Research Article

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On the supremal version of the Alt–Caffarelli minimization problem

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Abstract: This is a companion paper to our recent work [Bernoulli free boundary problem for the infinity Laplacian, preprint (2018), https://arxiv.org/abs/1804.08573]. Here we consider its variational side, which corresponds to the supremal version of the Alt–Caffarelli minimization problem.

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1 Introduction

Given a non-empty open bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, we consider the free boundary problem

$$m_\Lambda := \min\{J_\Lambda(u) := \|\nabla u\|_\infty + \Lambda|\{u > 0\}| : u \in \mathrm{Lip}_1(\Omega)\}, \tag{P$_\Lambda$}$$

where Λ is a positive constant, $|\{u > 0\}|$ denotes the Lebesgue measure of the set $\{x \in \Omega : u(x) > 0\}$, and

$$\mathrm{Lip}_1(\Omega) := \{u \in W^{1,\infty}(\Omega) : u \geq 0 \text{ in } \Omega,\ u = 1 \text{ on } \partial\Omega\}. \tag{1.1}$$

This may be viewed as the supremal version of the Alt–Caffarelli minimization problem, which for p-growth energies reads

$$\min\Big\{\int_\Omega |\nabla u|^p\,dx + \Lambda|\{u > 0\}| : u \in W^{1,p}(\Omega),\ u = 1 \text{ on } \partial\Omega\Big\}. \tag{1.2}$$

In both the minimization problems $(\mathrm{P})_\Lambda$ and (1.2), the free boundary is given by the set

$$F(u) := \partial\{u > 0\} \cap \Omega.$$

Clearly, the free boundary will be non-empty only if the parameter Λ is taken sufficiently large so that the measure term becomes active in the competition between the two addenda in the energy functional. The gradient term makes the difference: the integral functional appearing in (1.2) is converted into the supremal functional $\|\nabla u\|_\infty$ in $(\mathrm{P})_\Lambda$.

Problem (1.2) has a long history: starting from the groundbreaking paper [1], where it was introduced in the linear case $p = 2$, it has been widely studied in later works for any $p \in (1, +\infty)$ (see for instance [8, 12,

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13, 17]). In particular, the topic which has been object of a thorough investigation is the regularity of the free boundary, which has been settled to be locally analytic except for a $\mathcal{H}^{n-1}$-negligible singular set [1, 13].

Among the motivations behind problem (1.2), of chief importance is that its minimizers solve for a suitable constant $c > 0$ the "Bernoulli problem for the p-Laplacian", namely,

$$\begin{cases} \Delta_p u = 0 & \text{in } \{u > 0\} \cap \Omega, \\ u = 1 & \text{on } \partial\Omega, \\ |\nabla u| = c & \text{on } \partial\{u > 0\} \cap \Omega. \end{cases} \tag{1.3}$$

This overdetermined system, which is named after Daniel Bernoulli (in particular, from his law in hydrodynamics), has many physical and industrial applications, not only in fluid dynamics, but also in other contexts such as optimal insulation and electro-chemical machining. For a more precise description of the applied side, including several references, see [15, Section 2].

In our recent paper [11], we have studied the existence and uniqueness of solutions to the following "Bernoulli problem for the ∞-Laplacian":

$$\begin{cases} \Delta_\infty u = 0 & \text{in } \{u > 0\} \cap \Omega, \\ u = 1 & \text{on } \partial\Omega, \\ |\nabla u| = \lambda & \text{on } \partial\{u > 0\} \cap \Omega. \end{cases} \tag{1.4}$$

Among further recent works about free boundary value problems ruled by the infinity Laplacian, let us quote [2, 3].

Recall that the ∞-Laplacian is the degenerated nonlinear operator defined by

$$\Delta_\infty u := \nabla^2 u \nabla u \cdot \nabla u \quad \text{for all } u \in C^2(\Omega).$$

It is well known since Bhattacharya, DiBenedetto and Manfredi [7] that solutions to

$$\Delta_\infty u = 0 \tag{1.5}$$

must be intended in the viscosity sense since the differential operator is not in divergence form. Moreover, it is a widely recognized fact that (1.5) may be seen as the fundamental PDE of the calculus of variations in L^∞, i.e., an analogue of the Euler–Lagrange equation when considering variational problems for supremal functionals, the simplest of which is the L^∞-norm of the gradient. Indeed, as proved by Jensen [16], a function u is a viscosity solution to (1.5) in an open set A under a Dirichlet boundary datum $g \in C^0(\partial A)$ if and only if it is an absolutely minimizing Lipschitz extension of g (this property, usually shortened as AML, means that $u = g$ on ∂A and, for every $U \Subset A$, u minimizes the L^∞-norm of the gradient on U among functions v which agree with u on ∂U). After Jensen, variational problems for supremal functionals have been object of several works, among which we quote, with no attempt of completeness, [4–6, 9].

In this perspective, the aim of the present work is to investigate the variational side of problem (1.4), which is precisely the new free boundary problem of supremal type $(\mathrm{P})_\Lambda$.

Our results are presented in the next section, which is divided into three parts:

- In Section 2.1, we study the existence and uniqueness of non-constant solutions to $(\mathrm{P})_\Lambda$ on convex domains (see Theorems 1 and 8). In particular, we introduce the "variational ∞-Bernoulli constant"

  $$\Lambda_{\Omega,\infty} := \inf\{\Lambda > 0 : (\mathrm{P})_\Lambda \text{ admits a non-constant solution}\}, \tag{1.6}$$

  and we give a geometric characterization of it, involving the family of parallel sets of Ω. This formula is obtained with the help of the Brunn–Minkowski inequality and allows to explicitly compute the value of $\Lambda_{\Omega,\infty}$, at least for simple geometries. In particular, if we compare $\Lambda_{\Omega,\infty}$ with the "∞-Bernoulli constant", defined in [11] by

  $$\lambda_{\Omega,\infty} := \inf\{\lambda > 0 : (1.4) \text{ admits a non-constant solution}\},$$

  it turns out that $\Lambda_{\Omega,\infty} \geq \lambda_{\Omega,\infty}$; the computation on balls reveals that the inequality can be strict.

Still using its geometric characterization, we prove that $\Lambda_{\Omega,\infty}$ satisfies an isoperimetric inequality, namely, that it is minimal on balls under a volume constraint (see Theorem 6). This was inspired by a result in the same vein by Daners and Kawohl [12] for the variational p-Bernoulli constants associated with problem (1.2).

- In Section 2.2, we elucidate the relationship between solutions to problem $(\mathrm{P})_\Lambda$ and solutions to the Bernoulli problem (1.4) for the ∞-Laplacian by showing they are closely related to each other. More precisely, we answer the following natural questions (see Propositions 9 and 10):
  (Q_1) Let $\Lambda \geq \Lambda_{\Omega,\infty}$ so that problem $(\mathrm{P})_\Lambda$ admits a non-constant solution.
  Does a solution to problem $(\mathrm{P})_\Lambda$ solve problem (1.4) for some λ?
  (Q_2) Let $\lambda \geq \lambda_{\Omega,\infty}$ so that problem (1.4) admits a non-constant solution.
  Does a solution to problem (1.4) solve problem $(\mathrm{P})_\Lambda$ for some Λ?
- In Section 2.3, we show that problem $(\mathrm{P})_\Lambda$ can be obtained not merely in a heuristic way, but by performing a rigorous passage to the limit as $p \to +\infty$ in a family of minimum problems of the Alt–Caffarelli type for energies with p-growth (see Theorem 11). This result, which is somehow highly expected, is not straightforward. In fact, the passage to the limit as $p \to +\infty$ in problems of type (1.2) (suitably rescaled) provides a minimization problem with gradient constraint, similarly to what was shown by Kawohl and Shahgholian [17] for *exterior* Bernoulli problems. In order to arrive at problem $(\mathrm{P})_\Lambda$, one needs to perform a further minimization with respect to an additional parameter which plays the role of a multiplier for the gradient constraint.

The proofs of the results described above are given respectively in Sections 3, 4 and 5.

2 Results

2.1 Analysis of problem $(\mathrm{P})_\Lambda$

A major role in our study is played by the parallel sets of Ω and by the functions v_r defined for every $r \in [0, R_\Omega]$ (with $R_\Omega :=$ inradius of Ω) respectively by

$$\begin{aligned} \Omega_r &:= \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > r\}, \\ v_r(x) &:= \Big[1 - \frac{1}{r}\operatorname{dist}(x, \partial\Omega)\Big]_+, \quad x \in \overline{\Omega}. \end{aligned} \tag{2.1}$$

Clearly, for every $r \in (0, R_\Omega]$, we have

$$v_r \in \operatorname{Lip}_1(\Omega), \quad \|\nabla v_r\|_\infty = \frac{1}{r}, \quad \{v_r > 0\} \cap \Omega = D_r := \Omega \setminus \overline{\Omega_r},$$

where $\operatorname{Lip}_1(\Omega)$ is the set of functions defined in (1.1). (To be precise, we consider any function $u \in \operatorname{Lip}_1(\Omega)$ as a Hölder continuous function in $\overline{\Omega}$ since $u - 1 \in W_0^{1,p}(\Omega)$ for every $p > 1$.)

We are going to provide an explicit characterization for the variational infinity Bernoulli constant introduced in (1.6) and show that, for $\Lambda \geq \Lambda_{\Omega,\infty}$, non-constant solutions are precisely of the form (2.1).

Throughout the paper, we assume with no further mention that

$$\Omega \text{ is an open bounded } convex \text{ subset of } \mathbb{R}^n, \quad n \geq 2.$$

Moreover, since Ω will be fixed, for simplicity in the sequel, we simply write Λ_∞ in place of $\Lambda_{\Omega,\infty}$.

Theorem 1 (Identification of Λ_∞). *There exists a unique value r^* in the interval $(0, R_\Omega)$ such that*

$$\frac{|\partial\Omega_{r^*}|}{|\Omega_{r^*}|} = \frac{1}{r^*}, \tag{2.2}$$

and the variational infinity Bernoulli constant Λ_∞ defined in (1.6) *agrees with*

$$\Lambda^* := \frac{1}{r^*|\Omega_{r^*}|}.$$

More precisely,

- $(\mathrm{P})_\Lambda$ *uniquely admits the constant solution* $u \equiv 1$ *for* $\Lambda < \Lambda^*$*;*
- $(\mathrm{P})_\Lambda$ *admits the constant solution* $u \equiv 1$ *and the non-constant solution* v_{r^*} *for* $\Lambda = \Lambda^*$*;*
- $(\mathrm{P})_\Lambda$ *does not admit the constant solution* $u \equiv 1$ *and admits the non-constant solution* v_{r_Λ} *for* $\Lambda > \Lambda^*$*, where* r_Λ *is the smallest root of the equation* $\frac{1}{r^2|\partial\Omega_r|} = \Lambda$.

Moreover, for $\Lambda \geq \Lambda^*$*, any non-constant solution* v *to* $(\mathrm{P})_\Lambda$ *has the same Lipschitz constant and the same positivity set, i.e., it satisfies*

$$\|\nabla v\|_\infty = \frac{1}{r_\Lambda} \quad \text{and} \quad \{v > 0\} = D_{r_\Lambda} \tag{2.3}$$

(where $r_{\Lambda^*} = r^*$*).*

Remark 2 (Computation of m_Λ). The above result implies in particular that the infimum m_Λ of problem $(\mathrm{P})_\Lambda$ can be explicitly computed as

$$m_\Lambda = \begin{cases} \Lambda|\Omega| & \text{if } \Lambda \leq \Lambda_\infty, \\ \frac{1}{r_\Lambda} + \Lambda|D_{r_\Lambda}| & \text{if } \Lambda \geq \Lambda_\infty. \end{cases}$$

Remark 3 (Infinity harmonic solution to $(\mathrm{P})_\Lambda$). For every $\Lambda \geq \Lambda_\infty$, among solutions to $(\mathrm{P})_\Lambda$, there is exactly one which is infinity harmonic in its positivity set, namely the infinity harmonic potential w_{r_Λ} of D_{r_Λ}, defined as the unique solution to the Dirichlet boundary value problem

$$\begin{cases} \Delta_\infty w_{r_\Lambda} = 0 & \text{in } D_{r_\Lambda}, \\ w_{r_\Lambda} = 1 & \text{on } \partial\Omega, \\ w_{r_\Lambda} = 0 & \text{on } \partial\Omega_{r_\Lambda}. \end{cases} \tag{2.4}$$

Indeed, if $(\mathrm{P})_\Lambda$ admits an infinity harmonic solution, it agrees necessarily with w_{r_Λ} because its positivity set is uniquely determined by (2.3). To see that w_{r_Λ} is actually a solution to $(\mathrm{P})_\Lambda$, we observe that it has the same positivity set as v_{r_Λ}, and a Lipschitz constant not larger than v_{r_Λ} (because w_{r_Λ} has the AML property mentioned in the introduction). Hence, we necessarily have $J_\Lambda(v_{r_\Lambda}) = J_\Lambda(w_{r_\Lambda})$, yielding optimality.

Below, we give the explicit expression of Λ_∞ for some simple geometries (the ball in any space dimension and the square in dimension 2), and we show that it enjoys a nice isoperimetric property.

Example 4 (Ball). Let $\Omega = B_R \subset \mathbb{R}^n$ be the n-dimensional open ball of radius R centered at the origin. Then $R_\Omega = R$, and

$$\psi(r) := \frac{|\partial B_{R-r}|}{|B_{R-r}|} = \frac{n}{R-r}, \qquad r \in [0, R).$$

The unique solution to the equation $\psi(r) = \frac{1}{r}$ is $r^* = \frac{R}{n+1}$, and hence

$$\Lambda_\infty(B_R) = \frac{1}{r^*|\Omega_{r^*}|} = \frac{1}{\kappa_n r^*(R - r^*)^n} = \frac{(n+1)^{n+1}}{\kappa_n n^n R^{n+1}} \qquad (\kappa_n := |B_1|).$$

In particular, for $n = 2$, we get

$$\Lambda_\infty(B_R) = \frac{27}{4\pi R^3}.$$

Example 5 (Rectangle). Let $n = 2$ and $\Omega = Q_{a,b} := (0, a) \times (0, b)$, with $0 < b \leq a$. In this case, $R_\Omega = \frac{b}{2}$, and

$$\psi(r) := \frac{|\partial Q_{a-r,b-r}|}{|Q_{a-r,b-r}|} = 2\frac{a + b - 4r}{(a - 2r)(b - 2r)}, \qquad r \in [0, \tfrac{b}{2}).$$

The unique solution to the equation $\psi(r) = \frac{1}{r}$ in the interval $[0, \frac{b}{2})$ is

$$r^* = \frac{a + b - \sqrt{a^2 + b^2 - ab}}{6}$$

In particular, for the square $Q_a := Q_{a,a}$, we get $r^* = \frac{a}{6}$, and hence

$$\Lambda_\infty(Q_a) = \frac{27}{2a^3}.$$

Theorem 6 (Isoperimetric inequality). *Denoting by Ω^* a ball with the same volume as Ω, it holds*

$$\Lambda_\infty(\Omega) \geq \Lambda_\infty(\Omega^*),$$

with equality sign if and only if Ω is a ball.

Remark 7. As mentioned in the introduction, a result analogous to Theorem 6 has been obtained in [12] for a variational Bernoulli constant related to the Alt–Caffarelli minimization problems for p-growth energies. We wish to point out that Theorem 6 cannot be obtained simply by passing to the limit as $p \to +\infty$ in the isoperimetric inequality by Daners and Kawohl. After the discussion in Section 2.3, we will be in a position to give more details in this respect (see Remark 12).

We now turn our attention to the uniqueness of solutions to problem $(\mathrm{P})_\Lambda$. To that aim, we introduce the *singular radius of* Ω, defined as

$$r_{\mathrm{sing}} := \mathrm{dist}(\Sigma, \partial\Omega),$$

where Σ denotes the cut locus of Ω (i.e., the closure of the set of points where the distance from $\partial\Omega$ is not differentiable). Accordingly, since the map $\mathcal{R}\colon \Lambda \mapsto r_\Lambda$ (with r_Λ defined as in Theorem 1) turns out to be monotone decreasing from $[\Lambda^*, +\infty)$ to $[r^*, 0)$ (see Lemma 16 below), the value

$$\Lambda_{\mathrm{sing}} := \mathcal{R}^{-1}(r_{\mathrm{sing}})$$

is uniquely defined.

We point out that the radius r_{sing} may be smaller or larger than r^* according to the domain under consideration (and consequently Λ_{sing} may be larger or smaller than $\Lambda_\infty = \Lambda^*$). For instance, in dimension $n = 2$, if $\Omega = B_R$ (the ball of radius R), it holds

$$r_{\mathrm{sing}} = R > r^* = \frac{R}{3};$$

on the other hand, if $\Omega = Q_a$ (the square of side a), we have

$$r_{\mathrm{sing}} = 0 < r^* = \frac{a}{6}.$$

We are now in a position to discuss the uniqueness of solutions to $(\mathrm{P})_\Lambda$.

Theorem 8 (Uniqueness threshold). *We have the following:*

- *If $\Lambda_{\mathrm{sing}} \leq \Lambda_\infty$, then $(\mathrm{P})_\Lambda$ admits a unique solution (given by v_{r_Λ}) if and only if $\Lambda > \Lambda_\infty$.*
- *If $\Lambda_{\mathrm{sing}} > \Lambda_\infty$, then $(\mathrm{P})_\Lambda$ admits a unique solution (given by v_{r_Λ}) if and only if $\Lambda \geq \Lambda_{\mathrm{sing}}$.*

2.2 Relationship with the ∞-Bernoulli problem

We now examine the link between solutions to $(\mathrm{P})_\Lambda$ and solutions to (1.4). A first glance in this direction has been already given in Remark 3. A more precise answer to the questions (Q_1) and (Q_2) stated in the introduction is contained in the next two statements.

Proposition 9 (Solutions to $(\mathrm{P})_\Lambda$ versus solutions to (1.4)). *Let $\Lambda \geq \Lambda_\infty$. Among solutions to problem $(\mathrm{P})_\Lambda$, there is exactly one which solves (1.4) (for $\lambda = \frac{1}{r_\Lambda}$); it is given by the infinity harmonic potential w_{r_Λ} of D_{r_Λ}. In particular, when $(\mathrm{P})_\Lambda$ admits a unique solution, this one solves (1.4) (for $\lambda = \frac{1}{r_\Lambda}$), and it is given by $w_{r_\Lambda} = v_{r_\Lambda}$.*

Proposition 10 (Solutions to (1.4) versus solutions to $(\mathrm{P})_\Lambda$). *Let $\lambda \geq R_\Omega^{-1}$. Among solutions to problem (1.4), there is one which solves $(\mathrm{P})_\Lambda$ if and only if*

$$\lambda \geq \frac{1}{r^*}. \tag{2.5}$$

In this case, such solution is given by w_{r_Λ}, with $r_\Lambda = \frac{1}{\lambda}$, and agrees with v_{r_Λ} if and only if

$$\lambda \geq \max\left\{\frac{1}{r^*}, \frac{1}{r_{\mathrm{sing}}}\right\}.$$

2.3 Approximation by minima of p-energies

We now show that problem $(\mathrm{P})_\Lambda$ arises through an asymptotic analysis as $p \to +\infty$ of problems of type (1.2).

Let $\Lambda \geq \Lambda_\infty$ be fixed. For every $\lambda > 0$ and $p > 1$, let us consider the minimization problem

$$\min\{J_\Lambda^{p,\lambda}(u) : u \in W_1^{1,p}(\Omega)\}, \tag{2.6}$$

where the functionals $J_\Lambda^{p,\lambda}$ are defined by

$$J_\Lambda^{p,\lambda}(u) := \frac{1}{p}\int_\Omega \Big(\frac{|\nabla u|}{\lambda}\Big)^p \, dx + \lambda + \frac{p-1}{p}\Lambda|\{u > 0\}| \quad \text{for all } u \in W_1^{1,p}(\Omega) := 1 + W_0^{1,p}(\Omega).$$

Incidentally, let us recall from [13] that a solution to problem (2.6) exists, is non-negative in Ω and satisfies the overdetermined boundary value problem (1.3) with $c = \lambda\Lambda^{1/p}$ (provided the Neumann boundary condition on the free boundary is interpreted in a suitable weak sense).

If, for a given $\lambda > 0$, we consider a sequence of solutions $(u^{p_j,\lambda})_j$ to problem (2.6) for $p = p_j \to +\infty$, it is not difficult to see that, up to passing to a (not relabeled) subsequence, the functions $u^{p_j,\lambda}$ converge uniformly in $\overline{\Omega}$ to a function u^λ which is infinity harmonic in its positivity set and solves the variational problem

$$\min\{J_\Lambda^\lambda(u) : u \in \mathrm{Lip}_1(\Omega)\}, \tag{2.7}$$

the functionals J_Λ^λ being defined on $\mathrm{Lip}_1(\Omega)$ by

$$J_\Lambda^\lambda(u) := \begin{cases} \lambda + \Lambda|\{u > 0\}| & \text{if } \|\nabla u\|_\infty \leq \lambda, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.8}$$

The proof of this fact, which will be detailed in Section 5 (see Lemma 20), is similar to the one given by Kawohl and Shahgholian in the paper [17], where they deal with the asymptotics as $p \to +\infty$ of exterior p-Bernoulli problems. (Indeed, by computing a Γ-limit, they arrive precisely at a minimum problem with gradient constraint analogue to (2.7), in which they fix $\Lambda = 1$.)

Now we observe that the functionals J_Λ^λ are related to J_Λ by the equality

$$J_\Lambda(u) = \inf_{\lambda > 0} J_\Lambda^\lambda(u) \quad \text{for all } u \in \mathrm{Lip}_1(\Omega) \tag{2.9}$$

(and actually, the above infimum can be equivalently taken over $\lambda \geq \frac{1}{R_\Omega}$ since, for every $u \in \mathrm{Lip}_1(\Omega)$ with $\|\nabla u\|_\infty < \frac{1}{R_\Omega}$, it holds $|\{u > 0\}| = |\Omega|$).

The validity of (2.9) is precisely the reason why we put the constant addendum λ into the expression of the functionals $J_\Lambda^{p,\lambda}$; it can be interpreted as a sort of Lagrange multiplier for the gradient constraint $\|\nabla u\|_\infty \leq \lambda$ which appears when passing to the limit at fixed λ.

In the light of (2.9), it is now natural to guess how the value of m_Λ can be obtained from $J_\Lambda^{p,\lambda}$ in the limit as $p \to +\infty$; one has to take a double infimum, over $u \in W_1^{1,p}(\Omega)$ and over $\lambda \geq \frac{1}{R_\Omega}$.

Theorem 11 (p-approximation). *Let $\Lambda \geq \Lambda_\infty$. Then*

$$\lim_{p \to +\infty} \inf_{\lambda \geq 1/R_\Omega} \inf_{u \in W_1^{1,p}(\Omega)} J_\Lambda^{p,\lambda}(u) = m_\Lambda. \tag{2.10}$$

Remark 12. The variational p-Bernoulli constant Λ_p considered by Daners and Kawohl in [12] agrees with the infimum of positive λ such that problem (2.6) (with $\Lambda = 1$) admits a non-constant solution. In the limit as $p \to +\infty$, Λ_p does not converge to Λ_∞ (in fact, in [11], we proved that $\lim_{p\to+\infty} \Lambda_p = \frac{1}{R_\Omega}$), and this is precisely the reason why, as mentioned in Remark 7, Theorem 6 cannot be obtained by passing to the limit as $p \to +\infty$ in the isoperimetric inequality proved by Daners and Kawohl in [12]. In view of Theorem 11, one should not be surprised by the missed convergence of Λ_p to Λ_∞. Indeed, (2.10) suggests that, in order to find a p-approximation of Λ_∞, one should consider rather the constants $\widetilde{\Lambda}_p$ defined as the infimum of positive Λ such that the problem

$$\inf_{u \in W_1^{1,p}(\Omega)} \inf_{\lambda \geq 1/R_\Omega} J_\Lambda^{p,\lambda}(u)$$

admits a non-constant solution. Indeed, a straightforward formal computation shows that

$$\inf_{\lambda\geq 1/R_\Omega} J_\Lambda^{p,\lambda}(u) = \frac{p+1}{p}\|\nabla u\|_{L^p(\Omega)}^{p/(p+1)} + \frac{p-1}{p}\Lambda|\{u>0\}| \to J_\Lambda(u) \quad \text{as } p\to+\infty.$$

3 Proof of the results in Section 2.1

To start, we establish a simple existence result.

Lemma 13 (Existence of solutions to $(\mathrm{P})_\Lambda$). *Problem* $(\mathrm{P})_\Lambda$ *admits a solution for every* $\Lambda > 0$.

Proof. Let $\Lambda > 0$ and let $(u_j) \subset \mathrm{Lip}_1(\Omega)$ be a minimizing sequence for J_Λ. Clearly, it is not restrictive to assume that $J_\Lambda(u_j) \leq J_\Lambda(1) = \Lambda|\Omega|$, so that $\|\nabla u_j\|_\infty \leq \Lambda|\Omega|$ for every n. Hence, the sequence (u_j) is equi-Lipschitz. Since $u_j = 1$ on $\partial\Omega$ for every n, by Ascoli–Arzelá's theorem we deduce that there exist a subsequence (not relabeled) and a function $u \in \mathrm{Lip}_1(\Omega)$ such that $u_j \to u$ uniformly in $\overline{\Omega}$.

Since, for all $j \in \mathbb{N}$,

$$|u_j(x) - u_j(y)| \leq \|\nabla u_j\|_\infty |x-y| \quad \text{for all } x, y \in \overline{\Omega},$$

passing to the limit, we get

$$|u(x) - u(y)| \leq (\liminf_{n\to+\infty} \|\nabla u_j\|_\infty)|x-y| \quad \text{for all } x, y \in \overline{\Omega}$$

so that $\|\nabla u\|_\infty \leq \liminf_j \|\nabla u_j\|_\infty$.

Moreover, since $u_j(x) \to u(x)$ for every $x \in \overline{\Omega}$, it is easy to verify that

$$\chi_{\{u>0\}}(x) \leq \liminf_{j\to+\infty} \chi_{\{u_j>0\}}(x),$$

and hence, by Fatou's lemma, we deduce that

$$|\{u>0\}| = \int_\Omega \chi_{\{u>0\}} \leq \liminf_{j\to+\infty} \int_\Omega \chi_{\{u_j>0\}} = \liminf_{j\to+\infty} |\{u_j>0\}|.$$

In conclusion, J_Λ is lower semicontinuous with respect to the uniform convergence, and hence u is a minimum point for J_Λ. □

We establish now the first part in the statement of Theorem 1.

Lemma 14. *There exists a unique value* r^* *in the interval* $(0, R_\Omega)$ *satisfying equality* (2.2). *Moreover, the function* $r \mapsto r|\Omega_r|$ *attains its unique maximum on* $[0, R_\Omega]$ *precisely at* r^**:*

$$\max_{[0,R_\Omega]}(r|\Omega_r|) = r^*|\Omega_{r^*}|.$$

Remark 15. In the proof of Lemma 14 below (and also of the subsequent Lemma 16), we heavily make use of the Brunn–Minkowski inequality for the volume functional of the parallel sets Ω_r, and for their surface area measure as well. This motivates the convexity assumption made on the domain Ω.

Proof of Lemma 14. Let us prove the following claim:

$$\psi(r) := \frac{|\partial\Omega_r|}{|\Omega_r|}, \quad r \in [0, R_\Omega), \quad \text{is continuous and increasing, and} \quad \lim_{r\to R_\Omega-} \psi(r) = +\infty.$$

In fact, by the Brunn–Minkowski inequality, the functions $r \mapsto |\Omega_r|^{1/n}$, $r \mapsto |\partial\Omega_r|^{1/(n-1)}$ are concave in $[0, R_\Omega]$. (Here and in the following, $|\partial\Omega_r|$ denotes the $(n-1)$-dimensional Hausdorff measure of $\partial\Omega_r$.) Hence, ψ is continuous in $[0, R_\Omega)$, and the composition $r \mapsto \log|\Omega_r| = n\log|\Omega_r|^{1/n}$ is concave in the same interval. In particular, since $\frac{d}{dr}|\Omega_r| = -|\partial\Omega_r|$, we conclude that

$$\psi(r) = -\frac{d}{dr}\log|\Omega_r|$$

is increasing in $[0, R_\Omega)$. Finally, from the isoperimetric inequality, we have

$$\psi(r)^n = \frac{|\partial\Omega_r|^n}{|\Omega_r|^n} = \frac{|\partial\Omega_r|^n}{|\Omega_r|^{n-1}} \cdot \frac{1}{|\Omega_r|} \geq \frac{|\partial B_1|^n}{|B_1|^{n-1}} \cdot \frac{1}{|\Omega_r|} \to +\infty, \quad r \to R_\Omega^-.$$

The claim follows. As a consequence, there exists a unique value $r^* \in (0, R_\Omega)$ such that

$$\frac{|\partial\Omega_r|}{|\Omega_r|} < \frac{1}{r} \quad \text{for all } r \in (0, r^*), \qquad \frac{|\partial\Omega_r|}{|\Omega_r|} > \frac{1}{r} \quad \text{for all } r \in (r^*, R_\Omega). \tag{3.1}$$

Finally, in order to determine the maximum of the function $\varphi(r) := r|\Omega_r|$ on $[0, R_\Omega]$, we compute its first derivative

$$\varphi'(r) = |\Omega_r| - r|\partial\Omega_r| = \varphi(r)\Big(\frac{1}{r} - \frac{|\partial\Omega_r|}{|\Omega_r|}\Big).$$

By (3.1), it holds $\varphi'(r) > 0$ for $r \in (0, r^*)$ and $\varphi'(r) < 0$ for $r \in (r^*, R_\Omega)$, and hence φ attains its strict maximum at r^*. □

In the next lemma, which is the key step towards the completion of the proof of Theorem 1, we study the behavior of the function

$$f_\Lambda(r) := J_\Lambda(v_r) - J_\Lambda(1), \quad r \in (0, R_\Omega], \tag{3.2}$$

where v_r is defined by (2.1), and we analyze in particular the value of

$$\mu_\Lambda := \min_{(0,R_\Omega]} f_\Lambda(r).$$

Lemma 16. *There exist two values $0 < \Lambda' < \Lambda^*$, with*

$$\Lambda^* := \frac{1}{r^*|\Omega_{r^*}|},$$

with r^* given by Lemma 14, such that the following holds:

(a) *For every $\Lambda \in (0, \Lambda']$, the function f_Λ is strictly monotone decreasing (in particular, $\mu_\Lambda = f_\Lambda(R_\Omega) = \frac{1}{R_\Omega} > 0$).*
(b) *For every $\Lambda \geq \Lambda'$, there exist points $0 < r_\Lambda \leq \rho_\Lambda < R_\Omega$ such that*

$$r_{\Lambda'} = \rho_{\Lambda'}, \quad \Lambda \mapsto r_\Lambda \text{ strictly decreasing}, \quad \Lambda \mapsto \rho_\Lambda \text{ strictly increasing},$$
$$f'_\Lambda(r_\Lambda) = f'_\Lambda(\rho_\Lambda) = 0, \quad f'_\Lambda(r) > 0 \iff r \in (r_\Lambda, \rho_\Lambda).$$

In particular, for $\Lambda = \Lambda'$, the map f_Λ admits a flex at $r_{\Lambda'} = \rho_{\Lambda'}$, whereas, for every $\Lambda > \Lambda'$, f_Λ admits a local minimum at r_Λ and a local maximum at ρ_Λ.

(c) *For every $\Lambda \in (0, \Lambda^*)$, it holds $\mu_\Lambda > 0$ (and $\mu_\Lambda = \min\{\frac{1}{R_\Omega}, f_\Lambda(r_\Lambda)\}$).*
(d) *For every $\Lambda \geq \Lambda^*$, it holds $\mu_\Lambda = f_\Lambda(r_\Lambda) \leq 0$, and $\mu_\Lambda = 0$ if and only if $\Lambda = \Lambda^*$.*

Remark 17. By inspection of the proof of Lemma 16 given hereafter, it turns out that, for every $\Lambda \geq \Lambda'$, the radius r_Λ can be identified as stated in Theorem 1, namely as the smallest root in $(0, R_\Omega]$ of the equation

$$\frac{1}{r^2} = \Lambda|\partial\Omega_r|.$$

Proof of Lemma 16. A direct computation shows that

$$J_\Lambda(v_r) = \|\nabla v_r\|_\infty + \Lambda(|\{v_r > 0\}|) = \frac{1}{r} + \Lambda(|\Omega| - |\Omega_r|), \quad r \in (0, R_\Omega],$$

and hence

$$f_\Lambda(r) = \frac{1}{r} - \Lambda|\Omega_r|, \quad f'_\Lambda(r) = -\frac{1}{r^2} + \Lambda|\partial\Omega_r|, \quad r \in (0, R_\Omega].$$

We have

$$f'_\Lambda(r) > 0 \iff \sqrt[n-1]{\Lambda|\partial\Omega_r|} > r^{-2/(n-1)}.$$

Since, by the Brunn–Minkowski theorem, the map $r \mapsto \sqrt[n-1]{|\partial\Omega_r|}$ is decreasing and concave in $[0, R_\Omega]$, whereas $r \mapsto r^{-2/(n-1)}$ is decreasing and convex in $(0, R_\Omega]$, there exists a unique $\Lambda' > 0$ such that (a) and (b) hold.

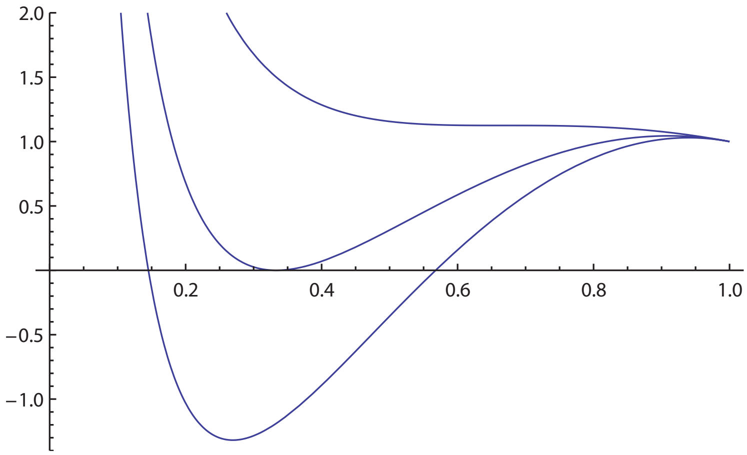

Figure 1: Plot of the map f_Λ when Ω is the unit two-dimensional ball, for $\Lambda = \Lambda' \simeq 1.07$ (top), $\Lambda = \Lambda^* \simeq 2.15$ (center) and $\Lambda = 3$ (bottom)

Observe that $f_0(r) = \frac{1}{r}$ is positive and monotone decreasing in $(0, R_\Omega]$, whereas, for every $r \in (0, R_\Omega)$, the map $\Lambda \mapsto f_\Lambda(r)$ is affine with strictly negative slope. By continuity, there exists a value $\Lambda^* > \Lambda'$ such that

$$\min_{r\in(0,R_\Omega]} f_\Lambda(r) > 0 \quad \text{for all } \Lambda < \Lambda^*, \qquad \min_{r\in(0,R_\Omega]} f_{\Lambda^*}(r) = 0, \qquad \min_{r\in(0,R_\Omega]} f_\Lambda(r) < 0 \quad \text{for all } \Lambda > \Lambda^*.$$

Moreover, if $r^* \in \operatorname{argmin} f_{\Lambda^*}$, then the pair (r^*, Λ^*) satisfies the conditions

$$f_\Lambda(r) = 0, \quad f'_\Lambda(r) = 0,$$

i.e.,

$$\frac{1}{r} - \Lambda|\Omega_r| = 0, \quad -\frac{1}{r^2} + \Lambda|\partial\Omega_r| = 0.$$

From the first equation, we have $\Lambda = (r|\Omega_r|)^{-1}$; substituting into the second equation, we get the condition

$$\frac{|\partial\Omega_r|}{|\Omega_r|} = \frac{1}{r}. \tag{3.3}$$

By Lemma 14, there exists a unique $r^* \in (0, R_\Omega)$ satisfying (3.3) so that $\Lambda^* = (r^*|\Omega_{r^*}|)^{-1}$.

The properties stated in (c) and (d) follow. □

Example 18. When $\Omega = B_R \subset \mathbb{R}^n$, for $r \in (0, R]$, we have

$$f_\Lambda(r) = \frac{1}{r} - \kappa_n\Lambda(R - r)^n, \quad f'_\Lambda(r) = -\frac{1}{r^2} + n\kappa_n\Lambda(R - r)^{n-1}.$$

An explicit computation gives

$$\Lambda' = \frac{1}{4}\left(\frac{n}{n-1}\right)^{n-1}\Lambda^*,$$

where the value of Λ^* has already been computed in Example 4.

The graph of f_Λ in the case $n = 2$ and $R = 1$, for the three choices $\Lambda = \Lambda' \simeq 1.07$, $\Lambda = \Lambda^* \simeq 2.15$ and $\Lambda = 3$, is shown in Figure 1.

Proof of Theorem 1. The existence of a unique value r^* satisfying (2.2) has already been proved in Lemma 14. Let now $\Lambda > 0$. By Lemma 13, the functional J_Λ admits a minimizer $v \in \mathrm{Lip}_1(\Omega)$. Since $v = 1$ on $\partial\Omega$ and $|v(x) - v(y)| \le \|\nabla v\|_\infty |x - y|$ for every $x, y \in \overline{\Omega}$, we observe that

$$\{v > 0\} \supseteq D_r \quad \text{with } r := \frac{1}{\|\nabla v\|_\infty}. \tag{3.4}$$

In particular,

$$\text{if} \quad \|\nabla v\|_\infty < \frac{1}{R_\Omega}, \quad \text{then} \quad \{v > 0\} = \Omega. \tag{3.5}$$

Then we distinguish two cases.

- If $\|\nabla v\|_\infty < \frac{1}{R_\Omega}$, then we conclude that necessarily $v \equiv 1$ and $\Lambda \le \Lambda^*$ by (3.4), for otherwise $J_\Lambda(v_{r_\Lambda}) < J_\Lambda(1)$ by Lemma 16.
- If $\|\nabla v\|_\infty \ge \frac{1}{R_\Omega}$, let $\overline{r} := \frac{1}{\|\nabla v\|_\infty} \in (0, R_\Omega]$. The function $v_{\overline{r}}$, defined in (2.1), has the same Lipschitz constant as v. Moreover, since $\{v_{\overline{r}} > 0\} = D_{\overline{r}}$, by (3.5), we deduce that $J_\Lambda(v_{\overline{r}}) \le J_\Lambda(v)$. Since v is a minimizer, then equality must hold, and hence $\{v > 0\} = D_{\overline{r}}$.

Then, recalling the definition (3.2) of the function f_Λ, we have

$$\min_{(0,R_\Omega]} f_\Lambda \le f_\Lambda(\overline{r}) = J_\Lambda(v_{\overline{r}}) - J_\Lambda(1) = J_\Lambda(v) - J_\Lambda(1) \le \min_{(0,R_\Omega]} f_\Lambda.$$

Hence, we have $\min_{(0,R_\Omega]} f_\Lambda = J_\Lambda(v) - J_\Lambda(1) \le 0$. By Lemma 16, this implies that $\Lambda \ge \Lambda^*$, and $\overline{r} = r_\Lambda$.

From the above analysis, it follows that $\Lambda_\infty = \Lambda^*$ and that, for every $\Lambda \ge \Lambda^*$, v_{r_Λ} is a non-constant solution to $(\mathrm{P})_\Lambda$ (and any other non-constant solution has the same Lipschitz constant and the same positivity set as v_{r_Λ}). □

Proof of Theorem 6. By Theorem 1, Lemma 14 and the explicit computation in Example 4, we have to prove that

$$\max_{r\in[0,R_\Omega]} r|\Omega_r| \le \frac{\kappa_n n^n R^{n+1}}{(n+1)^{n+1}}, \quad \text{with } R = \left(\frac{|\Omega|}{\kappa_n}\right)^{1/n}.$$

By the Brunn–Minkowski inequality, the function $\gamma(r) := |\Omega_r|^{1/n}$ is concave in $[0, R_\Omega]$, and hence we have $\gamma(r) \le \gamma(0) + r\gamma'_+(0)$, i.e.,

$$|\Omega_r|^{1/n} \le |\Omega|^{1/n} - \frac{r}{n}|\Omega|^{-1+1/n}|\partial\Omega| \quad \text{or} \quad |\Omega_r| \le |\Omega|\left(1 - \frac{r}{n}\frac{|\partial\Omega|}{|\Omega|}\right)^n \quad \text{for all } r \in [0, R_\Omega].$$

(For related inequalities, see [10, § 3].) Hence,

$$r|\Omega_r| \le r|\Omega|\left(1 - \frac{r}{n}\frac{|\partial\Omega|}{|\Omega|}\right)^n =: \varphi(r) \quad \text{for all } r \in [0, R_\Omega].$$

It is easy to check that φ attains its maximum at $r_0 := \frac{n}{n+1}\frac{|\Omega|}{|\partial\Omega|}$, and hence

$$\max_{[0,R_\Omega]} \varphi = \varphi(r_0) = \frac{n^{n+1}}{(n+1)^{n+1}} \cdot \frac{|\Omega|^2}{|\partial\Omega|}.$$

By the isoperimetric inequality and the definition of R, we have

$$\frac{|\Omega|^2}{|\partial\Omega|} = |\Omega|^{(n+1)/n}\frac{|\Omega|^{(n-1)/n}}{|\partial\Omega|} \le \kappa_n^{(n+1)/n} R^{n+1} \frac{1}{n\kappa_n^{1/n}} = \frac{\kappa_n R^{n+1}}{n},$$

with equality if and only if Ω is a ball, and finally,

$$\max_{r\in[0,R_\Omega]} r|\Omega_r| \le \varphi(r_0) \le \frac{\kappa_n n^n R^{n+1}}{(n+1)^{n+1}},$$

with equality if and only if Ω is a ball. □

Proof of Theorem 8. We are going to prove the following two facts:
(a) If $\Lambda \ge \Lambda_\infty$ and $\Lambda < \Lambda_{\mathrm{sing}}$, there are at least two distinct solutions.
(b) If $\Lambda > \Lambda_\infty$ and $\Lambda \ge \Lambda_{\mathrm{sing}}$, there is a unique solution.
Let us check first that the statement easily follows from (a) and (b).

- *Case* $\Lambda_{\mathrm{sing}} \le \Lambda_\infty$. If there is a unique solution, it must be $\Lambda > \Lambda_\infty$ (otherwise, both v_{r^*} and the constant function 1 are solutions). Vice versa, if $\Lambda > \Lambda_\infty$, by (b), there is a unique solution.
- *Case* $\Lambda_{\mathrm{sing}} > \Lambda_\infty$. If there is a unique solution, it must be $\Lambda \ge \Lambda_{\mathrm{sing}}$ (otherwise, by (a), there would be at least two solutions). Vice versa, if $\Lambda \ge \Lambda_{\mathrm{sing}}$, by (b), there is a unique solution.

Let us now prove (a). Let $\Lambda \ge \Lambda_\infty$ and $\Lambda < \Lambda_{\mathrm{sing}}$. Then, recalling from Lemma 16 that the map $\Lambda \mapsto r_\Lambda$ is monotone decreasing, we infer that $r_\Lambda > r_{\mathrm{sing}}$ so that v_{r_Λ} is not everywhere differentiable in D_{r_Λ}. On the other hand, the function w_{r_Λ} defined by (2.4), which is a minimizer of J_Λ by Corollary 3, is differentiable everywhere in D_{r_Λ} (see [14]). Hence, we have at least two different solutions, v_{r_Λ} and w_{r_Λ}.

Finally, let us prove (b). Let $\Lambda > \Lambda_\infty$ and $\Lambda \geq \Lambda_{\text{sing}}$. Since $\Lambda > \Lambda_\infty$, by Theorem 1, any solution v satisfies $\{v > 0\} \cap \Omega = D_{r_\Lambda}$. Let us prove that $v = v_{r_\Lambda}$.

Assume by contradiction that there exists a point $x \in D_{r_\Lambda}$ such that $v(x) \neq v_{r_\Lambda}(x)$. Since $r_\Lambda \leq r_{\text{sing}}$, the distance function $\operatorname{dist}(\cdot, \partial\Omega)$ from the boundary of Ω is differentiable everywhere in D_{r_Λ}. Therefore, the point x admits a unique projection $y \in \partial\Omega$ such that $|x - y| = \operatorname{dist}(x, \partial\Omega)$. Setting $\nu := \frac{x-y}{|x-y|}$ and $y_t := y + t\nu$, we have

$$y_t \in D_{r_\Lambda} \quad \text{for all } t \in (0, r_\Lambda), \qquad z := y_{r_\Lambda} \in \partial\Omega_{r_\Lambda}.$$

Since $v_{r_\Lambda}(z) = v(z) = 0$, if $v(x) > v_{r_\Lambda}(x)$, then we would have

$$v(x) - v(z) > v_{r_\Lambda}(x) - v_{r_\Lambda}(z) = \frac{|x - z|}{r_\Lambda}, \tag{3.6}$$

and hence $\|\nabla v\|_\infty > \frac{1}{r_\Lambda}$, a contradiction. Similarly, if $v(x) < v_{r_\Lambda}(x)$, since $v_{r_\Lambda}(y) = v(y) = 1$, then we would have

$$v(y) - v(x) > v_{r_\Lambda}(y) - v_{r_\Lambda}(x) = \frac{|x - y|}{r_\Lambda}, \tag{3.7}$$

and again $\|\nabla v\|_\infty > \frac{1}{r_\Lambda}$, a contradiction.

Notice that, to obtain the last equality in formulas (3.6) and (3.7), we have used the identity

$$\operatorname{dist}(\xi, \partial\Omega) = r_\Lambda - |\xi - z|$$

holding for every ξ in the segment $[y, z]$ because $\operatorname{dist}(\cdot, \partial\Omega)$ is differentiable in D_{r_Λ} for $r_\Lambda \leq r_{\text{sing}}$. □

4 Proofs of the results in Section 2.2

Before giving the proofs of Propositions 9 and 10, we need to recall a result from our paper [11] about the Bernoulli problem (1.4) on convex domains (therein, also the more general case of non-convex domains is considered).

We first resume a few preliminary definitions. A solution u to (1.4) is called non-trivial if the set $\{u = 0\}$ has non-empty interior. Given $r \in (0, R_\Omega)$, we define w_r as the infinity harmonic potential of $\overline{\Omega}_r$, namely the unique solution to

$$\begin{cases} \Delta_\infty w_r = 0 & \text{in } D_r := \Omega \setminus \overline{\Omega}_r, \\ w_r = 1 & \text{on } \partial\Omega, \\ w_r = 0 & \text{in } \overline{\Omega}_r. \end{cases}$$

Finally, still for $r \in (0, R_\Omega)$, we introduce the subset of D_r defined by

$$\widehat{D}_r := \bigcup_{y \in \partial\Omega_r} \{]y, z[\,:\, z \in \Pi_{\partial\Omega}(y)\},$$

where $]y, z[$ denotes the open (i.e., without the endpoints) segment joining y to z, and $\Pi_{\partial\Omega}(y)$ denotes the set of projections of y onto $\partial\Omega$.

Theorem 19 (See [11]). *The following statements hold.*

(a) *For every $\lambda > \frac{1}{R_\Omega}$, the function $w_{1/\lambda}$ is the unique non-trivial solution to problem* (1.4)*; moreover, it satisfies the estimates*

$$1 - \lambda \operatorname{dist}(x, \partial\Omega) \leq w_{1/\lambda}(x) \leq \lambda \operatorname{dist}(x, \partial\Omega_{1/\lambda}) \quad \text{in } \overline{D}_{1/\lambda}, \quad \text{with equalities in } \widehat{D}_{1/\lambda}.$$

(b) *For every $\lambda \in (0, \frac{1}{R_\Omega}]$, problem* (1.4) *does not admit non-trivial solutions.*

Proof of Proposition 9. From Corollary 3, we know that, for every $\Lambda \geq \Lambda_\infty$, the infinity harmonic potential w_{r_Λ} of D_{r_Λ} is a solution to problem $(\mathrm{P})_\Lambda$. Moreover, by Theorem 19, the function w_{r_Λ} solves problem (1.4) (for $\lambda = \frac{1}{r_\Lambda}$).

In order to prove that there are no other solutions to $(P)_\Lambda$ which solve (1.4), it is enough to recall that the positivity set of any solution to problem $(P)_\Lambda$ is given by D_{r_Λ} (cf. (2.3) in Theorem 1); since clearly a solution to $(P)_\Lambda$ needs to be infinity harmonic in its positivity set in order to solve (1.4), it agrees necessarily with w_{r_Λ}.

Finally, since we know from Theorem 1 and from Corollary 3 that v_{r_Λ} and w_{r_Λ} are both solutions to $(P)_\Lambda$, in case of uniqueness, we conclude that $v_{r_\Lambda} = w_{r_\Lambda}$; moreover, from the first part of the statement already proved, we deduce that such function solves (1.4) (for $\lambda = \frac{1}{r_\Lambda}$). □

Proof of Proposition 10. By Proposition 9, among solutions to problem $(P)_\Lambda$, there is one which solves problem (1.4) (for $\lambda = \frac{1}{r_\Lambda}$) if and only if $\Lambda \geq \Lambda_\infty$ (and in this case, it is given precisely by the function w_{r_Λ}). Then it is enough to observe that, by the continuity and decreasing monotonicity of the map $\mathcal{R}\colon \Lambda \mapsto r_\Lambda$, the condition $\Lambda \geq \Lambda_\infty$ is equivalent to (2.5) (being $\lambda = \frac{1}{r_\Lambda}$).

The last part of the statement follows from Theorem 19, combined with the fact that $\widehat{D}_r = D_r$ holds if and only if $r \leq r_{\mathrm{sing}}$. □

5 Proofs of the results in Section 2.3

Let $\Lambda \geq \Lambda_\infty$. As explained in Section 2.3, our first step towards the proof of Theorem 11 is the study of the asymptotics of a sequence of solutions to problem (2.6), for a fixed λ, in the limit as $p = p_j \to +\infty$.

To that aim, it is useful to notice preliminarily that the minimum of the functionals J_Λ^λ defined in (2.8) can be explicitly computed as

$$\min_{\mathrm{Lip}_1(\Omega)} J_\Lambda^\lambda = \begin{cases} \lambda + \Lambda|\Omega| & \text{if } \lambda \in [0, \frac{1}{R_\Omega}), \\ \lambda + \Lambda|D_{1/\lambda}| & \text{if } \lambda \geq \frac{1}{R_\Omega}. \end{cases} \tag{5.1}$$

Indeed, if $\lambda < \frac{1}{R_\Omega}$, then $\{u > 0\} = \overline{\Omega}$ for every $u \in \mathrm{Lip}_1(\Omega)$ with $\|\nabla u\|_\infty \leq \lambda$; on the other hand, for $\lambda \geq \frac{1}{R_\Omega}$, arguing as in the proof of Theorem 1, we see that a minimizer is given by the function $v_{1/\lambda} = [1 - \lambda\,\mathrm{dist}(\cdot\,, \partial\Omega)]_+$.

Figure 2 represents the plot of the map $\lambda \mapsto \min J_\Lambda^\lambda$ when Ω is the unit two-dimensional ball, for three different values of Λ.

Lemma 20 (Convergence of minimizers at fixed λ). *Let $\Lambda \geq \Lambda_\infty$. Let $(u^{p_j,\lambda})_j$ be a sequence of solutions to problem* (2.6) *for a given $\lambda > 0$ and $p = p_j \to +\infty$. Then, up to passing to a (not relabeled) subsequence, we have*

$$u^{p_j,\lambda} \rightharpoonup u^\lambda \quad \text{weakly in } W^{1,q}(\Omega) \quad \text{for all } q > 1, \qquad u^{p_j,\lambda} \to u^\lambda \quad \text{uniformly in } \overline{\Omega}, \tag{5.2}$$

where u^λ is a solution to problem (2.7) *and is infinity harmonic in its positivity set.*

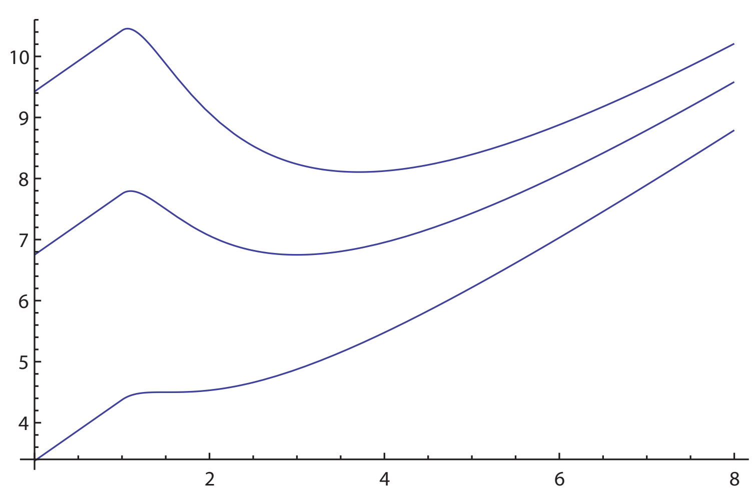

Figure 2: Plot of the map $\lambda \mapsto \min J_\Lambda^\lambda$ in (5.1) when Ω is the unit two-dimensional ball, for $\Lambda = \Lambda' \simeq 1.07$ (top), $\Lambda = \Lambda^* \simeq 2.15$ (center) and $\Lambda = 3$ (bottom)

Furthermore,

- *if* $\lambda \in [0, \frac{1}{R_\Omega})$, *then* $u^\lambda \equiv 1$ *and* $u^{p_j,\lambda} \equiv 1$ *for j large enough;*
- *if* $\lambda \geq \frac{1}{R_\Omega}$, *then*

$$\{u^\lambda > 0\} = D_{1/\lambda} \cup \partial\Omega, \quad \|\nabla u^\lambda\|_\infty = \lambda.$$

Before the proof of Lemma 20, we make a useful observation.

Remark 21. For every $u \in W_1^{1,p}(\Omega)$ and every fixed λ, the map $p \mapsto J_\Lambda^{p,\lambda}(u)$ is monotone non-decreasing. We omit the proof of this property, which can be found in [17, Proposition 1]. Moreover, it is readily checked that the limit as $p \to +\infty$ of $J_\Lambda^{p,\lambda}(u)$ is given precisely by the number $J_\Lambda^\lambda(u) \in [0, +\infty]$ defined in (2.8).

Proof of Lemma 20. Since

$$\frac{1}{p}\int_\Omega \Big(\frac{|\nabla u^{p,\lambda}|}{\lambda}\Big)^p \, dx + \lambda \leq J_\Lambda^{p,\lambda}(u^{p,\lambda}) \leq J_\Lambda^{p,\lambda}(1) = \lambda + \frac{p-1}{p}\Lambda|\Omega| \leq \lambda + \Lambda|\Omega|,$$

we get $\|\nabla u^{p,\lambda}\|_p \leq \lambda(p\Lambda|\Omega|)^{1/p}$. For every fixed exponent $q \in (1, +\infty)$, by the Hölder inequality, for every $p > q$, it holds

$$\|\nabla u^{p,\lambda}\|_q \leq \|\nabla u^{p,\lambda}\|_p |\Omega|^{\frac{p-q}{pq}} \leq \lambda(p\Lambda|\Omega|)^{\frac{1}{p}} |\Omega|^{\frac{p-q}{pq}} = \lambda(p\Lambda)^{\frac{1}{p}} |\Omega|^{\frac{1}{q}} \leq C, \tag{5.3}$$

where $C > 0$ is a constant independent of p.

Therefore, the family $(u^{p,\lambda})_p$ is uniformly bounded in $W^{1,q}(\Omega)$ for every $q > 1$. Using a diagonal argument, we can construct an increasing sequence $p_j \to +\infty$ satisfying (5.2) for some u^λ. Moreover, from (5.3), we deduce that $\|\nabla u^\lambda\|_\infty \leq \lambda$. Since $u^\lambda = 1$ on $\partial\Omega$, we conclude that $u^\lambda \in \mathrm{Lip}_1(\Omega)$ and $\|\nabla u^\lambda\|_\infty \leq \lambda$.

The fact that u^λ is ∞-harmonic in its positivity set is a standard consequence of the fact that the functions $u^{p_j,\lambda}$ are p_j-harmonic in their positivity set, with $p_j \to +\infty$; see, for instance, the arguments in [18, proof of Theorem 1].

Let us prove that u^λ is a minimizer of J_Λ^λ in $\mathrm{Lip}_1(\Omega)$. Let us fix $\varepsilon > 0$. Since $u^{p_j,\lambda} \to u^\lambda$ uniformly in $\overline{\Omega}$, there exists an index $j_\varepsilon \in \mathbb{N}$ such that

$$|\{u^\lambda > 0\}| \leq |\{u^{p_j,\lambda} > 0\}| + \varepsilon \quad \text{for all } j > j_\varepsilon;$$

moreover, for every $j > j_\varepsilon$, we have

$$J_\Lambda^{p_j,\lambda}(u^{p_j,\lambda}) \geq \lambda + \frac{p_j - 1}{p_j}\Lambda|\{u^{p_j,\lambda} > 0\}|.$$

So we obtain

$$\begin{aligned} J_\Lambda^{p_j,\lambda}(u^\lambda) &\leq \frac{1}{p_j}|\{u^\lambda > 0\}| + \lambda + \frac{p_j - 1}{p_j}\Lambda|\{u^\lambda > 0\}| \\ &\leq \frac{1}{p_j}|\Omega| + \lambda + \frac{p_j - 1}{p_j}\Lambda(|\{u^{p_j,\lambda} > 0\}| + \varepsilon) \\ &\leq \frac{1}{p_j}|\Omega| + J_\Lambda^{p_j,\lambda}(u^{p_j,\lambda}) + C\varepsilon. \end{aligned}$$

For every $u \in \mathrm{Lip}_1(\Omega)$, passing to the limit as $j \to +\infty$ (cf. Remark 21), we obtain

$$J_\Lambda^\lambda(u^\lambda) = \lim_{j\to+\infty} J_\Lambda^{p_j,\lambda}(u^\lambda) \leq \liminf_{j\to+\infty} J_\Lambda^{p_j,\lambda}(u^{p_j,\lambda}) \leq \liminf_{j\to+\infty} J_\Lambda^{p_j,\lambda}(u) = J_\Lambda^\lambda(u)$$

so that u^λ is a minimizer of J_Λ^λ in $\mathrm{Lip}_1(\Omega)$.

If $\lambda \in [0, \frac{1}{R_\Omega})$, then $u^\lambda \equiv 1$ (because u^λ is ∞-harmonic in Ω, with $u^\lambda = 1$ on $\partial\Omega$). Since $u^{p_j,\lambda} \to 1$ uniformly for j large enough, then $\{u^{p_j,\lambda} > 0\} = \overline{\Omega}$, and hence $u^{p_j,\lambda} \equiv 1$.

It remains to prove that, for $\lambda \geq \frac{1}{R_\Omega}$, the positivity set $\{u^\lambda > 0\}$ coincides with $D_{1/\lambda} \cup \partial\Omega$, and $\|\nabla u^\lambda\|_\infty = \lambda$. Since $\|\nabla u^\lambda\|_\infty \leq \lambda$, we have $\{u^\lambda > 0\} \supseteq D_{1/\lambda} \cup \partial\Omega$. On the other hand, since u^λ is a minimizer of J_Λ^λ in $\mathrm{Lip}_1(\Omega)$, it minimizes $|\{u > 0\}|$ among all functions $u \in \mathrm{Lip}_1(\Omega)$ with $\|\nabla u\|_\infty \leq \lambda$, which implies (by taking the competitor $v_{1/\lambda}$) that $|\{u^\lambda > 0\}| \leq |D_{1/\lambda}|$. We infer that $\{u^\lambda > 0\} = D_{1/\lambda} \cup \partial\Omega$.

Finally, as a consequence of (5.1) and the equality $\{u^\lambda > 0\} = D_{1/\lambda} \cup \partial\Omega$, we obtain $\|\nabla u^\lambda\|_\infty = \lambda$. ☐

Proof of Theorem 11. Observe that the limit as $p \to +\infty$ in (2.10) does exist, thanks to Remark 21. Let $u^{p,\lambda}$ and u^λ be as in Lemma 20.

For every $\lambda \geq \frac{1}{R_\Omega}$, let $v_{1/\lambda}(x) := [1 - \lambda \operatorname{dist}(x, \partial\Omega)]_+$, $x \in \overline{\Omega}$. Clearly,

$$\{v_{1/\lambda} > 0\} \cap \Omega = D_{1/\lambda} = \{u^\lambda > 0\} \cap \Omega, \quad |\nabla v_{1/\lambda}| = \lambda \quad \text{a.e. in } D_{1/\lambda}$$

so that $J_\Lambda^{p,\lambda}(u^\lambda) \leq J_\Lambda^{p,\lambda}(v_{1/\lambda})$. Observe that

$$J_\Lambda^{p,\lambda}(v_{1/\lambda}) = \frac{1}{p}|D_{1/\lambda}| + \lambda + \frac{p-1}{p}\Lambda|D_{1/\lambda}| \leq \frac{1}{p}|\Omega| + \lambda + \Lambda|D_{1/\lambda}|. \tag{5.4}$$

Hence,

$$\inf_{\lambda \geq 1/R_\Omega} J_\Lambda^{p,\lambda}(u^{p,\lambda}) \leq \inf_{\lambda \geq 1/R_\Omega} J_\Lambda^{p,\lambda}(u^\lambda) \leq \inf_{\lambda \geq 1/R_\Omega} J_\Lambda^{p,\lambda}(v_{1/\lambda}) \leq \frac{1}{p}|\Omega| + \frac{1}{r_\Lambda} + \Lambda|D_{r_\Lambda}|. \tag{5.5}$$

Notice carefully that, in the last inequality above, we have exploited the assumption $\Lambda \geq \Lambda_\infty$, and we have used Lemma 16 (in particular, the definition of r_Λ given therein, and part (d) of the statement).

In view of the upper bound obtained in (5.5), and taking into account that $J_\Lambda^{p,\lambda}(u^{p,\lambda}) > \lambda$ for every $\lambda > 0$, we see that the infimum w.r.t. λ in (2.10) can be taken for $\lambda \in I = [\frac{1}{R_\Omega}, C_\Lambda]$, being $C_\Lambda := |\Omega| + \frac{1}{r_\Lambda} + \Lambda|D_{r_\Lambda}|$.

From the explicit form of $J_\Lambda^{p,\lambda}(v_{1/\lambda})$ in (5.4), we also get

$$J_\Lambda^{p,\lambda}(v_{1/\lambda}) \geq \frac{p-1}{p}(\lambda + \Lambda|D_{1/\lambda}|) \geq \frac{p-1}{p}\Big(\frac{1}{r_\Lambda} + \Lambda|D_{r_\Lambda}|\Big),$$

which, together with (5.4) and (5.5), gives

$$\lim_{p \to +\infty} \inf_{\lambda \geq 1/R_\Omega} J_\Lambda^{p,\lambda}(v_{1/\lambda}) = \frac{1}{r_\Lambda} + \Lambda|D_{r_\Lambda}|. \tag{5.6}$$

Let $(\lambda_k) \subset I$ be an enumeration of the rational points in I. Since the map $\lambda \mapsto J_\Lambda^{p,\lambda}(u)$ is continuous, it is easy to check that

$$\inf\Big\{J_\Lambda^{p,\lambda}(u^{p,\lambda}) : \lambda \geq \frac{1}{R_\Omega}\Big\} = \inf\{J_\Lambda^{p,\lambda_k} : k \in \mathbb{N}\} \quad \text{for all } p > 1.$$

Using Lemma 20 and a diagonal argument, we can construct a sequence $p_j \to +\infty$ such that, for every $q > 1$, for $j \to +\infty$,

$$u^{p_j,\lambda_k} \rightharpoonup u^{\lambda_k} \text{ weakly in } W^{1,q}(\Omega), \quad u^{p_j,\lambda_k} \to u^{\lambda_k} \text{ uniformly in } \overline{\Omega} \quad \text{for all } k \in \mathbb{N}.$$

For every $j \in \mathbb{N}$, let us choose $k_j \in \mathbb{N}$ such that

$$\inf\{J_\Lambda^{p_j,\lambda}(u^{p_j,\lambda}) : \lambda \geq 1/R_\Omega\} \leq J_\Lambda^{p_j,\lambda_{k_j}}(u^{p_j,\lambda_{k_j}}) + \frac{1}{p_j}. \tag{5.7}$$

Upon extracting a further subsequence (not relabeled), we can assume that $\lambda_{k_j} \to \overline{\lambda} \in I$ and that (again using Lemma 20)

$$u^{p_j,\overline{\lambda}} \rightharpoonup u^{\overline{\lambda}} \text{ weakly in } W^{1,q}(\Omega), \quad u^{p_j,\overline{\lambda}} \to u^{\overline{\lambda}} \text{ uniformly in } \overline{\Omega}.$$

Claim. *For every $\varepsilon > 0$, there exists $j_\varepsilon \in \mathbb{N}$ such that*

$$|\{u^{\lambda_{k_j}} > 0\}| = |D_{1/\lambda_{k_j}}| \leq |\{u^{p_j,\lambda_{k_j}} > 0\}| + \varepsilon \quad \textit{for all } j \geq j_\varepsilon.$$

Proof of the claim. Let $j_0 \in \mathbb{N}$ be such that

$$|D_{1/\lambda_{k_j}}| \leq |D_{1/\overline{\lambda}}| + \frac{\varepsilon}{2} \quad \text{for all } j \geq j_0. \tag{5.8}$$

Since $u^{p_j,\overline{\lambda}} \to u^{\overline{\lambda}}$ uniformly in $\overline{\Omega}$, there exists $j_\varepsilon \in \mathbb{N}$, $j_\varepsilon \geq j_0$, such that

$$|D_{1/\overline{\lambda}}| = |\{u^{\overline{\lambda}} > 0\}| \leq |\{u^{p_j,\overline{\lambda}} > 0\}| + \frac{\varepsilon}{2} \quad \text{for all } j \geq j_\varepsilon. \tag{5.9}$$

From (5.8) and (5.9) the claim follows.

Let us fix $\varepsilon > 0$. Using the claim, for every $(p, \lambda) = (p_j, \lambda_{k_j})$ with $j \geq j_\varepsilon$, we have

$$J_\Lambda^{p,\lambda}(u^{p,\lambda}) \geq \lambda + \frac{p-1}{p}\Lambda(|D_{1/\lambda}| - \varepsilon),$$

and hence

$$J_\Lambda^{p,\lambda}(u^{p,\lambda}) \leq J_\Lambda^{p,\lambda}(u^\lambda) \leq J_\Lambda^{p,\lambda}(v_{1/\lambda}) = \frac{1}{p}|D_{1/\lambda}| + \lambda + \frac{p-1}{p}\Lambda|D_{1/\lambda}| \leq \frac{1}{p}|\Omega| + \frac{p-1}{p}\Lambda\varepsilon + J_\Lambda^{p,\lambda}(u^{p,\lambda}).$$

Finally, from (5.6) and (5.7), we conclude that (2.10) holds. □

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