Research Article

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Existence via regularity of solutions for elliptic systems and saddle points of functionals of the calculus of variations

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Abstract: The core of this paper concerns the existence (via regularity) of weak solutions in $W_0^{1,2}$ of a class of elliptic systems such as

$$\begin{cases} -\operatorname{div}((A+\varphi)\nabla u) = f, \\ -\operatorname{div}(M(x)\nabla\varphi) = \frac{1}{2}|\nabla u|^2, \end{cases}$$

deriving from saddle points of integral functionals of the type

$$J(v,\psi) = \frac{1}{2}\int_\Omega (A+\psi_+)|\nabla v|^2 - \frac{1}{2}\int_\Omega M(x)\nabla\psi\nabla\psi - \int_\Omega fv.$$

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1 Introduction

Let Ω be a bounded, open subset of $\mathbb{R}^N$, with $N > 2$, and let $M(x)$ be a measurable matrix such that

$$M(x)\xi\xi \geq \alpha|\xi|^2, \quad |M(x)| \leq \beta, \tag{1.1}$$

for almost every x in Ω and for every ξ in $\mathbb{R}^N$ with $0 < \alpha \leq \beta$. We also suppose that $A > 0$ is a positive real number and that f is a function belonging to $L^m(\Omega)$, $m \geq \frac{2N}{N+2}$. Let us consider the functional

$$J(v,\psi) = \frac{1}{2}\int_\Omega (A+\psi_+)|\nabla v|^2 - \frac{1}{2}\int_\Omega M(x)\nabla\psi\nabla\psi - \int_\Omega fv \tag{1.2}$$

with $v \in W_0^{1,2}(\Omega)$, $\psi \in W_0^{1,2}(\Omega)$ and

$$\psi_+|\nabla v|^2 \in L^1(\Omega).$$

Observe that the functional $J(\cdot,\psi)$ has a minimum for every ψ, while the functional $J(v,\cdot)$ has a maximum for every v.

Then it is reasonable to think that there exists a saddle point (u,φ) for J, which is formally a solution of the system

$$\begin{cases} -\operatorname{div}((A+\varphi)\nabla u) = f, \\ -\operatorname{div}(M(x)\nabla\varphi) = \frac{1}{2}|\nabla u|^2. \end{cases} \tag{1.3}$$

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However, there are several difficulties hidden in the previous sentence: the existence of a saddle point, the nondifferentiability of J and the fact that the second equation has an L^1 right-hand side. For these reasons, classical min-max theorems (see [2, 6, 9]) cannot be applied, so that we directly study system (1.3) instead of the functional J, proving first the existence in $W_0^{1,2}(\Omega)$ of a weak solution (u, φ) and then that it is a saddle point of J.

Our main result is the following theorem, proved in Section 2.

Theorem 1.1. *Assume that* (1.1) *holds and let f be a function in $L^{2_*}(\Omega)$, where $2_* = \frac{2N}{N+2}$. Then there exists a weak solution (u, φ) of the system* (1.3)*. More precisely:*

(i) *u and φ belong to $W_0^{1,2}(\Omega)$, u^3 and $\varphi^{3/2}$ belong to $W_0^{1,2}(\Omega)$, $\varphi|\nabla u|$ belongs to $L^2(\Omega)$, and $|\nabla u|^2 w$ belongs to $L^1(\Omega)$ for every w in $W_0^{1,2}(\Omega)$.*

(ii) *We have*

$$\int_\Omega (A + \varphi)\nabla u \nabla v = \int_\Omega fv \quad \text{for all } v \in W_0^{1,2}(\Omega). \tag{1.4}$$

(iii) *We have that $\varphi \geq 0$ and*

$$\int_\Omega M(x)\nabla\varphi\nabla w = \frac{1}{2}\int_\Omega |\nabla u|^2 w \quad \text{for all } w \in W_0^{1,2}(\Omega). \tag{1.5}$$

Remark 1.2. We point out the *regularizing effect* of the system: the solution φ belongs to $W_0^{1,2}(\Omega)$, despite the poor summability of the right-hand side of the equation it satisfies. Also, even though in principle u only belongs to $W_0^{1,2}(\Omega)$, so that $|\nabla u|^2$ is in $L^1(\Omega)$ and not better, the product $|\nabla u|^2 w$ belongs to $L^1(\Omega)$ for every w in $W_0^{1,2}(\Omega)$ (recall that $W_0^{1,2}(\Omega)$ functions are not, in general, bounded).

There are not many results about saddle points for integral functionals depending on two independent variables. A small comparison with our results is possible with the functional

$$I(z, \eta) = \frac{1}{2}\int_\Omega M(x)\nabla z\nabla z - \frac{A}{2r}\int_\Omega M(x)\nabla\eta\nabla\eta + \frac{A}{r}\int_\Omega \eta^+|z|^r - \int_\Omega fz,$$

where the coupling between z and η is in the zero order term. The existence of a saddle point for I (and of a solution of the related Euler–Lagrange system) is proved in [4]. Also the solutions of the Euler–Lagrange system for I enjoy the regularizing effect.

Remark 1.3. The fact that the solution u, due to the coupling of the equations of the system, is more regular than the solution of the single equation (for example, not only u, but also u^3 belongs to $W_0^{1,2}(\Omega)$) may lead to think that there exist solutions also under weaker summability assumptions on f, such as f belonging to $L^m(\Omega)$ for some $m < 2_*$. There is, however, a main obstruction to this fact: if f does not belong to $L^{2_*}(\Omega)$, one may not expect to have solutions u in $W_0^{1,2}(\Omega)$; therefore, $|\nabla u|^2$ would not belong to $L^1(\Omega)$, and so the second equation of (1.3) loses meaning. Therefore, in some sense, the assumption f in $L^{2_*}(\Omega)$ is the minimal one in order to have existence. Also note that if f belongs to $L^m(\Omega)$, with $m < 2_*$, the functional $J(\,\cdot\,, \psi)$ has no longer a minimum on $W_0^{1,2}(\Omega)$, so that also the concept of saddle point for J is lost if f is not sufficiently summable.

Remark 1.4. The coefficient $\frac{1}{2}$ which multiplies the quadratic gradient term in the second equation of (1.3) can be exchanged with any other real number $\lambda > 0$. Indeed, if (v, φ) is a solution of

$$\begin{cases} -\operatorname{div}((A + \varphi)\nabla v) = \sqrt{2\lambda} f, \\ -\operatorname{div}(M(x)\nabla\varphi) = \frac{1}{2}|\nabla v|^2 \end{cases}$$

and $u = \frac{v}{\sqrt{2\lambda}}$, then (u, φ) is a solution of

$$\begin{cases} -\operatorname{div}((A + \varphi)\nabla u) = f, \\ -\operatorname{div}(M(x)\nabla\varphi) = \lambda|\nabla u|^2. \end{cases} \tag{1.6}$$

Since the summability of $\sqrt{2\lambda} f$ does not depend on λ, once we have proved an existence result for (1.3) under the assumption of data f in $L^{2_*}(\Omega)$, we will have an existence result for (1.6) under the same assumption on f.

Once we have proved Theorem 1.1, we can prove, in Section 3, that the functional J defined above has a saddle point.

Theorem 1.5. *Under the assumptions of Theorem 1.1 and by assuming that $M(x)$ is symmetric, the solution (u, φ) of the system (1.3) is a saddle point of the functional J, defined in (1.2), that is,*

$$J(u, \psi) \le J(u, \varphi) \le J(z, \varphi) \tag{1.7}$$

for every z in $W_0^{1,2}(\Omega)$ such that $\varphi|\nabla z|^2$ belongs to $L^1(\Omega)$, and for every ψ in $W_0^{1,2}(\Omega)$.

Our final result will prove that the increased summability of u and φ given by Theorem 1.1 under the assumption f in $L^{2_*}(\Omega)$ can also be improved if f belongs to $L^m(\Omega)$, with $m > 2_*$. More precisely, we will prove the following:

(i) If $m > \frac{N}{2}$, then u belongs to $L^\infty(\Omega)$, while φ^ρ belongs to $W_0^{1,2}(\Omega)$ for every $\rho \ge 1$ (see Lemma 2.2 and Lemma 2.4, below).

(ii) There exists an increasing sequence m_k, with $2_* < m_k < \frac{N}{2}$, such that if f belongs to $L^{m_k}(\Omega)$, then u belongs to $L^{3 \cdot m_k^{**}}(\Omega)$ and φ belongs to $L^{3 \cdot m_k^{**}/2}(\Omega)$ (see Section 4 for the precise statement of this result).

Remark 1.6. In all the results of this paper, every result proved for u has a counterpart for φ in the sense that "φ almost behaves like u^2"; for example, in Theorem 1.1, u^3 belongs to $W_0^{1,2}(\Omega)$, and $\varphi^{3/2}$ belongs to the same space.

To give an idea of the reason why this happens, let us consider a one-dimensional reduction of the functional J, i.e., the function of two real variables

$$H(x, y) = \frac{1}{2}(A + y)x^2 - \frac{1}{2}y^2 - Bx,$$

where $A > 0$ and $B > 0$. It is easy to see that H has a unique critical point (x_0, y_0) with $y_0 = \frac{1}{2}x_0^2$ and x_0 such that $\frac{1}{2}x_0^3 + Ax_0 = B$, and that (x_0, y_0) is a saddle point for H. Therefore, as far as the critical saddle point (u, φ) of J is concerned, it is reasonable to expect that φ approximately behaves like u^2 since $y_0 \approx x_0^2$.

See also the beginning of Section 4 for further remarks on this topic.

2 Proof of Theorem 1.1

In the following, we will make frequent use of the function of one real variable (here $k > 0$):

$$T_k(s) = \max(-k, \min(s, k)).$$

We will prove the result in several steps. We begin proving that, if f belongs to $L^m(\Omega)$, with $m > \frac{N}{2}$, then an "approximate" system has a solution.

Lemma 2.1. *Assume that (1.1) holds and let f belong to $L^m(\Omega)$, with $m > \frac{N}{2}$. Let n be in $\mathbb{N}$. Then there exists a couple (u_n, φ_n) of functions such that the following hold:*

(i) *u_n belongs to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ and*

$$\int_\Omega (A + T_n(\varphi_n))\nabla u_n \nabla v = \int_\Omega fv \quad \text{for all } v \in W_0^{1,2}(\Omega). \tag{2.1}$$

(ii) *$\varphi_n \ge 0$ belongs to $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$, is such that $T_k(\varphi_n)$ belongs to $W_0^{1,2}(\Omega)$ for every $k \ge 0$, and*

$$\int_\Omega M(x)\nabla \varphi_n \nabla T_k(\varphi_n - \eta) \le \frac{1}{2}\int_\Omega |\nabla u_n|^2 T_k(\varphi_n - \eta) \tag{2.2}$$

for every $k \ge 0$ and every η in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Proof. Let f be a function as in the statement and let n be in $\mathbb{N}$. Let r be such that $1 < r < \frac{N}{N-2}$, and for $0 \le \varphi$ in $L^r(\Omega)$ let w in $W_0^{1,2}(\Omega)$ be the unique weak solution of

$$\int_\Omega (A + T_n(\varphi))\nabla w \nabla v = \int_\Omega f v \quad \text{for all } v \in W_0^{1,2}(\Omega).$$

Since $m > 2_*$, one has

$$\|w\|_{W_0^{1,2}(\Omega)} \le C\|f\|_{L^m(\Omega)} = R, \tag{2.3}$$

while (since $m > \frac{N}{2}$), by a result of Stampacchia (see [8]), one has

$$\|w\|_{L^\infty(\Omega)} \le C\|f\|_{L^m(\Omega)}. \tag{2.4}$$

Define then ψ to be the unique function (see [3], where ψ is called entropy solution) such that $T_k(\psi)$ belongs to $W_0^{1,2}(\Omega)$ for every $k \ge 0$ and that

$$\int_\Omega M(x)\nabla\psi\nabla T_k(\psi - \eta) \le \frac{1}{2}\int_\Omega |\nabla w|^2 T_k(\psi - \eta)$$

for every $k \ge 0$ and for every $\eta \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. We recall that $\psi \ge 0$ belongs to $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$ and that

$$\|\psi\|_{W_0^{1,q}(\Omega)} \le C_q \||\nabla w|^2\|_{L^1(\Omega)} = C_q\|w\|^2_{W_0^{1,2}(\Omega)} \le C_q\|f\|^2_{L^m(\Omega)},$$

where C_q is a constant depending on $q < \frac{N}{N-1}$ and we have used (2.3) in the last passage. In particular, by the Sobolev embedding, if q is such that $q^* = r$, we have

$$\|\psi\|_{L^r(\Omega)} \le C_r R^2 = Q(R).$$

Therefore, if $S : L^r(\Omega) \to L^r(\Omega)$ is defined by $S(\varphi) = \psi$, the ball of $L^r(\Omega)$ of radius $Q(R)$ is invariant for S. If $\{\varphi_\rho\}$ is bounded in $L^r(\Omega)$, it is easy to prove that one can extract from $S(\varphi_\rho)$ a sequence strongly convergent in $L^r(\Omega)$ since the sequence $\{S(\varphi_\rho)\}$ is bounded in $W_0^{1,q}(\Omega)$. Furthermore, if φ_ρ strongly converges to φ in $L^r(\Omega)$, it is easy to prove that w_ρ strongly converges to w (the unique solution) in $W_0^{1,2}(\Omega)$, so that $S(\varphi_\rho)$ strongly converges to $S(\varphi)$ in $L^r(\Omega)$. Therefore, by the Schauder fixed point theorem, for every n in $\mathbb{N}$ there exists $\varphi_n \ge 0$ such that $S(\varphi_n) = \varphi_n$, and so the functions u_n and φ_n satisfy (2.1) and (2.2), as desired. □

Our next result deals with the sequence $\{(u_n, \varphi_n)\}$ given by the previous lemma, proving that it has a limit, which solves the system (1.3) if f belongs to $L^m(\Omega)$, with $m > \frac{N}{2}$.

Lemma 2.2. *Assume that* (1.1) *holds and let f belong to $L^m(\Omega)$, with $m > \frac{N}{2}$. Then there exists a couple (u, φ) of functions such that the following hold:*

(i) *u belongs to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, φ belongs to $W_0^{1,2}(\Omega)$, $\varphi|\nabla u|$ belongs to $L^2(\Omega)$ and $|\nabla u|^2 w$ belongs to $L^1(\Omega)$ for every w in $W_0^{1,2}(\Omega)$.*

(ii) *u is such that*

$$\int_\Omega (A + \varphi)\nabla u \nabla v = \int_\Omega f v \quad \text{for all } v \in W_0^{1,2}(\Omega). \tag{2.5}$$

(iii) *$\varphi \ge 0$ is such that*

$$\int_\Omega M(x)\nabla\varphi\nabla w = \frac{1}{2}\int_\Omega |\nabla u|^2 w \quad \text{for all } w \in W_0^{1,2}(\Omega). \tag{2.6}$$

Proof. Here, and in the following, we will denote by $C(f)$ and $C(f, A)$, some possibly different positive constants which only depend on (some powers of) the norm of f in $L^m(\Omega)$ and on A.

Let (u_n, φ_n) be the couple of functions given by Lemma 2.1. We begin with some *a priori* estimates on both u_n and φ_n. Recalling (2.4), we have that

$$\|u_n\|_{L^\infty(\Omega)} \le C(f). \tag{2.7}$$

Furthermore, choosing $v = u_n$ in (2.1), we have

$$\int_\Omega (A + T_n(\varphi_n))|\nabla u_n|^2 \le \int_\Omega f u_n \le C(f),$$

so that

$$\{u_n\} \text{ is bounded in } W_0^{1,2}(\Omega) \text{ and } \{T_n(\varphi_n)|\nabla u_n|^2\} \text{ is bounded in } L^1(\Omega). \tag{2.8}$$

Finally, choosing $\eta = 0$ and $k = n$ in (2.2), we obtain, using (2.8) and (1.1), that

$$\alpha \int_\Omega |\nabla T_n(\varphi_n)|^2 \le \frac{1}{2} \int_\Omega T_n(\varphi_n)|\nabla u_n|^2 \le C(f), \tag{2.9}$$

so that $\{T_n(\varphi_n)\}$ is bounded in $W_0^{1,2}(\Omega)$. Notice now that since $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$, $\{|\nabla u_n|^2\}$ is bounded in $L^1(\Omega)$. Then, by the results of [3], $\{\varphi_n\}$ is bounded in $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$. Hence, φ_n converges, up to a subsequence still denoted by φ_n, to some function φ. But then, by $\{T_n(\varphi_n)\}$ being bounded in $W_0^{1,2}(\Omega)$, we have that φ belongs to $W_0^{1,2}(\Omega)$ as well.

We now choose $v = u_n T_n(\varphi_n)$ in (2.1) to obtain

$$\int_\Omega (A + T_n(\varphi_n))|\nabla u_n|^2 T_n(\varphi_n) + \int_\Omega (A + T_n(\varphi_n)\nabla u_n \nabla T_n(\varphi_n) u_n = \int_\Omega f u_n T_n(\varphi_n),$$

which implies (by dropping a positive term and using (2.7) and (2.9)) that

$$\begin{aligned} \int_\Omega T_n(\varphi_n)^2|\nabla u_n|^2 &\le C(f)\|T_n(\varphi_n)\|_{L^{m'}(\Omega)} + \int_\Omega (A + T_n(\varphi_n))|\nabla u_n||\nabla T_n(\varphi_n)||u_n| \\ &\le C(f) + \int_\Omega (A + T_n(\varphi_n))|\nabla u_n||\nabla T_n(\varphi_n)||u_n|, \end{aligned}$$

where in the last passage we have used that $m' < \frac{N}{N-2} < 2^*$. Using the Young and Hölder inequalities in last term of the right-hand side of the previous inequality, we have

$$\begin{aligned} &\int_\Omega (A + T_n(\varphi_n))|\nabla u_n||\nabla T_n(\varphi_n)||u_n| \\ &\quad \le A\|u_n\|_{L^\infty(\Omega)}\|u_n\|_{W_0^{1,2}(\Omega)}\|T_n(\varphi_n)\|_{W_0^{1,2}(\Omega)} + \frac{1}{2}\int_\Omega T_n(\varphi_n)^2|\nabla u_n|^2 + C\|u_n\|^2_{L^\infty(\Omega)}\|T_n(\varphi_n)\|^2_{W_0^{1,2}(\Omega)} \\ &\quad \le \frac{1}{2}\int_\Omega T_n(\varphi_n)^2|\nabla u_n|^2 + C(f, A). \end{aligned}$$

Therefore, after simplifying equal terms, we have

$$\int_\Omega T_n(\varphi_n)^2|\nabla u_n|^2 \le C(f, A). \tag{2.10}$$

Summing up, as a consequence of the previous estimates and convergences, we have that

$$\begin{cases} u_n \text{ weakly converges to } u \text{ in } W_0^{1,2}(\Omega), \\ T_n(\varphi_n) \text{ weakly converges to } \varphi \text{ in } W_0^{1,2}(\Omega) \end{cases} \tag{2.11}$$

and that there exists Y in $(L^2(\Omega))^N$ such that (recall (2.10))

$$T_n(\varphi_n)\nabla u_n \text{ weakly converges to } Y \text{ in } (L^2(\Omega))^N.$$

Let now Ψ be a $C_0^1(\Omega)$ function; from the above convergence we have that

$$\lim_{n\to+\infty} \int_\Omega T_n(\varphi_n)\nabla u_n \nabla\Psi = \int_\Omega Y\nabla\Psi.$$

On the other hand, since $\{\nabla u_n\}$ weakly converges in $(L^2(\Omega))^N$, while $T_n(\varphi_n)\nabla\Psi$ is strongly convergent in $(L^2(\Omega))^N$ by the Rellich theorem, we have

$$\lim_{n\to+\infty}\int_\Omega T_n(\varphi_n)\nabla u_n\nabla\Psi = \lim_{n\to+\infty}\int_\Omega \nabla u_n[\nabla\Psi T_n(\varphi_n)] = \int_\Omega \nabla u\nabla\Psi\varphi.$$

Therefore, $Y = \varphi\nabla u$, and so

$$T_n(\varphi_n)\nabla u_n \text{ weakly converges to } \varphi\nabla u \text{ in } (L^2(\Omega))^N.$$

Using this convergence, and the weak convergence of u_n to u in $W_0^{1,2}(\Omega)$ (see (2.11)), we can pass to the limit in (2.1), to find that

$$\int_\Omega (A+\varphi)\nabla u\nabla v = \int_\Omega fv \quad \text{for all } v \in W_0^{1,2}(\Omega). \tag{2.12}$$

Recall that u is such that it belongs to $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ and that $\varphi|\nabla u|$ belongs to $L^2(\Omega)$. Taking $v = u$ in (2.12), we then have

$$\int_\Omega (A+\varphi)|\nabla u|^2 = \int_\Omega fu.$$

On the other hand, choosing $v = u_n$ in (2.1), we have

$$\lim_{n\to+\infty}\int_\Omega (A+T_n(\varphi_n))|\nabla u_n|^2 = \lim_{n\to+\infty}\int_\Omega f_n u_n = \int_\Omega fu.$$

Therefore,

$$\lim_{n\to+\infty}\int_\Omega (A+T_n(\varphi_n))|\nabla u_n|^2 = \int_\Omega (A+\varphi)|\nabla u|^2.$$

From the positivity of the functions, this implies that

$$(A+T_n(\varphi_n))|\nabla u_n|^2 \text{ strongly converges to } (A+\varphi)|\nabla u|^2 \text{ in } L^1(\Omega),$$

which in turn implies that $\{u_n\}$ strongly converges to u in $W_0^{1,2}(\Omega)$. Hence, since $|\nabla u_n|^2$ is strongly convergent in $L^1(\Omega)$ and (see (2.11)) $T_n(\varphi_n)$ weakly converges to φ in $W_0^{1,2}(\Omega)$, we can pass to the limit in (2.2) to have that φ belongs to $W_0^{1,2}(\Omega)$ and

$$\int_\Omega M(x)\nabla\varphi\nabla T_k(\varphi-\eta) \le \frac{1}{2}\int_\Omega |\nabla u|^2 T_k(\varphi-\eta) \tag{2.13}$$

for every $k \ge 0$ and for every η in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$.

Since φ belongs to $W_0^{1,2}(\Omega)$, we now prove that it does not only satisfy (2.13), but that it also is a weak solution. In the first step, we prove that (2.6) holds for test functions w in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$; then we prove that test functions only belonging to $W_0^{1,2}(\Omega)$ are allowed.

To see that, let η be in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, let k be greater than the norm of η in $L^\infty(\Omega)$, and let $h > 0$. Then, since $T_h(\varphi)-\eta$ belongs to $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, one has

$$\int_\Omega M(x)\nabla\varphi\nabla T_k(\varphi - T_h(\varphi)+\eta) \le \frac{1}{2}\int_\Omega |\nabla u|^2 T_k(\varphi - T_h(\varphi)+\eta),$$

that is,

$$\int_{\{|G_h(\varphi)+\eta|\le k\}} M(x)\nabla\varphi\nabla G_h(\varphi) + \int_{\{|G_h(\varphi)+\eta|\le k\}} M(x)\nabla\varphi\nabla\eta \le \frac{1}{2}\int_\Omega |\nabla u|^2 T_k(\varphi - T_h(\varphi)+\eta).$$

The first term is positive, so that it can be dropped, while for the third, by recalling the choice of k, one has

$$\lim_{h\to+\infty}\frac{1}{2}\int_\Omega |\nabla u|^2 T_k(\varphi - T_h(\varphi)+\eta) = \frac{1}{2}\int_\Omega |\nabla u|^2 T_k(\eta) = \frac{1}{2}\int_\Omega |\nabla u|^2\eta.$$

As for the second term, we remark that

$$\lim_{h\to+\infty} M(x)\nabla\varphi\nabla\eta\chi_{\{|G_h(\varphi)+\eta|\le k\}} = M(x)\nabla\varphi\nabla\eta$$

and that

$$M(x)\nabla\varphi\nabla\psi\chi_{\{|G_h(\varphi)+\eta|\le k\}} \le \beta|\nabla\varphi||\nabla\eta| \in L^1(\Omega)$$

since both φ and η belong to $W_0^{1,2}(\Omega)$. Therefore, by the Lebesgue theorem,

$$\lim_{h\to+\infty} \int_{\{|G_h(\varphi)+\eta|\le k\}} M(x)\nabla\varphi\nabla\eta = \int_\Omega M(x)\nabla\varphi\nabla\eta.$$

Thus,

$$\int_\Omega M(x)\nabla\varphi\nabla\eta \le \frac{1}{2}\int_\Omega |\nabla u|^2\eta \quad \text{for all } \eta \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega).$$

Choosing $\eta = \pm\psi$, with ψ in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, we then obtain that $\varphi \ge 0$ belongs to $W_0^{1,2}(\Omega)$ and is such that

$$\int_\Omega M(x)\nabla\varphi\nabla\psi = \frac{1}{2}\int_\Omega |\nabla u|^2\psi \quad \text{for all } \psi \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega). \tag{2.14}$$

Let now $0 \le w \in W_0^{1,2}(\Omega)$ and use $\psi = T_n(w)$ in (2.14) to obtain

$$\int_\Omega M(x)\nabla\varphi\nabla T_n(w) = \frac{1}{2}\int_\Omega |\nabla u|^2 T_n(w).$$

We can pass to the limit on the left-hand side since $T_n(w)$ strongly converges in $W_0^{1,2}(\Omega)$, and on the right-hand side thanks to the Levi monotone convergence theorem. Thus we have

$$\int_\Omega M(x)\nabla\varphi\nabla w = \frac{1}{2}\int_\Omega |\nabla u|^2 w$$

for every positive w in $W_0^{1,2}(\Omega)$; by linearity, we thus have that (2.6) holds true, and that $|\nabla u|^2 w$ belongs to $L^1(\Omega)$ for every w in $W_0^{1,2}(\Omega)$, as desired. □

Remark 2.3. We remark that the fact that (2.6) holds true can also be proved by using a result by Brezis and Browder (see [5]); indeed, since φ is in $W_0^{1,2}(\Omega)$, we have that $-\operatorname{div}(M(x)\nabla\varphi)$ belongs to $W^{-1,2}(\Omega)$. Since it is positive, we have

$$\int_\Omega M(x)\nabla\varphi\nabla w = \langle -\operatorname{div}(M(x)\nabla\varphi), w\rangle = \frac{1}{2}\int_\Omega |\nabla u|^2 w$$

for every positive w in $W_0^{1,2}(\Omega)$; this implies (2.6) by writing every w in $W_0^{1,2}(\Omega)$ as $w = w_+ - w_-$.

In order to proceed with the proof of the main result, we need to prove further properties on the solution (u, φ) given by the previous lemma.

Lemma 2.4. *Assume that* (1.1) *holds and let f belong to $L^m(\Omega)$, with $m > \frac{N}{2}$. Then the function φ given by Lemma 2.2 is such that φ^ρ belongs to $W_0^{1,2}(\Omega)$ for every $\rho \ge 1$. Furthermore, for every ρ and σ in $\mathbb{N}$, one can choose $v = u^\sigma\varphi^\rho$ and $w = u^\sigma\varphi^\rho$ in* (2.5) *and* (2.6), *that is,*

$$\int_\Omega (A+\varphi)\nabla u\nabla(u^\sigma\varphi^\rho) = \int_\Omega f u^\sigma\varphi^\rho \tag{2.15}$$

and

$$\int_\Omega M(x)\nabla\varphi\nabla(u^\sigma\varphi^\rho) = \frac{1}{2}\int_\Omega |\nabla u|^2 u^\sigma\varphi^\rho, \tag{2.16}$$

with all the integrals involved being finite.

Proof. Here, and in the following, we will denote by $C(f, A, \gamma)$ possibly different constants depending on (powers of) the norm of f in $L^m(\Omega)$, on A, and on (powers of) a real number γ.

Recall that, as a consequence of Lemma 2.2, we have that u belongs to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, that φ belongs to $W_0^{1,2}(\Omega)$, and that

$$\|u\|_{W_0^{1,2}(\Omega)} + \|\varphi\|_{W_0^{1,2}(\Omega)} + \|u\|_{L^\infty(\Omega)} \le C\|f\|_{L^m(\Omega)} = C(f). \tag{2.17}$$

Let $\gamma \ge 2$ and let $k > 0$ large enough. Choose $v = uT_k(\varphi)^{\gamma-1}$ in (2.5) to obtain

$$\int_\Omega (A + \varphi)|\nabla u|^2 T_k(\varphi)^{\gamma-1} \le \int_\Omega f|u|T_k(\varphi)^{\gamma-1} + (\gamma - 1)\int_\Omega (A + T_k(\varphi))|\nabla u||\nabla T_k(\varphi)|T_k(\varphi)^{\gamma-2}.$$

Dropping a positive term on the left-hand side and recalling (2.17), we have

$$\int_\Omega |\nabla u|^2 T_k(\varphi)^\gamma \le C(f)\Big(\int_\Omega T_k(\varphi)^{(\gamma-1)m'}\Big)^{\frac{1}{m'}} + \gamma\int_\Omega (A + T_k(\varphi))|\nabla u||\nabla T_k(\varphi)|T_k(\varphi)^{\gamma-2}. \tag{2.18}$$

We now estimate a part of the right-hand side of the above inequality, by supposing that $k \ge A$ and by using (2.17):

$$\begin{aligned}\gamma A\int_\Omega |\nabla u||\nabla T_k(\varphi)||T_k(\varphi)|^{\gamma-2} &= \gamma A\int_{\{\varphi<A\}} |\nabla u||\nabla T_k(\varphi)||T_k(\varphi)|^{\gamma-2} + \gamma A\int_{\{\varphi\ge A\}} |\nabla u||\nabla T_k(\varphi)||T_k(\varphi)|^{\gamma-2}\\ &\le \gamma A^{\gamma-1}\int_\Omega |\nabla u||\nabla T_k(\varphi)| + \gamma\int_\Omega |\nabla u||\nabla T_k(\varphi)|T_k(\varphi)^{\gamma-1}\\ &\le C(f, A, \gamma) + \gamma\int_\Omega |\nabla u||\nabla T_k(\varphi)|T_k(\varphi)^{\gamma-1}.\end{aligned}$$

Inserting this inequality in (2.18), we obtain

$$\begin{aligned}\int_\Omega |\nabla u|^2 T_k(\varphi)^\gamma &\le C(f, A, \gamma) + C(f)\Big(\int_\Omega T_k(\varphi)^{(\gamma-1)m'}\Big)^{\frac{1}{m'}} + \gamma\int_\Omega |\nabla u||\nabla T_k(\varphi)|T_k(\varphi)^{\gamma-1}\\ &\le C(f, A, \gamma) + C(f)\Big(\int_\Omega T_k(\varphi)^{(\gamma-1)m'}\Big)^{\frac{1}{m'}} + \frac{1}{2}\int_\Omega |\nabla u|^2 T_k(\varphi)^\gamma + \frac{\gamma^2}{2}\int_\Omega |\nabla T_k(\varphi)|^2 T_k(\varphi)^{\gamma-2}.\end{aligned} \tag{2.19}$$

We now have, if $k > \frac{\gamma}{\alpha}$ with α given by (1.1),

$$\begin{aligned}\frac{\gamma^2}{2}\int_\Omega |\nabla T_k(\varphi)|^2 T_k(\varphi)^{\gamma-2} &\le \frac{\gamma^2}{2}\int_{\{\varphi<\frac{\gamma}{\alpha}\}} |\nabla T_k(\varphi)|^2 T_k(\varphi)^{\gamma-2} + \frac{\gamma^2}{2}\int_{\{\varphi\ge\frac{\gamma}{\alpha}\}} |\nabla T_k(\varphi)|^2 T_k(\varphi)^{\gamma-2}\\ &\le \frac{1}{2}\alpha^{2-\gamma}\gamma^\gamma\int_\Omega |\nabla T_k(\varphi)|^2 + \frac{\alpha\gamma}{2}\int_\Omega |\nabla T_k(\varphi)|^2 T_k(\varphi)^{\gamma-1}\\ &= C(f, A, \gamma) + \frac{\alpha\gamma}{2}\int_\Omega |\nabla T_k(\varphi)|^2 T_k(\varphi)^{\gamma-1}.\end{aligned}$$

Thus, inserting this inequality in (2.19) and simplifying equal terms, we have

$$\frac{1}{2}\int_\Omega |\nabla u|^2 T_k(\varphi)^\gamma \le C(f, A, \gamma) + C(f)\Big(\int_\Omega T_k(\varphi)^{(\gamma-1)m'}\Big)^{\frac{1}{m'}} + \frac{\alpha\gamma}{2}\int_\Omega |\nabla T_k(\varphi)|^2 T_k(\varphi)^{\gamma-1}. \tag{2.20}$$

We now choose $w = T_k(\varphi)^\gamma$ in (2.6), to obtain, after using (1.1),

$$\alpha\gamma\int_\Omega |\nabla T_k(\varphi)|^2 T_k(\varphi)^{\gamma-1} \le \gamma\int_\Omega M(x)\nabla\varphi\nabla T_k(\varphi)T_k(\varphi)^{\gamma-1} = \frac{1}{2}\int_\Omega |\nabla u|^2 T_k(\varphi)^\gamma.$$

Using this inequality in (2.20), we have

$$\int_\Omega |\nabla T_k(\varphi)|^2 T_k(\varphi)^{\gamma-1} \leq C(f,A,\gamma) + C(f,\gamma)\Big(\int_\Omega T_k(\varphi)^{(\gamma-1)m'}\Big)^{\frac{1}{m'}} + \frac{1}{2}\int_\Omega |\nabla T_k(\varphi)|^2 T_k(\varphi)^{\gamma-1},$$

which implies, after simplifying equal terms, that

$$\int_\Omega |\nabla T_k(\varphi)|^2 T_k(\varphi)^{\gamma-1} \leq C(f,A,\gamma) + C(f,\gamma)\Big(\int_\Omega T_k(\varphi)^{(\gamma-1)m'}\Big)^{\frac{1}{m'}}. \tag{2.21}$$

We now observe that, since $m > \frac{N}{2}$, we have that

$$(\gamma-1)m' < \frac{N(\gamma-1)}{N-2} = \frac{2^*(\gamma-1)}{2} < \frac{2^*(\gamma+1)}{2}.$$

Therefore, by the Hölder inequality and the Sobolev embedding,

$$\Big(\int_\Omega T_k(\varphi)^{(\gamma-1)m'}\Big)^{\frac{1}{m'}} \leq C\Big(\int_\Omega T_k(\varphi)^{\frac{2^*(\gamma+1)}{2}}\Big)^{\frac{2}{2^*}\frac{\gamma-1}{\gamma+1}} \leq C\Big(\int_\Omega |\nabla[T_k(\varphi)]^{\frac{\gamma+1}{2}}|^2\Big)^{\frac{\gamma-1}{\gamma+1}}.$$

Using this inequality in (2.21), we obtain

$$\frac{4}{(\gamma+1)^2}\int_\Omega |\nabla[T_k(\varphi)]^{\frac{\gamma+1}{2}}|^2 \leq C(f,A,\gamma) + C(f,\gamma)\Big(\int_\Omega |\nabla[T_k(\varphi)]^{\frac{\gamma+1}{2}}|^2\Big)^{\frac{\gamma-1}{\gamma+1}},$$

which then yields, by the Young inequality (which can be applied since $\frac{\gamma-1}{\gamma+1} < 1$), that

$$\int_\Omega |\nabla[T_k(\varphi)]^{\frac{\gamma+1}{2}}|^2 \leq C(f,A,\gamma) + \frac{1}{2}\int_\Omega |\nabla[T_k(\varphi)]^{\frac{\gamma+1}{2}}|^2.$$

Therefore,

$$\frac{(\gamma+1)^2}{4}\int_\Omega |\nabla T_k(vp)|^2 T_k(\varphi)^{\gamma-1} = \int_\Omega |\nabla[T_k(\varphi)]^{\frac{\gamma+1}{2}}|^2 \leq C(f,A,\gamma).$$

Letting k tend to infinity in the above identity and setting $\rho = \frac{\gamma+1}{2}$, we then obtain that

$$\int_\Omega |\nabla[\varphi^\rho]|^2 \leq C(f,A,\rho),$$

which implies that φ^ρ belongs to $W_0^{1,2}(\Omega)$ for every $\rho \geq \frac{3}{2}$ (recall that $\gamma \geq 2$). If $\rho = 1$, the result follows from (2.17), while if $1 < \rho < \frac{3}{2}$, we have

$$\begin{aligned}\int_\Omega |\nabla[\varphi]^\rho|^2 &= \rho^2 \int_{\{\varphi<1\}} |\nabla\varphi|^2\varphi^{2(\rho-1)} + \rho^2\int_{\{\varphi\geq 1\}} |\nabla\varphi|^2\varphi^{2(\rho-1)} \\ &\leq C\int_\Omega |\nabla T_1(\varphi)|^2 + C\int_\Omega |\nabla\varphi|^2\varphi \leq C + C\int_\Omega |\nabla[\varphi]^{\frac{3}{2}}|^2 \leq C.\end{aligned}$$

We now prove (2.15); to do that, we choose $v = u_+^\sigma T_k(\varphi)^\rho$, with $k > 0$ in (2.5), to obtain that

$$\sigma\int_\Omega (A+\varphi)|\nabla u|^2 u_+^{\sigma-1}T_k(\varphi)^\rho + \rho\int_\Omega (A+T_k(\varphi))\nabla u\nabla T_k(\varphi)u_+^\sigma T_k(\varphi)^{\rho-1} = \int_\Omega f u_+^\sigma T_k(\varphi)^\rho.$$

Recalling that u_+ belongs to $L^\infty(\Omega)$ and that both φ^ρ and $\varphi^{\rho+1}$ belong to $W_0^{1,2}(\Omega)$, we have that

$$\lim_{k\to+\infty}\rho\int_\Omega (A+T_k(\varphi))\nabla u\nabla T_k(\varphi)u_+^\sigma T_k(\varphi)^{\rho-1} = \rho\int_\Omega (A+\varphi)\nabla u\nabla\varphi u_+^\sigma\varphi^{\rho-1},$$

while, since φ^ρ belongs to $L^{m'}(\Omega)$ (by the Sobolev embedding, φ^ρ is in $L^\eta(\Omega)$ for every $\eta \geq 1$), we have

$$\lim_{k\to+\infty} \int_\Omega f u_+^\sigma T_k(\varphi)^\rho = \int_\Omega f u_+^\sigma \varphi^\rho .$$

As for the first term, we remark that it is positive and increasing with respect to k. Therefore, using the Levi monotone convergence theorem, we have

$$\lim_{k\to+\infty} \sigma \int_\Omega (A+\varphi)|\nabla u|^2 u_+^{\sigma-1} T_k(\varphi)^\rho = \sigma \int_\Omega (A+\varphi)|\nabla u|^2 u_+^{\sigma-1}\varphi^\rho .$$

Summing up, we have that

$$\int_\Omega (A+\varphi)\nabla u \nabla(u_+^\sigma \varphi^\rho) = \int_\Omega f u_+^\sigma \varphi^\rho .$$

Repeating the same calculations with $v = u_-^\sigma T_k(\varphi)^\rho$ and then subtracting the results, we obtain (2.15).

In order to prove (2.16), we choose $w = u_+^\sigma T_k(\varphi)^\rho$, with $k > 0$ in (2.6), to obtain

$$\rho \int_\Omega M(x)\nabla\varphi\nabla T_k(\varphi) u_+^\sigma T_k(\varphi)^{\rho-1} + \sigma \int_\Omega M(x)\nabla\varphi\nabla u u_+^{\sigma-1} T_k(\varphi)^\rho = \frac{1}{2}\int_\Omega |\nabla u|^2 u_+^\sigma T_k(\varphi)^\rho .$$

Once again, the properties of φ and the fact that u_+ belongs to $L^\infty(\Omega)$, allow to prove that

$$\lim_{k\to+\infty} \rho \int_\Omega M(x)\nabla\varphi\nabla T_k(\varphi) u_+^\sigma T_k(\varphi)^{\rho-1} = \rho \int_\Omega M(x)\nabla\varphi\nabla\varphi u_+^\sigma \varphi^{\rho-1}$$

and that

$$\lim_{k\to+\infty} \sigma \int_\Omega M(x)\nabla\varphi\nabla u u_+^{\sigma-1} T_k(\varphi)^\rho = \sigma \int_\Omega M(x)\nabla\varphi\nabla u u_+^{\sigma-1}\varphi^\rho ,$$

while the Levi monotone convergence theorem implies that

$$\lim_{k\to+\infty} \frac{1}{2}\int_\Omega |\nabla u|^2 u_+^\sigma T_k(\varphi)^\rho = \frac{1}{2}\int_\Omega |\nabla u|^2 u_+^\sigma \varphi^\rho .$$

Therefore, we have

$$\int_\Omega M(x)\nabla\varphi\nabla(u_+^\sigma\varphi^\rho) = \frac{1}{2}\int_\Omega |\nabla u|^2 u_+^\sigma\varphi^\rho ,$$

which then yields that (2.16) holds once combined with the same formula for u_-. □

We are now ready to prove Theorem 1.1, that is, the existence result, under minimal summability assumptions on the datum f.

Proof of Theorem 1.1. Let n in $\mathbb{N}$ and let f_n be a sequence of functions in $L^\infty(\Omega)$ strongly convergent to f in $L^{2_*}(\Omega)$ and such that

$$\|f_n\|_{L^{2_*}(\Omega)} \leq \|f\|_{L^{2_*}(\Omega)}; \tag{2.22}$$

take, for example, $f_n = T_n(f)$. Let u_n and φ_n be the solutions given by Lemma 2.2 with datum f_n.

Choosing u_n as test function in (2.15), we have

$$A\int_\Omega |\nabla u_n|^2 + \int_\Omega |\nabla u_n|^2\varphi_n = \int_\Omega (A+\varphi_n)|\nabla u_n|^2 = \int_\Omega f_n u_n \leq \|f\|_{L^{2_*}(\Omega)}\|u_n\|_{L^{2^*}(\Omega)},$$

where we have used (2.22) in the last passage. Thus, thanks to the Sobolev embedding, we have that

$$\int_\Omega |\nabla u_n|^2 + \int_\Omega |\nabla u_n|^2\varphi_n \leq C\|f\|^2_{L^{2_*}(\Omega)}. \tag{2.23}$$

Choosing φ_n as test function in (2.16) and using (1.1), we have

$$\alpha\int_\Omega |\nabla\varphi_n|^2 \le \int_\Omega M(x)\nabla\varphi_n\nabla\varphi_n = \frac{1}{2}\int_\Omega |\nabla u_n|^2\varphi_n,$$

which, together with (2.23) implies that

$$\int_\Omega |\nabla\varphi_n|^2 \le C\|f\|^2_{L^{2_*}(\Omega)}. \tag{2.24}$$

We now use Lemma 2.4, by choosing u_n^4 as test function in (2.16). We obtain

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2 u_n^4 = 4\int_\Omega M(x)\nabla\varphi_n\nabla u_n u_n^3 \le \frac{1}{4}\int_\Omega |\nabla u_n|^2 u_n^4 + C\int_\Omega |\nabla\varphi_n|^2 u_n^2.$$

Therefore,

$$\int_\Omega |\nabla u_n|^2 u_n^4 \le C\int_\Omega |\nabla\varphi_n|^2 u_n^2. \tag{2.25}$$

We now choose $\varphi_n u_n^2$ as test function in (2.16) to have that

$$\begin{aligned}\alpha\int_\Omega |\nabla\varphi_n|^2 u_n^2 &\le \frac{1}{2}\int_\Omega |\nabla u_n|^2\varphi_n u_n^2 + 2\beta\int_\Omega |\nabla\varphi_n||\nabla u_n|\varphi_n|u_n|.\\ &\le \int_\Omega |\nabla u_n|^2\varphi_n u_n^2 + C\int_\Omega |\nabla\varphi_n|^2\varphi_n.\end{aligned} \tag{2.26}$$

We now choose u_n^3 as test function in (2.15) to obtain

$$3A\int_\Omega |\nabla u_n|^2 u_n^2 + 3\int_\Omega |\nabla u_n|^2\varphi_n u_n^2 = \int_\Omega f_n u_n^3,$$

which implies

$$\int_\Omega |\nabla u_n|^2 u_n^2 + \int_\Omega |\nabla u_n|^2\varphi_n u_n^2 \le C\int_\Omega |f_n||u_n|^3. \tag{2.27}$$

Choosing φ_n^2 as test function in (2.16), we obtain

$$2\alpha\int_\Omega |\nabla\varphi_n|^2\varphi_n \le 2\int_\Omega M(x)\nabla\varphi_n\nabla\varphi_n\varphi_n = \frac{1}{2}\int_\Omega |\nabla u_n|^2\varphi_n^2,$$

which implies

$$\int_\Omega |\nabla\varphi_n|^2\varphi_n \le \frac{1}{4\alpha}\int_\Omega |\nabla u_n|^2\varphi_n^2. \tag{2.28}$$

We now choose $u_n\varphi_n$ as test function in (2.15) to obtain

$$\int_\Omega (A+\varphi_n)|\nabla u_n|^2\varphi_n \le \int_\Omega |f_n||u_n|\varphi_n + \int_\Omega (A+\varphi_n)|\nabla u_n||\nabla\varphi_n||u_n|,$$

which implies (by dropping a positive term and using (2.24) and (2.27))

$$\begin{aligned}\int_\Omega |\nabla u_n|^2\varphi_n^2 &\le C\int_\Omega |f_n||u_n|\varphi_n + C\int_\Omega |\nabla u_n||\nabla\varphi_n||u_n| + C\int_\Omega |\nabla u_n||\nabla\varphi_n||u_n|\varphi_n.\\ &\le C\int_\Omega |f_n||u_n|\varphi_n + C\int_\Omega |\nabla u_n|^2 u_n^2 + C\int_\Omega |\nabla\varphi_n|^2 + C\int_\Omega |\nabla u_n||\nabla\varphi_n||u_n|\varphi_n\\ &\le C\int_\Omega |f_n||u_n|\varphi_n + C\int_\Omega |f_n||u_n|^3 + C\|f\|^2_{L^{2_*}(\Omega)} + C\int_\Omega |\nabla u_n|^2\varphi_n u_n^2 + 2\alpha\int_\Omega |\nabla\varphi_n|^2\varphi_n\\ &\le C\int_\Omega |f_n||u_n|\varphi_n + C\int_\Omega |f_n||u_n|^3 + C\|f\|^2_{L^{2_*}(\Omega)} + 2\alpha\int_\Omega |\nabla\varphi_n|^2\varphi_n.\end{aligned} \tag{2.29}$$

Inserting (2.29) in (2.28), we thus get

$$\int_\Omega |\nabla\varphi_n|^2\varphi_n \le C\int_\Omega |f_n||u_n|\varphi_n + C\int_\Omega |f_n||u_n|^3 + C\|f\|^2_{L^{2_*}(\Omega)} + \frac{1}{2}\int_\Omega |\nabla\varphi_n|^2\varphi_n,$$

which implies that

$$\int_\Omega |\nabla\varphi_n|^2\varphi_n \le C\int_\Omega |f_n||u_n|\varphi_n + C\int_\Omega |f_n||u_n|^3 + C\|f\|^2_{L^{2_*}(\Omega)}. \tag{2.30}$$

Using (2.27) and (2.30) in (2.26), we thus obtain

$$\int_\Omega |\nabla\varphi_n|u_n^2 \le C\int_\Omega |f_n||u_n|\varphi_n + C\int_\Omega |f_n||u_n|^3 + C\|f\|^2_{L^{2_*}(\Omega)},$$

which, once used in (2.25), yields

$$\int_\Omega |\nabla u_n|^2u_n^4 \le C\int_\Omega |f_n||u_n|\varphi_n + C\int_\Omega |f_n||u_n|^3 + C\|f\|^2_{L^{2_*}(\Omega)}. \tag{2.31}$$

We now define $v_n = u_n^3$ and $\psi_n = \varphi_n^{3/2}$. With these definitions (2.31) and (2.30) become, after adding them,

$$\int_\Omega |\nabla v_n|^2 + \int_\Omega |\nabla\psi_n|^2 \le C\int_\Omega |f_n||v_n|^{\frac{1}{3}}\psi_n^{\frac{2}{3}} + C\int_\Omega |f_n||v_n| + C\|f\|^2_{L^{2_*}(\Omega)}. \tag{2.32}$$

We now remark that since

$$\frac{1}{2_*} + \frac{1}{3}\frac{1}{2^*} + \frac{2}{3}\frac{1}{2^*} = \frac{1}{2_*} + \frac{1}{2^*} = 1,$$

we have, by the Hölder inequality, the Sobolev embedding, and by (2.22),

$$\begin{aligned} C\int_\Omega |f_n||v_n|^{\frac{1}{3}}\psi_n^{\frac{2}{3}} + C\int_\Omega |f_n||v_n| &\le C\|f_n\|_{L^{2_*}(\Omega)}\|v_n\|^{\frac{1}{3}}_{W_0^{1,2}(\Omega)}\|\psi_n\|^{\frac{2}{3}}_{W_0^{1,2}(\Omega)} + C\|f_n\|_{L^{2_*}(\Omega)}\|v_n\|_{W_0^{1,2}(\Omega)} \\ &\le C\|f\|_{L^{2_*}(\Omega)}\|v_n\|^{\frac{1}{3}}_{W_0^{1,2}(\Omega)}\|\psi_n\|^{\frac{2}{3}}_{W_0^{1,2}(\Omega)} + C\|f\|_{L^{2_*}(\Omega)}\|v_n\|_{W_0^{1,2}(\Omega)}. \end{aligned}$$

Therefore, if we define $X_n = \|v_n\|_{W_0^{1,2}(\Omega)}$ and $Y_n = \|\psi_n\|_{W_0^{1,2}(\Omega)}$, inequality (2.32) can be rewritten as

$$X_n^2 + Y_n^2 \le C\|f\|_{L^{2_*}(\Omega)}X_n^{\frac{1}{3}}Y_n^{\frac{2}{3}} + C\|f\|_{L^{2_*}(\Omega)}X_n + C\|f\|^2_{L^{2_*}(\Omega)}.$$

We now remark that, since $\frac{1}{2} + \frac{1}{6} + \frac{1}{3} = 1$, by the Young inequality we have that

$$C\|f\|_{L^{2_*}(\Omega)}X_n^{\frac{1}{3}}Y_n^{\frac{2}{3}} \le C\|f\|^2_{L^{2_*}(\Omega)} + \frac{1}{4}X_n^2 + \frac{1}{2}Y_n^2,$$

while

$$C\|f\|_{L^{2_*}(\Omega)}X_n \le C\|f\|^2_{L^{2_*}(\Omega)} + \frac{1}{4}X_n^2,$$

so that

$$X_n^2 + Y_n^2 \le C\|f\|^2_{L^{2_*}(\Omega)}.$$

Therefore, recalling the definition of X_n and Y_n, and of v_n and ψ_n, we have that

$$\int_\Omega |\nabla[u_n]^3|^2 \le C\|f\|_{L^{2_*}(\Omega)}, \qquad \int_\Omega |\nabla[\varphi_n]^{3/2}|^2 \le C\|f\|_{L^{2_*}(\Omega)}. \tag{2.33}$$

Furthermore, recalling (2.29), we have that

$$\int_\Omega |\nabla u_n|^2\varphi_n^2 \le C\|f\|^2_{L^{2_*}(\Omega)}. \tag{2.34}$$

Thus, thanks to the previous estimates (see (2.23), (2.24) and (2.33)), we have that $\{u_n\}$, $\{\varphi_n\}$, $\{u_n^3\}$ and $\{\varphi_n^{3/2}\}$ are bounded in $W_0^{1,2}(\Omega)$, and (by recalling (2.34)) the sequence $\{|\nabla u_n|\varphi_n\}$ is bounded in $L^2(\Omega)$. Reasoning as in the proof of Lemma 2.6, we easily see that, if u and φ are the weak limits of u_n and φ_n, respectively, we have that

$$\varphi_n \nabla u_n \text{ weakly converges to } \varphi \nabla u \text{ in } (L^2(\Omega))^N.$$

This convergence allows us to pass to the limit in the identities

$$\int_\Omega (A + \varphi_n)\nabla u_n \nabla v = \int_\Omega f_n v, \tag{2.35}$$

which hold for every v in $W_0^{1,2}(\Omega)$, to obtain that

$$\int_\Omega (A + \varphi)\nabla u \nabla v = \int_\Omega f v \quad \text{for all } v \in W_0^{1,2}(\Omega). \tag{2.36}$$

Choosing $v = u$ in the above identity and $v = u_n$ in (2.35), we have that

$$\lim_{n\to+\infty} \int_\Omega (A + \varphi_n)|\nabla u_n|^2 \overset{(2.35)}{=} \lim_{n\to+\infty} \int_\Omega f_n u_n = \int_\Omega f u \overset{(2.36)}{=} \int_\Omega (A + \varphi)|\nabla u|^2,$$

which implies (since the functions are positive) that

$$(A + \varphi_n)|\nabla u_n|^2 \text{ strongly converges to } (A + \varphi)|\nabla u|^2 \text{ in } L^1(\Omega).$$

In particular, u_n strongly converges to u in $W_0^{1,2}(\Omega)$. This allows us to pass to the limit in the identities

$$\int_\Omega M(x)\nabla \varphi_n \nabla w = \frac{1}{2}\int_\Omega |\nabla u_n|^2 w,$$

which hold (in particular) for every w in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, to have that

$$\int_\Omega M(x)\nabla \varphi \nabla w = \frac{1}{2}\int_\Omega |\nabla u|^2 w \quad \text{for all } w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega).$$

Reasoning as in the proof of Lemma 2.2 and using that φ belongs to $W_0^{1,2}(\Omega)$, we finally have that

$$\int_\Omega M(x)\nabla \varphi \nabla w = \frac{1}{2}\int_\Omega |\nabla u|^2 w \quad \text{for all } w \in W_0^{1,2}(\Omega). \qquad \square$$

3 Saddle points

In this section, we prove Theorem 1.5. In order to do that, we will strongly use the existence result of Theorem 1.1, in particular the summability results on u and φ, and the fact that $W_0^{1,2}(\Omega)$ functions are allowed as test functions in both the equations of the system.

Proof of Theorem 1.5. Choosing $w = \psi - \varphi$ in (1.5), we have that

$$\int_\Omega M(x)\nabla \varphi \nabla(\psi - \varphi) = \frac{1}{2}\int_\Omega |\nabla u|^2(\psi - \varphi),$$

which implies

$$\frac{1}{2}\int_\Omega |\nabla u|^2(\psi - \varphi) = \frac{1}{2}\int_\Omega M(x)\nabla \psi \nabla \psi - \frac{1}{2}\int_\Omega M(x)\nabla(\psi - \varphi)\nabla(\psi - \varphi) - \frac{1}{2}\int_\Omega M(x)\nabla \varphi \nabla \varphi.$$

Therefore, rearranging terms and dropping a negative term, we have (recall that $|\nabla u|^2\psi$ belongs to $L^1(\Omega)$)

$$\frac{1}{2}\int_\Omega |\nabla u|^2\psi - \frac{1}{2}\int_\Omega M(x)\nabla\psi\nabla\psi \le \frac{1}{2}\int_\Omega |\nabla u|^2\varphi - \frac{1}{2}\int_\Omega M(x)\nabla\varphi\nabla\varphi,$$

that is,

$$\begin{aligned} J(u,\psi) &= \frac{1}{2}\int_\Omega (A+\psi)|\nabla u|^2 - \frac{1}{2}\int_\Omega M(x)\nabla\psi\nabla\psi - \int_\Omega fu \\ &\le \frac{1}{2}\int_\Omega (A+\varphi)|\nabla u|^2 - \frac{1}{2}\int_\Omega M(x)\nabla\varphi\nabla\varphi - \int_\Omega fu = J(u,\varphi) \end{aligned}$$

for every ψ in $W_0^{1,2}(\Omega)$, which is one half of (1.7).

Choosing $v = u - z$ in (1.4), we have

$$\int_\Omega (A+\varphi)\nabla u\nabla(u-z) = \int_\Omega f(u-z),$$

which implies

$$\frac{1}{2}\int_\Omega (A+\varphi)|\nabla u|^2 + \Big[\frac{1}{2}\int_\Omega (A+\varphi)|\nabla u|^2 - \int_\Omega (A+\varphi)\nabla u\nabla z\Big] = \int_\Omega f(u-z).$$

This latter identity implies, if z is such that $\varphi|\nabla z|^2$ belongs to $L^1(\Omega)$, that

$$\frac{1}{2}\int_\Omega (A+\varphi)|\nabla u|^2 + \frac{1}{2}\int_\Omega (A+\varphi)\nabla(u-z)\nabla(u-z) + \frac{1}{2}\int_\Omega (A+\varphi)|\nabla z|^2 = \int_\Omega f(u-z),$$

which then yields, by dropping a positive term, that

$$\frac{1}{2}\int_\Omega (A+\varphi)|\nabla u|^2 - \int_\Omega fu \le \frac{1}{2}\int_\Omega (A+\varphi)|\nabla z|^2 - \int_\Omega fz$$

for every z in $W_0^{1,2}(\Omega)$ such that $\varphi|\nabla z|^2$ belongs to $L^1(\Omega)$. From this inequality, it easily follows that

$$J(u,\varphi) \le J(z,\varphi)$$

for every z in $W_0^{1,2}(\Omega)$ such that $\varphi|\nabla z|^2$ belongs to $L^1(\Omega)$, i.e., the other half of (1.7). □

4 Increased summability results

In Theorem 1.1, we have proved that if f belongs to $L^{2_*}(\Omega)$, then there exists a solution (u,φ) with u in $L^{3\cdot 2^*}(\Omega)$ and φ in $L^{3\cdot 2^*/2}(\Omega)$; on the other hand, in Lemma 2.4 we have proved that if f belongs to $L^m(\Omega)$, with $m > \frac{N}{2}$, then u belongs to $L^\infty(\Omega)$, while φ belongs to $L^s(\Omega)$ for every $s \ge 1$.

If one looks at the regularities of both u and φ and at Remark 1.6, it looks like the role of φ is to be approximately equal to u^2, so that one wonders whether this is a fact, or a coincidence. To see that this latter may not be the case, let us go back to the functional J, in the case $M(x) = I$, and let us calculate $J(v, kv^2)$ with k being a real number. We have

$$\begin{aligned} J(v,kv^2) &= \frac{1}{2}\int_\Omega (A+kv^2)|\nabla v|^2 - 2k^2\int_\Omega |\nabla v|^2v^2 - \int_\Omega fv \\ &= \frac{A}{2}\int_\Omega |\nabla v|^2 + \frac{1}{2}(k-4k^2)\int_\Omega |\nabla v|^2v^2 - \int_\Omega fv. \end{aligned}$$

Choosing k in $(0, \frac{1}{4})$, we thus have

$$J(v, kv^2) = \frac{1}{2}\int_\Omega (A + Bv^2)|\nabla v|^2 - \int_\Omega fv$$

for some positive real number B. This functional is both coercive and weakly lower semicontinuous on $W_0^{1,2}(\Omega)$, and its minimum w can be proved to be a solution of

$$-\operatorname{div}((A + Bw^2)\nabla w) + Bw|\nabla w|^2 = f.$$

We recall (see [1, 7]) that, if f belongs to $L^m(\Omega)$, with $2_* \le m < \frac{N}{2}$, then w belongs to $L^{3\cdot m^{**}}(\Omega)$. Indeed, supposing that $f \ge 0$ so that $w \ge 0$, and choosing $T_k(w)^{2\gamma-1}$ as test function (with $k > 0$ and $\gamma > \frac{1}{2}$), we obtain

$$\begin{aligned}(2\gamma - 1)\int_\Omega (A + BT_k(w)^2)|\nabla T_k(w)|^2 T_k(w)^{2\gamma-2} + B\int_\Omega |\nabla w|^2 T_k(w)^{2\gamma} &= \int_\Omega fT_k(w)^{2\gamma-1} \\ &\le \|f\|_{L^m(\Omega)}\Big(\int_\Omega T_k(w)^{(2\gamma-1)m'}\Big)^{\frac{1}{m'}}.\end{aligned}$$

Dropping the first term, which is positive, we then have, since $|\nabla w|^2 \ge |\nabla T_k(w)|^2$, the following:

$$\frac{B}{(\gamma+1)^2}\int_\Omega |\nabla T_k(w)^{\gamma+1}|^2 = \int_\Omega |\nabla T_k(w)|^2 T_k(w)^{2\gamma} \le \|f\|_{L^m(\Omega)}\Big(\int_\Omega T_k(w)^{(2\gamma-1)m'}\Big)^{\frac{1}{m'}}.$$

This implies, by the Sobolev embedding,

$$C\Big(\int_\Omega T_k(w)^{(\gamma+1)2^*}\Big)^{\frac{2}{2^*}} \le \|f\|_{L^m(\Omega)}\Big(\int_\Omega T_k(w)^{(2\gamma-1)m'}\Big)^{\frac{1}{m'}}.$$

We now choose γ such that

$$(\gamma + 1)2^* = (2\gamma - 1)m' \quad \Longleftrightarrow \quad \gamma = \frac{2^* + m'}{2m' - 2^*}. \tag{4.1}$$

It is easy to see that $\gamma > \frac{1}{2}$ and that

$$(\gamma + 1)2^* = 3\frac{Nm}{N - 2m} = 3 \cdot m^{**}.$$

Hence, after simplifying equal terms, we have

$$\Big(\int_\Omega T_k(w)^{3\cdot m^{**}}\Big)^{\frac{1}{m^{**}}} \le \|f\|_{L^m(\Omega)}.$$

Letting k tend to infinity and using the Levi monotone convergence theorem, we thus have

$$\Big(\int_\Omega w^{3\cdot m^{**}}\Big)^{\frac{1}{m^{**}}} \le \|f\|_{L^m(\Omega)},$$

as desired.

Thus, since the role of the second variable φ in the functional is played here by kv^2, one can see why it is that φ behaves like u^2. Furthermore, we have on u (which here is w) an increased summability result, since it belongs to $L^{3\cdot m^{**}}(\Omega)$, instead of the linear case summability $L^{m^{**}}(\Omega)$ which corresponds to $B = 0$.

Let us now turn to the actual system, where φ is not comparable with u^2. We already know, in the limiting case $m = 2_*$, that u belongs to $L^{3\cdot 2^*}(\Omega) = L^{3\cdot m^{**}}(\Omega)$, so that we wonder whether this result holds for every $2_* < m < \frac{N}{2}$. As we will see, this is true for some (infinitely many) values of m, for which u and φ are more summable: u belongs to $L^{3\cdot m^{**}}(\Omega)$ and φ belongs to $L^{3\cdot m^{**}/2}(\Omega)$.

Theorem 4.1. *Let $\gamma \geq 2$ be an integer and let m be such that $(2\gamma - 1)m' = (\gamma + 1)2^*$. If (u, φ) is the solution of system (1.3) given by Theorem 1.1, then*

$$u \text{ belongs to } L^{3\cdot m^{**}}(\Omega), \quad \varphi \text{ belongs to } L^{3\cdot m^{**}/2}(\Omega).$$

Remark 4.2. The result of the previous theorem only holds for a discrete set $\mathcal{S}$ of values of the summability exponent m; more precisely, since $m' = \frac{(\gamma+1)2^*}{2\gamma-1}$, then

$$\mathcal{S} = \Big\{ m = \frac{2N(\gamma + 1)}{3N - 2 + 4\gamma}, \ \gamma \in \mathbb{N}, \ \gamma \geq 2 \Big\}.$$

Note that every m in $\mathcal{S}$ satisfies $2_* \leq m < \frac{N}{2}$. Clearly, if m is strictly between 2_* and $\frac{N}{2}$, but is not in $\mathcal{S}$, then a function in $L^m(\Omega)$ belongs to $L^{m_k}(\Omega)$, where m_k is the largest element of $\mathcal{S}$ smaller than m. Therefore, by Theorem 4.1, u belongs to $L^{3\cdot m_k^{**}}(\Omega)$ and φ belongs to $L^{3\cdot m_k^{**}/2}(\Omega)$. As stated before, we conjecture that the result of Theorem 4.1 holds for every m in the interval $[2_*, \frac{N}{2})$, with $3 \cdot m^{**}$ and $3 \cdot m^{**}/2$ as summability exponents for u and φ, but our technique is confined to the choice of integer values for the parameter γ. Also observe that the function which links m to γ in the statement of the theorem is the same given by (4.1) in the estimates for w.

The proof of Theorem 4.1 follows from the following result.

Lemma 4.3. *Let $\gamma \geq 2$ be an integer and let m be such that $(2\gamma - 1)m' = (\gamma + 1)2^*$. If (u, φ) is the solution of system (1.3) given by Theorem 1.1, then for every $1 \leq \eta \leq \gamma$ we have*

$$\int_\Omega |\nabla u|^2 u^{2\eta} \leq C(\|f\|_{L^m(\Omega)}), \quad \int_\Omega |\nabla \varphi|^2 \varphi^{\eta-1} \leq C(\|f\|_{L^m(\Omega)}), \tag{4.2}$$

and

$$\int_\Omega |\nabla u||\nabla \varphi|\varphi^k |u|^{2\eta-2k-1} \leq C(\|f\|_{L^m(\Omega)}) \tag{4.3}$$

for every $0 \leq k \leq \eta - 1$.

Remark 4.4. Observe that in (4.3) the sum of twice the exponent of φ and the exponent of u is constant, and equal to $2\eta - 1$. This is consistent with the fact that, in principle, φ behaves like u^2, so that increasing the exponent of φ by 1 decreases the exponent of u by 2.

Remark 4.5. Observe that a consequence of (4.2), written for $\eta = \gamma$, is that both $|u|^{\gamma+1}$ and $\varphi^{(\gamma+1)/2}$ belong to $W_0^{1,2}(\Omega)$, a result which has already been proved in Theorem 1.1 if $\gamma = 2$.

Proof of Lemma 4.3. We begin by observing that since $m' = \frac{(\gamma+1)2^*}{2\gamma-1}$, the assumption $\gamma \geq 2$ implies that $m' \leq 2^*$, so that $m \geq 2_*$. In what follows, to shorten notation we will denote by $C(f)$ any constant depending on the norm of f in $L^m(\Omega)$. We remark that since Ω is bounded, the norm of f in $L^s(\Omega)$ can be controlled by (a constant times) the norm of f in $L^m(\Omega)$ if $1 \leq s < m$.

The proof is by induction on η. If $\eta = 1$, inequalities (4.2) and (4.3) are equivalent to

$$\int_\Omega |\nabla u|^2 u^2 + \int_\Omega |\nabla \varphi|^2 + \int_\Omega |\nabla u||\nabla \varphi||u| \leq C(f),$$

which is what we are going to prove. Recalling that, by Theorem 1.1, the norm of φ in $W_0^{1,2}(\Omega)$ is controlled by the norm of f in $L^{2_*}(\Omega)$ (hence in $L^m(\Omega)$), we can control the second term above. On the other hand, by the Hölder inequality we have, if $\varepsilon > 0$, that

$$\int_\Omega |\nabla u||\nabla \varphi||u| \leq C\varepsilon \int_\Omega |\nabla u|^2 u^2 + \frac{C}{\varepsilon} \int_\Omega |\nabla \varphi|^2, \tag{4.4}$$

and, using u^2 as test function in (2.16) (i.e., choosing $\rho = 0$ and $\sigma = 2$), we have

$$\frac{1}{2} \int_\Omega |\nabla u|^2 u^2 = 2 \int_\Omega M(x)\nabla \varphi \nabla u u \leq 2\beta \int_\Omega |\nabla u||\nabla \varphi||u|.$$

Hence, from (4.4) it follows that

$$\int_\Omega |\nabla u||\nabla \varphi||u| \le C\varepsilon \int_\Omega |\nabla u||\nabla \varphi||u| + \frac{C}{\varepsilon}\int_\Omega |\nabla \varphi|^2,$$

which implies (by choosing ε small enough) that

$$\int_\Omega |\nabla u||\nabla \varphi||u| \le C \int_\Omega |\nabla \varphi|^2 \le C(f).$$

Finally,

$$\int_\Omega |\nabla u|^2 u^2 \le 4\beta \int_\Omega |\nabla u||\nabla \varphi||u| \le C(f),$$

and the case $\eta = 1$ is complete.

The case $\eta = 2$ has been dealt with in the proof of Theorem 1.1, so that we now can tackle the general case. Suppose that (4.2) and (4.3) hold for $\eta - 1$; this in particular implies that

$$\int_\Omega |\nabla u||\nabla \varphi|\varphi^h |u|^{2\eta-2h-3} \le C(f) \tag{4.5}$$

for every $0 \le h \le \eta - 2$. Let now $0 \le k \le \eta - 1$; define $Q_0 = 0$, $Q_\eta = 0$ and

$$Q_k = \int_\Omega |\nabla u||\nabla \varphi|\varphi^k |u|^{2\eta-2k-1}, \quad k = 1, \dots, \eta - 1.$$

Define also

$$F_k = \int_\Omega |f|\varphi^k |u|^{2\eta-2k-1}$$

for $0 \le k \le \eta - 1$. We are going to prove that, for every $1 \le k \le \eta - 1$ and if ε is small enough (depending on the data of the problem),

$$Q_k \le \frac{C}{\varepsilon} F_{k-1} + C\varepsilon F_k + \frac{C}{\varepsilon} Q_{k-1} + C\varepsilon Q_{k+1} + \frac{1}{\varepsilon} C(f). \tag{4.6}$$

We begin with the case $1 < k < \eta - 1$; by the Young inequality, we have

$$Q_k \le \frac{C}{\varepsilon} \int_\Omega |\nabla u|^2 \varphi^k u^{2\eta-2k} + C\varepsilon \int_\Omega |\nabla \varphi|^2 \varphi^k u^{2\eta-2k-2}. \tag{4.7}$$

Thanks to (2.15), with $\rho = k$ and $\sigma = 2\eta - 2k + 1$, we have, by dropping a positive term,

$$(2\eta - 2k + 1) \int_\Omega |\nabla u|^2 \varphi^k u^{2\eta-2k} \le \int_\Omega |f|\varphi^{k-1} |u|^{2\eta-2k+1} + (k-1) \int_\Omega |\nabla u||\nabla \varphi|(A + \varphi)\varphi^{k-2} |u|^{2\eta-2k+1},$$

which implies

$$\int_\Omega |\nabla u|^2 \varphi^k u^{2\eta-2k} \le C F_{k-1} + C Q_{k-1} + C \int_\Omega |\nabla u||\nabla \varphi|\varphi^{k-2} |u|^{2\eta-2k+1}.$$

Observe now that $0 \le k - 2 < \eta - 2$; therefore, from (4.5) with $h = k - 2$ we have that

$$\int_\Omega |\nabla u||\nabla \varphi|\varphi^{k-2} |u|^{2\eta-2k+1} \le C(f),$$

from which it follows that

$$\int_\Omega |\nabla u|^2 \varphi^k u^{2\eta-2k} \le C F_{k-1} + C Q_{k-1} + C(f). \tag{4.8}$$

On the other hand, thanks to (2.16) with $\rho = k + 1$ and $\sigma = 2\eta - 2k - 2$, we have

$$\alpha(k+1)\int_\Omega |\nabla\varphi|^2\varphi^k u^{2\eta-2k-2} \le \frac{1}{2}\int_\Omega |\nabla u|^2\varphi^{k+1}u^{2\eta-2k-2} + \beta(2\eta-2k-2)\int_\Omega |\nabla u||\nabla\varphi|\varphi^{k+1}|u|^{2\eta-2k-3},$$

which implies

$$\int_\Omega |\nabla\varphi|^2\varphi^k u^{2\eta-2k-2} \le C\int_\Omega |\nabla u|^2\varphi^{k+1}u^{2\eta-2k-2} + CQ_{k+1}. \tag{4.9}$$

Using (2.16) with $\rho = k$ and $\sigma = 2\eta - 2k - 1$, we have (by dropping a positive term)

$$(2\eta-2k-1)\int_\Omega |\nabla u|^2\varphi^{k+1}u^{2\eta-2k-2} \le \int_\Omega |f|\varphi^k|u|^{2\eta-2k-1} + k\int_\Omega |\nabla u||\nabla\varphi|(A+\varphi)\varphi^{k-1}|u|^{2\eta-2k-1},$$

from which it follows that

$$\int_\Omega |\nabla u|^2\varphi^{k+1}u^{2\eta-2k-2} \le CF_k + CQ_k + C\int_\Omega |\nabla u||\nabla\varphi|\varphi^{k-1}|u|^{2\eta-2k-1}. \tag{4.10}$$

Since $0 < k - 1 < \eta - 2$, we can apply (4.5) with $h = k - 1$ to have that

$$\int_\Omega |\nabla u||\nabla\varphi|\varphi^{k-1}|u|^{2\eta-2k-1} \le C(f).$$

Using this estimate in (4.10), we thus obtain that

$$\int_\Omega |\nabla u|^2\varphi^{k+1}u^{2\eta-2k-2} \le CF_{k+1} + CQ_k + C(f). \tag{4.11}$$

Inserting (4.11) in (4.9), we thus have

$$\int_\Omega |\nabla\varphi|^2\varphi^k u^{2\eta-2k-2} \le CF_{k+1} + CQ_k + CQ_{k+1} + C(f). \tag{4.12}$$

Using (4.8) and (4.12), inequality (4.7) becomes

$$Q_k \le \frac{C}{\varepsilon}F_{k-1} + \frac{C}{\varepsilon}Q_{k-1} + \frac{1}{\varepsilon}C(f) + C\varepsilon F_{k+1} + C\varepsilon Q_k + C\varepsilon Q_{k+1} + \varepsilon C(f).$$

Choosing ε small enough so that the term $C\varepsilon Q_k$ on the right-hand side can be absorbed by the one on the left-hand side, we have that (4.6) holds true.

We now turn to the case $k = 1$; since the techniques are similar to the case $1 < k < \eta - 1$, we will not repeat every passage in detail. Thus, again by (2.15), (2.16), the Young inequality, and (4.5), we have

$$\begin{aligned}
Q_1 &= \int_\Omega |\nabla u||\nabla\varphi|\varphi|u|^{2\eta-3} \\
&\le \frac{C}{\varepsilon}\int_\Omega |\nabla u|^2\varphi u^{2\eta-2} + C\varepsilon\int_\Omega |\nabla\varphi|^2\varphi u^{2\eta-4} \\
&\le \frac{C}{\varepsilon}\int_\Omega |f||u|^{2\eta-1} + C\varepsilon\int_\Omega |\nabla u|^2\varphi^2u^{2\eta-4} + C\varepsilon\int_\Omega |\nabla u||\nabla\varphi|\varphi^2|u|^{2\eta-5} \\
&\le \frac{C}{\varepsilon}F_0 + C\varepsilon\int_\Omega |f|\varphi|u|^{2\gamma-3} + C\varepsilon\int_\Omega |\nabla u||\nabla\varphi|(A+\varphi)|u|^{2\eta-3} + C\varepsilon Q_2 \\
&\le \frac{C}{\varepsilon}F_0 + C\varepsilon F_1 + C\varepsilon Q_1 + C\varepsilon\int_\Omega |\nabla u||\nabla\varphi||u|^{2\eta-3} + C\varepsilon Q_2 \\
&\le \frac{C}{\varepsilon}F_0 + C\varepsilon F_1 + C\varepsilon Q_1 + C\varepsilon Q_2 + C\varepsilon C(f),
\end{aligned}$$

which then yields (4.6) for $k = 1$ choosing ε small enough so that the term $C\varepsilon Q_1$ on the right-hand side can be absorbed by the one on the left-hand side. If $k = \eta - 1$, we have, still by (2.15), (2.16), the Young inequality, and (4.5),

$$\begin{aligned}
Q_{\eta-1} &\le \frac{C}{\varepsilon}\int_\Omega |\nabla u|^2\varphi^{\eta-1}u^2 + C\varepsilon|\nabla\varphi|^2\varphi^{\eta-1} \\
&\le \frac{C}{\varepsilon}\int_\Omega |f|\varphi^{\eta-2}|u|^3 + \frac{C}{\varepsilon}\int_\Omega |\nabla u||\nabla\varphi|(A+\varphi)\varphi^{\eta-3}|u|^3 + C\varepsilon|\nabla u|^2\varphi^\eta \\
&\le \frac{C}{\varepsilon}F_{\eta-2} + \frac{C}{\varepsilon}Q_{\eta-2} + \frac{C}{\varepsilon}\int_\Omega |\nabla u||\nabla\varphi|\varphi^{\eta-3}|u|^3 + C\varepsilon\int_\Omega |f|\varphi^{\eta-1}|u| + C\varepsilon\int_\Omega |\nabla u||\nabla\varphi|(A+\varphi)\varphi^{\eta-2}|u| \\
&\le \frac{C}{\varepsilon}F_{\eta-2} + \frac{C}{\varepsilon}Q_{\eta-2} + \frac{1}{\varepsilon}C(f) + C\varepsilon F_{\eta-1} + C\varepsilon Q_{\eta-1} + C\varepsilon\int_\Omega |\nabla u||\nabla\varphi|\varphi^{\eta-3}|u|^3 \\
&\le \frac{C}{\varepsilon}F_{\eta-2} + C\varepsilon F_{\eta-1} + \frac{C}{\varepsilon}Q_{\eta-2} + C\varepsilon Q_{\eta-1} + \frac{1}{\varepsilon}C(f),
\end{aligned}$$

which as before yields (4.6) by choosing ε small enough in order to absorb the term $C\varepsilon Q_{\eta-1}$ on the right on the left-hand side.

We now iteratively use (4.6), starting from the one which holds for $k = \eta - 1$. To simplify notations, we will denote by $\mathcal{F}$ any linear combination of the F_j's. Also, let $\delta > 0$ and write (4.6) for $\varepsilon = \delta$ (recall that $Q_\eta = 0$ by definition) as

$$Q_{\eta-1} \le \mathcal{F} + \frac{C}{\delta}Q_{\eta-2} + C(f).$$

On the other hand, again from (4.6) written for $k = \eta - 2$ and $\varepsilon = \delta^2$, we have

$$Q_{\eta-2} \le \mathcal{F} + \frac{C}{\delta^2}Q_{\eta-3} + C\delta^2 Q_{\eta-1} + C(f).$$

Inserting the first in the second, we obtain

$$Q_{\eta-2} \le \mathcal{F} + \frac{C}{\delta^2}Q_{\eta-3} + C\delta Q_{\eta-2} + C(f),$$

which then implies, if δ is small enough, that

$$Q_{\eta-2} \le \mathcal{F} + \frac{C}{\delta^2}Q_{\eta-3} + C(f).$$

On the other hand (we are supposing here that there is “enough space” to have $\eta - 4 \ge 1$), again from (4.6) written with $\varepsilon = \delta^3$, we have

$$Q_{\eta-3} \le \mathcal{F} + \frac{C}{\delta^3}Q_{\eta-4} + C\delta^3 Q_{\eta-2} + C(f),$$

so that

$$Q_{\eta-3} \le \mathcal{F} + \frac{C}{\delta^3}Q_{\eta-4} + C\delta Q_{\eta-3} + C(f).$$

Hence, choosing again δ small enough,

$$Q_{\eta-3} \le \mathcal{F} + \frac{C}{\delta^3}Q_{\eta-4} + C(f).$$

Going on, we obtain

$$Q_2 \le \mathcal{F} + \frac{C}{\delta^{\eta-2}}Q_1 + C(f).$$

Since $Q_0 = 0$, from (4.6) written for $k = \eta - 1$ and $\varepsilon = \delta^{\eta-1}$, we have

$$Q_1 \le \mathcal{F} + C\delta^{\eta-1}Q_2 + C(f),$$

so that

$$Q_1 \le \mathcal{F} + C\delta Q_1 + C(f),$$

which (finally) yields, by choosing δ small enough,

$$Q_1 \le \mathcal{F} + C(f).$$

Inserting this inequality in the one for Q_2, we thus get that

$$Q_2 \le \mathcal{F} + C(f),$$

and, retracing our steps, we thus have that

$$Q_k \le \mathcal{F} + C(f) \quad \text{for all } k = 1, \dots, \eta - 1.$$

In other words, we have proved that

$$\int_\Omega |\nabla u||\nabla\varphi|\varphi^k |u|^{2\eta-2k-1} \le \mathcal{F} + C(f) \quad \text{for all } k = 1, \dots, \eta - 1. \tag{4.13}$$

Now we are almost finished. Taking $u^{2\eta}$ as test function in (2.16), we obtain, thanks to the Young inequality,

$$\frac{1}{2}\int_\Omega |\nabla u|^2 u^{2\eta} = 2\eta \int_\Omega \nabla\varphi\nabla u u^{2\eta-1} \le \frac{1}{4}\int_\Omega |\nabla u|^2 u^{2\eta} + C\int_\Omega |\nabla\varphi|^2 u^{2\eta-2},$$

so that

$$\int_\Omega |\nabla u|^2 u^{2\eta} \le C\int_\Omega |\nabla\varphi|^2 u^{2\eta-2}.$$

But, again from (2.16) and (2.15), and from (4.13),

$$\begin{aligned}\int_\Omega |\nabla\varphi|^2 u^{2\eta-2} &\le C\int_\Omega |\nabla u|^2 \varphi u^{2\eta-2} + C\int_\Omega |\nabla u||\nabla\varphi|\varphi |u|^{2\eta-3} \\ &\le C\int_\Omega |f||u|^{2\eta-1} + C\int_\Omega |\nabla u||\nabla\varphi|\varphi|u|^{2\eta-3} \\ &\le CF_0 + \mathcal{F} + C(f) = \mathcal{F} + C(f),\end{aligned}$$

so that

$$\int_\Omega |\nabla u|^2 u^{2\eta} \le \mathcal{F} + C(f). \tag{4.14}$$

Furthermore, again by (2.16), (2.15), (4.13), and (4.5),

$$\begin{aligned}\int_\Omega |\nabla\varphi|^2\varphi^{\eta-1} \le C\int_\Omega |\nabla u|^2\varphi^\eta &\le C\int_\Omega |f|\varphi^{\eta-1}|u| + C\int_\Omega |\nabla u||\nabla\varphi|(A+\varphi)\varphi^{\eta-2}|u| \\ &\le CF_{\eta-1} + \mathcal{F} + C(f) = \mathcal{F} + C(f).\end{aligned} \tag{4.15}$$

Putting together (4.14) and (4.15), we have proved that there exist $\alpha_0, \dots, \alpha_{\eta-1}$ in $\mathbb{R}$ such that

$$\int_\Omega |\nabla u|^2 u^{2\eta} + \int_\Omega |\nabla\varphi|^2\varphi^{\eta-1} \le \sum_{k=0}^{\eta-1}\alpha_k \int_\Omega |f|\varphi^k|u|^{2\eta-2k-1} + C(f).$$

We now define $v = u^{\eta+1}$ and $\psi = \varphi^{\frac{\eta+1}{2}}$; the above inequality thus becomes

$$\int_\Omega |\nabla v|^2 + \int_\Omega |\nabla\psi|^2 \le C\sum_{k=0}^{\eta-1}\alpha_k \int_\Omega |f|\psi^{\frac{2k}{\eta+1}}|v|^{\frac{2\eta-2k-1}{\gamma+1}} + C(f).$$

Now observe that, by the assumption $(2\gamma-1)m' = (\gamma+1)2^*$, we have

$$m' = \frac{(\gamma+1)2^*}{2\gamma-1}.$$

Since the function

$$s \mapsto \frac{(s+1)2^*}{2s-1}$$

is decreasing, and since $1 \le \eta \le \gamma$, we thus have that

$$m' \le \frac{(\eta+1)2^*}{2\eta-1} \qquad \Longleftrightarrow \qquad \frac{2\eta-1}{(\eta+1)2^*} \le \frac{1}{m'}.$$

Therefore,

$$\frac{1}{m} + \frac{2k}{2^*(\eta+1)} + \frac{2\eta-2k-1}{2^*(\eta+1)} = \frac{1}{m} + \frac{2\eta-1}{2^*(\eta+1)} \le \frac{1}{m} + \frac{1}{m'} = 1,$$

so that the Hölder inequality implies that

$$\int_\Omega |\nabla v|^2 + \int_\Omega |\nabla \psi|^2 \le C\|f\|_{L^m(\Omega)} \sum_{k=0}^{\eta-1} \alpha_k \|\psi\|_{L^{2^*}(\Omega)}^{\frac{2k}{\eta+1}} \|v\|_{L^{2^*}(\Omega)}^{\frac{2\eta-2k-1}{\eta+1}} + C(f).$$

Hence, by the Sobolev inequality, we have

$$\|v\|^2_{W_0^{1,2}(\Omega)} + \|\psi\|^2_{W_0^{1,2}(\Omega)} \le C\|f\|_{L^m(\Omega)} \sum_{k=0}^{\eta-1} \alpha_k \|\psi\|_{W_0^{1,2}(\Omega)}^{\frac{2k}{\eta+1}} \|v\|_{W_0^{1,2}(\Omega)}^{\frac{2\eta-2k-1}{\eta+1}} + C(f).$$

Lemma 4.6 then implies that

$$\|v\|^2_{W_0^{1,2}(\Omega)} + \|\psi\|^2_{W_0^{1,2}(\Omega)} \le C(f).$$

Recalling the definition of v and ψ, we thus have

$$\int_\Omega |\nabla u|^2 |u|^{2\eta} + \int_\Omega |\nabla \varphi|^2 \varphi^{\eta-1} \le C(f),$$

which is (4.2) for η. Once we have proved (4.2), we have that

$$F_k \le C(f) \quad \text{for every } k = 0, \dots, \eta-1,$$

since we can use the Hölder and Sobolev inequalities. Therefore, from (4.13), it follows that

$$\int_\Omega |\nabla u||\nabla \varphi| \varphi^k |u|^{2\eta-2k-1} \le C(f) \quad \text{for every } k = 0, \dots, \eta-1,$$

which is (4.3) for η, and the result is proved. □

Theorem 4.1 is a straightforward consequence of Lemma 4.3.

Proof of Theorem 4.1. Starting from (4.2) written for $\rho = \gamma$, and by using the Sobolev embedding, we have that

$$\int_\Omega |u|^{(\gamma+1)2^*} + \int_\Omega \varphi^{\frac{(\gamma+1)2^*}{2}} \le C(f).$$

Using the relation between γ and m, we have

$$(\gamma+1)2^* = 3 \cdot m^{**},$$

and the result is proved. □

Lemma 4.6. *Let X and Y be positive real numbers such that*

$$X^2 + Y^2 \le \sum_{k=0}^{\eta-1} \beta_k X^{\frac{2k}{\eta+1}} Y^{\frac{2\eta-2k-1}{\eta+1}} + C$$

for some positive real numbers $\beta_0, \dots, \beta_{\eta-1}$. Then there exists a constant M, which depends on C and on the β_k's, such that

$$0 \le X \le M, \quad 0 \le Y \le M.$$

Proof. We begin by recalling that if $s > 0$ and $t > 0$ are such that $s + t < 2$, then for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$X^s Y^t \leq \varepsilon X^2 + \varepsilon Y^2 + C_\varepsilon.$$

Indeed, by a first Young inequality, we have

$$X^s Y^t \leq \varepsilon X^2 + C_\varepsilon Y^{\frac{2t}{2-s}},$$

so that a second Young inequality yields the result, with $\frac{2t}{2-s} < 2$ by assumption. Thus, since

$$\frac{2k}{\eta + 1} + \frac{2\eta - 2k - 1}{\gamma + 1} = \frac{2\eta - 1}{\eta + 1} < 2,$$

we have (by choosing $\varepsilon = \varepsilon_k = \frac{1}{2\eta\beta_k}$), that

$$X^{\frac{2k}{\eta+1}} Y^{\frac{2\eta-2k-1}{\eta+1}} \leq \frac{1}{2\eta\beta_k} X^2 + \frac{1}{2\eta\beta_k} Y^2 + C_k,$$

which implies

$$X^2 + Y^2 \leq \frac{1}{2} X^2 + \frac{1}{2} Y^2 + \sum_{k=0}^{\eta-1} \beta_k C_k + C,$$

and the proof is complete. □

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