GROUP AND POLYNOMIAL IDENTITIES IN GROUP RINGS

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To Antonio Giambruno on his 70th birthday.

Abstract. In the 80s Brian Hartley conjectured that if the unit group, $\mathcal{U}(FG)$, of a torsion group ring FG satisfies a group identity, then FG satisfies a polynomial identity. The aim of this survey is to review the most relevant results that arose from the proof of this conjecture and discuss some recent developments and open questions concerning $*$-group identities for $\mathcal{U}(FG)$ and group identities for the subgroup of its unitary units.

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1. Introduction

Throughout let F be a field of characteristic $p \geq 0$ and G a group. Write $\mathcal{U}(FG)$ for the unit group of the group ring FG. We say that a subset $S \subseteq \mathcal{U}(FG)$ satisfies a *group identity* if there exists a non-trivial reduced word $w(x_1, \ldots, x_n)$ in the free group on countably many generators, $\langle x_1, x_2, \ldots \rangle$, such that $w(g_1, \ldots, g_n) = 1$ for all $g_1, \ldots, g_n \in S$.

In the attempt to connect the algebraic structure of FG with the group structure of its unit group, Brian Hartley made the following conjecture.

Conjecture 1.1. *Let G be a torsion group. If $\mathcal{U}(FG)$ satisfies a group identity, then FG satisfies a polynomial identity.*

We recall that a subset $V \subseteq FG$ satisfies a *polynomial identity* if there exists a non-zero element $f(x_1, \ldots, x_m)$ in the free algebra on non-commuting indeterminates $F\{x_1, x_2, \ldots\}$ such that $f(a_1, \ldots, a_m) = 0$ for all $a_i \in V$.

When F is infinite, after a first result of Gonçalves and Mandel [13] dealing with the special case of semigroup identities (that is, identities of the form $x_{i_1} \cdots x_{i_k} = x_{j_1} \cdots x_{j_l}$), Giambruno, Jespers and Valenti [5] confirmed Hartley's Conjecture under the assumption that G does not contain elements of p-power order if $p > 0$. Some years later Giambruno, Sehgal and Valenti [10] were able to remove the hypothesis on G.

Finally, modifying the original proof of [10], Liu [22] positively answered the question for fields of any size.

Many years before, Isaacs and Passman (see Corollaries 5.3.8 and 5.3.10 of [24]) characterized group rings satisfying a polynomial identity. Recalling

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that a group G is called p-abelian if its commutator subgroup, G', is a finite p-group, and that 0-abelian means abelian, their result was as follows.

Theorem 1.2. *The group ring FG satisfies a polynomial identity if, and only if, G has a p-abelian subgroup of finite index.*

From this theorem one deduces that if FG satisfies a polynomial identity, then $\mathcal{U}(FG)$ does not necessarily satisfy a group identity. In fact, one of the easiest consequences of the solution of Conjecture 1.1 is that, in characteristic 0, $\mathcal{U}(FG)$ satisfies a group identity if, and only if, G is abelian (see Corollary 1.2.21 of [17]). Thus, if you take any finite non-abelian group G, then $\mathbb{C}G$ satisfies a polynomial identity, but $\mathcal{U}(\mathbb{C}G)$ does not satisfy a group identity.

Anyway, the positive solution of Hartley's Conjecture was the crucial step leading to the establishment of necessary and sufficient conditions for $\mathcal{U}(FG)$ to satisfy a group identity. This was first done by Passman [25] for infinite fields and then by Liu and Passman [23] for arbitrary fields.

Theorem 1.3. *Let $p > 0$ and G a torsion group. If G' is a p-group, then the following are equivalent:*

(i) *$\mathcal{U}(FG)$ satisfies a group identity;*
(ii) *$\mathcal{U}(FG)$ satisfies the group identity $(x_1^{-1}x_2^{-1}x_1x_2)^{p^r} = 1$ for some positive integer r;*
(ii) *FG satisfies a polynomial identity and G' has bounded exponent.*

Theorem 1.4. *Let $p > 0$ and G a torsion group. If G' is not a p-group, then the following are equivalent:*

(i) *$\mathcal{U}(FG)$ satisfies a group identity;*
(ii) *$\mathcal{U}(FG)$ has bounded exponent;*
(ii) *FG satisfies a polynomial identity, G has bounded exponent and F is finite.*

It turns out that the solution for finite fields is different if G' is not a p-group, but, in any case, if the unit group of a torsion group ring satisfies a group identity, then it satisfies an identity of a particularly nice form.

Once that the torsion case was settled, it was natural to investigate what happens when the group G contains elements of infinite order. Here the situation is much more complicated because of the difficulty in handling the torsion-free part of the group. Indeed, for any such result, a restriction will occur for the sufficiency, pending a positive answer to the following celebrated conjecture by Kaplansky.

Conjecture 1.5. *Let G be a torsion-free group. Then $\mathcal{U}(FG)$ contains only trivial units, that is, units of the form λg, where $0 \neq \lambda \in F$ and $g \in G$.*

In this setting Hartley's Conjecture is not expected to hold in general. In fact, if G is isomorphic to the direct product of infinitely many copies of a non-abelian torsion-free nilpotent group, for any field F, $\mathcal{U}(FG)$ has only trivial units, and hence is nilpotent, but, according to Theorem 1.2, FG does not satisfy a polynomial identity. But it is true if one restricts the assumptions on G, as deduced from Theorem 5.5 of [12].

Theorem 1.6. *Let $p > 0$ and G a group with an element of infinite order and infinitely many p-elements. If $\mathcal{U}(FG)$ satisfies a group identity, then FG satisfies a polynomial identity.*

In [12] Giambruno, Sehgal and Valenti proved more, classifying group rings of non-torsion groups whose group of units satisfies a group identity. In more detail, they proved that under this assumption the torsion elements of G form a subgroup, T. For the converse, a suitable restriction on G/T was required, namely that it is a *unique product group*, that is, for every pair of non-empty sets H_1 and H_2 of G/T there exists an element $g \in G/T$ that can be uniquely written as $g = h_1 h_2$ with $h_i \in H_i$, in order to force the units of $F(G/T)$ to be trivial. For the complete result (which is quite technical and split in several cases), we refer to the original paper or to Chapter 1 of [17]. We confine ourselves to report here the only part which does not require any restriction on the torsion-free part of G, as summarized in Theorem 1.5.16 of [17].

Theorem 1.7. *Let $p > 0$ and G a group with an element of infinite order and let the p-elements of G have unbounded exponent. Then $\mathcal{U}(FG)$ satisfies a group identity if, and only if, FG satisfies a polynomial identity and G' is a p-group of bounded exponent.*

All these results allowed researchers to solve problems open for decades, such as, for instance, the classification of group rings whose unit group is solvable, concluded by A. Bovdi [3] after a series of papers over many years beginning with Bateman [2]. For these and other results of the same type we refer to the monograph [17]. The aim of this note is to present some recent developments concerning group identities for symmetric and unitary units (with respect to an involution) of FG and discuss some open questions which naturally arise in these frameworks. To this end, we recall that for what concerns symmetric units one can find a comprehensive outline of the known results for the classical involution in [17]. For this reason, in the sequel we avoid reporting in detail what is already collected in that book.

2. $*$-GROUP IDENTITIES FOR $\mathcal{U}(FG)$

Assume that the group G is endowed with an involution $*$. The F-linear extension of $*$ to FG is an involution of FG, also denoted by $*$. An element $\alpha \in FG$ is said to be *symmetric* (with respect to $*$) if $\alpha^* = \alpha$. Write FG^+ for the set of symmetric elements and $\mathcal{U}^+(FG)$ for the set of symmetric units of FG. One of the problems of main interest is to understand if group identities satisfied by $\mathcal{U}^+(FG)$ can be lifted to $\mathcal{U}(FG)$ or, if this is not the case, how they influence the structure of FG.

Since the 90s a lot of attention has been devoted to the *classical involution* of FG, that is, the one induced from the map $g \mapsto g^{-1}$ on G. In the last decade Giambruno, Polcino Milies and Sehgal considered involutions as above other than the classical one, and recently they framed the problem in a different setting, inspired by a classical result of Amitsur. Specifically, let R be an F-algebra with F-linear involution $*$. We say that R satisfies a $*$-polynomial identity if there exists a non-zero element $f(x_1, x_1^*, \ldots, x_m, x_m^*)$ in the free algebra with involution $F\{x_1, x_1^*, x_2, x_2^*, \ldots\}$ such that $f(a_1, a_1^*, \ldots, a_m, a_m^*) = 0$ for all $a_i \in R$. Obviously, if the symmetric elements of R satisfy the polynomial identity $f(x_1, \ldots, x_m)$, then R satisfies the $*$-polynomial identity $f(x_1 + x_1^*, \ldots, x_m + x_m^*)$. Amitsur [1] proved that also the converse is true: indeed, if R satisfies a $*$-polynomial identity, then R satisfies a polynomial identity and, consequently, also R^+ does the same.

Following this direction, in [8] they considered $*$-group identities for $\mathcal{U}(FG)$. We say that $\mathcal{U}(FG)$ satisfies a $*$-group identity if there exists a non-trivial word $w(x_1, x_1^*, \ldots, x_n, x_n^*)$ in the free group with involution $\langle x_1, x_1^*, x_2, x_2^*, \ldots \rangle$ such that $w(a_1, a_1^*, \ldots, a_n, a_n^*) = 1$ for all $a_i \in \mathcal{U}(FG)$. It is clear that if $\mathcal{U}^+(FG)$ satisfies the group identity $w(x_1, \ldots, x_n)$, then $\mathcal{U}(FG)$ satisfies the $*$-group identity $w(x_1 x_1^*, \ldots, x_n x_n^*)$. In the same paper the torsion case was settled proving a quite surprising result for the formulation of which we need the notion of an *SLC-group*. We recall that a non-abelian group G is said to be an LC-group (for *lack of commutativity*) if, for any pair of commuting elements $g, h \in G$, at least one among g, h and gh is central. According to Proposition III.3.6 of [15], G is an LC-group with a unique non-identity commutator if, and only if, $G/\zeta(G) \cong C_2 \times C_2$, where $\zeta(G)$ is the center of G. An LC-group G with involution $*$ is called a *special LC-group*, or *SLC-group*, if it has a unique non-identity commuator z, and for all $g \in G$ we have $g^* = g$ if $g \in \zeta(G)$, and otherwise $g^* = gz$. In group ring theory SLC-groups have a special role, since they occur in the characterization of group rings whose symmetric elements commute, as proved by Jespers and Ruiz Marin in [16]: in more detail, they stated that if $p \neq 2$ and G is a non-abelian group with involution linearly extended to FG, then FG^+ is commutative if, and only if, G is an SLC-group. In particular, if $*$ is the classical involution, G is a Hamiltonian 2-group.

The main result of [8], as formulated in the survey paper [18], is the following.

Theorem 2.1. *Let F be an infinite field of characteristic $p \neq 2$ and G a torsion group with involution linearly extended to FG. Then the following are equivalent:*

(i) *$\mathcal{U}(FG)$ satisfies a $*$-group identity;*
(ii) *$\mathcal{U}^+(FG)$ satisfies a group identity;*
(iii) *one of the following occurs:*
  (a) *$\mathcal{U}(FG)$ satisfies a group identity,*
  (b) *$p = 0$ and G is an SLC-group, or*
  (c) *$p > 2$, FG satisfies a polynomial identity, and G contains a $*$-invariant normal p-subgroup N of bounded exponent such that G/N is an SLC-group.*

For the sake of completeness, we recall that, under the same assumptions, Giambruno, Polcino Milies and Sehgal [7] had already provided necessary and sufficient condition so that $\mathcal{U}^+(FG)$ satisfies a group identity, and the same result was previously established by Giambruno, Sehgal and Valenti [11] for the classical involution.

According to the above statements, $*$-group identities on $\mathcal{U}(FG)$ do not force group identities on $\mathcal{U}(FG)$, but Hartley's Conjecture remains true under the weaker assumption that $\mathcal{U}(FG)$ satisfies a $*$-group identity.

Very recently in [9] the non-torsion case was investigated. Here the situation is much more involved and an analogue of Theorem 2.1 was proven with some restrictions upon G. For the rest of the section, let us denote by T the set of torsion elements of G, and by P that of p-elements, respectively. For the semiprime case the result is the following.

Theorem 2.2. *Let F be an infinite field of characteristic $p \neq 2$ and G a group with involution $*$ linearly extended to FG. Assume that G contains no 2-elements and T is a subgroup of G. If FG is semiprime and $\mathcal{U}(FG)$ satisfies a $*$-group identity, then*

(a) *T is an abelian p'-subgroup such that every idempotent of FT is central in FG (and, consequently, every subgroup of T is normal in G), and*

(b) *G/T satisfies a $*$-group identity.*

Conversely, if (a) holds and G/T is a unique product group satisfying a group identity, then $\mathcal{U}(FG)$ satisfies a group identity.

Assume now that F, G and $\mathcal{U}(FG)$ are as in Theorem 2.2, but FG is not necessarily semiprime. As stressed at the beginning of Section 4 of [9], P is a (normal) subgroup of G. The solution for the general case is split in two parts, just according to the structure of P, as shown in the following

Theorem 2.3. *Let F be an infinite field of characteristic $p \neq 2$ and G a group with involution $*$ linearly extended to FG. Assume that G contains no 2-elements, T is a subgroup of G and FG is not semiprime.*

(1) *If $\mathcal{U}(FG)$ satisfies a $*$-group identity and P is of bounded exponent, then*
   (a) *T/P is abelian and every idempotent of $F(T/P)$ is central in $F(G/P)$, and*
   (b) *G/T satisfies a $*$-group identity.*
   Conversely, if (a) holds and G/T is a unique product group satisfying a group identity, then $\mathcal{U}(FG)$ satisfies a group identity.

(2) *If $\mathcal{U}(FG)$ satisfies a $*$-group identity and P is of unbounded exponent, then*
   (a') *FG satisfies a polynomial identity, and*
   (b') *G' has bounded p-power exponent.*
   Conversely, if G satisfies (a') and (b'), then $\mathcal{U}(FG)$ satisfies a group identity.

Of particular interest is the following

Corollary 2.4. *Let F and G be as in Theorem 2.3. If P is of unbounded exponent, then $\mathcal{U}(FG)$ satisfies a $*$-group identity if, and only if, $\mathcal{U}(FG)$ satisfies a group identity.*

A first question naturally arises from the above mentioned results.

Question 2.5. *Let F be an infinite field of characteristic $p \neq 2$ and G a group with involution $*$ (and, if it helps, with no 2-elements) linearly extended to FG. Assume that $\mathcal{U}(FG)$ satisfies a $*$-group identity (or, if it helps, that $\mathcal{U}^+(FG)$ satisfies a group identity). Is it true that T is a subgroup of G?*

Sehgal and Valenti in [26] gave a positive answer in the case in which the symmetric units of FG with respect to the classical involution satisfy a group identity, and characterized when this happens (under the same restrictions on G/T discussed before Theorem 1.7). Among other things (for the complete result we refer to the original paper or to Chapter 2 of [17]), they proved that

- the statement of Theorem 1.6 remains true under the weaker assumption that $\mathcal{U}^+(FG)$ satisfies a group identity, and
- if FG is as in Theorem 1.7, $\mathcal{U}^+(FG)$ satisfies a group identity if, and only if, $\mathcal{U}(FG)$ satisfies a group identity.

In particular, unlike in [9], they allow the presence of 2-elements in G. This generates a second question.

Question 2.6. *How to modify the above Theorems if G contains 2-elements, even under the assumption that T is a subgroup of G (and, if it helps, that $\mathcal{U}^+(FG)$ satisfies a group identity).*

Let us briefly discuss the possible obstacles just in the semiprime case. First of all, we must be careful with Remark 3.3 of [9], as it will not work if g has even order. This becomes an issue in Lemma 3.8 of [9], as in the final part of the proof, we cannot be sure that $1 + \pi$ is not a zero divisor. But in any case, that result cannot extend to the case where H is an SLC-group. Indeed, assuming that F is algebraically closed, we see that FH must include non-commutative matrix rings among its Wedderburn components, and therefore it has idempotents that are not even central in FH, let alone in FG. And such a case can arise. If H is any finite SLC-group, then let $G = H \times \langle x \rangle$, where x is a symmetric element of infinite order. Then G is an SLC-group, and hence $\mathcal{U}^+(FG)$ is commutative. Why is this important? Let us come back to the classical involution. In Theorem 4 of [26] it was established that, if $p = 0$ and $\mathcal{U}^+(FG)$ satisfies a group identity, then T is either abelian or a Hamiltonian 2-group and every idempotent of FT is central in FG. According to the previously discussed characterization of group rings whose symmetric elements commute, one could expect to generalize the result of [26] just replacing Hamiltonian 2-groups with SLC-groups, but, as seen above, this cannot be the case.

3. Group identities for unitary units of FG

It is natural to ask the same questions discussed for symmetric units of FG for the subgroup of its *unitary units*

$$Un(FG) := \{\alpha \mid \alpha \in FG,\ \alpha\alpha^* = 1\}.$$

But the picture is not as clear here, and just a few general results have been proved.

Assume for the rest of this section that FG is equipped with the classical involution. In [14] Gonçalves and Passman studied group rings whose unitary units contain no non-abelian free subgroup (in the paper they call a group satisfying this property 2-*related*). In more detail, suppose that G is finite. If F is an absolute field, that is, algebraic over a finite field, then $\mathcal{U}(FG)$ is locally finite and, hence, $Un(FG)$ contains no non-abelian free subgroup. Therefore it has sense to ask the question when F is non-absolute (in other words, when either $p = 0$ or $p > 0$ and F has an element trascendental over its prime subfield). The main result they proved is the following

Theorem 3.1. *Let F be a non-absolute field of characteristic $p \neq 2$ and G a finite group. Then $Un(FG)$ contains no non-abelian free subgroup if, and only if,*

(a) *G has a normal Sylow p-subgroup P (by convention, $P = \{1\}$ if $p = 0$), and*
(b) *either $\overline{G} := G/P$ is abelian or it has an abelian subgroup $\overline{A}$ of index 2. Furthermore, if the latter occurs, then either $\overline{G} = \overline{A} \rtimes \langle \overline{y} \rangle$ is dihedral, or $\overline{A}$ is an elementary abelian 2-group.*

Obviously, if $Un(FG)$ contains a non-abelian free subgroup, it cannot satisfy a group identity. This means that if one wants to classify group rings whose unitary units satisfy a group identity, one has to concentrate on the class of groups appearing in Theorem 3.1. This inspired the work of Giambruno and Polcino Milies [14], where they reached this objective when this identity is 2-*free*, that is, it does not vanish on elements of order 2 (for instance nilpotency and the bounded Engel condition), provided F a field of characteristic 0 and G a torsion group, as shown in the following

Theorem 3.2. *Let $p = 0$ and T the set of torsion elements of G. If $Un(FG)$ satisfies a group identity which is 2-free, then T is a subgroup of G and one of the following conditions holds:*

(a) *T is abelian,*
(b) *$A := \langle g \,|\, g \in T,\ o(g) \neq 2 \rangle$ is a normal abelian subgroup of G and $(T \setminus A)^2 = \{1\}$, or*
(c) *T contains an elementary abelian 2-subgroup B of index 2.*

Conversely, if G is a torsion group satisfying one of the above conditions, then $Un(FG)$ satisfies a group identity.

Going through the details of [14], we notice that one of the main issues is the relation beetween the existence of free groups in $Un(FG)$ and the nilpotency of the Lie algebra of skew elements, $FG^- := \{x \,|\, x \in FG,\ x^* = -x\}$, of FG (that should not be surprising if one looks at the general linear group, as stressed in the Introduction of [14]). More generally, the Lie structure of FG^- (which has been exetensively investigated in the last decades: for an overview we refer to [20]) seems to deeply influence the structure of $Un(FG)$ and Lee, Sehgal and Spinelli used Lie methods as a main tool to explore the conditions under which the subgroup of unitary units of FG satisfies certain group identities ([19] and [21]).

In particular, in [21] they studied the question of when $Un(FG)$ is both bounded Engel and solvable (as a natural extension of what done in [19]). Of course, every nilpotent group satisfies these properties, but even the bounded Engel property and solvability together are not enough to guarantee nilpotence (see Section 4 of [27] for examples): according to a classical result of Gruenberg, under these hypotheses one can only conclude that it is locally nilpotent. However, Fisher, Parmenter and Sehgal [4] showed that if FG is not modular (recall that FG is said to be modular if $p > 0$ and G has an element of order p), then whenever $\mathcal{U}(FG)$ is both bounded Engel and solvable, it is also nilpotent. Inspired by this result, Lee, Sehgal and Spinelli asked if it is sufficient to assume that the unitary units are both bounded Engel and solvable, in order to prove that the entire unit group is nilpotent. They showed that, under certain restrictions upon the field, this is the case.

Theorem 3.3. *Let F be an infinite field of characteristic $p > 2$ and G a group such that FG is modular. Then the following are equivalent:*

(i) *$Un(FG)$ is bounded Engel and solvable;*
(ii) *$\mathcal{U}(FG)$ is nilpotent;*
(iii) *G is nilpotent and p-abelian.*

When FG is not modular, one has

Theorem 3.4. *Let $p \neq 2$ and G a torsion group such that FG is not modular and G has no elements of order 2. Then $Un(FG)$ is bounded Engel and solvable if, and only if, G is abelian.*

However, restricting the field suitably, we obtain

Theorem 3.5. *Let F be an algebraically closed field of characteristic $p \neq 2$ and G a group such that FG is not modular. Then the following are equivalent:*

(i) *$Un(FG)$ is bounded Engel and solvable;*
(ii) *$\mathcal{U}(FG)$ is nilpotent;*
(iii) *G is nilpotent and the torsion elements of G are central.*

The assumption on the field in the above statement is not imposed frivolously; indeed, in [19], it was pointed out that if F is the field of 5 elements and G is the dihedral group of order 8, then $Un(FG)$ is nilpotent; however, $\mathcal{U}(FG)$ is neither bounded Engel nor solvable (see Theorems 5.2.1 and 6.2.2 of [22]). Thus Theorem 3.4 fails for an arbitrary field if we allow 2-elements.

In particular, we see that if F is algebraically closed and $p \neq 2$, then $\mathcal{U}(FG)$ is nilpotent whenever $Un(FG)$ is nilpotent. This is quite different from the situation for the symmetric units, where there are counterexamples (recall that when G is isomorphic to a Hamiltonian 2-group, $\mathcal{U}^+(FG)$ is commutative, but, according to Proposition 4.2.6 of [17], $\mathcal{U}(FG)$ is not nilpotent). Anyway, the state of the art for what concerns unitary units is still too fragmentary and we are very far from a knowledge comparable with that of symmetric units. A classification of when $Un(FG)$ satisfies a group identity would be a very appreciable result, but, at the moment, the tools to attack it still seem unclear. Maybe, as in the ordinary case, a decisive step could be to give an answer to the following question, which is an analogue of Hartley's Conjecture,

Question 3.6. *Let G be a torsion group (and, if it helps, F an infinite or algebraically closed field of characteristic $p \neq 2$). Is it true that if $Un(FG)$ satisfies a group identity, then FG satisfies a polynomial identity?*

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