load sharing models are well known and recurrently studied in the reliability literature (see, e.g., Spizzichino [30], Rychlik and Spizzichino [23], and the references cited therein). In particular the joint and marginal distributions of the order statistics have been studied in some details. For what concerns the special case of exchangeability see also Kamps [14].
In the literature it is also assumed that the m.c.h.r. functions do not depend on the order of failures, however it is interesting here to extend such a definition to a generalized class of models in which instead also the order of failure times may influence the m.c.h.r. functions. Actually some of the existing results on load sharing models can be easily extended to this class.

Definition 22. *The joint distribution of the random variables $T_1, \ldots, T_r$ is an* **Order Dependent Load Sharing model (ODLS)** *if it is absolutely continuous, and the m.c.h.r. functions do not depend on the failure times, i.e., for any $k = 0, 1, \ldots, r-1$, there exist $\binom{r}{k}k!(r-k)$ functions $v \mapsto \lambda_{j|j_1,\ldots,j_k}(v)$ such that, for any $0 < v_1 < \cdots < v_k < v$, and $(j_1, \ldots, j_k) \in \Pi_k([r])$*

$$\lambda_{j|j_1,\ldots,j_k}(v|v_1, \cdots, v_k) = \lambda_{j|j_1,\ldots,j_k}(v).$$

The model is said simply **Load Sharing (LS) model** *when the m.c.h.r. functions depend neither on the failure times nor on the order of failures, i.e., for any $k = 0, 1, \ldots, r\text{-}1$, there exist $\binom{r}{k}$ $(r-k)$ functions $v \mapsto \lambda_{j|\{j_1,\ldots,j_k\}}(v)$ such that, for any $0 < v_1 < \cdots < v_k < v$, and $(j_1, \ldots, j_k) \in \Pi_k([r])$*

$$\lambda_{j|j_1,\ldots,j_k}(v|v_1, \cdots, v_k) = \lambda_{j|\{j_1,\ldots,j_k\}}(v).$$

When furthermore the functions $v \mapsto \lambda_{j|j_1,\ldots,j_k}(v) = \lambda_{j|j_1,\ldots,j_k}$ (respectively, the functions $v \mapsto \lambda_{j|\{j_1,\ldots,j_k\}}(v)$) are constant w.r.t. time v, then the model $T_1, \ldots, T_r$ is said an **Order Dependent Time Homogeneous Load Sharing model** *(ODTHLS) (respectively, a* **Time Homogeneous Load Sharing model** *(THLS)).*

Clearly a Load Sharing model is also an Order Dependent Load Sharing model. To distinguish between the two cases, we will sometimes say that a model is a ***strictly Order Dependent Load Sharing model*** when the m.c.h.r. functions do depend on the order. From an engineering-oriented viewpoint, strictly ODLS models do not seem very significant for applications in the field of reliability. However models with this property may emerge in different fields, as shown in De Santis, Spizzichino [8] in the analysis of aggregation paradoxes. See also Example 29 below, where the condition of load sharing must be limited to strictly ODTHLS models on the purpose of finding among uniform frailty models (see (76)) those which are minimally stable, without falling in the exchangeable case.

Concerning the analysis of the minimal stability, it is important to stress that, in general, d-marginal models of load sharing models are not load sharing. This fact entails that the m.c.h.r. functions $\lambda^A_{j|\emptyset}$ are in general not easy to compute, even in the exchangeable case. In the latter case however we will be able to compute such functions explicitly by using the results of Section 4. On the contrary in the non-exchangeable case, we will give minimal stability conditions in terms of the m.c.h.r. functions $\lambda_{j|j_1,\ldots,j_{k-1}}$.

Exchangeable THLS models. An exchangeable load sharing model clearly cannot be strictly order dependent, in that its m.c.h.r. functions are such that $\mu(t|k; t_1, \ldots, t_k) = \mu(t|k)$. Furthermore it is time homogeneous if and only if for any $k = 0, 1, .., r-1$ there exists a constant $L(r-k)$ such that $\Lambda_{j_1,\ldots,j_k}(t) = L(r-k)$ and

$$\mu(k) = \frac{L(r-k)}{r-k}. \tag{58}$$

In such a case it is easily seen that

$$\overline{G}_{k:r}(t) = \mathbb{P}\big(\tfrac{X_0}{L(r)} + \tfrac{X_1}{L(r-1)} + \cdots + \tfrac{X_{k-1}}{L(r-(k-1))} > t\big)$$

where $X_i \sim EXP(1)$, $i = 0, 1, 2\ldots, r-1$, are independent random variables (see in particular Spizzichino [30], Kamps [14], Cramer and Kamps [4], and references therein). In other words, for any $k = 1, .., r$, the distribution of $T_{k:r}$ coincides with the distribution of the sum of k independent exponential distributions of parameters $\gamma_1 = L(r), \dots, \gamma_k = L(r-(k-1))$. In the literature such a distribution is known as Generalized Erlang or Hypoexponential distribution: for a fixed vector $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_r) \in \mathbb{R}_+^r$,

$$\overline{G}_k^{\boldsymbol{\gamma}}(t) := \mathbb{P}\Big(\sum_{j=1}^{k} \tfrac{Y_j}{\gamma_j} > t\Big) \tag{59}$$

for $Y_1, \dots, Y_r$, independent and standard exponential random variables.

When $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_r)$ is such that $\gamma_i \neq \gamma_j$ for all $i \neq j$, the above distribution is referred to as Hyperexponential, and furthermore (see, e.g., Cramer and Kamps [4] and references therein) the survival function and the probability density are respectively given by

$$\overline{G}_k^{\boldsymbol{\gamma}}(t) = \sum_{j=1}^{k} \left(\prod_{h\in[k]\setminus\{j\}} \frac{\gamma_h}{\gamma_h - \gamma_j} \right) e^{-\gamma_j t},$$

and

$$g_k^{\boldsymbol{\gamma}}(t) = \sum_{j=1}^{k} \left(\prod_{h\in[k]\setminus\{j\}} \frac{\gamma_h}{\gamma_h - \gamma_j} \right) \gamma_j e^{-\gamma_j t}.$$

Therefore, denoting by $\boldsymbol{L}$ the vector

$$\boldsymbol{L} := \big(L(r), L(r-1), \dots, L(1)\big),$$

in the exchangeable case we can write

$$\overline{G}_{k:r}(t) = \overline{G}_k^{\boldsymbol{L}}(t). \tag{60}$$

Furthermore, on the one hand formula (9) takes the special form

$$\overline{G}(t) = \frac{1}{r}\sum_{k=1}^{r} \mathbb{P}\big(\tfrac{X_0}{L(r)} + \tfrac{X_1}{L(r-1)} + \dots + \tfrac{X_{k-1}}{L(r-(k-1))} > t\big) = \frac{1}{r}\sum_{k=1}^{r} \overline{G}_k^{\boldsymbol{L}}(t). \tag{61}$$

On the other hand, taking into account (11), for any $A \subset [r]$, with $|A| = d$, one has

$$\begin{aligned}\mathbb{P}\big(T_{1:A} > t\big) = \delta_d\big(\overline{G}(t)\big) &= \frac{d}{(r)_d}\sum_{k=1}^{r-d+1} (r-k)_{d-1}\, \mathbb{P}\big(\tfrac{X_0}{L(r)} + \tfrac{X_1}{L(r-1)} + \dots + \tfrac{X_{k-1}}{L(r-(k-1))} > t\big) \\ &= \frac{d}{(r)_d}\sum_{k=1}^{r-d+1} (r-k)_{d-1}\, \overline{G}_k^{\boldsymbol{L}}(t)\end{aligned} \tag{62}$$

and consequently, recalling the notation in (47), (53) becomes

$$\mu^{[d]}(t|0) = \frac{1}{d}\,\frac{\sum_{k=1}^{r-d+1}(r-k)_{d-1}\, g_k^{\boldsymbol{L}}(t)}{\sum_{k=1}^{r-d+1}(r-k)_{d-1}\, \overline{G}_k^{\boldsymbol{L}}(t)}. \tag{63}$$

In particular, assuming that $L(i) \neq L(j)$ for $i \neq j$, and setting

$$\vartheta_{\ell,k}^{\boldsymbol{L}} := \prod_{h\in\{0,\dots,k-1\}\setminus\{\ell\}} \frac{L(r-h)}{L(r-h) - L(r-\ell)},$$

one has

$$\begin{aligned}\overline{G}(t) &= \frac{1}{r}\sum_{k=1}^{r}\sum_{j=1}^{k}\left(\prod_{h\in[k]\setminus\{j\}} \frac{L(r-(h-1))}{L(r-(h-1)) - L(r-(j-1))}\right) e^{-L(r-(j-1))t} \\ &= \frac{1}{r}\sum_{\ell=0}^{r-1}\left(\sum_{k=\ell+1}^{r} \vartheta_{\ell,k}^{\boldsymbol{L}}\right) e^{-L(r-\ell)t}, \\ \mathbb{P}\big(T_{1:A} > t\big) &= \frac{d}{(r)_d}\sum_{\ell=0}^{r-d}\left(\sum_{k=\ell+1}^{r-d+1} (r-k)_{d-1}\vartheta_{\ell,k}^{\boldsymbol{L}}\right) e^{-L(r-\ell)t}\end{aligned}$$

and

$$\mu^{[d]}(t|0) = \frac{1}{d}\,\frac{\sum_{\ell=0}^{r-d}\left(\sum_{k=\ell+1}^{r-d+1}(r-k)_{d-1}\,\vartheta_{\ell,k}^{\boldsymbol{L}}\right)\,L(r-\ell)\,e^{-L(r-\ell)t}}{\sum_{\ell=0}^{r-d}\left(\sum_{k=\ell+1}^{r-d+1}(r-k)_{d-1}\,\vartheta_{\ell,k}^{\boldsymbol{L}}\right)\,e^{-L(r-\ell)t}}$$

Note that, when $d = r$, then we obviously get that the function $t \mapsto \mu^{[r]}(t|0)$ is constant and $\mu^{[r]}(t|0) = \frac{1}{r}\,L(r)$, whereas for $d < r$ the function $t \mapsto \mu^{[d]}(t|0)$ is not constant.
This fact is somehow related to the afore-mentioned circumstance that the d-dimensional marginal distributions of a load sharing model is generally not load sharing.

Before passing to the non-exchangeable case, we observe that in the present THLS exchangeable case the function $\Psi(t;[r],\mathbf{j})$ defined in (34) can be explicitly computed:

$$\begin{aligned}\Psi(t;[r],\mathbf{j}) &= \mathbb{P}\big(T_{j_1} < T_{j_2} < \cdots < T_{j_d} \le t < T_i,\ \forall i \notin \{j_1,\ldots,j_d\}\big)\\ &= \frac{1}{d!}\,\mathbb{P}\big(T_j \le t < T_i,\ \forall j \in \{j_1,\ldots,j_d\},\ \forall i \notin \{j_1,\ldots,j_d\}\big)\\ &= \frac{1}{d!}\,\frac{1}{\binom{r}{d}}\,\mathbb{P}\big(N(t)=d\big) = \frac{1}{(r)_d}\,\big[\overline{G}_{d+1}^{\boldsymbol{L}}(t) - \overline{G}_{d}^{\boldsymbol{L}}(t)\big], \end{aligned} \tag{64}$$

where we have used the notation introduced in (59). In view of Proposition 16, expression (64) turns out to be useful also in the analysis of minimal stability conditions. Indeed even for any ODTHLS model one has

$$\begin{aligned}\Psi(t;[r],\mathbf{j}) &= \mathbb{P}\big(T_{j_1} < T_{j_2} < \cdots < T_{j_d} \le t < T_i,\ \forall i \notin \{j_1,\ldots,j_d\}\big)\\ &= \prod_{\ell=1}^{d}\lambda_{j_\ell|j_1,\ldots,j_{\ell-1}}\cdot\int_0^t ds_d\int_0^{s_d} ds_{d-1}\cdots\int_0^{s_2} ds_1\\ &\qquad\qquad\left[e^{-(t-s_d)\Lambda_{j_1,\ldots,j_d}}\prod_{\ell=1}^{d}e^{-(s_\ell-s_{\ell-1})\Lambda_{j_1,\ldots,j_{\ell-1}}}\right] \end{aligned} \tag{65}$$

$$= \prod_{\ell=1}^{d}\frac{\lambda_{j_\ell|j_1,\ldots,j_{\ell-1}}}{\Lambda_{j_1,\ldots,j_{\ell-1}}}\,\big[\overline{G}_{d+1}^{\boldsymbol{\Lambda}}(t) - \overline{G}_{d}^{\boldsymbol{\Lambda}}(t)\big], \tag{66}$$

where $\boldsymbol{\Lambda} = \big(\Lambda_\emptyset, \Lambda_{j_1}, \ldots, \Lambda_{j_1,\ldots,j_k}, \ldots, \Lambda_{j_1,\ldots,j_{r-1}}\big)$.

In the following Example 23, for the case $r = 3$, we will analyze two different THLS models: an exchangeable THLS model and a minimally stable (non-exchangeable) THLS model which is trivially not strictly order dependent. On this purpose we will compute the survival functions of minima on sets of size $d = 1, 2, 3$ and the marginal survival functions of order statistics. We will focus both on the common features and on the differences between the two models.

Example 23. *We start by considering the non-exchangeable model: let $r = 3$ and let T_1, T_2, T_3 be lifetimes jointly distributed according to a THLS model with m.c.h.r. functions given as follows:*

$$\begin{aligned}&\lambda_{1|\emptyset}(t) = \lambda_{2|\emptyset}(t) = \lambda_{3|\emptyset}(t) = \frac{1}{3},\\ &\lambda_{3|1}(t) = \gamma,\ \lambda_{2|1}(t) = 1-\gamma,\quad \lambda_{1|2}(t) = \gamma,\ \lambda_{3|2} = 1-\gamma\quad \lambda_{2|3}(t) = \gamma,\ \lambda_{1|3}(t) = 1-\gamma,\end{aligned}$$

for a fixed value $\gamma \in \left(\frac{1}{2}, 1\right)$ and finally

$$\lambda_{1|2,3}(t) = \lambda_{1|3,2}(t) = \lambda_{2|1,3}(t) = \lambda_{2|3,1}(t) = \lambda_{3|1,2}(t) = \lambda_{3|2,1}(t) = 2.$$

For this model one has

$$(i)\quad \Lambda_\emptyset = 1,\quad \Lambda_{j_1} = 1,\quad \Lambda_{j_1,j_2} = 2,\qquad \textit{for any } j_1 \ne j_2 \in \{1,2,3\}.$$

Furthermore we consider lifetimes $\widetilde{T}_1, \widetilde{T}_2, \widetilde{T}_3$ jointly distributed according to the exchangeable THLS model defined by

$$(ii) \quad \widetilde{\Lambda}_\emptyset = L(3) = 1,\ \widetilde{\Lambda}_{j_1} = L(2) = 1,\ \widetilde{\Lambda}_{j_1,j_2} = L(1) = 2,\ \textit{for any } j_1 \neq j_2 \in \{1,2,3\},$$

or, equivalently (recall (58) *with $r = 3$), such that $\mu(0) = \frac{1}{3},\ \mu(1) = \frac{1}{2},\ \mu(2) = 2$.*

As we are going to see below, the two models share the same marginal survival functions of the order statistics. Before checking this property, we point out a main difference between the two models: for the exchangeable model one obviously has

$$\mathbb{P}\big(\widetilde{T}_{j_1} < \widetilde{T}_{j_2} < \widetilde{T}_{j_3}\big) = \frac{1}{3!}, \quad \textit{for any } (j_1, j_2, j_3) \in \Pi([3]),$$

whereas, for instance, one has

$$\frac{1-\gamma}{3} = \mathbb{P}(T_1 < T_2 < T_3) < \mathbb{P}(T_1 < T_3 < T_2) = \frac{\gamma}{3}.$$

The above inequality is implied by the observation that for THLS models one has (see, e.g., Spizzichino [30])

$$\mathbb{P}(T_{j_1} < T_{j_2} < T_{j_3}) = \frac{\lambda_{j_1|\emptyset}}{\Lambda_\emptyset} \frac{\lambda_{j_2|j_1}}{\Lambda_{j_1}} \frac{\lambda_{j_3|j_1,j_2}}{\Lambda_{j_1,j_2}}.$$

The above inequality shows also that T_1, T_2, T_3 is a non-exchangeable THLS model. However the random variables T_1, T_2, T_3 are minimally stable. Indeed by the previous values in (i), and by Eq. (65)*, one has:*
for any $j_1 \in \{1,2,3\}$,

$$\begin{aligned}\Psi\big(t; [3], j_1\big) &= \mathbb{P}\big(T_{j_1} \le t, T_j > t, j \neq j_1\big)\\ &= \int_0^t \lambda_{j_1|\emptyset} e^{-s\Lambda_\emptyset}\, e^{-(t-s)\Lambda_{j_1}}\, ds = \int_0^t \frac{1}{3} e^{-s}\, e^{-(t-s)}\, ds = \frac{1}{3}\, t\, e^{-t};\end{aligned} \tag{67}$$

for any $(j_1, j_2) \in \Pi_2([3])$,

$$\begin{aligned}\Psi\big(t; [3], (j_1, j_2)\big) &= \mathbb{P}\big(T_{j_1} \le T_{j_2} \le t, T_{j_3} > t\big)\\ &= \int_0^t ds \int_0^s ds'\, e^{-(t-s)\Lambda_{j_1,j_2}}\, \lambda_{j_1|\emptyset}\lambda_{j_2|j_1}\, e^{-s'\Lambda_\emptyset}\, e^{-(s-s')\Lambda_{j_1}}\, ds\\ &= \frac{1}{3}\lambda_{j_2|j_1}\, e^{-2t} \int_0^t s\, e^s\, ds = \frac{1}{3}\lambda_{j_2|j_1}\, \big(e^{-t} t - e^{-t} + e^{-2t}\big).\end{aligned}$$

Taking into account that, for any (j_1, j_2)

$$\lambda_{j_2|j_1} + \lambda_{j_1|j_2} = \gamma + (1-\gamma) = 1,$$

we may apply Proposition 16 to conclude that T_1, T_2, T_3 are minimally stable. Furthermore we get that, for any $(j_1, j_2, j_3) \in \Pi\big([3]\big)$,

$$\mathbb{P}\big(T_{j_1} \le t, T_{j_2} \le t, T_{j_3} > t\big) = \frac{1}{3}\,\big(e^{-t} t - e^{-t} + e^{-2t}\big). \tag{68}$$

From the previous computations, and in particular (67) *and* (68)*, we get explicitly the following survival functions*

$$\begin{aligned}\mathbb{P}(T_{1:\{1,2,3\}} > t) &= \mathbb{P}(T_1 > t, T_2 > t, T_3 > t) = e^{-t};\\ \mathbb{P}(T_{1:\{1,2\}} > t) &= \mathbb{P}(T_1 > t, T_2 > t) = \mathbb{P}(T_1 > t, T_2 > t, T_3 > t)\\ &\quad + \mathbb{P}(T_1 > t, T_2 > t, T_3 \le t) = e^{-t} + \frac{1}{3}\, t\, e^{-t} = e^{-t}\left(1 + \frac{t}{3}\right),\\ \overline{G}(t) = \mathbb{P}(T_1 > t) &= \mathbb{P}(T_1 > t, T_2 > t, T_3 > t) + \mathbb{P}(T_1 > t, T_2 > t, T_3 \le t)\\ &\quad + \mathbb{P}(T_1 > t, T_3 > t, T_2 \le t) + \mathbb{P}(T_1 > t, T_2 \le t, T_3 \le t)\\ &= e^{-t} + 2\,\frac{t}{3}\, e^{-t} + \frac{1}{3}\,\big(t\, e^{-t} - e^{-t} + e^{-2t}\big) = \frac{2}{3}\, e^{-t} + t\, e^{-t} + \frac{1}{3}\, e^{-2t}.\end{aligned}$$

To see that the families of the marginal survival functions of the order statistics coincide for the two models, we take into account that in general, even with $r \geq 3$,

$$\overline{G}_{1:r}(t) = e^{-\int_0^t \Lambda_\emptyset(s)\,ds} = \mathbb{P}(N(t) = 0), \quad \textit{and} \quad \overline{G}_{\ell:r}(t) = \mathbb{P}(N(t) \leq \ell - 1).$$

We notice furthermore that, for minimally stable models, Eq. (17) *(with $h = r - k$ and $H = \{k+1, \dots, r\}$) and Eq.* (35) *imply that*

$$\begin{aligned}\mathbb{P}(N(t) = k) &= \binom{r}{k} \mathbb{P}\big(T_i \leq t,\ \forall\, i \in \{1, 2, .., k\},\ T_j > t,\ \forall j \in \{k+1, \dots, r\}\big) \\ &= \binom{r}{k} \sum_{(j_1, \dots, j_k) \in \Pi([k])} \Psi\big(t; [r], (j_1, j_2, .., j_k)\big), \quad 1 \leq k \leq r.\end{aligned}$$

Thus, comparing Eq. (66) *with Eq.* (64) *for minimal THLS models, and taking into account that $\boldsymbol{\Lambda} = \boldsymbol{L}$, i.e., $\Lambda_{j_1,\dots,j_k} = \widetilde{\Lambda}_{j_1,\dots,j_k} = L(r-k)$, we obtain the afore mentioned conclusion, i.e., that for both models one has*

$$\overline{G}_{1:3}(t) = e^{-\Lambda_\emptyset t} = e^{-t}, \quad \overline{G}_{2:3}(t) = e^{-t}\,(1+t), \quad \overline{G}_{3:3}(t) = 2\,e^{-t}\,t + e^{-2t}.$$

We will see moreover that even the respective joint distributions of the order statistics do coincide for the two models. Actually the latter circumstance is a consequence of the condition that the functions $(j_1, \dots, j_k) \mapsto \Lambda_{j_1,\dots,j_k}$ are constant, only depending on k (see Remark 27 below and condition (80) *in Example 32 in the Appendix).*

Minimally stable ODTHLS models. We start our discussion with a simple necessary condition for minimal stability of ODTHLS models.

Lemma 24. *Let $T_1, \dots, T_r$ be an ODTHLS model. If $T_1, \dots, T_r$ are minimally stable then necessarily*

$$\lambda_{i|\emptyset} = \frac{\Lambda_\emptyset}{r} \quad \textit{and} \quad \Lambda_i = \Lambda_1, \quad \forall\, i \in [r]$$

Proof. By Proposition 5 we know that when $T_1, \dots, T_r$ are minimally stable, for any $t > 0$ the probabilities $\mathbb{P}(T_i \leq t,\ T_j > t, \forall j \neq i)$ necessarily assume the same value for any $i \in [r]$. Taking into account that (see (65) and (66))

$$\mathbb{P}(T_i \leq t,\ T_j > t, \forall j \neq i) = \lambda_{i|\emptyset} \int_0^t e^{-\Lambda_\emptyset s}\, e^{-\Lambda_i (t-s)} ds$$

one immediately gets that

$$\mathbb{P}(T_i \leq t,\ T_j > t, \forall j \neq i) = \begin{cases} \lambda_{i|\emptyset}\, t\, e^{-\Lambda_\emptyset t} & \text{if } \Lambda_i = \Lambda_\emptyset. \\ \lambda_{i|\emptyset} \dfrac{e^{-\Lambda_i t} - e^{-\Lambda_\emptyset t}}{\Lambda_\emptyset - \Lambda_i} & \text{if } \Lambda_i \neq \Lambda_\emptyset, \end{cases}$$

If there exists $i_0 \in [r]$ such that $\Lambda_{i_0} = \Lambda_\emptyset$ then necessarily $\Lambda_i = \Lambda_\emptyset$, for any $i \in [r]$. Consequently, also $\lambda_{i|\emptyset} = \lambda_{i_0|\emptyset}$.

Viceversa if there exists $i_0 \in [r]$ such that $\Lambda_{i_0} \neq \Lambda_\emptyset$, then necessarily $\Lambda_i \neq \Lambda_\emptyset$, for any $i \in [r]$. Furthermore one necessarily has

$$\lambda_{i|\emptyset} \frac{e^{-\Lambda_i t} - e^{-\Lambda_\emptyset t}}{\Lambda_\emptyset - \Lambda_i} = \lambda_{1|\emptyset} \frac{e^{-\Lambda_1 t} - e^{-\Lambda_\emptyset t}}{\Lambda_\emptyset - \Lambda_1}.$$

Then the thesis follows by the linear independence of the functions $t \mapsto e^{at}$ for different values of $a \in \mathbb{R}$. ☐

In the next result (see Proposition 25 below) we show that the survival functions $\overline{G}_{h:r}$ of a minimally stable ODTHLS model is a mixture of Hypoexponential distributions.

Before stating formally our result we need to introduce some further notation. Let $T_1, \dots, T_r$ be an ODTHLS model. For any permutation $\mathbf{j} = (j_1, \dots, j_r) \in \Pi[r]$ we will write $\boldsymbol{\Lambda}_{j_1,\dots,j_r}$ to denote the vector

$$\boldsymbol{\Lambda}_{j_1,\dots,j_r} := (\Lambda_\emptyset, \Lambda_{j_1}, \dots, \Lambda_{j_1,\dots,j_k}, \dots, \Lambda_{j_1,\dots,j_{r-1}}).$$

We will also use the shorter notation $\boldsymbol{\Lambda}_{\mathbf{j}}$ instead of $\boldsymbol{\Lambda}_{j_1,\dots,j_r}$.

Then we consider the partition of $\Pi[r]$ generated by the equivalence relation

$$\mathbf{j} \sim \mathbf{j}' \quad \Leftrightarrow \quad \boldsymbol{\Lambda}_{j_1,\dots,j_r} = \boldsymbol{\Lambda}_{j'_1,\dots,j'_r}.$$

If we define

$$\mathcal{L} := \left\{ \boldsymbol{L} : \exists\, \mathbf{j} \in \Pi([r]) \text{ with } \boldsymbol{\Lambda}_{\mathbf{j}} = \boldsymbol{L} \right\} \tag{69}$$

then the elements of the partition may be labeled by the vectors $\boldsymbol{L} \in \mathcal{L}$:

$$\Pi([r]) = \bigcup_{\boldsymbol{L} \in \mathcal{L}} \Pi([r]; \boldsymbol{L}),$$

where

$$\Pi([r]; \boldsymbol{L}) := \left\{ \mathbf{j} \in \Pi([r]) \text{ such that } \boldsymbol{\Lambda}_{\mathbf{j}} = \boldsymbol{L} \right\}. \tag{70}$$

For our purposes it is convenient to label the coordinates of the vectors in $\mathcal{L}$ as follows:

$$\boldsymbol{L} = \big(L(r), L(r-1), \dots, L(1)\big).$$

When $T_1, \dots, T_r$ are minimally stable, then, in view of Lemma 24, $L(r)$ takes on the same value for any $\boldsymbol{L} \in \mathcal{L}$, and the same happens for $L(r-1)$. More precisely one has

$$L(r) = \Lambda_\emptyset, \quad L(r-1) = \Lambda_1 = \Lambda_i, \ \forall\, i \in [r].$$

Proposition 25. *Let $T_1, \dots, T_r$ be a minimally stable ODTHLS model. Then, with the notation introduced above, the survival functions $\overline{G}_{\ell:r}$, $\ell = 1, \dots, r$, can be obtained as the following mixture of Hypoexponential survival functions*

$$\overline{G}_{\ell:r}(t) = \sum_{\boldsymbol{L} \in \mathcal{L}} \frac{|\Pi([r]; \boldsymbol{L})|}{r!}\, \overline{G}^{\boldsymbol{L}}_{\ell}(t). \tag{71}$$

Before giving the proof of the previous proposition, it is convenient to present the following remarks.

Remark 26. *The previous expression* (71) *of $\overline{G}_{\ell:r}(t)$ is alternative to the expression* (44) *given in Proposition 18. The main difference is that in* (44) *we need to know explicitly the hazard rates $\Lambda^{[h]}_\emptyset$ of the minima, but we need not to know explicitly the m.c.h.r. functions $\lambda_{j|j_1,\dots,j_k}$, while, viceversa, in* (71) *we need to know explicitly the m.c.h.r. functions, but we need not to compute the hazard rates $\Lambda^{[h]}_\emptyset$ of the minima.*

Remark 27. *Note that (see* (60)*) the functions $\overline{G}^{\boldsymbol{L}}_{k}(t)$ are the survival functions of the order statistics of an exchangeable THLS model with $\mu(k) = L(r-k)/(r-k)$. Therefore the r.h.s. of (71) can be interpreted as a mixture of the survival functions of exchangeable models. In the model T_1, T_2, T_3 of Example 23 the mixture turns out to be degenerate, since the set $\mathcal{L}$ is the singleton $\{(1, 1, 2)\}$. The latter circumstance explains the reason why the two models in Example 23 share the same family $\{\overline{G}_{1:3}, \overline{G}_{2:3}, \overline{G}_{3:3}\}$.*

Proof of Proposition 25. Consider a random permutation $\sigma_1, \dots, \sigma_r$, uniformly distributed in $\Pi([r])$. Then $T_{\sigma_1}, \dots, T_{\sigma_r}$ is the symmetrized model of $T_1, \dots, T_r$, denoted by $\widetilde{T}_1, \dots, \widetilde{T}_r$ in Remark 11.

Clearly the two models share the same distributions for the order statistics, and the joint distribution of $\widetilde{T}_1, \dots, \widetilde{T}_r$ is the mixture over $\boldsymbol{L} \in \mathcal{L}$ of the exchangeable THLS models with m.h.c.r. functions

$$\mu(k) = \frac{L(r-k)}{r-k},$$

with mixture weights given by $\frac{|\Pi([r];\boldsymbol{L})|}{r!}$. For any $\boldsymbol{L} \in \mathcal{L}$, the survival functions of the order statistics of the above exchangeable THLS model is $\overline{G}^{\boldsymbol{L}}_{h:r}(t)$, and therefore the marginal survival functions of the order statistics of the model $\widetilde{T}_1,\ldots, \widetilde{T}_r$ are given by

$$\mathbb{P}(\widetilde{T}_{h:r} > t) = \sum_{\boldsymbol{L}\in\mathcal{L}} \frac{|\Pi([r];\boldsymbol{L})|}{r!}\,\overline{G}^{\boldsymbol{L}}_{h:r}(t),$$

whence the thesis follows. □

Similarly to (71) one obtains that for minimally stable ODTHLS models the marginal survival function and the survival functions of the minima are mixtures over $k \in [r]$ and $\boldsymbol{L} \in \mathcal{L}$ of $\overline{G}^{\boldsymbol{L}}_k(t)$. More precisely by (9) one gets immediately that

$$\overline{G}(t) = \frac{1}{r}\sum_{k=1}^{r}\sum_{\boldsymbol{L}\in\mathcal{L}} \frac{|\Pi([r];\boldsymbol{L})|}{r!}\,\overline{G}^{\boldsymbol{L}}_k(t) \tag{72}$$

and, furthermore, by (11) one gets

$$\mathbb{P}\big(T_{1:A} > t\big) = \delta_d\big(\overline{G}(t)\big) = \frac{d}{(r)_d}\sum_{k=1}^{r-d+1}(r-k)_{d-1}\sum_{\boldsymbol{L}\in\mathcal{L}} \frac{|\Pi([r];\boldsymbol{L})|}{r!}\overline{G}^{\boldsymbol{L}}_k(t). \tag{73}$$

As a generalization of the arguments presented in Example 23, we now characterize the set of all minimally stable ODTHLS models T_1, T_2, T_3 in terms of the m.c.h.r. functions.

Example 28 (Minimally stable ODTHLS with $r = 3$). *Let us consider the ODTHLS model with T_1, T_2, T_3 whose joint distribution is given in terms of the m.c.h.r. functions $\lambda_{j_1|\emptyset}$, $\lambda_{j_2|j_1}$, $\lambda_{j_3|j_1,j_2}$, $j_1, j_2, j_3 \in \{1,2,3\}$, $j_2 \neq j_1$, $j_3 \neq j_1, j_2$.*
We are going to prove that T_1, T_2, T_3 are minimally stable if and only if conditions **(A1)** *and* **(A2)** *below hold, together with either condition* **(A3)** *or condition* **(A3)**$'$, *where*
(A1) *there exists a value $L(3)$ such that*

$$\lambda_{1|\emptyset} = \lambda_{2|\emptyset} = \lambda_{3|\emptyset} = \frac{L(3)}{3};$$

(A2) *there exists a value $L(2)$ such that*

$$\Lambda_1 = \lambda_{2|1} + \lambda_{3|1} = \Lambda_2 = \lambda_{1|2} + \lambda_{3|2} = \Lambda_3 = \lambda_{1|3} + \lambda_{2|3} = L(2);$$

(A3) *there exists a value $L(1)$ such that*

$$\lambda_{j_3|j_1,j_2} = L(1), \quad \forall (j_1, j_2, j_3) \in \Pi([3]),$$

and there exist two values γ_1 and γ_2 (possibly equal) such that

$$\{\lambda_{j_2|j_1}, \lambda_{j_1|j_2}\} = \{\gamma_1, \gamma_2\}, \quad \text{for any } \{j_1, j_2\}, \tag{74}$$

and

$$\gamma_1 + \gamma_2 = L(2); \tag{75}$$

(A3)$'$ *there exist two values $L'(1) \neq L''(1)$ such that*

$$\lambda_{j_3|j_1,j_2} \in \big\{L'(1),\ L''(1)\big\}, \quad \forall (j_1, j_2, j_3) \in \Pi([3]),$$

there exist two values γ_1 and γ_2 (possibly equal) such that (74) *and* (75) *hold, and furthermore*

$$\lambda_{j_2|j_1} = \gamma_1,\ \lambda_{j_3|j_1,j_2} = L'(1) \quad \text{if and only if} \quad \lambda_{j_1|j_2} = \gamma_2,\ \lambda_{j_3|j_2,j_1} = L''(1).$$

Note that the model is strictly order dependent only under condition **(A3)′**.

Conditions **(A1)** *and* **(A2)** *are the necessary conditions of Lemma 24 with* $L(3) := \Lambda_\emptyset$ *and* $L(2) := \Lambda_1$ *and guarantee that the following probabilities (see* (65) *and* (66)*) take the same value for any* $j_1 \in \{1,2,3\}$,

$$\Psi\big(t;[3],j_1\big) = \mathbb{P}\big(T_{j_1} \le t, T_j > t, j \ne j_1\big) = \int_0^t \lambda_{j_1|\emptyset} e^{-s\Lambda_\emptyset}\, e^{-(t-s)\Lambda_{j_1}}\, ds.$$

Furthermore for any $(j_1, j_2) \in \Pi_2([3])$*, we get*

$$\begin{aligned}\Psi\big(t;[3],(j_1,j_2)\big) &= \mathbb{P}\big(T_{j_1} \le T_{j_2} \le t, T_{j_3} > t\big)\\ &= \int_0^t ds \int_0^s ds'\, e^{-(t-s)\Lambda_{j_1,j_2}}\, \lambda_{j_1|\emptyset}\lambda_{j_2|j_1}\, e^{-s'\Lambda_\emptyset}\, e^{-\Lambda_{j_1}(s-s')}\, ds\end{aligned}$$

whereas

$$\Psi\big(t;[3],(j_2,j_1)\big) = \int_0^t ds \int_0^s ds' \lambda_{j_2|\emptyset}\lambda_{j_1|j_2}\, e^{-\Lambda_{j_1,j_2}(t-s)}\, e^{-\Lambda_\emptyset s'}\, e^{-\Lambda_{j_2}(s-s')}.$$

Proposition 16 guarantees that T_1, T_2, T_3 *are minimally stable if and only if for any* $t > 0$ *the following probabilities take on the same value for any* $\{j_1, j_2\} \subset \{1,2,3\}$*:*

$$\mathbb{P}\big(T_{j_1} \le t, T_{j_2} \le t, T_{j_3} > t\big) = \Psi\big(t;[3],(j_1,j_2)\big) + \Psi\big(t;[3],(j_2,j_1)\big).$$

Taking into account the necessary conditions **(A1)** *and* **(A2)**, *and that when* $r = 3$, *then* $\Lambda_{j_1,j_2} = \lambda_{j_3|j_1,j_2}$ *the previous condition is equivalent to requiring that, for any* $t > 0$ *the following sums take on the same value for any* $\{j_1, j_2\} \subset \{1,2,3\}$*:*

$$\begin{aligned}&\lambda_{j_2|j_1} \int_0^t ds \int_0^s ds'\, e^{-\lambda_{j_3|j_1,j_2}(t-s)}\, e^{-L(3)s'}\, e^{-L(2)(s-s')}\\ &\quad + \lambda_{j_1|j_2} \int_0^t ds \int_0^s ds'\, e^{-\lambda_{j_3|j_2,j_1}(t-s)}\, e^{-L(3)s'}\, e^{-L(2)(s-s')}.\end{aligned}$$

In its turn the above requirement is equivalent to either condition **(A3)** *or* **(A3)′**.

As a generalization of the previous example, in Example 32 (see the Appendix), we characterize the minimal stability property for ODTHLS models whose $\mathcal{L}$ is a singleton.

Among load sharing models, an interesting subclass is the class of the so-called *uniform frailty models*, whose m.c.h.r. functions are such that, for any $k = 0, 1, 2, \dots, r-1$,

$$\lambda_{j|j_1,\dots,j_k} = \frac{\Lambda_{j_1,\dots,j_k}}{r-k}, \quad \forall (j_1,\dots,j_k) \in \Pi_k([r]). \tag{76}$$

We conclude this section by analyzing the property of minimal stability for the model of the previous Example 28, under the additional assumption of uniform frailty.

Example 29. *Let us consider the model* T_1, T_2, T_3 *of the previous Example 28. If besides minimal stability, we impose the uniform frailty condition, then the model* T_1, T_2, T_3 *turns out to be non-exchangeable only if it is strictly order dependent. Indeed when condition* **(A3)** *holds, then the additional uniform frailty condition implies that* $\gamma_1 = \gamma_2 = L(2)/2$ *and therefore the model is exchangeable: for any* $(j_1, j_2, j_3) \in \Pi([3])$

$$\lambda_{j|\emptyset} = \frac{L(3)}{3}, \quad \lambda_{j_2|j_1} = \frac{L(2)}{2}, \quad \lambda_{j_3|j_1,j_2} = L(1).$$

On the contrary, when condition **(A3)′** *holds, the model is strictly order dependent. Then the uniform frailty and the minimal stability conditions together become: for any* $(j_1, j_2, j_3) \in \Pi([3])$

$$\lambda_{j_1|\emptyset} = \frac{L(3)}{3}, \quad \lambda_{j_2|j_1} = \frac{L(2)}{2}, \quad \{\lambda_{j_3|j_1,j_2}, \lambda_{j_3|j_2,j_1}\} = \{L'(1), L''(1)\}.$$

A More general examples

Example 30 (A procedure to construct DD, not exchangeable, n-dimensional copulas). *Our aim is to prove a generalization of Example 13. We start by proving that, given a DD* $n-1$*-dimensional copula* C_{n-1}*, then it is possible to construct a* n*-dimensional copula* C_n *which is DD. Subsequently we show that by using this construction recursively, starting with a* 2*-dimensional not symmetric copula, the copulas* C_n *are not exchangeable.*

Given the DD copula C_{n-1}*, we define a* n*-dimensional copula* C_n *as*

$$\begin{aligned} &C_n(u_1, \dots, u_{n-1}, u_n) \\ :=&\frac{1}{n}\left[C_{n-1}(u_1, \dots, u_{n-1}) \cdot u_n + C_{n-1}(u_2, \dots, u_{n-1}, u_n) \cdot u_1 + \cdots \right. \\ &\left. \cdots + C_{n-1}(u_{n-1}, u_n, u_1, \dots, u_{n-3}) \cdot u_{n-2} + C_{n-1}(u_n, u_1, \dots, u_{n-2}) \cdot u_{n-1}\right], \end{aligned}$$

Namely C_n *is obtained as the symmetric mixture of the copulas over the* n *cyclic permutations of* $(1, 2, \dots, n)$

$$\sigma_1 = (1, 2, \dots, n), \text{ and } \sigma_k = (k, k+1, \dots, n, 1, .., k-1), \quad 2 \le k \le n,$$

$$C_n(u_1, \dots, u_{n-1}, u_n) := \frac{1}{n}\sum_{k=1}^{n} C_{n-1}(u_{\sigma_k(1)}, \dots, u_{\sigma_k(n-1)}) \cdot u_{\sigma_k(n)}. \tag{77}$$

It is easy to see that C_n *is DD, with diagonal sections*

$$\delta_n^{C_n}(u) = C_n(u, \dots, u, u) = \delta_{n-1}^{C_{n-1}}(u) \cdot u, \quad \delta_1^{C_n}(u) = u$$

and

$$\delta_d^{C_n}(u) = \frac{d}{n}\,\delta_{d-1}^{C_{n-1}}(u) \cdot u + \left(1 - \frac{d}{n}\right)\delta_d^{C_{n-1}}(u), \quad 2 \le d \le n-1. \tag{78}$$

Indeed when $(u_1, \dots, u_{n-1}, u_n) = (u\mathbf{e}_A + \mathbf{e}_{A^c})$*, with* $A \subset [n]$*, and* $|A| = d$*, then, for any* $k = 1, \dots, n$*, one can write the* $n-1$*-dimensional vectors appearing in* (77) *as*

$$(u_{\sigma_k(1)}, u_{\sigma_k(2)}, \dots, u_{\sigma_k(n-1)}) \equiv (u_k, u_{k+1}, \dots, u_n, \dots, u_{k-2}) = (u\mathbf{e}_B + \mathbf{e}_{B^c}),$$

where B *is a suitable subset of* $[n-1]$*. Furthermore the cardinality* $|B|$ *takes on either the value* d *or the value* $d-1$*, depending on the value of* u_{k-1}*, namely*

$$|B| = \begin{cases} d & \text{if } u_{\sigma_k(n)} \equiv u_{k-1} = 1 \\ d-1 & \text{if } u_{\sigma_k(n)} \equiv u_{k-1} = u, \end{cases}$$

where we have used the convention that $u_0 = u_n$*.*

Starting from (78) *one can easily prove that*

$$\delta_d^{C_n}(u) = \alpha_{n,d}\, u^d + (1 - \alpha_{n,d})\, C(u,u) u^{d-2}, \quad 1 \le d \le n,$$

with

$$\alpha_{n,1} = 1, \quad \alpha_{n,d} = \frac{d}{n}\alpha_{n-1,d-1} + \left(1 - \frac{d}{n}\right)\alpha_{n-1,d}.$$

Starting from a fixed permutation $(\pi(1), \dots, \pi(n)) \notin \{\sigma_k,\ k = 1, \dots, n\}$, *a similar procedure can be used to construct another DD n-dimensional copula (possibly different from* C_n*), by using the n cyclic permutations* $\sigma_k \in \Pi_{\mathcal{C}}(1, 2, \dots, n)$*:*

$$C_{n,\pi}(u_1, \dots, u_{n-1}, u_n) := \frac{1}{n} \sum_{k=1}^{n} C_{n-1}\left(u_{\pi(\sigma_k(1))}, u_{\pi(\sigma_k(2))}, \dots, u_{\pi(\sigma_k(n-1))}\right) \cdot u_{\pi(\sigma_k(n))}.$$

When the procedure is implemented recursively starting with a fixed permutation $\pi \in \Pi([n])$*, and with a not-symmetric copula* $C_2(u, v) = C(u, v)$*, as in Example 13, denote by* $U_1, \dots, U_n$ *the random variables associated to the copula of* $C_{n,\pi}$, $n \geq 3$. *Then for any fixed* $i = 1, \dots, n$ *the 2-dimensional marginals of* U_i, U_{i+1} *(with the convention that* $U_{n+1} = U_0$*) are obtained recursively as*

$$C_{n,\pi}(u, v, \overbrace{1, \dots, 1}^{n-2}) = \frac{1}{n}\left[(n-2)C_{n-1,\pi}(u, v, \overbrace{1, \dots, 1}^{n-3}) + 2uv\right], \quad u, v \in [0, 1],$$

and therefore are all equal, i.e.,

$$C_{n,\pi}\left(u\mathbf{e}_{\{i\}} + v\mathbf{e}_{\{i+1\}} + \mathbf{e}_{[n]\setminus\{i,i+1\}}\right) = C_{n,\pi}(u, v, 1, \dots., 1).$$

Notice that therefore

$$C_{n,\pi}(u, v, 1, \dots., 1) \neq C_{n,\pi}(v, u, 1, \dots., 1),$$

so that the copulas $C_{n,\pi}$ *are not symmetric.*

Since we are particularly interested in examples with absolutely continuous joint and marginal distributions, observe that if we start this procedure with a absolutely continuous copula we get absolutely continuous copulas.

Furthermore we recall the class of absolutely continuous examples given in Navarro and Fernandez-Sanchez [17] (see in particular Proposition 1 therein). The class in [17] may be seen as a particular case of the larger class considered in the next example.

Example 31 (Negative mixtures of DD copulas are DD). *Suppose that* $D(u_1, \dots, u_r)$ *is an absolutely continuous exchangeable copula with probability density d such that*

$$0 < \underline{d}\,\rho(u_1, \dots, u_r) \leq d(u_1, \dots, u_r),$$

for some positive function ρ *and some positive constant* $\underline{d}$.
Let $C_i(u_1, \dots, u_r)$, $i = 1, 2$, *be two different copulas which are DD, but non-exchangeable, and absolutely continuous, with probability density* $c_i(u_1, \dots, u_r)$ *such that, for some positive constant* $\overline{c}$

$$0 \leq c_i(u_1, \dots, u_r) \leq \overline{c}\,\rho(u_1, \dots, u_r).$$

Assume also that the function $c_1(u_1, \dots, u_r) - c_2(u_1, \dots, u_r)$ *is not symmetric, and define*

$$K_\alpha(u_1, \dots, u_r) := D(u_1, \dots, u_r) + \alpha\left[C_1(u_1, \dots, u_r) - C_2(u_1, \dots, u_r)\right]. \tag{79}$$

If α *is strictly positive and sufficiently small, then* K_α *is an absolutely continuous DD copula, but not exchangeable.*

We now proceed with the proof of the previous statement.

The function K_α *defined in* (79) *has a density*

$$k_\alpha(u_1, \dots, u_r) = d(u_1, \dots, u_r) + \alpha\left[c_1(u_1, \dots, u_r) - c_2(u_1, \dots, u_r)\right],$$

such that the integral

$$\int_{[0,1]^n} k_\alpha(u_1,\dots,u_r)du_1\cdots du_r = 1.$$

Therefore k_α is a probability density, if and only if $k_\alpha(u_1,\dots,u_r) \geq 0$ for any $(u_1,\dots,u_r) \in (0,1)^r$. The condition $\overline{c}\alpha \leq \underline{d}$ implies that

$$k_\alpha(u_1,\dots,u_r) \geq \underline{d}\rho(u_1,\dots,u_r) + \alpha[0 - \overline{c}\,\rho(u_1,\dots,u_r)] \geq 0$$

The assumption on the densities may be weakened, for instance, it is clearly not necessary any assumption on the density of C_1. Furthermore the example could be generalized to the case

$$K_{\alpha_1,\dots,\alpha_m}(u_1,\dots,u_r) := D(u_1,\dots,u_r) + \sum_{k=1}^{m} \alpha_k \left[C_{1,k}(u_1,\dots,u_r) - C_{2,k}(u_1,\dots,u_r)\right],$$

with suitable conditions on α_i and $C_{i,k}$, for $k = 1,\dots,m$, $i = 1,2$.

Finally it is interesting to note that K_α is a negative mixture of copulas, and that negative mixture of i.i.d. random variable are linked to finite exchangeability, and the problem of extendibility.

Example 32. *(Minimally stable ODTHLS models sharing the joint distribution of the order statistics with an exchangeable THLS model) Let us assume that $T_1,..,T_r$, with $r \geq 3$, is an ODTHLS model described by the m.c.h.r. functions $\lambda_{j|j_1,\dots,j_{d-1}}$, $d = 1,\dots,r$, $(j_1,\dots,j_{d-1},j) \in \Pi([r])$ with the usual convention that when $d = 1$ then $\lambda_{j|j_1,\dots,j_{d-1}} = \lambda_{j|\emptyset}$. We are going to characterize all the minimally stable ODTHLS models in the particular case when the set $\mathcal{L}$ is the singleton $\{\boldsymbol{L} = (L(r), L(r-1),\dots,L(1))\}$, i.e.,*

$$\Lambda_{j_1,\dots,j_k} = \sum_{j\notin\{j_1,\dots,j_k\}} \lambda_{j|j_1,\dots,j_k} = L(r-k), \quad \forall\, k = 0,1,\dots,r-1. \tag{80}$$

By Proposition 25, for all the above minimally stable ODTHLS models, one has $\overline{G}_{k:r} = \overline{G}^{\boldsymbol{L}}_k$, $k = 1,2,\dots,r$, i.e., the same Hypoexponential marginal survival functions of the exchangeable THLS model (58). The characterization of the condition that $T_1,\dots,T_r$ are minimally stable is then a consequence of Proposition 16 together with the comparison between (66) and (64): the m.c.h.r. functions satisfy the following system of equations

$$\begin{cases} \sum_{j\notin\{j_1,\dots,j_{d-1}\}} \lambda_{j|j_1,\dots,j_{d-1}} = L(r-(d-1)), & \{j_1,\dots,j_{d-1}\} \subset [r] \\ \sum_{\mathbf{j}\in\Pi(I)} \prod_{h=1}^{d} \lambda_{j_h|j_1,\dots,j_{h-1}} = \frac{1}{\binom{r}{d}} \prod_{\ell=1}^{d} L(r-(\ell-1)), & \text{for any } I \subset [r], \text{ with } |I| = d, \\ \qquad d = 1,\dots,r. \end{cases}$$

This characterization yields that there exist infinitely many minimally stable ODTHLS models satisfying condition (80). This statement is a consequence of the observation that one can see the previous system as a family of nested linear systems:

$$\begin{cases} \sum_{j\in[r]} \lambda_{j|\emptyset} = L(r), \\ \lambda_{j|\emptyset} = \frac{1}{r}\, L(r); & j \in [r] \end{cases}$$

once $\lambda_{j|\emptyset}$ are given, then $\lambda_{j_2|j_1}$ are the solutions x_{j_1,j_2}, $j_2 \neq j_1$, of

$$\begin{cases} \sum_{j\neq j_1} \lambda_{j_1|\emptyset} x_{j_1,j} = L(r-1), & \forall\, j_1 \in [r] \\ \lambda_{j_1|\emptyset}\, x_{j_1,j_2} + \lambda_{j_2|\emptyset}\, x_{j_2,j_1} = \frac{1}{\binom{r}{2}}\, L(r)L(r-1); & \forall\, \{j_1,j_2\} \subset [r]; \end{cases}$$

once $\lambda_{j|\emptyset}$ and $\lambda_{j|j_1}$ are given, then $\lambda_{j_3|j_1,j_2}$ are the solutions x_{j_1,j_2,j_3} of

$$\begin{cases} \sum_{j\notin\{j_1,j_2\}} x_{j_1,j_2,j} = L(r-2), & \forall\, \{j_1,j_2\} \subset [r], \\ \sum_{(j_1,j_2,j_3)\in\Pi(\{k_1,k_2,k_3\})} \lambda_{j_1|\emptyset}\lambda_{j_2|j_1} x_{j_1,j_2,j_3} & \\ \qquad = \frac{1}{\binom{r}{3}}\, L(r)\, L(r-1)L(r-2); & \forall\, \{k_1,k_2,k_3\} \subset [r]; \end{cases}$$

and so on. Since the above nested systems always admit the solutions $x_{j_1,\dots,j_k,j} = \lambda_{j|j_1,\dots,j_k} = \frac{L(r-k)}{r-k}$*, then there are infinite solutions.*

We end this example by observing that under condition (80) *all the ODTHLS models (not necessarily minimally stable) share with the exchangeable THLS model* (58) *not only the marginal distributions but also the joint distribution of the order statistics. More precisely the joint distribution of* $(T_{1:r}, \dots, T_{r:r})$ *coincides with the joint distribution of*

$$\left(\frac{Y_0}{L(r)}, \frac{Y_0}{L(r)} + \frac{Y_1}{L(r-1)}, \dots, \frac{Y_0}{L(r)} + \frac{Y_1}{L(r-1)} + \cdots \frac{Y_{r-1}}{L(1)}\right), \tag{81}$$

where Y_k*,* $k = 0, 1, \dots, r-1$*, are i.i.d. standard exponential. Indeed, one can easily extend Corollary 3 in Rychlik and Spizzichino [23] for THLS models, to ODTHLS ones: for any permutation* $(j_1, \dots, j_r) \in \Pi([r])$*, the conditional joint distribution of* $(T_{1:r}, \dots, T_{r:r})$ *given the event* $\{T_{j_1} < T_{j_2} < \cdots < T_{j_r}\}$*, coincides with the law of*

$$\left(\frac{Y_0}{\Lambda_\emptyset}, \frac{Y_0}{\Lambda_\emptyset} + \frac{Y_1}{\Lambda_{j_1}}, \dots, \frac{Y_0}{\Lambda_\emptyset} + \frac{Y_1}{\Lambda_{j_1}} + \cdots + \frac{Y_{r-1}}{\Lambda_{j_1,\dots,j_{r-1}}}\right), \tag{82}$$

and by (80)*, the random vectors in Eq.'s* (81) *and* (82) *do coincide.*

Finally we observe that, as a consequence, these models satisfy also the following condition: for any permutation $(j_1, \dots, j_r) \in \Pi([r])$*,*

$$\mathbb{P}\left(T_{k:r} > t | T_{j_1} < T_{j_2} < \cdots < T_{j_r}\right) = \mathbb{P}\left(T_{k:r} > t\right) \tag{83}$$

Indeed, in the case of ODTHLS models, we have just seen that condition (80) *implies an even stronger property:*

$$\mathbb{P}\left(T_{k:r} > t_k, k = 1, 2, .., r | T_{j_1} < T_{j_2} < \cdots < T_{j_r}\right) = \mathbb{P}\left(T_{k:r} > t_k, k = 1, 2, \dots r\right). \tag{84}$$

Condition (83) *emerges in a natural way even in more general settings beyond load-sharing, as pointed out in Navarro et al. [19], where it has been referred to as a condition of* weak exchangeability *(see also Navarro et al. [18]).*
In the frame of load-sharing models, condition (80)*, (and therefore also* (84) *and* (83)*) emerges in De Santis and Spizzichino [8], where it plays an important role for the special type of problems studied therein.*

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