Maximum Principle and generalized principal eigenvalue for degenerate elliptic operators

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Abstract

We characterize the validity of the Maximum Principle in bounded domains for fully nonlinear degenerate elliptic operators in terms of the sign of a suitably defined generalized principal eigenvalue. Here, maximum principle refers to the non-positivity of viscosity subsolutions of the Dirichlet problem. This characterization is derived in terms of a new notion of generalized principal eigenvalue, which is needed because of the possible degeneracy of the operator, admitted in full generality. We further discuss the relations between this notion and other natural generalizations of the classical notion of principal eigenvalue, some of which had already been used in the literature for particular classes of operators.

1 Introduction

This paper is concerned with the Maximum Principle property for degenerate second order elliptic operators. Our aim is to characterize the validity of the Maximum Principle for arbitrary degeneracy of the operator - including the limiting cases of first and zero-order operators - in terms of the sign of a suitably defined generalized principal eigenvalue. Such a complete characterization is missing, as far as we know, even for the case of linear operators, which was of course our first motivation. Due to the possible loss of regularity, as well as of boundary conditions, which is caused by degeneracy of ellipticity, the appropriate framework to deal with this problem is, even in the linear case, that of viscosity solutions. This approach is of course not restricted to the linear case, so we study the question in the more general setting of homogeneous fully nonlinear degenerate elliptic operators $F(x, u, Du, D^2u)$.

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Let Ω be a bounded domain in $\mathbb{R}^N$ and $\mathcal{S}_N$ be the space of $n \times n$ symmetric matrices endowed with the usual partial order, with I being the identity matrix. A fully nonlinear operator $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N \to \mathbb{R}$ is said to be *degenerate elliptic* if F is non increasing in the matrix entry, see condition (H1) in the next section. The basic example to have in mind is that of linear operators in non divergence form

$$F(x, u, Du, D^2u) = -\mathrm{Tr}(A(x)D^2u) - b(x) \cdot Du - c(x)u, \quad x \in \Omega,$$

where $A(x)$ is nonnegative definite.

We are interested in the following version of the Maximum Principle, **MP** in short :

Definition 1.1. The operator F satisfies **MP** in Ω if every viscosity subsolution $u \in USC(\overline{\Omega})$ of the Dirichlet problem

$$\begin{cases} F(x, u, Du, D^2u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

satisfies $u \leq 0$ in $\overline{\Omega}$.

We denote by $USC(\overline{\Omega})$ the set of upper semicontinuous functions on $\overline{\Omega}$. It is worth pointing out that in the above definition both the PDE and the boundary conditions are understood in the viscosity sense (see Section 7 of [8]). Precisely, u is a subsolution of (1) if for all $\varphi \in C^2(\overline{\Omega})$ and $\xi \in \overline{\Omega}$ such that $(u - \varphi)(\xi) = \max_{\overline{\Omega}}(u - \varphi)$, it holds that

$$F(\xi, u(\xi), D\varphi(\xi), D^2\varphi(\xi)) \leq 0 \quad \text{if } \xi \in \Omega,$$

$$\max \left[u(\xi) \, , \, F(\xi, u(\xi), D\varphi(\xi), D^2\varphi(\xi)) \right] \leq 0 \quad \text{if } \xi \in \partial\Omega.$$

Note, in particular, that the validity of the **MP** property implies that viscosity subsolutions cannot be positive on $\partial\Omega$, namely, the inequality $u \leq 0$ on $\partial\Omega$ holds in the classical pointwise sense.

Before describing our results, let us recall some classical and more recent results concerning the Maximum Principle and the principal eigenvalue.

A standard result in the viscosity theory is that, under suitable continuity assumptions on the degenerate elliptic operator F, the Maximum Principle for viscosity subsolutions holds true if $r \mapsto F(x, r, p, X)$ is strictly increasing (see e.g. [8]). This is only a sufficient condition. It is well known that if Ω is a bounded smooth domain and F is a uniformly elliptic linear operator with smooth coefficients, then the validity of the Maximum Principle for classical subsolutions is equivalent to the positivity of the principal eigenvalue $\lambda_1(F, \Omega)$ associated with Dirichlet boundary condition. This eigenvalue is the bottom of the spectrum of the operator F acting on functions satisfying the Dirichlet boundary conditions. It follows from the Krein-Rutman theory that $\lambda_1(F, \Omega)$ is simple and the associated eigenfunction φ is positive in Ω. So, if $\lambda_1(F, \Omega) \leq 0$ then φ violates the Maximum Principle. As a consequence, if the Maximum Principle holds then the problem admits a positive strict supersolution.
The reverse implication is also true, but its proof is not completely straightforward since it

requires an analysis of how sub and supersolutions can vanish at the boundary. To this aim, one typically makes use of barriers and Hopf's lemma, for which uniform ellipticity or some other properties are required.
As we will see, the possibility of different behaviours of supersolutions at the boundary is one of the most delicate points one has to handle in order to deal with degenerate operators.

The connection between the Maximum Principle and the existence of positive strict supersolutions led the first author, L. Nirenberg and S. R. S. Varadhan to introduce in [3] the following notion of *generalized principal eigenvalue*:

$$\lambda_1(F,\Omega) := \sup\{\lambda \in \mathbb{R} \ : \ \exists \phi \in W^{2,N}_{loc}(\Omega), \ \phi > 0 \text{ in } \Omega, \ F[\phi] - \lambda\phi \geq 0 \text{ a.e. in } \Omega\}.$$

Here and henceforth we will write $F[\cdot](x)$ for short, or simply $F[\cdot]$, in place of $F(x, \cdot, D\cdot, D^2\cdot)$. Using this generalization, they were able to extend the characterization of the Maximum Principle for linear elliptic operators to the case of non-smooth domains, where the classical principal eigenvalue is not defined.

In [6], I. Birindelli and F. Demengel adapted the definition of [3] to a class of fully nonlinear operators F which are homogeneous of degree $\alpha > 0$, including some degenerate elliptic operators which are modeled on the example of the $p-$Laplacian. They defined the principal eigenvalue as

$$\lambda_1(F,\Omega) := \sup\{\lambda \in \mathbb{R} \ : \ \exists \phi \in LSC(\Omega), \ \phi > 0 \text{ and } F[\phi] - \lambda\phi^\alpha \geq 0 \text{ in } \Omega\}.$$

Here, $LSC(\Omega)$ denotes the set of lower semicontinuous functions on Ω.

Actually, in their earlier work [5], the same authors had defined the generalized principal eigenvalue in the following slightly different way:

$$\overline{\lambda}_1(F,\Omega) := \sup\{\lambda \in \mathbb{R} \ : \ \phi \in LSC(\Omega), \ \inf_\Omega \phi > 0, \ F[\phi] - \lambda\phi^\alpha \geq 0 \text{ in } \Omega\}.$$

The two notions coincide in the case treated in [6], but, as we will see in the proof of Proposition 2.1 part (i) below, this may not be the case in general. Let us mention that the non-equivalence between λ_1 and $\overline{\lambda}_1$, which in the cases considered in the present paper is due to the degeneracy of the operator, can occur when Ω is unbounded even for uniformly elliptic linear operators. The characterization of the Maximum Principle in terms of generalized principal eigenvalues such as λ_1 and $\overline{\lambda}_1$, as well as the study of their properties, for uniformly elliptic linear operators in unbounded domains is the object of the recent paper [4].

It turns out that the generalized principal eigenvalue λ_1 is not well suited to characterize the validity of **MP** for the general degenerate cases that we address in the present paper. This is showed in the next section, in which we also discuss the pertinence of other natural candidates, such as $\overline{\lambda}_1$, as well as the limit of the principal eigenvalues of the ε-viscosity regularized operators $F^\varepsilon = -\varepsilon\Delta + F$. None of those choices will be sufficient in order to characterize the **MP** property in the most general situation. Indeed, one of our main goals is to identify the right set of admissible functions, in the definition of a generalized principal eigenvalue, which can be suitable for general degenerate elliptic operators. Eventually, the right notion for our purposes turns out to be given by the following.

Definition 1.2. Given a domain Ω in $\mathbb{R}^N$ and an open set $\mathcal{O}$ such that $\overline{\Omega} \subset \mathcal{O}$, and a fully nonlinear degenerate elliptic operator F in $\mathcal{O}$, we define

$$\mu_1(F,\Omega) := \sup\{\lambda \in \mathbb{R} \ : \ \exists \Omega' \supset \overline{\Omega},\ \phi \in LSC(\Omega'),\ \phi > 0 \text{ and } F[\phi] - \lambda\phi^\alpha \geq 0 \text{ in } \Omega'\}.$$

Hence, the definition of the generalized principal eigenvalue μ_1 in a domain Ω requires the operator to be defined in a larger set. Equivalent formulations for $\mu_1(F,\Omega)$ are

$$\mu_1(F,\Omega) = \sup_{\Omega' \supset \overline{\Omega}} \lambda_1(F,\Omega') = \lim_{\varepsilon \to 0^+} \lambda_1(F,\Omega + B_\varepsilon),$$

where the last one follows from the monotonicity of $\lambda_1(F,\Omega)$ with respect to inclusion of the domains.

1.1 Hypotheses and main result

Throughout the paper, Ω is a bounded domain in $\mathbb{R}^N$, not necessarily smooth, and $\mathcal{O}$ is an open set such that $\overline{\Omega} \subset \mathcal{O} \subset \mathbb{R}^N$. We assume that $F : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N \to \mathbb{R}$ is a continuous function which satisfies the following hypotheses:

(H1) $F(x,r,p,X+Y) - F(x,r,p,X) \leq 0, \quad \forall (x,r,p,X,Y) \in \mathcal{O} \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N \times \mathcal{S}_N,\ Y \geq 0;$

(H2) $\exists \alpha > 0,\ F(x,\tau r,\tau p,\tau X) = \tau^\alpha F(x,r,p,X), \quad \forall (x,r,p,X) \in \mathcal{O} \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N,\ \tau \geq 0;$

(H3) $r \mapsto F(x,r,p,X)$ is continuous, uniformly with respect to $(x,p,X) \in \mathcal{O} \times \mathbb{R}^N \times \mathcal{S}_N$;

(H4) For all $R > 0$, there exists a function $\omega \in C([0,+\infty))$ with $\omega(0) = 0$ such that if $X, Y \in \mathcal{S}_N$ satisfy

$$\exists \sigma > 0, \quad -3\sigma \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\sigma \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

then

$$F(y,r,\sigma(x-y),Y) - F(x,r,\sigma(x-y),X) \leq \omega(\sigma|x-y|^2 + |x-y|), \quad \forall x,y \in \mathcal{O},\ |r| \leq R.$$

As it was established in [8], hypothesis (H4) is the key structure condition for the validity of the Comparison Theorem for viscosity solutions of degenerate elliptic equations. Let us emphasize that no regularity assumption is required on the set Ω.

We now state the main result of this article.

Theorem 1.3. *Under the assumptions (H1)-(H4), F satisfies the* **MP** *property in $\Omega \subset\subset \mathcal{O}$ if and only if $\mu_1(F,\Omega) > 0$.*

Some remarks on the statement of Theorem 1.3 are in order. Since our characterization of **MP** in Ω requires the operator F to be defined in some $\mathcal{O} \supset \overline{\Omega}$ then, if F is just defined in Ω and satisfies (H1)-(H4) there, in order to apply our result we need to extend it to an operator satisfying (H1)-(H4) in the larger domain $\mathcal{O}$. This is not completely satisfactory, even though the result itself ensures that the notion $\mu_1(F,\Omega)$ does not depend on the particular extension. A characterization expressed in terms of a more intrinsic notion, such as λ_1 or $\overline{\lambda}_1$, would be preferable. Proposition 2.1 below provides examples showing that **MP** is not guaranteed by $\lambda_1 > 0$ nor by $\overline{\lambda}_1 > 0$. However, in the case of $\overline{\lambda}_1$, the only examples we are able to construct do not satisfy (H4).

We leave it as an open problem to know whether μ_1 coincides with $\overline{\lambda}_1$ under the assumption (H4), and then whether $\mu_1(F,\Omega)$ can be replaced by $\overline{\lambda}_1(F,\Omega)$ in Theorem 1.3. In Theorems 4.2 and 4.4 below we show that, for a smooth domain Ω, this is true in two significant cases: if the operator admits barriers at each point of the boundary, and, for the case of linear operators, if in each connected component of $\partial\Omega$ the so–called Fichera condition is either always satisfied or always violated. The general case remains open.

Finally, let us point out that a generalized principal eigenfunction associated with μ_1 does not always exist (see Remark 2 in Section 3). This is due to the degeneracy of the operator and is true even in the linear case.

1.2 Examples

In this section we present some examples of operators to which Theorem 1.3 applies, recovering some known results. We further analyse the generalized principal eigenvalues in these particular cases. The examples are divided into classes, but none of them is intended to be exhaustive.

The standard sufficient condition

If an operator F satisfies $\min_{x\in\overline{\Omega}} F(x,r,0,0) > 0$ for all $r > 0$, then **MP** holds. This is an immediate consequence of the definition of viscosity subsolution. Notice that in this case $\overline{\lambda}_1(F,\Omega) > 0$ and, up to extending F outside Ω as a continuous function, $\mu_1(F,\Omega) > 0$ too.

First-order operators

Theorem 1.3 applies to the generalized eikonal operator $F[u] = -b(x)|Du| - c(x)u$, provided that $b \in W^{1,\infty}(\mathcal{O})$ and $c \in C(\mathcal{O})$. The Lipschitz-continuity of b is required for (H4) to hold. Furthermore, the result still holds for an operator $G[u] = F[u] - \mathrm{Tr}(A(x)D^2u)$, with $A = \Sigma^t\Sigma$ and $\Sigma \in W^{1,\infty}(\mathcal{O})$.

Another family of operators which can be considered is $F[u] = -b(x)\cdot Du|Du|^{\alpha-1} - c(x)|u|^{\alpha-1}u$, with $\alpha \geq 1$. The hypothesis on b is again $b \in W^{1,\infty}(\mathcal{O})$ if $\alpha = 1$, otherwise we need $(b(x)-b(y))\cdot(x-y) \leq 0$ for $x,y \in \mathcal{O}$.

Subelliptic operators

For several classes of subelliptic operators one can derive the sign of μ_1 and thus apply Theorem 1.3. For example, if the ellipticity of F is not degenerate in a direction ξ, in the sense that there exists $\beta > 0$ such that

$$F(x,r,p,X+\xi\otimes\xi) - F(x,r,p,X) \leq -\beta, \quad \forall x \in \mathcal{O},\ r \in \mathbb{R},\ p \in \mathbb{R}^N,\ X \in \mathcal{S}_N,$$

and the positive constants are supersolutions of $F = 0$ in $\mathcal{O}$, i.e., $F(x, 1, 0, 0) \geq 0$ in $\mathcal{O}$, then $\mu_1(F, \Omega) > 0$. This is seen by taking $\phi(x) = 1 - \varepsilon e^{\sigma \xi \cdot x}$, with σ large and then ε small.
The above conditions are satisfied for instance by the Grushin operator: $-\partial_{xx} - |x|^\alpha \partial_{yy}$, with $\alpha > 0$. This operator, which is a Hörmander operator if α is an even integer, belongs to the class of Δ_λ operators studied in [11]. The key property of such operators is the 2-homogeneity with respect to a group of dilations of $\mathbb{R}^N$. Concerning the generalized principal eigenvalues introduced before, this property yields $\mu_1(\Delta_\lambda, \Omega) = \lambda_1(\Delta_\lambda, \Omega)$ if Ω is starshaped with respect to the origin. Actually, the following much weaker scaling property is required on an operator F in order to have $\mu_1(F, \Omega) = \lambda_1(F, \Omega)$: there is a family of C^2-diffeomorphisms $(\vartheta_t)_{t>0}$ of $\mathbb{R}^N$ into itself and a function $\psi : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$\forall t > 1, \quad \overline{\Omega} \subset \vartheta_t(\Omega), \qquad \lim_{t \to 1} \psi(t) = 1,$$

and

$$\forall u \in C^2(\vartheta_t(\Omega)), \quad F[u \circ \vartheta_t](x) = \psi(t) F[u](\vartheta_t(x)), \quad x \in \Omega.$$

Indeed for $\phi \in LSC(\Omega)$, the above condition implies (in the viscosity sense)

$$F[\phi \circ \vartheta_t^{-1}](x) = (\psi(t))^{-1} F[\phi](\vartheta_t^{-1}(x)), \quad x \in \vartheta_t(\Omega).$$

It follows from the definition of λ_1 that the mapping $t \mapsto \lambda_1(F, \vartheta_t(\Omega))$ is lower semicontinuous at $t = 1$. Hence, since for $t > 1$, $\lambda_1(F, \vartheta_t(\Omega)) \leq \mu_1(F, \Omega) \leq \lambda_1(F, \Omega)$, we infer that $\mu_1(F, \Omega) = \lambda_1(F, \Omega)$.

Parabolic operators

It is well known that the classical Maximum Principle holds for uniformly parabolic linear operators of the type $\partial_t u - \mathrm{Tr}(A(t, x) D^2 u) - b(t, x) \cdot Du - c(t, x) u$, with $t > 0$, $x \in \Omega$. Uniformly parabolic means that $A \geq \alpha I$, for some positive constant α. Note that a crucial difference with the elliptic case is that the Maximum Principle holds even if the zero order term c is positive and very large. One can interpret the validity of the Maximum Principle as a consequence of the positivity of the principal eigenvalues. Indeed, considering the function $\phi = e^{\sigma t}$ and letting $\sigma \to +\infty$, one finds that in this case all notions of principal eigenvalues introduced in Section 1 are equal to $+\infty$. However, the parabolic Maximum Principle cannot be derived right away from Theorem 1.3 due to the unboundedness of the domain.

1-*homogeneous, uniformly elliptic operators*

A fully nonlinear operator $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N \to \mathbb{R}$ is said to be *uniformly elliptic* if there exists $\alpha > 0$ such that

$$F(x, r, p, X + Y) - F(x, r, p, X) \leq -\alpha \mathrm{Tr}(Y), \quad \forall (x, r, p, X, Y) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N \times \mathcal{S}_N.$$

An important role in the theory of fully nonlinear, uniformly elliptic operators is played by 1-homogeneous operators, that is, operators satisfying (H2) with $\alpha = 1$. Besides linear operators, this class includes Pucci, Bellman and Isaacs operators. The latter class, which is the most general, fulfils (H1)-(H4) under suitable regularity conditions on the coefficients.

Being uniformly elliptic, it also satisfies the hypotheses of Theorem 4.2 below, which implies that the **MP** property is characterized by the sign of $\overline{\lambda}_1$ if Ω is smooth.

Several works have already addressed the question of the validity of the Maximum Principle and the existence of the principal eigenfunction. Among them, let us cite the papers [9] and [18] for the Pucci operator and [19] for more general operators, including the Bellman one. In [19], the simplicity of the principal eigenvalue is further obtained. The method used in the above mentioned papers differs from the one of [5] and ours. It follows the line of the classical proof based on the Krein-Rutman theory. This is possible because of the uniform ellipticity of the operator, which makes the $W^{2,N}$ estimates available and avoids the direct use of the definition of viscosity solution.

1-*homogeneous, degenerate elliptic operators*

Two examples of fully nonlinear degenerate operators are

$$-\mathcal{P}_k(D^2u) := -\eta_{N-k+1}(D^2u) - \ldots - \eta_N(D^2u),$$

k being an integer between 1 and N and $\eta_1(D^2u) \leq \eta_2(D^2u) \leq \ldots \leq \eta_N(D^2u)$ being the ordered eigenvalues of the matrix D^2u, and the degenerate maximal Pucci operator

$$-\mathcal{M}^+_{0,1}(D^2u) := -\sum_{i=1}^{N} \max(\eta_i(D^2u), 0) = -\sup_{A\in\mathcal{S}_N,\ 0\leq A\leq I} \mathrm{Tr}(AD^2u).$$

The operator $-\mathcal{P}_N$ has been used by F. R. Harvey and H. B. Lawson to characterize the validity of the Maximum Principle for operators only depending on the Hessian: Theorem 2.1 of [15] states, with a geometrical terminology, that an operator $F : \mathcal{S}_N \to \mathbb{R}$ satisfies the **MP** if and only if $F(X) \leq 0 \ \Rightarrow \ -\mathcal{P}_N(X) \leq 0$. Notice that if F satisfies such a property then the function $\phi(x) := k - |x|^2$, with $k > \sup\{|x|^2 \ : \ x \in \Omega\}$, satisfies $\phi > 0$ in $\overline{\Omega}$ and $F(D^2\phi) > 0$, whence $\mu_1(F, \Omega) > 0$.

Both $\mathcal{P}_k$ and $-\mathcal{M}^+_{0,1}$ have positive principal eigenvalue μ_1 and satisfy the **MP**. In addition, they admit continuous barriers at every point of the boundary of a smooth domain, in the sense of Definition 4.1 below. Therefore Theorem 4.2 implies that μ_1 coincides with $\overline{\lambda}_1$.

The p and the infinity Laplacian

The p and the infinity Laplacian are defined respectively by

$$\Delta_p u := \mathrm{div}(|Du|^{p-2}Du), \quad p > 1, \qquad \Delta_\infty u := \frac{Du}{|Du|}D^2u\frac{Du}{|Du|}.$$

These definitions have a meaning, in the viscosity sense, if the gradient is nonzero. One has to extend them in suitable way to get a general definition. Both the operators $F = -\Delta_p, -\Delta_\infty$, with the possible addition of a degenerate elliptic operator sharing the same homogeneity property, fit with the hypotheses (H1)-(H4) of Theorem 1.3 above.

The characterization of the Maximum Principle for the p-Laplacian was derived by I. Birindelli and F. Demengel in [5] and [6], using the principal eigenvalue $\overline{\lambda}_1$ and λ_1 respectively. The fact that $\lambda_1 = \overline{\lambda}_1 = \mu_1$ in that case is due to the validity of the Hopf lemma and the existence of barriers. The result for the infinity Laplacian is due to P. Juutinen [17] and is expressed in terms of $\overline{\lambda}_1$. The existence of barrier is crucial also in this case. We remark that, owing to the particular structure of the infinity-Laplacian, barriers exist without assuming any regularity of $\partial\Omega$. Finally, the existence of a generalized principal eigenfunction is proved in [6] and [17], but not its simplicity.

2 Exploring other notions of generalized principal eigenvalue

In this section we show that the validity of the **MP** is not characterized by the positivity of λ_1, nor by that of other natural notions of generalized principal eigenvalue. One is the quantity $\overline{\lambda}_1(F,\Omega)$ defined before. Another natural candidate is

$$\lambda_*(F,\Omega) := \liminf_{\varepsilon\to 0^+} \lambda^\varepsilon,$$

where λ^ε denotes the classical Dirichlet principal eigenvalue of the regularized operator $-\varepsilon\Delta+F$ in Ω. If Ω is smooth and F is a uniformly elliptic linear operator with smooth coefficients, then the notions λ_1, $\overline{\lambda}_1$, μ_1, λ_* coincide. In the general case we only have that $\mu_1 \leq \overline{\lambda}_1 \leq \lambda_1$.

We now show that, even in the linear case, the sign of λ_1, λ_*, $\overline{\lambda}_1$ do not characterize the validity of the **MP** for degenerate elliptic operators.

Proposition 2.1. *For each of the following conditions:*

(i) $\lambda_1(F,\Omega) > 0$,

(ii) $\lambda_*(F,\Omega) > 0$,

(iii) $\overline{\lambda}_1(F,\Omega) > 0$,

there exists a degenerate elliptic linear operator F with smooth coefficients in Ω that does not satisfies the **MP** *property and yet satisfies that condition. Moreover, for cases (i) and (ii), such operator satisfies (H4).*

Proof. (i) Let $F[u] = \frac{x}{2}u' - u$ and $\Omega = (0,1)$. The function $u(x) = x(1-x)$ violates the **MP**, but $\lambda_1(F,\Omega) = +\infty$ (as it is seen by taking $\phi(x) = x^n$ in the definition, with $n \to +\infty$). We remark that in this case $\overline{\lambda}_1(F,\Omega) = \mu_1(F,\Omega) \leq 0$ by Theorems 1.3 and 4.4 below.

(ii) Let $F[u] = -2xu'$ and $\Omega = (0,1)$. For $\varepsilon > 0$ and $\phi \in C^2(\Omega)$, we have that

$$-\varepsilon\phi'' + \left(\frac{x^2}{\varepsilon} + 1\right)\phi = e^{\frac{x^2}{2\varepsilon}}(-\varepsilon D^2 + F)\left[\phi e^{-\frac{x^2}{2\varepsilon}}\right].$$

As a consequence, λ^ε coincides with the Dirichlet principal eigenvalue of the operator $-\varepsilon u''+\left(\frac{x^2}{\varepsilon}+1\right)u$, which is greater than or equal to 1. This shows that $\lambda_*(F,\Omega) \geq 1$. On the other hand, the indicator function of $\{0\}$ violates **MP**, because, as one can readily check, any smooth function φ touching it from above at some $x_0 \in [0,1)$ satisfies $F[\varphi](x_0) = 0$.

(iii) For this case, we give two examples, one with a first order and one with a second order operator. The operator $F[u] = -\sqrt{x}u'$ does not satisfies **MP** in $\Omega = (0,1)$, as it is seen by taking u equal to the indicator function of $\{0\}$. But, taking $\phi(x) = 2 - \sqrt{x}$ in the definition of $\overline{\lambda}_1$ yields $\overline{\lambda}_1(F,\Omega) \geq 1/4$.

An example of the second order is provided by $F[u] = -xu''$ and $\Omega = (0,1)$. As before, the indicator function of $\{0\}$ violates **MP**. On the other hand, the function $\phi(x) = 1 + \sqrt{x}$ satisfies

$$F[\phi] = \frac{1}{4\sqrt{x}} \geq \frac{1}{4} \geq \frac{1}{8}\phi \quad \text{in } (0,1),$$

whence $\overline{\lambda}_1(F,\Omega) \geq 1/8$. □

Remark 1. The two operators used as examples for case (iii) do not satisfy hypothesis (H4), hence Theorem 1.3 does not apply to them. Nevertheless, they do not violate the conclusion of the theorem because, as one can check, $\mu_1(F,\Omega) = 0$ in both cases, independently of the extension of F outside Ω. This seems to suggest that hypothesis (H4) in Theorem 1.3 could be relaxed.

Proposition 2.1 involves linear operators, for which the notion of viscosity solution could appear artificial. However, one cannot characterize the validity of the Maximum Principle for C^2 solutions in terms of the signs of λ_1, $\overline{\lambda}_1$, μ_1 or λ_*. Indeed any C^2 (or even C^0) subsolution of the equation $F[u] := x^2 u = 0$ in $\Omega = (-1,1)$ is necessarily nonpositive, but it is not hard to check that $\lambda_1(F,\Omega) = \overline{\lambda}_1(F,\Omega) = \mu_1(F,\Omega) = \lambda_*(F,\Omega) = 0$. Also, notice that the operators used in the proof of Proposition 2.1 would still yield the result under the additional requirement that $\phi \in C^2(\Omega)$ in the definitions of λ_1 and $\overline{\lambda}_1$.

The case (ii) in Proposition 2.1 shows that, for degenerate elliptic operators, the notion of generalized principal eigenvalue is unstable with respect to perturbations of the operator. Thus, owing to Theorem 1.3, the same is in some sense true for the **MP** property. We now present an example that exhibits the instability of the notions λ_1, $\overline{\lambda}_1$, μ_1 with respect to perturbations of the operator and approximations of the domain from inside. Let F be the operator defined by $F[u] = -xu'$, $x \in \Omega = (0,1)$. It turns out that

$$\lambda_1(F,\Omega) = \overline{\lambda}_1(F,\Omega) = \mu_1(F,\Omega) = 0,$$

$$\forall \varepsilon > 0, \quad \lambda_1(F - \varepsilon D,\Omega) = \overline{\lambda}_1(F - \varepsilon D,\Omega) = \mu_1(F - \varepsilon D,\Omega) = +\infty,$$

$$\forall \Omega' \subset\subset \Omega, \quad \lambda_1(F,\Omega') = \overline{\lambda}_1(F,\Omega') = \mu_1(F,\Omega') = +\infty.$$

The instability of the principal eigenvalue is one of the main differences with the uniformly elliptic case. In particular, the stability with respect to interior perturbations of the domain is crucial in the arguments of [3]. Its validity is based on the Harnack inequality, which is not available in the general degenerate elliptic case.

It is straightforward to check that μ_1 is stable with respect to perturbations of the domain from outside. If the same property holds for λ_1, $\overline{\lambda}_1$ then they coincide with μ_1. Proposition 2.1 shows that this is not always the case.

3 Proof of Theorem 1.3

We start by proving that the condition $\mu_1(F,\Omega) > 0$ is sufficient for the **MP** property to hold.

Proposition 3.1. *If (H1)-(H4) hold and $\mu_1(F,\Omega) > 0$ then F satisfies* **MP** *in Ω.*

Proof. If $\mu_1(F,\Omega) > 0$ then there exist $\lambda > 0$, $\Omega' \supset \overline{\Omega}$ and $\phi \in LSC(\Omega')$ such that

$$\phi > 0, \quad F[\phi] - \lambda\phi^\alpha \geq 0 \quad \text{in } \Omega'.$$

Up to shrinking Ω', it is not restrictive to assume that $\phi \in LSC(\overline{\Omega'})$ and $\phi > 0$ in $\overline{\Omega'}$. Assume by contradiction that (1) admits a subsolution u which is positive somewhere in $\overline{\Omega}$. We claim that the function $\tilde{u}$ defined by

$$\tilde{u}(x) := \begin{cases} \max(u(x), 0) & \text{if } x \in \overline{\Omega} \\ 0 & \text{otherwise,} \end{cases}$$

satisfies $F[\tilde{u}] \leq 0$ in Ω'. Indeed, if ψ is a smooth function touching $\tilde{u}$ from above at some $x_0 \in \Omega'$, then either $\tilde{u}(x_0) = 0$, or $\tilde{u}(x_0) = u(x_0) > 0$ and $x_0 \in \overline{\Omega}$. In the first case ψ has a local minimum at x_0 and then $F[\psi](x_0) \leq 0$ by (H1) and (H2), in the second case $F[\psi](x_0) \leq 0$ because u is a subsolution of (1). Next, up to replacing ϕ with $\left(\max_{\overline{\Omega}} \frac{\tilde{u}}{\phi}\right)\phi$, we can restrict the study to the case where $\max_{\overline{\Omega'}}(\tilde{u} - \phi) = 0$. Then, the standard doubling variable technique used to prove the comparison principle yields a contradiction (see Theorem 3.3 in [8]). Let us sketch the argument. Define the following function on $\overline{\Omega'} \times \overline{\Omega'}$:

$$\Phi(x,y) := \tilde{u}(x) - \phi(y) - \frac{n}{2}|x-y|^2.$$

Calling (x_n, y_n) a maximum point for Φ in $\overline{\Omega'} \times \overline{\Omega'}$, we see that

$$0 = \max_{x\in\overline{\Omega'}} \Phi(x,x) \leq \Phi(x_n,y_n) = \tilde{u}(x_n) - \phi(y_n) - \frac{n}{2}|x_n - y_n|^2.$$

It follows that $x_n - y_n = o(1)$ as $n \to \infty$. Whence, since $\tilde{u}(x_n) - \phi(y_n) \geq 0$, x_n and y_n converge (up to subsequences) to a point z where $\tilde{u} - \phi$ vanishes and $|x_n - y_n|^2 = o(n^{-1})$. In particular, $z \in \overline{\Omega}$. We can therefore apply Theorem 3.2 of [8] and find that

$$F(y_n, \phi(y_n), n(x_n - y_n), Y) - F(y_n, \tilde{u}(x_n), n(x_n - y_n), X) \geq \lambda\phi^\alpha(y_n),$$

for some $X, Y \in \mathcal{S}_N$ satisfying

$$-3n \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3n \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Since, as $n \to \infty$, $\tilde{u}(x_n) - \phi(y_n) = o(1)$, using (H3), (H4) we eventually derive

$$\lambda\phi^\alpha(y_n) \leq o(1) + \omega(n|x_n - y_n|^2 + |x_n - y_n|).$$

That is, $\phi(z) \leq 0$, which is a contradiction. □

Let us prove now that if F satisfies **MP** in Ω then $\mu_1(F,\Omega) > 0$. This is a consequence of the following general property of μ_1.

Proposition 3.2. *Under the assumptions (H1)-(H4), there exists a nonnegative subsolution $U \in USC(\overline{\Omega})$, $U \not\equiv 0$, of the problem*

$$\begin{cases} F[U] - \mu_1(F,\Omega)U^\alpha \leq 0 & \text{in } \Omega, \\ U \leq 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. We construct the subsolution U at the eigenlevel $\mu_1(F,\Omega)$ following the method of [6]: we solve the problem at level less than $\mu_1(F,\Omega)$ with a positive right-hand side (say equal to 1) and we show that as the level approaches $\mu_1(F,\Omega)$ the renormalized solutions tend to a function U satisfying the desired property. An extra difficulty with respect to [6] is that U could be positive somewhere on $\partial\Omega$, due to the lack of existence of barriers. In order to show that $U \leq 0$ on $\partial\Omega$ in the viscosity sense, we combine the above procedure with an external approximation of the domain Ω. This is the point where the definition of μ_1 is really exploited.

Let $(\Omega_n)_{n\in\mathbb{N}}$ be a family of smooth domains such that

$$\bigcap_{n\in\mathbb{N}} \overline{\Omega}_n = \overline{\Omega}, \qquad \forall n \in \mathbb{N}, \quad \overline{\Omega} \subset \Omega_{n+1} \subset \Omega_n \subset \mathcal{O}.$$

For $n \in \mathbb{N}$, we consider subsolutions of the equation

$$F[u] - \left(\mu_1(F,\Omega) - \frac{1}{n}\right) u^\alpha = 1 \quad \text{in } \mathcal{O}, \tag{2}$$

whose support is contained in $\overline{\Omega}_n$. Following Perron's method, we define

$$\forall x \in \mathcal{O}, \quad w_n(x) := \sup\{z(x) \ : \ z \in USC(\mathcal{O}) \text{ is a subsolution of (2)}, z = 0 \text{ outside } \overline{\Omega}_n\}.$$

The function w_n could possibly be infinite at some -and even any- point of $\overline{\Omega}_n$. Taking $z \equiv 0$ yields $w_n \geq 0$. We claim that

$$\lim_{n\to\infty} \sup_{\Omega_n} w_n = +\infty. \tag{3}$$

Assume by way of contradiction that (3) does not hold. Then $(w_n)_{n\in\mathbb{N}}$ satisfies (up to subsequences) $\sup_{\Omega_n} w_n \leq C$, for some C independent of n. For $n \in \mathbb{N}$, consider the lower and upper semicontinuous envelopes of w_n:

$$\forall x \in \mathcal{O}, \quad (w_n)_*(x) := \lim_{r\to 0^+} \inf_{|y-x|<r} w_n(y), \quad (w_n)^*(x) := \lim_{r\to 0^+} \sup_{|y-x|<r} w_n(y).$$

It follows from the standard theory (see Lemma 4.2 in [8]) that $(w_n)^*$ is a subsolution of (2). Since the function w_n vanishes outside $\overline{\Omega}_n$, its definition yields $w_n = (w_n)^*$. By Lemma 4.4 in [8], if $(w_n)_*$ fails to be a supersolution of (2) at some point in Ω_n then there exists a subsolution of (2) larger than w_n and still vanishing outside $\overline{\Omega}_n$, which contradicts the definition of w_n. Therefore, $F[(w_n)_*] - (\mu_1(F,\Omega) + 1/n)((w_n)_*)^\alpha \geq 1$ in Ω_n, and clearly $0 \leq (w_n)_* \leq w_n \leq C$. As a consequence,

$$F[(w_n)_*] - \left(\mu_1(F,\Omega) + \frac{1}{n}\right)((w_n)_*)^\alpha \geq -\frac{2}{n}((w_n)_*)^\alpha + 1 \geq 1 - \frac{2C^\alpha}{n} \quad \text{in } \Omega_n.$$

It follows that, for n large enough, $(w_n)_*$ satisfies $F[(w_n)_*] - (\mu_1(F,\Omega) + 1/n)((w_n)_*)^\alpha > 0$ in Ω_n, and by (H3) the same is true for $(w_n)_* + \varepsilon$, with $\varepsilon > 0$ small enough. This contradicts the definition of μ_1, hence (3) is proved. There exists then a family $(z_n)_{n\in\mathbb{N}}$, with $z_n \in USC(\mathcal{O})$ subsolution of (2) vanishing outside $\overline{\Omega}_n$, such that

$$\lim_{n\to\infty} \max_{\overline{\Omega}_n} z_n = +\infty.$$

Replacing z_n with its positive part, it is not restrictive to assume that $z_n \geq 0$. The functions u_n defined by

$$u_n(x) := \frac{z_n(x)}{\max_{\overline{\Omega}_n} z_n},$$

satisfy

$$u_n = 0 \text{ outside } \overline{\Omega}_n, \qquad \max_{\mathcal{O}} u_n = 1, \qquad F[u_n] - \left(\mu_1(F,\Omega) - \frac{1}{n}\right) u_n^\alpha \leq \left(\max_{\overline{\Omega}_n} z_n\right)^{-\alpha} \quad \text{in } \mathcal{O}.$$

Define the function U by setting

$$\forall x \in \mathcal{O}, \quad U(x) := \lim_{j\to\infty} \sup\{u_n(y) \ : \ n \geq j, \ |x-y| < 1/j\}.$$

By stability of viscosity subsolutions (see e.g. Remark 6.3 in [8]), we know that U satisfies $F[U] - \mu_1(F,\Omega)U^\alpha \leq 0$ in $\mathcal{O}$. Moreover, $U = 0$ outside $\overline{\Omega}$ and $\max_{\overline{\Omega}} U = 1$. It remains to show that U satisfies the Dirichlet condition on $\partial\Omega$ in the relaxed viscosity sense. Suppose that there exists $\xi \in \partial\Omega$, $\rho > 0$ and $\varphi \in C^2(\overline{\Omega})$ such that

$$U(\xi) > 0, \qquad \sup_{\overline{\Omega}\cap B_\rho(\xi)} (U - \varphi) = (U - \varphi)(\xi) = 0.$$

By continuity of F, we can assume that φ is a paraboloid, thus defined in the whole $\mathbb{R}^N$. Up to decreasing ρ if need be, we have that $\varphi > 0$ in $B_\rho(\xi)$. Since $U = 0$ outside $\overline{\Omega}$, we infer that $\sup_{\mathcal{O}\cap B_\rho(\xi)}(U - \varphi) = (U - \varphi)(\xi) = 0$, whence $F[\varphi](\xi) - \mu_1(F,\Omega)\varphi^\alpha(\xi) \leq 0$. □

As a corollary, we immediately deduce that if F satisfies **MP** in Ω then $\mu_1(F,\Omega) > 0$.

Remark 2. The function U constructed in the above proof is a good candidate for being the principal eigenfunction of F in Ω, i.e., a positive solution of

$$\begin{cases} F[U] - \mu_1(F,\Omega)U^\alpha = 0 & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

However, this is not true in general. There are indeed operators which do not admit a principal eigenfunction. It is clearly the case if $\mu_1(F,\Omega) = +\infty$, as for instance for the operator $F[u] = u'$. An example with $\mu_1(F,\Omega)$ finite is given by the operator $F[u] = x^2u'$ in $\Omega = (-1,1)$. Indeed, the indicator function of $\{0\}$ violates **MP**, and then $\mu_1(F,\Omega) \leq 0$ by Theorem 1.3. On the other hand, $\mu_1(F,\Omega) \geq 0$, as it is seen by taking $\phi \equiv 1$ in the definition. Hence, $\mu_1(F,\Omega) = 0$. But the unique solution of (4) is $U \equiv 0$.

4 Conditions for the equivalence between μ_1 and $\overline{\lambda}_1$

Theorem 1.3 provides a characterization of the **MP** property in terms of the sign of the generalized principal eigenvalue μ_1. We do not know if μ_1 can be replaced by the more intrinsic notion $\overline{\lambda}_1$, that is, if μ_1 and $\overline{\lambda}_1$ always have the same sign. This property reduces to the equivalence of μ_1 and $\overline{\lambda}_1$, because they satisfy

$$\forall \lambda \in \mathbb{R}, \quad \overline{\lambda}_1(F[u] + \lambda u^\alpha, \Omega) = \overline{\lambda}_1(F, \Omega) + \lambda, \quad \mu_1(F[u] + \lambda u^\alpha, \Omega) = \mu_1(F, \Omega) + \lambda.$$

Let us see what happens if we try to follow the arguments in the proof of Proposition 3.1 with $\mu_1(F, \Omega)$ replaced by $\overline{\lambda}_1(F, \Omega)$. The difference is that now the supersolution ϕ is only defined in Ω, but still has positive infimum. Setting $\phi(\xi) := \liminf_{x\to\xi} \phi(x)$ for $\xi \in \partial\Omega$, one sees that the arguments fail only if the points y_n used in the proof belong to $\partial\Omega$. This difficulty can be overcome if at any $\xi \in \partial\Omega$ one of the following occurs:

$$\text{any subsolution } u \in USC(\overline{\Omega}) \text{ of (1) satisfies } u(\xi) \leq 0, \tag{5}$$

$$\begin{array}{c}\text{any strictly positive supersolution } \phi \text{ of } F[\phi] = \lambda \phi^\alpha \text{ in } \Omega,\ \lambda > 0, \\ \text{is a supersolution in } \Omega \cup \Gamma, \text{ for some neighbourhood } \Gamma \text{ of } \xi.\end{array} \tag{6}$$

Indeed, the limit ξ of (a subsequence of) y_n cannot satisfy (5), but if (6) holds one can conclude exactly as in the the proof of Proposition 3.1.

This Section is devoted to establish sufficient conditions for either (5) or (6) to occurr, in order to have $\mu_1 = \overline{\lambda}_1$. Under suitable assumptions on F, the case (5) is guaranteed by the existence of a continuous *barrier* (see Definition 4.1 below). This is shown in Section 4.1. In Section 4.2 we show that, for linear operators, $\mu_1 = \overline{\lambda}_1$ if the boundary only contains connected components where the so–called Fichera condition is satisfied or violated.

4.1 Problems with barriers

Here is the definition of barrier.

Definition 4.1. We say that a point $\xi \in \partial\Omega$ admits a (continuous) *barrier* if there exists a ball B centred at ξ and a nonnegative function $w \in C(\overline{\Omega \cap B})$ vanishing at ξ and satisfying $F[w] \geq 1$ in $\Omega \cap B$.

We will need the following extra assumptions on F in an open neighbourhood V of $\partial\Omega$:

(H5) For all $R > 0$, $(r, p, X) \mapsto F(x, r, p, X)$ is uniformly continuous in $[0, R] \times \mathbb{R}^N \times \mathcal{S}_N$, uniformly with respect to $x \in \overline{\Omega} \cap V$.

(H6) For all $R > 0$, there exists $K > 0$ such that if $X, Y \in \mathcal{S}_N$ satisfy

$$\exists \sigma > 0, \quad -\sigma \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \sigma \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

then

$$F(y, r, p, Y) - F(x, r, p, X) \leq K(1 + |x-y||p| + \sigma|x-y|^2), \quad \forall x, y \in \overline{\Omega} \cap V,\ |r| \leq R,\ p \in \mathbb{R}^N.$$

Remark 3. Condition (H5) implies that the degree of homogeneity α in (H2) must be less than or equal to 1.

Overall, conditions (H4)–(H6) (or close variations) are often required in the context of comparison of viscosity solutions possibly discontinuous at the boundary, see e.g. assumptions (7.15)–(7.16) in [8], Section 7.

Theorem 4.2. *If (H1), (H2), (H4)-(H6) hold, Ω is smooth and every point $\xi \in \partial\Omega$ admits a barrier, then $\mu_1(F,\Omega) = \overline{\lambda}_1(F,\Omega)$.*

As explained before, in order to prove the result it is sufficient to show that (5) holds at every $\xi \in \partial\Omega$. Theorem 4.2 is then a consequence of the following

Proposition 4.3. *Assume that F satisfies (H1), (H2), (H5), (H6), that Ω is a smooth domain and that there exists $\xi \in \partial\Omega$ admitting a barrier. Then every subsolution $u \in USC(\overline{\Omega})$ of* (1) *satisfies $u(\xi) \le 0$.*

Proof. Let $w \in C(\overline{\Omega \cap B})$ be the barrier at ξ, provided by Definition 4.1. Conditions (H2), (H5) imply that, up to replacing w with $2w + k|x-\xi|^2$, with $k > 0$ small enough, it is not restrictive to assume that $w > 0$ outside the point ξ. We can also suppose without loss of generality that $w \ge 1 > u$ on $\overline{\Omega} \cap \partial B$. Assume by contradiction that $u(\xi) > 0$. For $\varepsilon > 0$, we set

$$w_\varepsilon = w + \varepsilon, \qquad k_\varepsilon := \max_{\overline{B\cap\Omega}} \frac{u}{w_\varepsilon}.$$

Let x_ε be a point where k_ε is attained. Since $k_\varepsilon \ge \frac{u(\xi)}{\varepsilon}$, we have that, as $\varepsilon \to 0^+$, $k_\varepsilon \to \infty$, whence $x_\varepsilon \to \xi$. Then, it makes sense to use $\nu(x_\varepsilon) := Dd(x_\varepsilon)$, where $d(x)$ is the signed distance function from $\partial\Omega$, positive inside Ω and smooth in a neighbourhood of $\partial\Omega$. We follow now the strategy of the strong comparison principle when comparing a continuous supersolution with a possibly discontinuous subsolution, see Theorem 7.9 in [8]. We consider the function

$$\Phi(x,y) = u(x) - k_\varepsilon w_\varepsilon(y) - |n(x-y) + \delta\nu(x_\varepsilon)|^2 - \delta|x - x_\varepsilon|^2 \qquad x, y \in B_\rho \cap \overline{\Omega},$$

where n, $\delta > 0$. Let then $(x_n, y_n) \in \overline{\Omega}$ be such that

$$\Phi(x_n, y_n) = \max_{\overline{B\cap\Omega}} \Phi(x,y).$$

Of course the two points also depend on δ, ε but we avoid to stress this fact to simplify the notation. We have that $\Phi(x_n,y_n) \ge \Phi(x_\varepsilon, x_\varepsilon) = -\delta^2$. Furthermore, for n large, $x_\varepsilon + \frac{\delta}{n}\nu(x_\varepsilon) \in B \cap \Omega$, hence $\Phi(x_n,y_n) \ge \Phi(x_\varepsilon, x_\varepsilon + \frac{\delta}{n}\nu(x_\varepsilon))$, which implies

$$|n(x_n - y_n) + \delta\nu(x_\varepsilon)|^2 + \delta|x_n - x_\varepsilon|^2 \le u(x_n) - k_\varepsilon w_\varepsilon(y_n) - u(x_\varepsilon) + k_\varepsilon w_\varepsilon(x_\varepsilon + \frac{\delta}{n}\nu(x_\varepsilon)).$$

Since w is continuous, we have $w_\varepsilon(x_\varepsilon + \frac{\delta}{n}\nu(x_\varepsilon)) = w_\varepsilon(x_\varepsilon) + o(1)$ as $n \to \infty$, hence, using also that $k_\varepsilon w_\varepsilon(x_\varepsilon) = u(x_\varepsilon)$, we deduce

$$|n(x_n - y_n) + \delta\nu(x_\varepsilon)|^2 + \delta|x_n - x_\varepsilon|^2 \le u(x_n) - k_\varepsilon w_\varepsilon(y_n) + o(1) \quad \text{as } n \to \infty.$$

We first use this inequality to infer that both x_n and y_n converge to x_ε as n tends to infinity. Then, together with the upper semicontinuity of u and the fact that $u - k_\varepsilon w \le 0$, it implies that $n(x_n - y_n) + \delta\nu(x_\varepsilon) = o(1)$ as $n \to \infty$. Since ν is continuous we eventually derive

$$y_n = x_n + \frac{\delta}{n}\nu(x_n) + o\left(\frac{1}{n}\right) \qquad \text{as } n \to \infty.$$

It follows that $y_n \in \Omega$ for n large. This allows us to use the equation of w as a supersolution. As far as u is concerned, we have

$$u(x_n) \ge \Phi(x_n, y_n) + k_\varepsilon w_\varepsilon(y_n) \ge \varepsilon k_\varepsilon - \delta^2 .$$

Choosing δ small enough, compared to $\varepsilon k_\varepsilon$, we get that $u(x_n) > 0$ so that we can use the equation of u at x_n even if $x_n \in \partial\Omega$. Usual viscosity arguments (see Theorem 3.2 in [8]) yield

$$F(y_n, k_\varepsilon w_\varepsilon(y_n), q, Y) - F(x_n, u(x_n), p, X) \ge k_\varepsilon^\alpha , \tag{7}$$

where $p = 2n(n(x_n - y_n) + \delta\nu(x_\varepsilon)) + 2\delta(x_n - x_\varepsilon)$, $q = 2n(n(x_n - y_n) + \delta\nu(x_\varepsilon))$, X, Y satisfy

$$-(2n^2 + \|A\|)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le A + \frac{1}{2n^2}A^2, \qquad A = \begin{pmatrix} 2n^2 I + 2\delta I & -2n^2 I \\ -2n^2 I & 2n^2 I \end{pmatrix}.$$

Since

$$\|A\| \le 4n^2 + 2\delta, \qquad A^2 \le 4n^2(2n^2 + \delta)\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 4\delta(\delta + n^2)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

we derive

$$-(6n^2 + 2\delta + \beta)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X - \beta I & 0 \\ 0 & -(Y + \beta I) \end{pmatrix} \le (6n^2 + 2\delta)\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \quad \beta = 4\delta + \frac{2\delta^2}{n^2}.$$

Hence, by (H6), there is K such that, for bounded r,

$$F(y_n, r, q, Y + \beta I) - F(x_n, r, q, X - \beta I) \le K[1 + (6n^2 + 2\delta + \beta)|x_n - y_n|^2 + |x_n - y_n|(|q| + 1)].$$

Since $-\delta^2 \le u(x_n) - k_\varepsilon w_\varepsilon(y_n) \le o(1)$ as $n \to \infty$, by (H5) we can choose δ small enough and n large in such a way that

$$F(y_n, k_\varepsilon w_\varepsilon(y_n), q, Y) - F(x_n, u(x_n), p, X) \le 2K.$$

Whence, by (7), $k_\varepsilon^\alpha \le 2K$, which is impossible since $k_\varepsilon \to \infty$ as $\varepsilon \to 0^+$. □

Remark 4. At least if F is linear, the existence of a global smooth barrier implies that $\mu_1 = \lambda_1$.

Namely, assume that there exists $v \in C^2$ such that $F[v] \ge 1$ in some neighbourhood of $\partial\Omega$ and $v = 0$ on $\partial\Omega$. Let $\lambda > 0$ be such that $F[\varphi] \ge \lambda\varphi$ for some $\varphi \in LSC(\Omega) \cap L^\infty(\Omega)$ such that $\varphi > 0$ in Ω. There exists $\rho > 0$ (only depending on λ) such that $F[v] \ge \lambda v + \frac{1}{2}$ if

$d(x) < \rho$. Let us take any small $\varepsilon > 0$; if ζ is a cut–off function such that $\zeta = 1$ if $d(x) < \rho$ and $\zeta = 0$ if $d(x) \geq 2\rho$, we claim that $w := \varphi + \delta(v+\varepsilon)\zeta$ is a supersolution of $F[w] \geq (\lambda - \varepsilon)w$ in Ω up to choosing a suitable δ. Indeed, if $d(x) \geq 2\rho$, since $\zeta = 0$ we have

$$F[w] = F[\varphi] \geq \lambda\varphi = \lambda w,$$

whereas, if $d(x) < \rho$, since $\zeta = 1$ we see that

$$F[w] \geq F[\varphi] + F[\delta(v+\varepsilon)] \geq \lambda\varphi + \delta\lambda v + \frac{1}{2}\delta + c(x)\delta\varepsilon \geq \lambda w + \delta(\frac{1}{2} - (\lambda + |c(x)|)\varepsilon) \geq \lambda w$$

if ε is small. In the set $\{\rho < d(x) < 2\rho\}$ we have $F[\delta(v+\varepsilon)\zeta] \geq -C(v,\zeta)\delta$ and $\inf \varphi > 0$, whence

$$F[w] \geq \lambda\varphi - C(v,\zeta)\delta \geq (\lambda - \varepsilon)w + \varepsilon\varphi - \tilde{C}\delta \geq (\lambda - \varepsilon)w$$

provided δ is sufficiently small. Finally, for any $\varepsilon > 0$ we can find δ such that $F[w] \geq (\lambda - \varepsilon)w$ in Ω, and since $w > 0$ in $\overline{\Omega}$ we deduce that $\lambda_1''(\Omega) \geq \lambda - \varepsilon$. Since λ was any value smaller than λ_1 and ε is arbitrary, we conclude that $\lambda_1'' \geq \lambda_1$, and therefore $\lambda_1'' = \lambda_1 = \mu_1$.

4.2 Linear operators

In this section F is a degenerate elliptic linear operator. Namely,

$$F[u] = -\mathrm{Tr}(A(x)D^2u) - b(x) \cdot Du - c(x)u, \quad x \in \mathcal{O},$$

with $A = \Sigma^t\Sigma$, $\Sigma : \overline{\Omega} \to \mathcal{S}_N$, $b : \overline{\Omega} \to \mathbb{R}^N$ and $c : \overline{\Omega} \to \mathbb{R}$. We will require that

$$\Sigma, b \in W^{1,\infty}(\overline{\Omega}), \qquad c \in C(\overline{\Omega}), \tag{8}$$

As shown in Example 3.6 of [8], the Lipschitz continuity of Σ is precisely the condition for the second order term to satisfy (H4). The Lipschitz continuity of b could be relaxed by

$$\exists K > 0, \quad \forall x, y \in \overline{\Omega}, \quad (b(x) - b(y)) \cdot (x - y) \geq -K|x-y|^2 \tag{9}$$

in order to fulfil (H4), but $b \in W^{1,\infty}(\overline{\Omega})$ is needed to have (H6).

We say that the *Fichera condition* is satisfied at a point $\xi \in \partial\Omega$ if one of the following two cases occurs:

$$Dd(\xi)A(\xi)Dd(\xi) > 0, \quad \text{or} \quad \begin{cases} Dd(\xi)A(\xi)Dd(\xi) = 0 \\ \mathrm{Tr}(A(\xi)D^2d(\xi)) + b(\xi) \cdot Dd(\xi) < 0, \end{cases}$$

where, as before, d is the signed distance function from $\partial\Omega$, positive inside Ω. This condition was introduced by G. Fichera in [10] in order to study the question whether the Dirichlet condition should be assumed or not at boundary points. See also [1] for a discussion of the same problem in terms of viscosity solutions.

Theorem 4.4. *If* (8) *holds, Ω is smooth and in every connected component of $\partial\Omega$ the Fichera condition is either always satisfied or always violated, then $\mu_1(F,\Omega) = \overline{\lambda}_1(F,\Omega)$.*

Remark 5. The previous result applies to a significant example, namely to the case that the domain Ω is invariant for the associated stochastic dynamics $dX_t = b(X_t)dt + \sqrt{2}\,dW_t$ defined in a standard probability space, being W_t a Wiener process in $\mathbb{R}^N$. In fact, it is well known that Ω is invariant (and, at the same time, $\overline{\Omega}$ is invariant) if and only if the Fichera condition is violated everywhere on the boundary, see e.g. [12], [13] and [7] for a complete discussion of this property even in non smooth domains.

Recall that the result would follow if we show that (5) or (6) hold at every $\xi \in \partial\Omega$. One can readily check that the Fichera condition implies that, for $\delta > 0$ small enough, the function $w(x) := \log(\delta + d(x)) - \log\delta$ is a barrier at ξ in the sense of Definition 4.1. Thus, by Proposition 4.3, (5) holds in the connected components where the Fichera condition is fulfilled. Let us show that (6) holds in the others. Since the Fichera condition does not involve the zero order term of the operator, we can restrict to $\lambda = 0$ in (6). Hence, the proof of Theorem 4.4 relies on the following result, which is essentially proved in [2], Lemma 4.1. For the sake of clarity, since there are minor differences in our setting, we provide a simple proof below.

Lemma 4.5. *Assume that* (8) *holds,* Ω *is smooth and the Fichera condition is not satisfied in an open subset* Γ *of* $\partial\Omega$ *(in the induced topology), that is,*

$$\forall \xi \in \Gamma, \quad Dd(\xi)A(\xi)Dd(\xi) = 0, \quad \mathrm{Tr}(A(\xi)D^2d(\xi)) + b(\xi)\cdot Dd(\xi) \geq 0.$$

Then, any supersolution $\phi \in LSC(\Omega)$ *of* $F = 0$ *in* Ω*, which is bounded from below, extended to* Γ *by setting*

$$\forall \xi \in \Gamma, \quad \phi(\xi) := \liminf_{\substack{x\to\xi \\ x\in\Omega}} \phi(x),$$

is a supersolution in $\Omega \cup \Gamma$*.*

Proof. In this statement, we use the convention that ϕ automatically satisfies the condition of being a supersolution at the points $\xi \in \Gamma$ where $\phi(\xi) = +\infty$. Let $\xi \in \Gamma$ and $\psi \in C^2(\Omega\cup\Gamma)$ be such that $(\phi - \psi)(\xi) = \min_{\overline{\Omega}\cap B}(\phi - \psi) = 0$, for some closed ball B (with positive radius) centered at ξ satisfying $B \cap \partial\Omega \subset \Gamma$. Our aim is to show that $F[\psi](\xi) \geq 0$. By usual arguments, it is not restrictive to assume that the above minimum is strict. Consider the family of functions $(\psi_\varepsilon)_{\varepsilon>0}$ defined in Ω by $\psi_\varepsilon(x) := \psi(x) + \varepsilon\log(d(x))$. Let $(x_\varepsilon)_{\varepsilon>0}$ in $\Omega \cap B$ be such $(\phi - \psi_\varepsilon)(x_\varepsilon) = \min_{\Omega\cap B}(\phi - \psi_\varepsilon)$, and let $\zeta \in \overline{\Omega} \cap B$ be the limit as $\varepsilon \to 0^+$ of (a subsequence of) x_ε. For $x \in \Omega \cap B$, we see that

$$\begin{aligned}(\phi-\psi)(x) &= \lim_{\varepsilon\to 0^+}(\phi-\psi_\varepsilon)(x) \geq \liminf_{\varepsilon\to 0^+}(\phi-\psi_\varepsilon)(x_\varepsilon) \geq (\phi-\psi)(\zeta) - \limsup_{\varepsilon\to 0^+}\varepsilon\log(d(x_\varepsilon)) \\ &\geq (\phi-\psi)(\zeta).\end{aligned}$$

Since this holds for any $x \in \Omega \cap B$, applying this inequality to a sequence of points along which ϕ tends to $\phi(\xi)$, we infer that $\zeta = \xi$, because $\phi - \psi$ has a strict minimum at ξ. This shows that $x_\varepsilon \to \xi$ as $\varepsilon \to 0^+$. In particular, since $x_\varepsilon \notin \partial B$, we deduce, being ϕ a supersolution in Ω, that

$$[-\mathrm{Tr}(AD^2\psi) - b\cdot D\psi - c\phi - \varepsilon(d^{-1}\mathrm{Tr}(AD^2d) - d^{-2}DdADd + d^{-1}b\cdot Dd)](x_\varepsilon) \geq 0.$$

This inequality reads as

$$[F[\psi] + c(\psi - \phi) - \varepsilon(d^{-1}\mathrm{Tr}(AD^2 d) - d^{-2} DdADd + d^{-1} b \cdot Dd)](x_\varepsilon) \geq 0.$$

For ε small enough, x_ε has a unique projection ξ_ε on $\partial\Omega$, the function d is smooth in a neighbourhood of x_ε and satisfies $Dd(x_\varepsilon) = (x_\varepsilon - \xi_\varepsilon)/|x_\varepsilon - \xi_\varepsilon| =: -\nu(\xi_\varepsilon)$. Up to decreasing ε, we have that $\xi_\varepsilon \in \Gamma$ because $x_\varepsilon, \xi_\varepsilon \to \xi$ as $\varepsilon \to 0$. It follows that $\Sigma(\xi_\varepsilon)\nu(\xi_\varepsilon) = 0$ and thus $DdADd(x_\varepsilon) = \|\Sigma(x_\varepsilon)\nu(\xi_\varepsilon)\|^2 \geq -l^2 d^2(x_\varepsilon)$, where l is the Lipschitz constant of Σ. On the other hand, the Lipschitz continuity of Σ and b imply the existence of a constant C such that

$$(\mathrm{Tr}(AD^2 d) + b \cdot Dd)(x_\varepsilon) \geq (\mathrm{Tr}(AD^2 d) + b \cdot Dd)(\xi_\varepsilon) - Cd(x_\varepsilon).$$

Since $\xi_\varepsilon \in \Gamma$, we have that $(\mathrm{Tr}(AD^2 d) + b \cdot Dd)(\xi_\varepsilon) \geq 0$ and then, using the above inequalities, we obtain

$$F[\psi](x_\varepsilon) \geq -\varepsilon(C + l^2) + \sup_\Omega |c|(\phi - \psi)(x_\varepsilon). \tag{10}$$

Since x_ε is a minimum point for $\phi - \psi_\varepsilon$, we have that

$$\forall x \in \Omega \cap B, \quad \phi(x_\varepsilon) - [\psi(x_\varepsilon) + \varepsilon \log(d(x_\varepsilon))] \leq \phi(x) - [\psi(x) + \varepsilon \log(d(x))].$$

Notice that $\log(d(x_\varepsilon)) < 0$ for ε small enough, whence

$$\forall x \in \Omega \cap B, \quad \limsup_{\varepsilon \to 0^+} (\phi - \psi)(x_\varepsilon) \leq (\phi - \psi)(x).$$

Choosing in place of x a sequence of points converging to ξ, along which ϕ tends to $\phi(\xi)$, we eventually infer that $(\phi - \psi)(x_\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Therefore, passing to the limit in (10) we deduce $F[\psi](\xi) \geq 0$, which concludes the proof. □

Remark 6. We do not know whether or not μ_1 and $\overline{\lambda}_1$ do coincide when $\partial\Omega$ has a connected component containing both points where the Fichera condition is satisfied and points where it is not. The problem is that positive supersolutions in Ω may not be supersolutions at the points ξ that satisfy the Fichera condition but belong to the boundary of the set where the Fichera condition does not hold. In such case, one could replace the perturbation $\varepsilon \log(d(x))$ used in the proof of Lemma 4.5 with $\varepsilon \log(|x - \xi|)$, and the perturbation terms could be controlled if the sequences $(x_n)_{n\in\mathbb{N}}$ converging to ξ on which ϕ tends to $\phi(\xi)$ satisfy $d(x_n, \partial\Omega) \gtrsim |x_n - \xi|$. This is the so-called *cone condition*, namely that the value of φ at $\partial\Omega$ may be reached along at least one sequence of points lying in a cone. The relevance of this condition for strong comparison results (i.e. comparison of viscosity solutions discontinuous at the boundary) was already pointed out before and specifically in connection with stochastic control problems, see [14], [2]. In particular, the conclusion of Lemma 4.5 would still hold if the cone condition is fulfilled at any point of the boundary (or at least at those points where a barrier does not exist).

References

[1] Barles G.; Burdeau J., The Dirichlet problem for semilinear second–order degenerate elliptic equatiions and applications to stochastic exit time control problems. *Comm. Partial Differential Equations* **20** (1995) no. 1-2, 129–178.

[2] Barles, G.; Rouy, E. A strong comparison result for the Bellman equation arising in stochastic exit time control problems and its applications. *Comm. Partial Differential Equations* **23** (1998), no. 11-12, 1995–2033.

[3] Berestycki, H.; Nirenberg, L.; Varadhan, S. R. S. The principal eigenvalue and maximum principle for second-order elliptic operators in general domains. *Comm. Pure Appl. Math.* **47** (1994), no. 1, 47–92.

[4] Berestycki, H.; Rossi, L. Generalizations and properties of the principal eigenvalue of elliptic operators in unbounded domains. *To appear in Comm. Pure Appl. Math.*

[5] Birindelli, I.; Demengel, F. First eigenvalue and Maximum Principle for fully nonlinear singular operators. *Adv. Differential Equations* **11** (2006), no. 1, 91–119.

[6] Birindelli, I.; Demengel, F. Eigenvalue, maximum principle and regularity for fully non linear homogeneous operators. *Commun. Pure Appl. Anal.* **6** (2007), no. 2, 335–366.

[7] Cannarsa, P.; Da Prato, G.; Frankowska, H. Invariant measures associated to degenerate elliptic operators. *Indiana Univ. Math. J.* **59** (2010), no. 1, 53–78.

[8] Crandall, M. G.; Ishii, H.; Lions, P.-L. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)* **27** (1992), no. 1, 1–67.

[9] Felmer, P. L.; Quaas, A. Positive radial solutions to a 'semilinear' equation involving the Pucci's operator. *J. Differential Equations* **199** (2004), no. 2, 376–393.

[10] Fichera, G. Sulle equazioni differenziali lineari ellittico-paraboliche del secondo ordine. *Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I. (8)* **5** (1956), 1–30.

[11] Franchi, B.; Lanconelli, E. Une métrique associée à une classe d'opérateurs elliptiques dégénérés. *Rend. Sem. Mat. Univ. Politec. Torino* (1983), no. Special Issue, 105–114 (1984). Conference on linear partial and pseudodifferential operators (Torino, 1982).

[12] Friedman, A. *Stochastic differential equations and applications*. Vol. 1 & 2. Probability and Mathematical Statistics, Vol. 28. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975-1976.

[13] Friedman, A.; Pinsky M.A. Asymptotic stability and spiraling properties for solutions of stochastic equations. *Trans. Amer. Math. Soc.* **186** (1973), 331–358.

[14] Katsoulakis M. A., Viscosity solutions of second order fully nonlinear elliptic equations with state constraints. *Indiana Univ. Math. J.* **43** (1994) n. 2, 493–519.

[15] Harvey, F. R.; Lawson, H. B. Characterizing the Strong Maximum Principle. *Preprint.* http://arxiv-web3.library.cornell.edu/abs/1309.1738

[16] Ishii, H. Perron's method for Hamilton-Jacobi equations. *Duke Math. J.* **55** (1987), no. 2, 369–384.

[17] Juutinen, P. Principal eigenvalue of a very badly degenerate operator and applications. *J. Differential Equations* **236** (2007), no. 2, 532–550.

[18] Quaas, A. Existence of a positive solution to a “semilinear” equation involving Pucci’s operator in a convex domain. *Differential Integral Equations* **17** (2004), no. 5-6, 481–494.

[19] Quaas, A.; Sirakov, B. Principal eigenvalues and the Dirichlet problem for fully nonlinear elliptic operators. *Adv. Math.* **218** (2008), no. 1, 105–135.