The Bisognano-Wichmann property on nets of standard subspaces, some sufficient conditions

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Abstract

We discuss the Bisognano-Wichmann property for local Poincaré covariant nets of standard subspaces. We provide a sufficient algebraic condition on the covariant representation ensuring the Bisognano-Wichmann and the Duality properties without further assumptions on the net. We call it modularity condition. It holds for direct integrals of scalar massive and massless representations. We present a class of massive modular covariant nets not satisfying the Bisognano-Wichmann property. Furthermore, we give an outlook on the relation between the Bisognano-Wichmann property and the Split property in the standard subspace setting.

1 Introduction

The Algebraic Quantum Field Theory (AQFT) is a very fruitful approach to study properties of quantum fields by using operator algebras. Models of local quantum field theories are described by nets of von Neumann algebras on a fixed spacetime, satisfying basic relativistic and quantum assumptions. A local Poincaré covariant net of von Neumann algebras on a fixed Hilbert space $\mathcal{H}$ is a local isotonic map $\mathcal{K} \ni \mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$ from the set $\mathcal{K}$ of open, connected bounded regions of the $1+3$ dimensional Minkowski spacetime $\mathbb{R}^{1+3}$ to von Neumann algebras $\mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$. It is assumed the existence of a unitary positive energy Poincaré representation U on $\mathcal{H}$ acting covariantly on $\mathcal{A}$ and of a unique (up to a phase) normalized U-invariant vector $\Omega \in \mathcal{H}$ which is cyclic for all the local algebras, namely the vacuum vector. The deep connection between the algebraic structure and the geometry of the model is a very fascinating fact. In [2, 3] Bisognano and Wichmann showed that any model coming from Wightman fields encloses within itself the information on its geometry. The authors proved that the modular operators related to algebras associated with wedge-shaped regions and the vacuum state have a geometrical meaning: they implement pure

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Lorentz transformations. This is expressed through the **Bisognano-Wichmann property (B-W)**:

$$U(\Lambda_W(2\pi t)) = \Delta^{-it}_{\mathcal{A}(W),\Omega} \tag{1}$$

for any wedge region W, where $t \mapsto \Lambda_W(t)$ is the one-parameter group of boosts associated with the wedge W, and $\Delta^{it}_{\mathcal{A}(W),\Omega}$ is the modular group of the von Neumann algebra $\mathcal{A}(W)$ generated by $\cup_{\mathcal{O}\subset W}\mathcal{A}(\mathcal{O})$ w.r.t. the vacuum vector Ω.

It can be stated something more. A von Neumann algebra net $\mathcal{A}$ is said to be **modular covariant** if

$$\Delta^{it}_{\mathcal{A}(W),\Omega}\mathcal{A}(O)\Delta^{-it}_{\mathcal{A}(W),\Omega} = \mathcal{A}(\Lambda_W(-2\pi t)O), \quad \text{for any } O \in \mathcal{K} \tag{2}$$

i.e. the modular group associated with any wedge algebra implements the covariant action of the associated one-parameter group of boosts on the local net. The modular covariance property is introduced in [5], it is weaker than the B-W property and it ensures that there exists a covariant unitary positive energy representation of the Poincaré group generated by the modular theory of the net algebras [18]. This marked one of the successes of the Tomita-Takesaki theory: once the algebra of the observables and the vacuum state are specified, the modular structure is determined and it has a geometrical meaning. Sufficient conditions on these properties are given in [6, 7, 8, 32].

The B-W property cares only about the modular theory of the algebraic model, which is contained in the real structure of the net. This can be described by using real standard subspaces of the Hilbert state space (cf. definition in Sect. 2.1). It is possible to characterize the standard subspaces with an analogue of the Tomita-Takesaki modular theory which coincides with the von Neumann algebra Tomita-Takesaki theory when one considers von Neumann algebras $A \subset \mathcal{B}(\mathcal{H})$ with a cyclic and separating vector $\Omega \in \mathcal{H}$ and the subspaces $H = \overline{A_{sa}\Omega} \subset \mathcal{H}$.

At this point it is natural to consider analogous nets of standard subspaces, which provide a very fruitful approach to QFT. For instance, they have a key role in finding localization properties of Infinite Spin particles, cf. Ref. [24] or finding new models in low dimensional quantum field theory [20, 25]. A geometrical approach to nets of standard subspaces is in [28].

The B-W property is essential to give a canonical structure to the first quantization nets, hence to the free fields (cf. [4]): given a particle, namely an irreducible, positive energy, unitary representation of the Poincaré group, there is a canonical way to build up the associated one-particle net of localized states and its second quantization free field. We make an analysis in the converse sense. Are those nets in some sense unique? At this level, the question is not well posed since there is some freeness in choosing the modular conjugation implementing the position and time reflection. We can ask when a (unitary, positive energy) Poincaré representation U is **modular, i.e. any net of standard subspaces it acts covariantly on satisfies the B-W property** (cf. Definition 3.3). In particular U would be implemented by the modular operators. This would give an answer to the necessity to assume the B-W property instead of deducing it by the assumptions.

An approach to this problem is to show the B-W property by exploiting the analyticity property of the wave functions as in [10, 26]. It is difficult to extend this analytic approach to more general representations as infinite direct sums, direct integrals or to the massless case.

We are going to present a purely algebraic argument giving a sufficient condition for the modularity of a large family of Poincaré representations as direct integrals of scalar representations, including the massless case. The idea comes from the following facts: the Lorentz group acts geometrically on the momentum space and one can check that on the mass shell it is possible to pointwise reconstruct the action of a Lorentz transformation, sending a wedge W to its causal complement W' just considering W-fixing transformations (see Remark 4.1). With this hint we introduce the following condition on a (unitary, positive energy) representation U of the Poincaré group, called **modularity condition** (cf. Definition 3.3). We ask that the von Neumann algebra, generated by translations and the Lorentz subgroup of transformations fixing W is large enough to enclose the transformations sending W onto W'. In Theorem 3.4 we show that the modularity condition is sufficient to prove that U is modular. In particular, for any U-covariant standard subspace net H the quotient between the two sides of (1) is the identity. This condition does neither depend on the net nor on multiplicity of the representation and passes to some direct integrals. Our analysis holds for scalar Poincaré representations in any spacetime $\mathbb{R}^{1+s}$, with $s \geq 3$, see Remark 4.6. A comment on the massless finite helicity case is given in Remark 4.5. We rely on the idea that the modular covariance has to be a natural assumption in every Quantum Field Theory satisfying basic relativistic and quantum hypotheses.

Known counterexamples to modular covariance seem very artificial as they imply a breakdown of Poincaré covariance, see [34, 19]. In Sect. 5, we give an explicit example of a massive Poincaré covariant standard subspace net, which is modular covariant not satisfying the B-W property. The massless case was treated in [24]. These kinds of general counterexamples clarify what kind of settings may prevent the identification of the covariant representation with the modular symmetries.

The relation between the split and the modular covariance properties is an interesting problem. In [14], Doplicher and Longo proved that if the dual net of local von Neumann algebras associated with the scalar generalized free field with Källen-Lehmann measure μ has the split property, then μ is purely atomic and concentrated on isolated points. In Sect. 6, we give an outlook in the standard subspace setting of the problem and of this result.

The paper is organized as follows. In Sect. 2 we introduce the one-particle setting and we recall relevant results on the real subspace structure, on Poincaré representations and on nets of standard subspaces. In Sect. 3 we present our algebraic condition and its properties. We prove it for general scalar representations in Sect. 4. In Sect. 5 we show the massive counterexample to the B-W property. In Sect.6 it is discussed the relation between the B-W and the Split properties in this one-particle setting.

This paper was reviewed in [27].

2 One-particle net

2.1 Standard subspaces

Firstly, we recall some definitions and basic results on standard subspaces and (generalized) one-particle models. A linear, real, closed subspace H of a complex Hilbert space $\mathcal{H}$ is called **cyclic** if $H + iH$ is dense in $\mathcal{H}$, **separating** if $H \cap iH = \{0\}$ and **standard** if it is cyclic and separating.

Given a standard subspace H the associated **Tomita operator** S_H is defined to be the closed the anti-linear involution with domain $H + iH$, given by:

$$S_H : H + iH \ni \xi + i\eta \mapsto \xi - i\eta \in H + iH, \qquad \xi, \eta \in H,$$

on the dense domain $H + iH \subset \mathcal{H}$. The polar decomposition

$$S_H = J_H \Delta_H^{1/2}$$

defines the positive self-adjoint **modular operator** Δ_H and the anti-unitary **modular conjugation** J_H. In particular, Δ_H is invertible and

$$J_H \Delta_H J_H = \Delta_H^{-1}.$$

If H is a real linear subspace of $\mathcal{H}$, the *symplectic complement* of H is defined by

$$H' \equiv \{\xi \in \mathcal{H} \ ; \ \Im(\xi, \eta) = 0, \forall \eta \in H\} = (iH)^{\perp_{\mathbb{R}}} \ ,$$

where $\perp_{\mathbb{R}}$ denotes the orthogonal in $\mathcal{H}$ viewed as a real Hilbert space with respect to the real part of the scalar product on $\mathcal{H}$. H' is a closed, real linear subspace of $\mathcal{H}$. If H is standard, then $H = H''$. It is a fact that H is cyclic (resp. separating) iff H' is separating (resp. cyclic), thus H is standard iff H' is standard and we have

$$S_{H'} = S_H^* \ .$$

Fundamental properties of the modular operator and conjugation are

$$\Delta_H^{it} H = H, \quad J_H H = H' \ , \qquad t \in \mathbb{R} \ .$$

We shall call the one-parameter, strongly continuous group $t \mapsto \Delta_H^{it}$, the **modular group** of H (cf. [30]).

There is a 1-1 correspondence between Tomita operators and standard subspaces.

Proposition 2.1. [22, 23]. *The map*

$$H \longmapsto S_H \tag{3}$$

is a bijection between the set of standard subspaces of $\mathcal{H}$ and the set of closed, densely defined, anti-linear involutions on $\mathcal{H}$. The inverse of the map (3) *is*

$$S \longmapsto \ker(\mathbf{1} - S).$$

Furthermore, this map is order-preserving, namely

$$H_1 \subset H_2 \quad \Leftrightarrow \quad S_{H_1} \subset S_{H_2},$$

and we have $S_H^* = S_{H'}$.

As a consequence,

$$\left\{\begin{array}{c}\text{Standard}\\ \text{subspaces}\\ H\subset\mathcal{H}\end{array}\right\}\overset{1:1}{\longleftrightarrow}\left\{\begin{array}{c}\text{closed, dens. def.}\\ \text{anti-linear}\\ \text{involutions S}\end{array}\right\}\overset{1:1}{\longleftrightarrow}\left\{\begin{array}{c}(J,\Delta)\text{ pairs of an}\\ \text{anti-unitary}\\ \text{involution and a}\\ \text{positive self-adjoint}\\ \text{operator on }\mathcal{H}\text{ s.t.}\\ J\Delta J=\Delta^{-1}\end{array}\right\}$$

Here is a basic result on the standard subspace modular theory.

Lemma 2.2. [22, 23]. *Let $H, K \subset \mathcal{H}$ be standard subspaces and $U \in \mathcal{U}(\mathcal{H})$ be a unitary operator on $\mathcal{H}$ such that $UH = K$. Then $U\Delta_H U^* = \Delta_K$ and $UJ_H U^* = J_K$.*

The following is the analogue of the Takesaki theorem for standard subspaces.

Lemma 2.3. [22, 23]. *Let $H \subset \mathcal{H}$ be a standard subspace, and $K \subset H$ be a closed, real linear subspace of H.*

If $\Delta_H^{it} K = K$, $\forall t \in \mathbb{R}$, then K is a standard subspace of $\mathcal{K} \equiv \overline{K + iK}$ and $\Delta_H|_K$ is the modular operator of K on $\mathcal{K}$. Moreover, if K is a cyclic subspace of $\mathcal{H}$, then $H = K$.

The following is the Borchers theorem in the standard subspace setting.

Theorem 2.4. [22, 23]. *Let $H \subset \mathcal{H}$ be a standard subspace, and $U(t) = e^{itP}$ be a one-parameter unitary group on $\mathcal{H}$ with generator $\pm P > 0$, such that $U(t)H \subset H$, $\forall t \geq 0$. Then,*

$$\begin{cases} \Delta_H^{is} U(t) \Delta_H^{-is} = U(e^{\mp 2\pi s} t) \\ J_H U(t) J_H = U(-t) \end{cases} \qquad \forall t, s \in \mathbb{R}. \tag{4}$$

In other words, the last theorem claims that if there is a one-parameter unitary group $t \mapsto U(t)$, with a positive generator, which properly translates a standard subspace, then (4) establishes the unique (up to multiplicity), positive energy representation of the translation-dilation group.

The converse of the Borchers theorem can be stated in the following way.

Theorem 2.5. [4]. *Let H be a standard space in the Hilbert space $\mathcal{H}$ and $U(t) = e^{itP}$ a one-parameter group of unitaries on $\mathcal{H}$ satisfying:*

$$\Delta_H^{is} U(t) \Delta_H^{-is} = U(e^{\mp 2\pi s} t) \quad \textit{and} \quad J_H U(t) J_H = U(-t), \qquad \forall t, s \in \mathbb{R}$$

The following are equivalent:

1. *$U(t)H \subset H$ for $t \geq 0$;*
2. *$\pm P$ is positive.*

2.2 The Minkowski space and the Poincaré group

Let $\mathbb{R}^{1+3}$ be the Minkowski space, i.e. a four-dimensional real manifold endowed with the metric

$$(x,y) = x_0 y_0 - \sum_{i=1}^{3} x_i y_i .$$

In a 4-vector $x = (x_0, x_1, x_2, x_3)$ x_0 and $\{x_i\}_{i=1,2,3}$ are the time and space coordinates, respectively. The Minkowski space has a causal structure induced by the metric. The causal complement of a region O is given by $O' = \{x \in \mathbb{R}^{1+3} : (x-y)^2 < 0, \forall y \in O\}$, where $(x-y)^2 = (x-y, x-y)$ refers to the norm induced by the metric. A **causally closed** region is such that $O = O''$.

We shall denote with $\mathcal{P}$ the **Poincaré group**, i.e the inhomogeneous symmetry group of $\mathbb{R}^{1+3}$. It is the semidirect product of the Lorentz group $\mathcal{L}$, the homogeneous Minkowski symmetry group, and the $\mathbb{R}^4$-translation group , i.e. $\mathcal{P} = \mathbb{R}^4 \rtimes \mathcal{L}$. It has four connected components, as $\mathcal{L}$ has four connected components, and we shall indicate with $\mathcal{P}_+^\uparrow = \mathbb{R}^4 \rtimes \mathcal{L}_+^\uparrow$ the connected component of the identity. One usually refers to $\mathcal{P}_+^\uparrow$ as the **proper ortochronous** (connected component of the) **Poincaré group**. $\mathcal{L}_+^\uparrow$ is not simply connected. The $\mathcal{L}_+^\uparrow$ universal covering $\widetilde{\mathcal{L}}_+^\uparrow$ is $\mathrm{SL}(2,\mathbb{C})$, thus the $\mathcal{P}_+^\uparrow$-universal covering $\widetilde{\mathcal{P}}_+^\uparrow$ is isomorphic to $\mathbb{R}^4 \rtimes \mathrm{SL}(2,\mathbb{C})$. Let $\Lambda : \mathbb{R}^4 \rtimes \mathrm{SL}(2,\mathbb{C}) \ni (a, A) \longmapsto \Lambda(a, A) \in \mathcal{P}_+^\uparrow$ be the **covering map**.

Unitary positive energy representations of the (universal covering of the) Poincaré group belong to three families: massive, massless finite helicity and massless infinite spin. Massive representations are labelled by a mass parameter $m \in (0, +\infty)$ and a spin parameter $s \in \frac{\mathbb{N}}{2}$; massless finite helicity and infinite spin representations have zero mass and are labelled by an helicity parameter $h \in \frac{\mathbb{Z}}{2}$ and a couple (κ, ϵ) where $\kappa \in \mathbb{R}^+$ is the radius and $\epsilon \in \{0, \frac{1}{2}\}$ the bose/fermi alternative, respectively.

We shall indicate with $\mathcal{P}_+$ the subgroup of $\mathcal{P}$ generated by $\mathcal{P}_+^\uparrow$ and the space and time reflection θ. Consider the automorphism α of the Poincaré group θ generated by the adjoint action of the θ-reflection. The **proper Poincaré group** $\mathcal{P}_+$ is generated as a semidirect product

$$\mathcal{P}_+^\uparrow \rtimes_\alpha \mathbb{Z}_2$$

through the α-action. It is well known (see for example [33]) that any irreducible representation of the Poincaré group U, except for finite helicity, extends to an (anti-)unitary representation of $\widetilde{U}$ of $\widetilde{\mathcal{P}}_+^\uparrow$, i.e.

$$\widetilde{U}(g) \text{ is } \begin{cases} \text{is unitary} & g \in \widetilde{\mathcal{P}}_+^\uparrow \\ \text{is anti-unitary} & g \in \widetilde{\mathcal{P}}_+^{\downarrow} \dot{=} \theta \cdot \mathcal{P}_+^\uparrow \end{cases}$$

At this point it is necessary to fix some notations about regions and isometries of the Minkowski spacetime. A **wedge-shaped region** $W \subset \mathbb{R}^{1+3}$ is an open region of the form gW_1 where $g \in \mathcal{P}_+^\uparrow$ and $W_1 = \{x \in \mathbb{R}^{1+3} : |x_0| < x_1\}$. The set of wedges is denoted by $\mathcal{W}$. Let $\mathcal{W}_0 \subset \mathcal{W}$ be the subset of wedges the form gW_1 where $g \in \mathcal{L}_+^\uparrow$. Note that if $W \in \mathcal{W}$ (or $W \in \mathcal{W}_0$), then $W' \in \mathcal{W}$ (resp. $W' \in \mathcal{W}_0$). For every wedge region $W \in \mathcal{W}$ there exists a unique one-parameter group of Poincaré boosts $t \mapsto \Lambda_W(t)$ preserving W, i.e. $\Lambda_W(t)W = W$ for every $t \in \mathbb{R}$. It is defined by the adjoint action of a $g \in \mathcal{P}_+^\uparrow$ such that $gW_1 = W$ on Λ_{W_1}, where $\Lambda_{W_1}(t)x = (x_0 \cosh t + x_1 \sinh t, x_0 \sinh t + x_1 \cosh t, x_2, x_3)$. Let $t \mapsto \lambda_W(t)$ be the (unique, one-parameter group) lift to the covering group. We shall denote with $W_\alpha \in \mathcal{W}$ the wedge along x_α-axis, i.e. $W_\alpha = \{x \in \mathbb{R}^{1+3} : |x_0| < x_\alpha\}$ with $\alpha = 1, 2, 3$. Let $t \mapsto \Lambda_\alpha(t)$ and $\theta \mapsto R_\alpha(\theta)$ be respectively the boosts and the rotations of $\mathcal{P}_+^\uparrow$ fixing W_α and $t \mapsto \lambda_j(t)$ and $\theta \mapsto r_j(\theta)$ be the corresponding lifts to $\mathrm{SL}(2, \mathbb{C})$. Note that $\lambda_j(t) = e^{\sigma_j t/2}$ and $r_j(\theta) = e^{i\sigma_j \theta/2}$, where $\{\sigma_i\}_{i=1,2,3}$ are the Pauli matrices.

2.3 One-particle nets

Let U be a unitary positive energy representation of the Poincaré group $\widetilde{\mathcal{P}}_+^\uparrow$ on an Hilbert space $\mathcal{H}$. We shall call a **U-covariant (or Poincaré covariant) net of standard subspaces on wedges** a map

$$H : \mathcal{W} \ni W \longmapsto H(W) \subset \mathcal{H},$$

associating to every wedge in $\mathbb{R}^{1+3}$ a closed real linear subspace of $\mathcal{H}$, satisfying the following properties:

1. **Isotony:** If $W_1, W_2 \in \mathcal{W}$ and $W_1 \subset W_2$ then $H(W_1) \subset H(W_2)$;
2. **Poincaré Covariance:** $U(g)H(W) = H(gW)$, $\forall g \in \widetilde{\mathcal{P}}_+^\uparrow$, $\forall W \in \mathcal{W}$;
3. **Positivity of the energy:** the joint spectrum of translations in U is contained in the forward light cone $V_+ = \{x \in \mathbb{R}^{1+3} : x^2 = (x, x) \geq 0 \text{ and } x_0 \geq 0\}$
4. **Reeh-Schlieder property (R-S):** if $W \in \mathcal{W}$, then $H(W)$ is a cyclic subspace of $\mathcal{H}$;
5. **Twisted Locality:** there exists a self-adjoint unitary operator $\Gamma \in U(\widetilde{\mathcal{P}}_+^\uparrow)'$, s.t. $\Gamma H(W) = H(W)$ for any $W \in \mathcal{W}$ and if $W_1 \subset W_2'$ then
$$BH(W_1) \subset H(W_2)',$$
with $B = \dfrac{1 + i\Gamma}{1 + i}$.

We shall indicate a U-covariant net of standard subspaces on wedges $W \mapsto H(W)$ satisfying 1.-5. with the couple (U, H). This is the setting we are going to study the following two properties:

6. **Bisognano-Wichmann property**: if $W \in \mathcal{W}$, then
$$U(\lambda_W(2\pi t)) = \Delta_{H(W)}^{-it}, \qquad \forall t \in \mathbb{R}; \tag{5}$$
7. **Duality property**: if $W \in \mathcal{W}$, then $H(W)' = BH(W')$.

Clearly, if U factors through $\mathcal{P}_+^\uparrow$ then the two expressions of the B-W property (1) and (5) coincide.

The relations between the modular theory of the wedge subspaces and the twisted operator are expressed by the following proposition.

Proposition 2.6. *The following hold*

$$[\Delta_{H(W)}, B] = 0$$

$$J_{H(W)} B J_{H(W)} = B^*$$

Proof. As $\Gamma H(W) = H(W)$ then, by Lemma 2.2, $\Gamma \Delta_{H(W)} \Gamma^* = \Delta_{H(W)}$ and $\Gamma J_{H(W)} \Gamma^* = J_{H(W)}$. A straightforward computation concludes the argument. □

Proposition 2.7. *Wedge duality is consequence of the B-W property.*

Proof. By Proposition 2.6 $\Delta_{H(W)} = \Delta_{BH(W)}$ and by covariance

$$H(W') = U(\Lambda_W(-2\pi t))H(W') = \Delta^{it}_{H(W)} H(W').$$

By twisted locality $BH(W') \subset H(W)'$ and Lemma 2.3 we get the thesis. □

It is possible to define closed real linear subspaces associated with bounded causally closed regions as follows

$$H(O) \doteq \bigcap_{\mathcal{W} \ni W \supset O} H(W). \tag{6}$$

This defines a net of real subspaces on causally closed regions $O \mapsto H(O)$. Note that $H(O)$ is not necessarily cyclic. If H is a net satisfying 1.-7. assumptions and $H(O)$ is cyclic, then

$$H(W) = \overline{\sum_{O \subset W} H(O)}$$

by Lemma 2.3. If $H(O)$ is cyclic for any double cone O, we say that the net $O \mapsto H(O)$ satisfies the **R-S property for double cones**.

In [4], Brunetti, Guido and Longo showed that there is a 1-1 correspondence between (anti-)unitary, positive energy representations of $\mathcal{P}_+$ and covariant nets of standard subspaces satisfying the B-W property. For the sake of completeness, we recall their theorem and we present the proof in the fermionic case which is not contained in the original paper.

Theorem 2.8. [4]. *There is a 1-1 correspondence between:*

a. (Anti-)unitary positive energy representations of $\widetilde{\mathcal{P}}_+$.

b. Local nets of standard subspaces satisfying 1-7.

Proof. Consider the automorphism of the Poincaré group $\widetilde{\mathcal{P}}_+$ generated by the adjoint action of j_3, one of the two $\widetilde{\mathcal{P}}_+$ elements implementing the $x_0 - x_3$ reflection. One can check in $\widetilde{\mathcal{P}}_+$ that

$$j_3(a, A)j_3 = (j_3 a, \sigma_3 A \sigma_3), \qquad \forall (a, A) \in \mathbb{R}^4 \rtimes \mathrm{SL}(2, \mathbb{C}) \tag{7}$$

and

$$r_1(\pi)\lambda_3(t)r_1(\pi)^{-1} = \lambda_3(-t) \qquad \text{and} \qquad r_1(\pi)j_3r_1(\pi)^{-1} = -j_3, \tag{8}$$

(e.g. cf. Appendix in [26]). Consider an (irreducible) fermionic representation U of $\widetilde{\mathcal{P}}_+$, namely a $\widetilde{\mathcal{P}}_+$-representation which does not factor through $\mathcal{P}_+$. In particular $U(2\pi) = -\mathbf{1}$. Since U lifts to a representation of the Lie algebra of $\widetilde{\mathcal{P}}_+^\uparrow$ on the Gärding domain and by relations (7) and (8) we get

$$U(R_1(\pi))K_3U(R_1(\pi))^* = -K_3, \tag{9}$$

$$U(R_1(\pi))J_3U(R_1(\pi))^* = -J_3, \tag{10}$$

where $U(\lambda_3(t)) = e^{iK_3t}$ and J_3 denotes $U(j_3)$ (choose one of the two possible choices for $U(j_3)$ in $\widetilde{\mathcal{P}}_+$). The anti-unitary operator $J_3 = U(j_3)$ and the self-adjoint operator $\Delta_{W_3} = e^{2\pi K_3}$ satisfy $J_3\Delta_{W_3}J_3 = \Delta_{W_3}^{-1}$. Hence, it is possible to define an anti-unitary involution $S_{W_3} = J_3\Delta_{W_3}^{1/2}$, and, by Proposition 2.1, a subspace $H(W_3)$ associated with the W_3 wedge. Clearly S_{W_3} is the Tomita operator $S_{H(W_3)}$ of the subspace $H(W_3)$. By covariance, a map of standard subspaces $\mathcal{W} \ni W \longmapsto H(W) \subset \mathcal{H}$ is well defined. Indeed, for any wedge W, $S_{H(W)}$ is the Tomita operator determining $H(W)$, defined as $S_{H(W)} = U(g)S_{H(W_3)}U(g)^*$, with $g \in \mathcal{P}_+^\uparrow$ such that $gW_3 = W$. Note that $S_{H(W)} = S_{H(r(2\pi)W)}$ and $S_{H(W)}$ is well defined. This clarify the ambiguity in the choice of J_3. Furthermore, by covariance and relations (9) and (10), as $S_{H(W)} = J_W\Delta_W^{1/2}$ then $S_{H(W')} = -J_W\Delta_{H(W)}^{-1/2}$. It easily follows that $H(W') = iH(W)'$. This ensures twisted locality and duality as we can define $\Gamma \doteq U(2\pi) \in U(\widetilde{\mathcal{P}}_+^\uparrow)'$ and $B = \frac{1-i}{1+i}\cdot\mathbf{1} = -i\cdot\mathbf{1}$ is the twist operator, i.e. $H(W') = BH(W)'$ where $W \in \mathcal{W}$.

Positivity of the energy, Poincaré covariance, B-W and R-S properties are ensured by construction. Isotony follows as in [4] by positivity of the energy and Theorem 2.5. □

3 A modularity condition for the Bisognano-Wichmann property

We define the following subgroups of $\widetilde{\mathcal{P}}_+^\uparrow$:

- $G_W^0 \doteq \{A \in \mathrm{SL}(2,\mathbb{C}) : \Lambda(A)W = W\}$, where $W \in \mathcal{W}_0$. It is the subgroup of $\widetilde{\mathcal{L}}_+^\uparrow$ elements fixing W through the covering homomorphism Λ.
- $G_W = \langle G_W^0, \mathcal{T}\rangle$, with $W \in \mathcal{W}_0$, where $\mathcal{T}$ is the $\mathbb{R}^{1+3}$-translation group. G_W is the group generated by G_W^0 and $\mathcal{T}$.
- For a general wedge $W \in \mathcal{W}$, G_W^0 and G_W are defined by the transitive action of $\mathcal{P}_+^\uparrow$ on wedges.

Let $W \in \mathcal{W}$. Consider the strongly continuous map

$$Z_{H(W)} : \mathbb{R} \ni t \mapsto \Delta_{H(W)}^{it}U(\lambda_W(2\pi t)). \tag{11}$$

It has to be the identity map if the B-W property (5) holds.

Proposition 3.1. *Let (U, H) be a Poincaré covariant net of standard subspaces. Then, for every $W \in \mathcal{W}$, the map $t \mapsto Z_{H(W)}(t)$ defines a one-parameter group and*

$$Z_{H(W)}(t) \in U(G_W)', \qquad \forall t \in \mathbb{R}.$$

Proof. As $\mathcal{P}_+^\uparrow$ acts transitively on wedges, there is no loss of generality if we fix $W = W_3$ and consider $G_W = G_3 \subset \widetilde{\mathcal{P}}_+^\uparrow$. As $\Lambda_3(t)W_3 = W_3$ for any $t \in \mathbb{R}$ then $U(\lambda_3(t))H(W_3) = H(W_3)$ and, by Lemma 2.2, $\Delta_{H(W_3)}$ commutes with $U(\lambda_3(t))$. In particular, $t \mapsto Z_{H(W_3)}(t)$ defines a unitary one-parameter group.

By positivity of the energy and Theorem 2.4, $\Delta_{H(W_3)}^{-it}$ has the same commutation relations as boosts $U(\lambda_3(2\pi t))$ w.r.t. translations. Indeed, translations in x_1 and x_2 directions commute with $\Delta_{H(W_3)}^{it}$ since they fix $H(W_3)$, and with $U(\lambda_3(t))$. Translations along directions $\mathrm{v}_+ = (1, 0, 0, 1)$ and $\mathrm{v}_- = (-1, 0, 0, 1)$ have, respectively, positive and negative generators and $U(t)H(W_3) \subset H(W_3)$ for any $t > 0$. Then, by Theorem 2.4

$$\Delta_{H(W_3)}^{is} U_\pm(t) \Delta_{H(W_3)}^{-is} = U_\pm(e^{\mp 2\pi s} t) \qquad s, t \in \mathbb{R},$$

as well as

$$U(\lambda_{W_3}(-2\pi t)) U_\pm(t) U(\lambda_{W_3}(2\pi t)) = U_\pm(e^{\mp 2\pi s} t), \qquad s, t \in \mathbb{R}$$

where $U_\pm(t) = U(t \cdot \mathrm{v}_\pm)$. Translations along x_1, x_2, v_+ and v_- generate $\mathbb{R}^4$ translations and as a consequence $Z_{H(W_3)} \in U(\mathcal{T})'$.

Any element $g \in G_3^0$ fixes the standard subspace $H(W_3)$, hence by Lemma 2.2, $U(g)$ commutes with the modular operator $\Delta_{H(W_3)}$. Furthermore, as g fixes W_3, then $U(g)$ also commutes with $U(\lambda_3)$. We conclude that $Z_{H(W_3)} \in U(G_3)'$. □

Note that $G_3^0 = \langle r_3, \lambda_3, r(2\pi) \rangle$, where $r(2\pi)$ is the 2π rotation.

Proposition 3.2. *Let (U, H) be a Poincaré covariant net of standard subspaces. Let $W \in \mathcal{W}$, and $r_W \in \widetilde{\mathcal{P}}_+^\uparrow$ be such that $\Lambda(r_W)W = W'$. Assume that $Z_{H(W)}$ commutes with $U(r_W)$, then the B-W and Duality properties hold.*

Proof. The map

$$\mathbb{R} \ni t \mapsto Z_{H(W)}(t)$$

is a unitary, one-parameter, s.o.-continuous group by Proposition 3.1. Now by hypothesis and covariance

$$Z_{H(W')}(t) = U(r_W) Z_{H(W)}(t) U(r_W)^* = Z_{H(W)}(t)$$

where

$$Z_{H(W')}(t) = U(\lambda_W(-2\pi t)) \Delta_{H(W')}^{it}.$$

We find that

$$Z_{H(W)}(2t) = Z_{H(W)}(t) Z_{H(W)}(t) = Z_{H(W)}(t) Z_{H(W')}(t) = \Delta_{H(W)}^{it} \Delta_{H(W')}^{it}$$

and since $Z_{H(W)}(2t)$ is an automorphism of $H(W)$ and

$$\begin{aligned}\Delta_{H(W)}^{-it} Z_{H(W)}(2t) H(W) = \Delta_{H(W')}^{it} H(W) \Leftrightarrow \\ \Delta_{H(W)}^{-it} H(W) = \Delta_{H(W')}^{it} H(W) \Leftrightarrow \\ H(W) = \Delta_{H(W')}^{it} H(W) \qquad \forall t \in \mathbb{R}.\end{aligned}$$

By locality $H(W) \subset (B\, H(W'))'$, Lemma 2.2 and Proposition 2.6, we have

$$\Delta_{(BH(W'))'}^{it} H(W) = \Delta_{BH(W')}^{-it} H(W) = \Delta_{H(W')}^{-it} H(W) = H(W)$$

and by Lemma 2.3 we conclude wedge duality,

$$H(W) = (B\, H(W'))'.$$

Furthermore, by the last condition for any $W \in \mathcal{W}$ then

$$\Delta_{H(W)} = \Delta_{H(W')}^{-1}$$

and the B-W property follows since

$$\begin{aligned}Z_{H(W)}(t) &= U(r_W) Z_{H(W)}(t) U(r_W)^* \\ &= U(\lambda_W(-2\pi t)) \Delta_{H(W)}^{-it} = Z_{H(W)}(-t)\end{aligned}$$

hence $Z_{H(W)}(t) = \mathbf{1}$. □

Now we state the properties we are interested in.

Definition 3.3. We shall say that a unitary, positive energy representation is **modular** if for any U-covariant net of standard subspaces H, namely any couple (U, H), then the B-W property holds.

Let $W \in \mathcal{W}$. A unitary, positive energy $\widetilde{\mathcal{P}}_+^\uparrow$-representation U satisfies the **modularity condition** if for an element $r_W \in \widetilde{\mathcal{P}}_+^\uparrow$ such that $\Lambda(r_W)W = W'$ we have that

$$U(r_W) \in U(G_W)''. \tag{MC}$$

Note that (MC) does neither depend on the choice of r_W, nor of W. Indeed if $\widetilde{r}_W \in \widetilde{\mathcal{P}}_+^\uparrow$ is another element such that $\Lambda(\widetilde{r}_W)W = W'$ then $r_W \widetilde{r}_W \in G_W$ and if (MC) holds for $U(r_W)$, then it holds for $U(\widetilde{r}_W)$. We conclude (MC) for any other wedge by the transitivity of the $\mathcal{P}_+^\uparrow$-action on wedge regions.

Now, we prove that the representations satisfying (MC) are modular.

Theorem 3.4. *Let U be a positive energy unitary representation of the Poincaré group $\widetilde{\mathcal{P}}_+^\uparrow$. If the condition* (MC) *holds on U, then any local U-covariant net of standard subspaces, namely any pair (U, H), satisfies the B-W and the Duality properties.*

Proof. Let (U, H) be a U-covariant net of standard subspaces, then $Z_{H(W)} \in U(G_W)'$ by Proposition 3.1. Then by assumptions $Z_{H(W)}$ commutes with $U(r_W)$ where $\Lambda(r_W)W = W'$, then we conclude the thesis by Proposition 3.2 □

Let U be a representation of $\widetilde{\mathcal{P}}_+^\uparrow$ acting on a standard subspace net $\mathcal{W} \ni W \mapsto H(W) \subset \mathcal{H}$. Assume that $J_{geo,W}$ is an anti-unitary operator extending U to a representation of $\widetilde{\mathcal{P}_+}$ through W-reflection and assume that modular covariance holds. As in [18], the algebraic $J_{H(W)}$ implements the wedge W reflection and, up to a $\widetilde{\mathcal{P}}_+^\uparrow$ element, the PCT operator (the proof in [18] can be straightforwardly adapted in the standard subspace net case). In this setting, let K_W be the W-boost generator on $\mathcal{H}$, the following operators

$$S_{geo,W} \doteq J_{geo,W} e^{-\pi K_W}, \qquad S_{alg,W} \doteq J_{H(W)} \Delta_{H(W)}^{1/2}, \tag{12}$$

can be called **geometric** and **algebraic Tomita operators**.

Corollary 3.5. *With the assumptions of Proposition 3.2,*

$$S_{geo,W} = C S_{alg,W}, \qquad \forall W \in \mathcal{W}$$

where $C \in U(\widetilde{\mathcal{P}}_+^\uparrow)'$

Proof. By duality, $J_{geo,W}$ and $J_{H(W)}$ both implement anti-unitarily $U(j_W)$, then $J_{geo} J_{H(W)} \in U(\mathcal{P}_+^\uparrow)'$. By the B-W property, $e^{-\pi K_W} = \Delta_W^{1/2}$, and we conclude. □

The B-W property and the $\widetilde{\mathcal{P}}_+$ covariance do not imply that there is a unique net undergoing the U-action. The conjugation operator can differ from the geometric conjugation by a unitary in $U(\mathcal{P}_+^\uparrow)'$. For instance, given an irreducible, (anti-)unitary $\mathcal{P}_+$-representation U where $U(j_W)$ implements the W-reflection, we have two U-covariant nets, according to the couples $\{(U(j_W), e^{-2\pi K_W})\}_{W\in\mathcal{W}}$ and $\{(-U(j_W), e^{-2\pi K_W})\}_{W\in\mathcal{W}}$ defining the wedge subspaces. If we just require $\widetilde{\mathcal{P}}_+^\uparrow$-covariance through a representation U, then for any modulus one complex number λ, the couples $\{(\lambda U(j_W), e^{-2\pi K_W})\}_{W\in\mathcal{W}}$ define U-covariant standard subspace nets.

Direct sums

The modularity property easily extends to direct integrals and multiples of representations as the following proposition shows.

Proposition 3.6. *Let U and $\{U_x\}_{x\in X}$ be unitary positive energy representations of $\widetilde{\mathcal{P}}_+^\uparrow$ satisfying* (MC).

Let $\mathcal{K}$ be an Hilbert space, then (MC) *holds for $U \otimes 1_{\mathcal{K}} \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$.*

Let (X, μ) be a standard measure space. Assume that $U_x|_{G_W}$ and $U_y|_{G_W}$ are disjoint for μ-a.e. $x \neq y$ in X. Then $U = \int_X U_x d\mu(x)$ satisfies (MC).

Proof. We can assume $W = W_3$. Let $\mathcal{U} \doteq U \otimes 1_K$, since $\mathcal{U}(G_3)' = U(G_3)' \otimes B(\mathcal{K})$ it follows that $\mathcal{U}(r_1(\pi)) = U(r_1(\pi)) \otimes 1$ commutes with $\mathcal{U}(G_3)'$, hence $\mathcal{U}(r_1(\pi)) \in \mathcal{U}(G_3)''$.

For the second statement, let

$$U(a,\Lambda) = \int_X^{\oplus} U_x(a,\Lambda) d\mu(x) \quad \text{acting on} \quad \mathcal{H} = \int_X^{\oplus} \mathcal{H}_x d\mu(x).$$

Then, by disjointness,

$$U(G_3)'' = \int_X^{\oplus} U_x(G_3)'' d\mu(x)$$

and $U(r_1) = \int_X^{\oplus} U(r_1(\pi)) d\mu(x)$ we have that $U(r_1(\pi)) \in U(G_3)''$ □

4 The scalar case

In this section we are going to show that (MC) holds for scalar representations, namely massive or massless 0-spin/helicity representations. The scalar representations have the following form

$$(U_{m,0}(a,A)\phi)(p) = e^{iap}\phi(\Lambda(A)^{-1}p), \qquad (a,A) \in \mathbb{R}^{1+3} \rtimes \tilde{\mathcal{L}}_+^{\uparrow} = \tilde{\mathcal{P}}_+^{\uparrow},$$

where

$$\phi \in \mathcal{H}_{m,0} \doteq L^2(\Omega_m, \delta(p^2 - m^2)\theta(p_0)d^4p),$$

and $\Omega_m = \{p = (p_0, \mathbf{p}) \in \mathbb{R}^{1+3} : p^2 = p_0^2 - \mathbf{p}^2 = m^2\}$ with $m \geq 0$. We recall that $U_{m,0}$ factors through $\mathcal{P}_+^{\uparrow}$.

Any momentum $p \in \Omega_m$ is a point in the dual group of $\mathcal{T}$ i.e. a character. We recall that $\mathcal{P}_+^{\uparrow}$ acts on Ω_m-characters as dual action of the adjoint action of $\mathcal{P}_+^{\uparrow}$ on $\mathcal{T}$. Clearly, $\mathcal{T}$ acts trivially and $\mathcal{L}_+^{\uparrow}$ acts geometrically on Ω_m, i.e. $(a,\Lambda)\cdot p = \Lambda^{-1}p$ with $(a,\Lambda) \in \mathbb{R}^{1+3} \rtimes \mathcal{L}_+^{\uparrow} = \mathcal{P}_+^{\uparrow}$.

We start with the following remark.

Remark 4.1. Fix $p = (p_0, p_1, p_2, p_3) \in \Omega_m$, $m > 0$,

$$R_1(\pi)p = (p_0, p_1, -p_2, -p_3)$$

can be obtained as a composition of a Λ_3-boost of parameter t_p and a R_3-rotation of parameter θ_p as

$$\begin{aligned} \Lambda_3(t_p)R_3(\theta_p)(p_0,p_1,p_2,p_3) &= \Lambda_3(t_p)(p_0,p_1,-p_2,p_3) \\ &= (p_0,p_1,-p_2,-p_3). \end{aligned} \tag{13}$$

Clearly t_p and θ_p depend on p. By (13), we deduce that G_3^0 orbits on Ω_m are not changed by $R_1(\pi)$. With $m = 0$ an analogue argument holds for all the orbits except $\{(p_0,0,0,p_0), p_0 \geq 0\}$ and $\{(p_0,0,0,-p_0), p_0 \geq 0\}$, i.e. there is no $g \in G_3$, such that $g(p_0,0,0,-p_0) = (p_0,0,0,p_0)$. On the other hand these orbits have null measure with the restriction of the Lebesgue measure to ∂V_+. This remark holds in $\mathbb{R}^{1+s}$ with $s > 2$.

Lemma 4.2. *Let $f \in L^\infty(\Omega_m)$ such that for every $g \in G_3^0$, $f(p) = f(gp)$ for a.e. $p \in \Omega_m$. Then $f(p) = h(p)$ for a.e. $p \in \Omega_m$ where $h \in L^\infty(\Omega_m)$ is constant on any $\overline{\{g\,p\}}_{g\in G_3}$ orbit.*

Proof. Any point $p = (p_0, p_1, p_2, p_3) \in \Omega_m$ (except the massless null measure sets $\{(p_0, 0, 0, p_0), p_0 \geq 0\}$ and $\{(p_0, 0, 0, -p_0), p_0 \geq 0\}$), can be identified with a radius $r = p_1^2 + p_2^2$, an angle $\theta \in [0, 2\pi]$ such that $(p_1, p_2) = r(\cos\theta, \sin\theta)$ and a parameter $t \in \mathbb{R}$ such that if $(p_0, p_3) = \sqrt{r^2 + m^2}(\cosh t, \sinh t)$. In particular G_3^0 orbits σ_r are labelled by $r \in \mathbb{R}^+$. On each orbit (with its invariant G_3^0 measure) the only positive measure set which is preserved by the G_3^0 action is the full orbit (up to a set of measure zero). This fact ensures that G_3^0 invariant functions have to be almost everywhere constant. Up to unitary equivalence, we can decompose the Hilbert space as $\int_{\mathbb{R}^+} L^2(\sigma_r, d\mu_r)dr$ where $d\mu_r$ is the G_3 invariant measure on σ_r. G_3 representations on different orbits are inequivalent, then $U(G_3)' = \int_{\mathbb{R}^+}(f(r) \cdot 1)dr$ and we conclude. □

Proposition 4.3. *Let U be a unitary, positive energy, irreducible scalar representation of the Poicaré group. Then U satisfies the modularity condition* (MC).

Proof. It is enough to consider W_3. Let $Z \in U(G_3)'$. Since U is a scalar representation then the translation algebra $\mathcal{T}'' = \{U(a) : a \in \mathbb{R}^4\}''$ is a MASA and $\mathcal{T}'' = L^\infty(\Omega_m)$. Indeed, the translation unitaries $(U(a)\phi)(p) = e^{iap}\phi(p)$ are multiplication operators and generate $L^\infty(\Omega_m)$ ultra-weakly. In particular $Z \in \mathcal{T}' = \mathcal{T}''$, hence it is a multiplication operator $Z = M_f$ by $f \in L^\infty(\Omega)$. Furthermore, Z has to commute with $U(\Lambda_3)$ and $U(R_3)$, hence $\forall t \in \mathbb{R}$ and $\forall \theta \in [0, 2\pi]$

$$U(\Lambda_3(t))ZU(\Lambda_3(t))^* = Z \quad \Leftrightarrow \quad f(\Lambda_3(t)^{-1}p) = f(p), \qquad \text{a.e.}\, p \in \Omega_m$$

and

$$U(R_3(\theta))ZU(R_3(\theta))^* = Z \quad \Leftrightarrow f(R_3(\theta)^{-1}p) = f(p) \qquad \text{a.e.}\, p \in \Omega_m.$$

We can assume $f(p)$ to be constant on any Λ_3 and R_3 orbit by Lemma 4.2.

Now observe that any momentum on the hyperboloid $p = (p_0, p_1, p_2, p_3)$ can be connected to the $R_1(\pi)p = (p_0, p_1, -p_2, -p_3)$ through a R_3-rotation and a Λ_3-boost as in Remark 4.1. It follows that $f(p) = f(R_1(-\pi)p)$, for every $p \in \Omega_m$. As a consequence

$$U(R_1(\pi))ZU(R_1(\pi))^* = Z \quad \text{as} \quad f(R_1(\pi)^{-1}p) = f(p), \qquad \forall p \in \Omega_m$$

and we conclude. □

Now, we can state the theorem.

Theorem 4.4. *Let $U = \int_{[0,+\infty)} U_m d\mu(m)$ where $\{U_m\}$ are (finite or infinite) multiples of the scalar representation of mass m, then U satisfies* (MC). *In particular if (U, H) is a U-covariant net of standard subspaces, then the Duality and the B-W properties hold.*

Proof. Unitary representations of $\mathcal{P}_+^\uparrow$ with different masses are disjoint, and they have disjoint restrictions to G_3 subgroup. The thesis becomes a consequence of Propositions 3.6, 4.3 and Theorem 3.4 . □

Remark 4.5. Proposition 4.3 straightforwardly holds also for irreducible massless finite helicity representations as they are induced from a one-dimensional representation of the little group. As a consequence, an irreducible nonzero helicity representation U cannot act covariantly on a net of standard subspaces on wedges H. Indeed, the B-W property must hold as in Proposition 4.3 and J_{W_3}, the modular conjugation of $H(W_3)$, would implement the j_3 reflection on U (cf. [18]). In particular, the PT operator defined by $\Theta = J_{W_3}U(R_3(\pi))$ extends U to an (anti-)unitary representation $\hat{U}$ of $\widetilde{\mathcal{P}}_+$ and acts covariantly on H. On the other hand nonzero helicity representations are not induced by a self-conjugate representation of the little group and do not extend to anti-unitary representations of $\widetilde{\mathcal{P}}_+$ [33]. This shows a contradiction.

Remark 4.6. Consider the $\mathbb{R}^{1+s}$ spacetime with $s \geq 3$ and let U be a scalar (unitary, positive energy) representation of $\mathcal{P}_+^\uparrow$. The one-parameter group $t \mapsto Z_{H(W_1)}(t)$ given in (11), is generated by the multiplication operator by a real function of the form $f(p_2^2 + ... + p_s^2)$. For each value of the radius $r = p_2^2 + ... + p_s^2$ there is a unique G_1-orbit on Ω_m which is fixed by any transformation $R \in \mathcal{P}_+^\uparrow$ such that $RW_1 = W_1'$, for instance $R_2(\pi)$. In particular, the analysis of this section extends to any Minkowski spacetime $\mathbb{R}^{1+s}$ with $s \geq 3$. It fails in $2+1$ spacetime dimensions as $R_2(\pi)$ does not preserve G_1-orbits.

5 Massive counter-examples

Borchers, in [6], showed that a unitary, positive energy Poincaré representation acting covariantly on a modular covariant von Neumann algebra net $\mathcal{A}$ in the vacuum sector, can only differ from the modular representation by a unitary representation of the Lorentz group.

Theorem 5.1. [6]. *Let $\mathcal{A}$ be a local quantum field theory von Neumann algebra net in the vacuum sector undergoing two different representations of the Poincaré group. Let U_0 be the representation implemented by wedge modular operators and U_1 be the second representation. Then there exists a unitary representation of the Lorentz Group $G(\Lambda)$ defined*

$$G(\Lambda) = U_1(a, \Lambda)U_0(a, \Lambda)^*.$$

Furthermore, $G(\Lambda)$ commutes with $U_0(a, \Lambda')$ for all $a \in \mathbb{R}^{1+3}$, $\Lambda, \Lambda' \in \mathcal{L}_+^\uparrow$ and the $G(\Lambda)$ adjoint action on $\mathcal{A}$ implements automorphisms of local algebras, i.e. maps any local algebra into itself.

With this hint it is possible to build up Poincaré covariant nets picturing the above situation: *modular covariance without the B-W property.* We are going to make explicit computations on this kind of counterexamples in order to understand what may prevent the B-W property. Here, we study the massive case. The massless case can be found in [24].

Consider $U_{m,s}$, the m-mass, s-spin, unitary, irreducible representation of the Poincaré group $\widetilde{\mathcal{P}}_+^\uparrow$ and $H : W \mapsto H(W)$ its canonical net of standard subspaces. Let $p \in \Omega_m$, $A_p \doteq \sqrt{\underset{\sim}{p}/m}$, where $\underset{\sim}{p} = p_0 \cdot \mathbf{1} + \sum_{i=1}^3 p_i\sigma_i$ is the $\mathrm{SL}(2,\mathbb{C})$ element implementing the boost sending the point $q_m = (m,0,0,0)$ to p. An explicit form of $U_{m,s}$ is the following

$$(U_{m,s}(a,A)\phi)(p) = e^{iap}\mathcal{D}^s(A_p^{-1}AA_{\Lambda^{-1}p})\phi(\Lambda(A)^{-1}p),$$

where $\mathcal{D}^s$ is the s-spin representation of $\mathrm{SU}(2)$ on the $2s+1$ dimensional Hilbert space $\mathfrak{h}_s$ and

$$\begin{aligned}\phi \in \mathcal{H}_{m,s} &\doteq \mathfrak{h}_s \otimes L^2(\Omega_m, \delta(p^2-m^2)\theta(p_0)d^4p)\\ &= \mathbb{C}^{2s+1} \otimes L^2(\Omega_m, \delta(p^2-m^2)\theta(p_0)d^4p).\end{aligned}$$

Let V be a real unitary, nontrivial, $\mathrm{SL}(2,\mathbb{C})$-representation on an Hilbert space $\mathcal{K}$, i.e. there exists an anti-unitary involution J on the Hilbert space $\mathcal{K}$, commuting with V such that the real vector space $K \subset \mathcal{K}$ of J-fixed vectors, is a standard subspace and

$$V(\mathrm{SL}(2,\mathbb{C}))K = K.$$

In particular $JK = K$ and $\Delta_K = 1$.

We can define the following net of standard subspaces,

$$K \otimes H : \mathcal{W} \ni W \to K \otimes H(W) \subset \mathcal{K} \otimes \mathcal{H}.$$

We can see two Poincaré representations acting covariantly on $K \otimes H$:

$$U_I(a,A) \equiv 1 \otimes U_{m,s}(a,A) \qquad A \in \mathrm{SL}(2,\mathbb{C}),\ a \in \mathbb{R}^{1+3}$$

and

$$U_V(a,A) \equiv V(A) \otimes U_{m,s}(a,A) \qquad A \in \mathrm{SL}(2,\mathbb{C}),\ a \in \mathbb{R}^{1+3}.$$

U_I is implemented by $K \otimes H$ modular operators and the B-W property holds w.r.t. U_I (cf. Lemma 2.6 in [24]). Note that for $s = 0$, U_I satisfies the condition (MC), hence the B-W property also by Theorem 3.4.

U_V decomposes in a direct sum of infinitely many inequivalent representations of mass m, i.e. infinitely many spins appear. Indeed, consider the Lorentz transformation $A_p = \sqrt{\underset{\sim}{p}/m}$ and the unitary operator on $\mathcal{H} \otimes \mathcal{K}$

$$W : \mathcal{K} \otimes \mathcal{H} \ni (p \mapsto \phi(p)) \longmapsto \big(p \mapsto (V(A_p^{-1}) \otimes 1_{\mathbb{C}^{2s+1}})\phi(p)\big) \in \mathcal{K} \otimes \mathcal{H} \tag{14}$$

as $\mathcal{K} \otimes \mathcal{H} = L^2(\Omega_m, \mathcal{K} \otimes \mathbb{C}^{2s+1}, \delta(p^2-m^2)\theta(p_0)d^4p)$. We can define a unitarily equivalent representation $U_V' = WU_VW^*$ as follows:

$$(U_V'(a,A)\phi)(p) = e^{ia\cdot p}(V(A_p^{-1}AA_{\Lambda(A)^{-1}p}) \otimes \mathcal{D}^s(A_p^{-1}AA_{\Lambda(A)^{-1}p}))\phi(\Lambda(A^{-1}p)).$$

It is easy to see that $A_p^{-1}AA_{\Lambda(A)^{-1}p} \in \mathrm{Stab}(m,0,0,0) = \mathrm{SU}(2)$.

As we are interested in the disintegration of U_V it is not restrictive to assume that V is irreducible. Unitary irreducible representations of $\mathrm{SL}(2,\mathbb{C})$, denoted by $V_{\rho,n}$, are labelled by pairs of numbers (ρ, n) such that $\rho \in \mathbb{R}$ and $n \in \mathbb{Z}_+$. The restriction of $V_{\rho,n}$ to SU(2) decomposes in $\bigoplus_{s=n/2}^{+\infty} \mathcal{D}^s$ (see, for example, [31, 12]). Thus, if $V = V_{\rho,n}$ is an irreducible SL$(2,\mathbb{C})$ representation then

$$U_V \simeq \bigoplus_{i=\frac{n}{2}}^{\infty} \bigoplus_{j=|s-i|}^{s+i} U_{m,j} \tag{15}$$

since U'_V, hence U_V, decomposes according to the $(V \otimes \mathcal{D}^s)|_{\mathrm{SU}(2)}$ decomposition into irreducible representations.

Note that any representation class appears with finite multiplicity and U_V does not satisfy the condition (MC). Furthermore, we can conclude that having modular covariance without the B-W property requires to the "wrong" representation the presence of an infinite family of inequivalent Poincaré representations possibly with finite multiplicity.

U_I cannot act covariantly on the U_V-canonical net H_V. We can see it explicitly. Let W_1 be the wedge in x_1 direction and $W = gW_1$ where $g \in \mathcal{L}_+^\uparrow$ and $W_1 \neq W$. If $U_I(g)H_V(W_1) = H_V(W)$ then $U_I(g)\Delta^{-it}_{H_V(W_1)}U_I(g)^* = \Delta^{-it}_{H_V(W)}$ where $\Delta_{H_V(W_1)}$ and $\Delta_{H_V(W)}$ are the modular operator of $H_V(W_1)$ and $H_V(W)$, respectively. As

$$\begin{aligned} U_I(g)\Delta^{-it}_{H_V(W_1)}U_I(g)^* = & \\ &= (1 \otimes U_{m,s}(g))(V(\Lambda_{W_1}(2\pi t)) \otimes U_{m,s}(\Lambda_{W_1}(2\pi t)))(1 \otimes U_{m,s}(g))^* \\ &= V(\Lambda_{W_1}(2\pi t)) \otimes U_{m,s}(g\Lambda_{W_1}(2\pi t))g^{-1}) \end{aligned}$$

but $\Delta^{-it}_{H_V(W)} = V(g\Lambda_{W_1}(2\pi t)g^{-1}) \otimes U_{m,s}(g\Lambda_{W_1}(2\pi t))g^{-1})$, we get the contradiction unless V is trivial.

The modular covariance and the identification of the geometric and the algebraic PCT operators imply the uniqueness of the covariant representation.

Proposition 5.2. *With the assumptions of Theorem 5.1 assume that the algebraic PCT operator defined in* (12) *implements the U_1-PCT operator too. Then $U_1 = U_0$.*

Proof. With the notations of Theorem 5.1, we consider the representation $U_1(a,\Lambda) = G(\Lambda)U_0(a,\Lambda)$ of $\widetilde{\mathcal{P}}_+^\uparrow$ on $\mathcal{H}$. G fixes any subspace $H(W) = \overline{\mathcal{A}(W)\Omega}$ and in particular $J_{H(W)}G(\Lambda)J_{H(W)} = G(\Lambda)$, $\forall\Lambda \in \mathcal{L}_+^\uparrow$. It follows that $J_{H(W)}$ has the correct commutation relations with U_1 if and only if $G = 1$. □

We have seen that given a Poincaré covariant net with the B-W property (U, H) we can find a second covariant Poincaré representation $\widetilde{U}$ if the commutant $U(\widetilde{\mathcal{P}}_+^\uparrow)'$ is large enough, i.e. when the Poincaré representation implemented by the modular operators has infinite multiplicity. Furthermore, there are no conditions on the spin of $\widetilde{U}$. In particular, counterexamples can produce wrong relations between spin and statistics. For instance assuming $U_I(r(2\pi)) = \mathbf{1}$ and $V(r(2\pi)) = -\mathbf{1}$ where $r(2\pi)$ is the 2π-rotation. The (MC) condition does not hold for U_V and a wrong spin-statistics relation may arise whenever the B-W property fails.

As we said in the introduction known counterexamples to modular covariance are very artificial and not all the basic relativistic and quantum assumptions are respected. It is interesting to look for natural counterexamples to modular covariance, in the class of representations excluded by this discussion, if they exist. In [15] it is shown that assuming finite one-particle degeneracy, then the Spin-Statistics Theorem holds. This accords with the analysis obtained by Mund in [26]. We expect that an algebraic proof for the B-W property can be established, without assuming finite multiplicity of sub-representations.

6 Outlook

Now we come to an outlook on the relation between the split and the B-W properties.

Definition 6.1. [14] Let $(\mathcal{N} \subset \mathcal{M}, \Omega)$ be a *standard inclusion* of von Neumann algebras, i.e. Ω is a cyclic and separating vector for N, M and $N' \cap M$.

A standard inclusion $(\mathcal{N} \subset \mathcal{M}, \Omega)$ is *split* if there exists a type I factor $\mathcal{F}$ such that $\mathcal{N} \subset \mathcal{F} \subset \mathcal{M}$.

A Poincaré covariant net $(\mathcal{A}, U, \Omega)$ satisfies the *split property* if the von Neumann algebra inclusion $(\mathcal{A}(O_1) \subset \mathcal{A}(O_2), \Omega)$ is split, for every compact inclusion of bounded causally closed regions $O_1 \Subset O_2$.

In a natural way one can define the split property for an inclusion of subspaces by second quantization: *let $K \subset H \subset \mathcal{H}$ be an inclusion of standard subspaces of an Hilbert space $\mathcal{H}$ such that $K' \cap H$ is standard, then the inclusion $K \subset H$ is split if its second quantization inclusion $R(K) \subset R(H)$ is standard split w.r.t. the vacuum vector $\Omega \in e^{\mathcal{H}}$.* The second quantization respects the lattice structure (cf. [1]). Here, we just deal with the Bosonic second quantization (see, for example, [21]).

We want a coherent first quantization version of the split property. We assume H and K to be **factor** subspaces, i.e. $H \cap H' = \{0\} = K \cap K'$. We need the following theorem.

Theorem 6.2. [16]. *Let H be a standard subspace and $R(H)$ be its second quantization.*

- *Second quantization factors are type I if and only if $\Delta_H|_{[0,1]}$ is a trace class operator where $\Delta_H|_{[0,1]}$ is the restriction of the H-modular operator Δ_H to the spectral subspace relative to the interval* $[0, 1]$
- *Second quantization factors which are not type I are type III.*

The canonical intermediate factor, for a standard split inclusion of von Neumann algebras $N \subset M$ is

$$\mathcal{F} = N \vee JNJ = M \cap JMJ$$

where J is the modular conjugation associated with the relative commutant algebra $(N' \cap M, \Omega)$, cf. [14]. The standard inclusion $(N \subset M, \Omega)$ is split iff $\mathcal{F}$ is type I. A canonical intermediate subspace can be analogously defined for standard split subspace inclusions $K \subset H$:

$$F = \overline{K + JK} = H \cap JH.$$

Second quantization of modular operators were computed in [13, 21, 17].

Consider the following proposition:

Proposition 6.3. *Let $K \subset H$ be an inclusion of standard subspaces, such that $K' \cap H$ is standard. Let J be the modular conjugation of the symplectic relative complement $K' \cap H$ and $\Delta = \Delta_F$ be the modular operator of the intermediate subspace $F = \overline{K + JK}$. Assume that F is a factor. Then the following statements are equivalent.*

1. *$R(K) \subset R(H)$ is a split inclusion.*

2. *the operator $\Delta|_{[0,1]}$ is trace class.*

Proof. (1. $\Rightarrow$ 2.) If the inclusion is split, then the intermediate canonical factor $R(K) \vee R(JK)$ is the second quantization of $\overline{K + JK}$. In particular by Theorem 6.2 the thesis holds. (2. $\Rightarrow$ 1.) As $\Delta|_{[0,1]}$ is trace class, then the second quantization of F is an intermediate type I factor between $R(K)$ and $R(H)$ by Theorem 6.2. Then the split property holds for the inclusion $R(H) \subset R(K)$. □

Let U be an (anti-)unitary representation of $\mathcal{P}_+$. We shall say that U is **split** if the canonical net associated with U satisfies the split property on bounded causally closed regions (defined through equation (6)), i.e. the inclusion

$$H(O_1) \subset H(O_2)$$

is split for every $O_1 \Subset O_2$ as above.

Scalar free fields satisfy Haag duality, thus (6) holds (see, for example, [29]). Furthermore, any scalar irreducible representation is split (cf. [11, 9]). The following theorem is the first quantization analogue of Theorem 10.2 in [14].

Theorem 6.4. *Let U be an (anti-)unitary representation of $\mathcal{P}_+$, direct integral of scalar representations. If U has the split property then $U = \int_0^{+\infty} U_m d\mu(m)$ where μ is purely atomic on isolated points and for each mass there can only be a finite multiple of $U_{m,0}$.*

Proof. As the B-W property holds, the net disintegrates according to the representation:

$$H = \int_0^{+\infty} H_m d\mu(m)$$

where H_m is the canonical net associated with U_m.
Fix a couple of bounded and causally closed regions $O_1 \Subset O_2 \subset \mathbb{R}^{1+3}$. Irreducible components satisfy the split property. In particular by Proposition 6.3, the restriction of the modular operator of the intermediate standard subspace F, defined as

$$F = \overline{H(O_1) + JH(O_1)} = \int_0^{+\infty} \overline{H_m(O_1) + J_m H_m(O_1)_m} d\mu(m)$$
$$= \int_0^{+\infty} F_m \, d\mu(m),$$

to the spectral subset $[0, 1]$, has to be trace class. It follows that μ has to be purely atomic and for each isolated mass (no finite accumulation point) there can only be a finite multiple of the scalar representation. □

Corollary 6.5. *Let (U, H) be a Poincaré covariant net of standard subspaces and U be a $\mathcal{P}_+^\uparrow$-split representation. Assume that U is a direct integral of scalar representations. Then the B-W and the duality properties hold.*

Proof. By Theorem 6.4 the disintegration of the covariant Poincaré representation is purely atomic on masses, concentrated on isolated points and for each mass there can only be a finite multiple of the scalar representation. The disintegration satisfies the condition (MC) and the thesis follows by Theorem 3.4. □

It remains an old interesting challenge to build up a more general bridge between the Split and the B-W properties. We expect this analysis to be generalized to finite multiples of spinorial representations.

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