Stochastic weighted variational inequalities in non-pivot Hilbert spaces with applications to a transportation model

Annamaria Barbagallo[a,*], Giovanni Scilla[a]

[a] *Department of Mathematics and Applications "R. Caccioppoli", University of Naples "Federico II", Via Cintia, Monte S. Angelo - 80126 Naples (ITALY)*

Abstract

A class of stochastic weighted variational inequalities in non-pivot Hilbert spaces is proposed. Hence existence and continuity results are proved. These theoretical results play an important role in order to introduce a new weighted transportation model with uncertainty. Moreover, they allow to establish the equivalence between the random weighted equilibrium principle and a suitable stochastic weighted variational inequality. At the end, a numerical model is discussed.

Keywords: Non-pivot Hilbert spaces, Stochastic weighted variational inequalities, Existence, Stochastic continuity, Traffic problem

1. Introduction

In the last years, more and more problems arising from Applied Mathematics, Economics and Engineering, but also from real life, have been modeled by variational inequalities. In particular, variational inequalities for example provide a unifying framework for the study of diverse problems as boundary value problems, equilibrium problems and game theory (see, for instance, [1], [2], [3]). Some years ago, the stochastic formulation for a special class of variational inequalities has been studied in [4].

*Corresponding author

Email addresses: annamaria.barbagallo@unina.it (Annamaria Barbagallo), giovanni.scilla@unina.it (Giovanni Scilla)

The paper deals with the introduction of a new class of variational inequalities, which we call *stochastic weighted variational inequalities*. More precisely, we give some useful definitions which we need to show some existence results. Moreover, by using the set convergence in Kuratowski's sense, we are able to obtain a continuity result. The introduction of this new class is motivated by the formulation of a more realistic transportation equilibrium model. More precisely, the random weighted traffic equilibrium problem is presented. Finally, thanks to the theoretical results, the random weighted equilibrium principle is characterized by a stochastic weighted variational inequality.

Firstly, the weighted variational inequalities have been introduced for giving the definition of weighted equilibrium flow (see [5]). Later, a systematic study on the weighted variational inequalities has been done in [6]. In particular, existence and regularity results have been obtained. Then, by using such theoretical theorems, a numerical scheme for the computation of equilibrium solutions has been presented. In [7], making use of the infinite-dimensional duality theory, the characterization of the weighted traffic equilibrium conditions in terms of Lagrange multipliers has been established. Recently, in [8], weighted quasi-variational inequalities have been proposed in order to study the weighted traffic model in which the travel demands depend on time and also on the equilibrium flow. Furthermore, it is worth to remind that recently a static *random* traffic model has been introduced in [9] and [10], where the authors characterize the random Wardrop equilibrium distribution by means of a stochastic variational inequality.

This paper is motivated by the fundamental role that the uncertainty has on network user's decisions. Under that light, the results obtained in this paper have a wide field of applicability and open to remarkable developments. We want to emphasize that a deterministic model is quite unrealistic. The need to develop a random traffic model arises because the path flows as well as the travel demand often vary over time in a non-regular and unpredictable manner. Such an uncertainty can be caused by several factors such as the particular hour of the day but also by a sudden accident or a maintenance work. This remark suggests the necessity to couple the deterministic dependence on time of all data with the probability. Moreover, as well as in [5] and [6], we introduce a system of weights which permit us to express the real time information in the duality pairing. In particular, such weights, acting on the path flows and the path cost function, are introduced in the functional space. Such a space cannot be identified with its topological dual. As a consequence, we have a *non-pivot Hilbert space*, in which the duality pairing depends implicitly on time. The advantage of this setting is the possibility to study models with more general flows and cost functions.

The outline of this paper is as follows. In Section 2 we fix the notation and introduce the weighted functional spaces we will deal with. We define the stochastic weighted variational inequality problem. We recall some topological and monotonicity properties of operators and the definition of set convergence in Kuratowski's sense, which will be useful to obtain a regularity result. In Section 3, we establish existence results for stochastic weighted variational inequalities. Section 4 is devoted to the study of the stochastic continuity of equilibrium solutions. In Section 5 we present in detail the random weighted traffic equilibrium model. In particular, we state a random weighted generalized Wardrop equilibrium principle and we show the characterization with a stocastic variational inequality. Finally, in Section 6 a numerical example is analyzed.

2. Basic concepts and setting of the problem

Let $T > 0$, let $\Omega \subset \mathbb{R}$ be open and let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space. We consider the product measure space $(\Omega_T, \mathcal{A}, \nu)$, where we set $\Omega_T :=]0, T[\times\Omega$, $\mathcal{A} := \mathcal{B}(]0, T[) \otimes \mathcal{F}$ and $\nu := \mathcal{L}^1 \otimes \mathcal{P}$, $\mathcal{B}(]0, T[)$ is the Borel σ-field of $]0, T[$ and $\mathcal{L}^1$ stands for the 1-dimensional Lebesgue measure on $]0, T[$. A point in Ω_T will be denoted by the couple (t, ω). If $X = X(\omega)$ is a random variable on Ω, then its expectation will be denoted by $\mathbb{E}(X)$. We assume the reader is familiar with basic notions as stochastic processes and stochastic continuity (see e.g. [11] for a general reference on these topics).

The measure space $(\Omega_T, \mathcal{A}, \nu)$ is naturally endowed with a metric structure $(\Omega_T, \|\cdot\|_2)$, where $\|\cdot\|_2$ is the Euclidean norm on $\mathbb{R}^2$. We denote by $\mathcal{C}_0(\Omega_T)$ the space of continuous functions with compact support on Ω_T. The following proposition, whose proof is immediate, shows that $\mathcal{C}_0(\Omega_T)$ can be endowed with an inner product as a pre-Hilbert space.

Proposition 2.1. *Let $p = p(t)$, $m = m(t)$ be continuous and strictly positive functions on $]0, T[$ called* weight *and* real time density*, respectively. Then the bilinear form defined on $\mathcal{C}_0(\Omega_T)$ by*

$$((u,v))_{p,m} = \mathbb{E}\Big(\int_0^T u(\xi,\omega)v(\xi,\omega)p(\xi)m(\xi)\,d\xi\Big)$$

is an inner product.

It's worth to note that, since the product space $(\Omega_T, \mathcal{A}, \nu)$ is σ-finite, then by Fubini and Tonelli's Theorem the previous inner product could be defined alternatively as

$$((u,v))_{p,m} = \int_0^T \mathbb{E}\Big(u(\xi,\omega)v(\xi,\omega)p(\xi)m(\xi)\Big)\,d\xi.$$

However, we observe that the space $\mathcal{C}_0(\Omega_T)$ endowed with the inner product $((\cdot,\cdot))_{p,m}$ is not a Hilbert space (as shown by a simple counterexample; see e.g. [12]). As a consequence, we give the following definition.

Definition 2.2. A completion of $\mathcal{C}_0(\Omega_T)$ with respect to the inner product $((u,v))_{p,m}$ is denoted by $L^2(\Omega_T, p, m)$.

This space is a non-pivot Hilbert space, since it has no sense to identify it with its topological dual (see [6] for details). Let us note that supposing p a weight, then we obtain that also p^{-1} is a weight so that we may define, correspondingly, the space $L^2(\Omega_T, p^{-1}, m)$ according to Definition 2.2.

In the following, we present a d-dimensional non-pivot Hilbert space. Let $\mathbf{p} = (p_1, \ldots, p_d)$ and $\mathbf{m} = (m_1, \ldots, m_d)$ be two d-tuples of continuous and strictly positive functions on $]0, T[$. Denoting by $X_k = L^2(\Omega_T, p_k, m_k)$ and

$X_k^* = L^2(\Omega_T, p_k^{-1}, m_k)$, the space

$$X = \prod_{k=1}^{d} X_k \tag{1}$$

is a non-pivot Hilbert space with respect to the inner product

$$((\mathbf{A}, \mathbf{B}))_X = ((\mathbf{A}, \mathbf{B}))_{\mathbf{p},\mathbf{m}} = \sum_{k=1}^{d} \mathbb{E}\Big(\int_0^T A_k(\xi,\omega) B_k(\xi,\omega) p_k(\xi) m_k(\xi)\, d\xi\Big).$$

The same assertion holds for the space

$$X^* = \prod_{k=1}^{d} X_k^* \tag{2}$$

with respect to

$$((\mathbf{A}, \mathbf{B}))_{X^*} = ((\mathbf{A}, \mathbf{B}))_{\mathbf{p^{-1}},\mathbf{m}} = \sum_{k=1}^{d} \mathbb{E}\Big(\int_0^T \frac{A_k(\xi,\omega) B_k(\xi,\omega) m_k(\xi)}{p_k(\xi)}\, d\xi\Big).$$

Furthermore, if we consider the bilinear form

$$\langle\langle \mathbf{f}, \mathbf{x}\rangle\rangle_{X^* \times X} = \langle\langle \mathbf{f}, \mathbf{x}\rangle\rangle_{\mathbf{m}} = \sum_{k=1}^{d} \mathbb{E}\Big(\int_0^T f_k(\xi,\omega) x_k(\xi,\omega) m_k(\xi)\, d\xi\Big), \tag{3}$$

then the following result holds.

Proposition 2.3. *The bilinear form (3) on $X^* \times X$ defines a duality between X^* and X. The duality mapping is given by $J(\mathbf{A}) = (p_1 A_1, \ldots, p_d A_d)$.*

Proof. We adapt to our purposes the argument of Proposition 3.1 in [5]. By Definition 2.2, we know that for each k, $X_k(\Omega_T) = \overline{\mathcal{C}_0(\Omega_T)}^{\{p_k, m_k\}}$ and that X is complete if for each k, X_k is complete. Then it is enough to take $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{C}_0^d(\Omega_T) := \mathcal{C}_0(\Omega_T) \times \mathcal{C}_0(\Omega_T) \times \cdots \times \mathcal{C}_0(\Omega_T)$ d-times. By using twice the Cauchy-Schwartz's inequality for finite sums and integrals we obtain

$$\begin{aligned}
\langle\langle \mathbf{A}, \mathbf{B}\rangle\rangle_{\mathbf{m}} &\leq \sum_{k=1}^{d} \mathbb{E}\Big(\int_0^T \Big| A_k(\xi,\omega)\sqrt{m_k(\xi)}\sqrt{p_k(\xi)} \frac{B_k(\xi,\omega)\sqrt{m_k(\xi)}}{\sqrt{p_k(\xi)}}\Big|\, d\xi\Big) \\
&\leq \sum_{k=1}^{d} \mathbb{E}\Big(\Big(\int_0^T A_k^2(\xi,\omega) m_k(\xi) p_k(\xi)\, d\xi\Big)^{\frac{1}{2}} \Big(\int_0^T \frac{B_k^2(\xi,\omega) m_k(\xi)}{p_k(\xi)}\, d\xi\Big)^{\frac{1}{2}}\Big) \\
&\leq \Big(\sum_{k=1}^{d} \mathbb{E}\Big(\int_0^T A_k^2(\xi,\omega) m_k(\xi) p_k(\xi)\, d\xi\Big)\Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{d} \mathbb{E}\Big(\int_0^T \frac{B_k^2(\xi,\omega) m_k(\xi)}{p_k(\xi)}\, d\xi\Big)\Big)^{\frac{1}{2}} \\
&\leq \|\mathbf{A}\|_{\mathbf{p},\mathbf{m}} \|\mathbf{B}\|_{\mathbf{p^{-1}},\mathbf{m}},
\end{aligned}$$

where $\|\cdot\|_{\mathbf{p},\mathbf{m}}$ and $\|\cdot\|_{\mathbf{p^{-1}},\mathbf{m}}$ denote the norm in X and X^*, respectively.

Now if $\mathbf{A} \in X$ then $\mathbf{pA} = (p_1 A_1, \ldots, p_d A_d) \in X^*$ and $\|\mathbf{pA}\|_{\mathbf{p^{-1}},\mathbf{m}} = \|\mathbf{A}\|_{\mathbf{p},\mathbf{m}}$ that means

$$\|\mathbf{B}\|_{\mathbf{p^{-1}},\mathbf{m}} = \sup_{\mathbf{A}\in X} \frac{\langle\langle \mathbf{A}, \mathbf{B}\rangle\rangle_{\mathbf{m}}}{\|\mathbf{A}\|_{\mathbf{p},\mathbf{m}}}.$$

As a consequence, $\langle\langle \cdot, \cdot\rangle\rangle_{\mathbf{m}}$ is a duality pairing and

$$((\mathbf{A}, \mathbf{B}))_{\mathbf{p},\mathbf{m}} = \sum_{k=1}^{d} \mathbb{E}\Big(\int_0^T A_k(\xi,\omega) B_k(\xi,\omega) p_k(\xi) m_k(\xi)\, d\xi\Big) = \langle\langle \mathbf{pA}, \mathbf{B}\rangle\rangle_{\mathbf{m}}.$$

$\square$

We are now ready to introduce the *stochastic weighted variational inequalities.*

Definition 2.4. Let $\mathbf{M}$ be a nonempty, closed and convex subset of X and let $\mathbf{A} :]0,T[\times \mathbf{M} \to X^*$ be a vector-function. The *stochastic weighted variational inequality* (SWVI) is the problem to find a (vector) stochastic process $\mathbf{u} = \mathbf{u}(t,\omega) \in \mathbf{M}$, such that

$$\begin{aligned}&\langle\langle \mathbf{A}(\mathbf{u}), \mathbf{v} - \mathbf{u}\rangle\rangle_{\mathbf{m}} = \\ &= \sum_{k=1}^{d} \mathbb{E}\Big(\int_0^T m_k(\xi) A_k(\xi, u_k(\xi,\omega))(v_k(\xi,\omega) - u_k(\xi,\omega))\, d\xi\Big) \geq 0, \quad \forall \mathbf{v} \in \mathbf{M}.\end{aligned} \tag{4}$$

2.1. Some tools

In this subsection we recall some useful definitions for the study of SWVIs. First, we recall some topological and monotonicity properties of operators necessary to obtain existence results, both in the monotone approach and without monotonicity assumptions. Let X and X^* be the non-pivot Hilbert spaces presented in (1) and (2), respectively, and $\langle\langle \cdot, \cdot\rangle\rangle_{\mathbf{m}}$ be the duality pairing defined by (3).

Let $\mathbf{M}$ be a subset of X. The following definition deals with two more general assumptions than monotonicity.

Definition 2.5. An operator $\mathbf{A} :]0,T[\times \mathbf{M} \to X^*$ is said to be

- *pseudomonotone in the sense of Karamardian* (*K-pseudomonotone*) on $\mathbf{M}$ if for every $\mathbf{u}, \mathbf{v} \in \mathbf{M}$

$$\langle\langle \mathbf{A}(\mathbf{v}), \mathbf{u}-\mathbf{v}\rangle\rangle_{\mathbf{m}} \geq 0 \Longrightarrow \langle\langle \mathbf{A}(\mathbf{u}), \mathbf{u}-\mathbf{v}\rangle\rangle_{\mathbf{m}} \geq 0;$$

- *strongly pseudomonotone with degree* $\alpha > 0$ *on* $\mathbf{M}$, uniformly with respect to $t \in]0, T[$, if there exists $\eta > 0$ such that, for every $\mathbf{u}, \mathbf{v} \in \mathbf{M}$,

$$\langle\langle \mathbf{A}(\mathbf{v}), \mathbf{u}-\mathbf{v}\rangle\rangle_{\mathbf{m}} \geq 0 \;\Rightarrow\; \langle\langle \mathbf{A}(\mathbf{u}), \mathbf{u}-\mathbf{v}\rangle\rangle_{\mathbf{m}} \geq \eta \|\mathbf{u}-\mathbf{v}\|_X^{\alpha},$$

where $\|\mathbf{z}\|_X = \sqrt{((\mathbf{z}, \mathbf{z}))_X}$.

The approach to existence without monotonicity assumptions requires some continuity and lower semi-continuity properties of the operator, as follows.

Definition 2.6. An operator $\mathbf{A} :]0, T[\times \mathbf{M} \to X^*$ is said to be

- *pseudomonotone in the sense of Brézis* (*B-pseudomonotone*) iff

  (a) for every sequence $\{\mathbf{u}_j\}$ weakly converging to $\mathbf{u}$ (shortly, $\mathbf{u}_j \rightharpoonup \mathbf{u}$) in $\mathbf{M}$ and such that $\limsup_j \langle\langle \mathbf{A}(\mathbf{u}_j), \mathbf{u}_j - \mathbf{u}\rangle\rangle_{\mathbf{m}} \leq 0$ it results that

$$\liminf_j \langle\langle \mathbf{A}(\mathbf{u}_j), \mathbf{u}_j - \mathbf{v}\rangle\rangle_{\mathbf{m}} \geq \langle\langle \mathbf{A}(\mathbf{u}), \mathbf{u}-\mathbf{v}\rangle\rangle_{\mathbf{m}}, \quad \forall \mathbf{v} \in \mathbf{M};$$

  (b) for every $\mathbf{v} \in \mathbf{M}$ the function $\mathbf{u} \to \langle\langle \mathbf{A}(\mathbf{u}), \mathbf{u}-\mathbf{v}\rangle\rangle_{\mathbf{m}}$ is lower bounded on the bounded subsets of $\mathbf{M}$.

- *hemicontinuous in the sense of Fan* (*F-hemicontinuous*) iff for all $\mathbf{v} \in \mathbf{M}$, the function

$$\mathbf{u} \longmapsto \langle\langle \mathbf{A}(\mathbf{u}), \mathbf{u}-\mathbf{v}\rangle\rangle_{\mathbf{m}}$$

  is weakly lower semi-continuous on $\mathbf{M}$;

We conclude the list with the following continuity property which usually is coupled with a monotonicity assumption in existence theorems.

Definition 2.7. Let $\mathbf{M} \subset X$ be convex. An operator $\mathbf{A} :]0,T[\times\mathbf{M} \to X^*$ is said to be *lower hemicontinuous along line segments* iff the function

$$\mathbf{z} \longmapsto \langle\langle \mathbf{A}(\mathbf{z}), \mathbf{u} - \mathbf{v} \rangle\rangle_{\mathbf{m}}$$

is lower semi-continuous for all $\mathbf{u}, \mathbf{v} \in \mathbf{M}$ on the line segments $[\mathbf{u}, \mathbf{v}]$.

In order to investigate the regularity properties of solutions to SWVIs, we will need a suitable notion of convergence for a family of closed sets. Let us recall the set convergence in Kuratowski's sense, that can be expressed as follows (see [13] for further details).

Definition 2.8. Let (X, d) be a metric space and $\mathbf{K} \subset X$ be a nonempty, closed and convex set. A sequence of nonempty, closed and convex sets $\{\mathbf{K}_j\}$ *converges to* $\mathbf{K}$ *in Kuratowski's sense*, as $j \to +\infty$, and we write $\mathbf{K}_j \xrightarrow{\mathcal{K}} \mathbf{K}$, if and only if

(i) for all $\mathbf{q} \in \mathbf{K}$, there exists a sequence $\{\mathbf{q}_j\}$ converging to $\mathbf{q}$ in X such that $\mathbf{q}_j \in \mathbf{K}_j$ for every $j \in \mathbb{N}$,

(ii) for all sequences $\{\mathbf{q}_j\}$ such that there exists $\{\mathbf{q}_{j_k}\} \subseteq \{\mathbf{q}_j\}$, $\mathbf{q}_{j_k} \in \mathbf{K}_{j_k}$, $\forall k \in \mathbb{N}$ and $\mathbf{q}_{j_k} \to \mathbf{q}$ in X, then the limit $\mathbf{q}$ belongs to $\mathbf{K}$.

3. Existence theorems

In this section we recall some existence results for variational inequalities in Banach spaces (see e.g. [14]). Let E be a reflexive Banach space over the reals, let $C \subseteq E$ be a nonempty, closed and convex set. Let $A : C \to E^*$ be a map to the dual space E^* equipped with the weak* topology, and let $\langle \cdot, \cdot \rangle$ denote the duality pairing between E and E^*. The first theorem deals with the existence without monotonicity hypotheses (see [14], Theorem 2.18).

Theorem 3.1. *Let C be a nonempty, closed and convex subset of E and let $A : C \to E^*$ be B-pseudomonotone or F-hemicontinuous. Let us assume that A satisfies the following condition*

- *there exist* $u_0 \in C$ *and* $R > \|u_0\|$ *such that*

$$\langle A(v), v - u_0 \rangle \geq 0, \ \forall\, v \in C \cap \{v \in E : \ \|v\| = R\}. \tag{5}$$

Then there exists $u \in C$ *such that*

$$\langle A(u), v - u \rangle \geq 0, \quad \forall v \in C. \tag{6}$$

The following result, instead, provides existence under monotonicity assumptions (see [14], Theorem 3.6).

Theorem 3.2. *Let* C *be a nonempty, closed and convex subset of* E. *Let* $A : C \to E^*$ *be a K-pseudomonotone and lower hemicontinuous along line segments map. Moreover, let us suppose that* A *satisfies condition (5). Then* (6) *admits solutions.*

We note that if in addition C is bounded, then condition (5) may be removed in the statements of Theorem 3.1 and 3.2. Moreover, if we replace the K-pseudomonotonicity assumption on A by the strongly pseudomonotonicity of degree $\alpha > 0$, then we get also the uniqueness of solutions to (6).

The previous results can be clearly enunciated also for non-pivot Hilbert spaces. More precisely, we obtain the following existence theorems.

Theorem 3.3. *Let* $\mathbf{M}$ *be a nonempty, closed and convex subset of non-pivot Hilbert space* X *and let* $\mathbf{A} : \mathbf{M} \to X^*$ *be B-pseudomonotone or F-hemicontinuous. Let us assume that* $\mathbf{A}$ *satisfies the following condition*

- *there exist* $\mathbf{u}_0 \in \mathbf{M}$ *and* $R > \|\mathbf{u}_0\|$ *such that*

$$\langle\langle \mathbf{A}(\mathbf{v}), \mathbf{v} - \mathbf{u}_0 \rangle\rangle_{\mathbf{m}} \geq 0, \ \textit{for every } \mathbf{v} \in \mathbf{M} \cap \{\mathbf{v} \in X : \ \|\mathbf{v}\| = R\}. \tag{7}$$

Then the stochastic weighted variational inequality

$$\langle\langle \mathbf{A}(\mathbf{u}), \mathbf{v} - \mathbf{u} \rangle\rangle_{\mathbf{m}} \geq 0, \quad \forall \mathbf{v} \in \mathbf{M}, \tag{8}$$

admits a solution.

Theorem 3.4. *Let* $\mathbf{M}$ *be a nonempty, closed and convex subset of non-pivot Hilbert space* X *and let* $\mathbf{A} : \mathbf{M} \to X^*$ *be a K-pseudomonotone map, which is lower hemicontinuous along line segments. Moreover, let us assume that* $\mathbf{A}$ *satisfies condition (7). Then* (8) *admits a solution.*

4. Stochastic continuity

The aim of this section is to show that a solution to (4) is stochastically continuous (see [6] for the deterministic counterpart). More precisely, we prove a stronger result, since we show that the stochastic process solving (4) is sample path continuous (see e.g. [11]); that is, if $\mathbf{u} = \mathbf{u}(t, \omega)$ is solution, then for almost any fixed $\omega \in \Omega$, the function $t \to \mathbf{u}(t, \omega)$ is continuous.

For this reason, we introduce the finite-dimensional stochastic weighted variational inequality associated to (4). We consider on $\mathbb{R}^d$ the norm

$$\|\mathbf{u}\|_{d,\mathbf{p},\mathbf{m}} = \sqrt{\sum_{k=1}^{d} u_k^2 p_k m_k}$$

with $\mathbf{p}, \mathbf{m} \in \mathbb{R}^d$, $p_k, m_k > 0, k = 1, \ldots, d$, and the bilinear form:

$$\langle \cdot, \cdot \rangle_{d,\mathbf{m}} : (\mathbb{R}^d, \|\cdot\|_{d,\mathbf{p}^{-1},\mathbf{m}}) \times (\mathbb{R}^d, \|\cdot\|_{d,\mathbf{p},\mathbf{m}}) \to \mathbb{R}$$

$$\langle \mathbf{u}, \mathbf{v} \rangle_{d,\mathbf{m}} = \sum_{k=1}^{d} u_k v_k m_k.$$

It can be immediately shown, with the same technique used in [5], that $\langle \cdot, \cdot \rangle_{d,\mathbf{m}}$ is a duality pairing between the normed spaces $(\mathbb{R}^d, \|\cdot\|_{d,\mathbf{p}^{-1},\mathbf{m}})$ and $(\mathbb{R}^d, \|\cdot\|_{d,\mathbf{p},\mathbf{m}})$.

Let $\mathbf{M}$ be a nonempty, closed and convex subset of X. Setting for any $(t, \omega) \in \Omega_T$,

$$\mathbf{M}(t, \omega) = \left\{ \mathbf{g}(t, \omega) \in \mathbb{R}^d : \ \mathbf{g} \in \mathbf{M} \right\},$$

we note that $\mathbf{M}(t, \omega)$ is nonempty, closed and convex, so that one can present the pointwise SWVI associated to (4):

Find a $\mathbf{u}(t,\omega) \in \mathbf{M}(t,\omega)$ such that

$$\langle \mathbf{A}(t,\mathbf{u}(t,\omega)), \mathbf{v}(t,\omega) - \mathbf{u}(t,\omega)\rangle_{d,\mathbf{m}(t)} \geq 0, \quad \forall \mathbf{v}(t,\omega) \in \mathbf{M}(t,\omega), \text{ a.e. in }]0,\mathrm{T}[, \ \mathcal{P} - \text{a.s.} \tag{9}$$

We can establish, under our assumptions, the next equivalence:
u is a solution to (4) $\Leftrightarrow$ $\mathbf{u}(t,\omega)$ is a solution to (9), a.e. in $]0,T[$, $\mathcal{P}$-a.s.

For this aim, we assume that the variational inequality problem holds in its integral formulation. If the pointwise formulation were false, we would have the existence of a measurable subset $Z \in \mathcal{A}$ with $\nu(Z) > 0$ and $\widetilde{\mathbf{v}}(t,\omega) \in \mathbf{M}(t,\omega)$ such that

$$\langle \mathbf{A}(t,\mathbf{u}(t,\omega)), \widetilde{\mathbf{v}}(t,\omega) - \mathbf{u}(t,\omega)\rangle_{d,\mathbf{m}(t)} < 0, \quad \forall (t,\omega) \in Z.$$

Setting

$$\mathbf{v}^*(t,\omega) = \begin{cases} \widetilde{\mathbf{v}}(t,\omega), & (t,\omega) \in Z, \\ \mathbf{u}(t,\omega), & (t,\omega) \in \Omega_T \setminus Z, \end{cases},$$

we would obtain

$$\begin{aligned} &\int_\Omega \int_0^T \langle \mathbf{A}(\xi,\mathbf{u}(\xi,\omega)), \mathbf{v}^*(\xi,\omega) - \mathbf{u}(\xi,\omega)\rangle_{d,\mathbf{m}(\xi)} \, d\xi \, d\mathcal{P}(\omega) \\ &= \iint_{\Omega_T \setminus Z} \langle \mathbf{A}(\xi,\mathbf{u}(\xi,\omega)), \mathbf{u}(\xi,\omega) - \mathbf{u}(\xi,\omega)\rangle_{d,\mathbf{m}(\xi)} \, d\xi \, d\mathcal{P}(\omega) \\ &+ \iint_Z \langle \mathbf{A}(\xi,\mathbf{u}(\xi,\omega)), \widetilde{\mathbf{v}}(\xi,\omega) - \mathbf{u}(\xi,\omega)\rangle_{d,\mathbf{m}(\xi)} \, d\xi \, d\mathcal{P}(\omega) < 0 \end{aligned}$$

which would be a contradiction.

The main result of this section is the following stochastic regularity theorem.

Theorem 4.1. *Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, let X and X^* be as in Section 2, $T > 0$, $(t,\omega) \in \Omega_T$ and let $\mathbf{M}(t,\omega)$ be a nonempty, closed, convex and bounded subset of $\mathbb{R}^d$ verifying Kuratowski's convergence assumptions with respect to t, namely: for any fixed $\omega \in \Omega$ and $t_j \to t$ as $j \to +\infty$ then*

$\mathbf{M}(t_j,\omega) \overset{\mathcal{K}}{\to} \mathbf{M}(t,\omega)$ in Kuratowski's sense; let $\mathbf{A} :]0,T[\times \mathbf{M} \to X^*$ be a continuous function and $\mathbf{A}(t,\cdot)$ strongly pseudo-monotone with degree $\alpha > 1$. Then the solution process $\mathbf{u} = \mathbf{u}(t,\omega)$ to (9) is stochastically continuous on $]0,T[$.

Proof. Let $\omega \in \Omega$ be fixed, and $\mathbf{u}(t_j,\omega)$ be the unique solution of the SWVI

$$\langle \mathbf{A}(t_j, \mathbf{u}(t_j,\omega)), \mathbf{v}(t_j,\omega) - \mathbf{u}(t_j,\omega)\rangle_{d,\mathbf{m}(t_j)} \geq 0, \quad \forall \mathbf{v}(t_j,\omega) \in \mathbf{M}(t_j,\omega),\ \forall j \in \mathbb{N}. \tag{10}$$

With fixed $(t,\omega) \in \Omega_T$, it suffices to show that, for any sequence of times $\{t_j\}$ in $]0,T[$ converging to t as $j \to +\infty$, we have that $\mathbf{u}(t_j,\omega) \to \mathbf{u}(t,\omega)$ as $j \to +\infty$.

A generalized version of Minty-Browder's lemma ensures that for every $(t,\omega) \in \Omega_T$ we have

$$\langle \mathbf{A}(t, \mathbf{v}(t,\omega)), \mathbf{v}(t,\omega) - \mathbf{u}(t,\omega)\rangle_{d,\mathbf{m}(t)} \geq 0, \quad \forall \mathbf{v}(t,\omega) \in \mathbf{M}(t,\omega).$$

Now, making use of the Kuratowski's set convergence assumption with respect to t on $\mathbf{M}(t,\omega)$, we have that for any $\mathbf{u}(t,\omega) \in \mathbf{M}(t,\omega)$, there exists a sequence $\{\mathbf{z}(t_j,\omega)\}_{j\in\mathbb{N}}$ such that $\mathbf{z}(t_j,\omega) \in \mathbf{M}(t_j,\omega)$ for j large enough and $\mathbf{z}(t_j,\omega) \to \mathbf{u}(t,\omega)$. The continuity of function $\mathbf{A}$ implies that $\mathbf{A}(t_j,\mathbf{z}(t_j,\omega)) \to \mathbf{A}(t,\mathbf{u}(t,\omega))$. If we set, for j large enough, $\mathbf{v}(t_j,\omega) = \mathbf{z}(t_j,\omega)$ in (10), we have

$$\langle \mathbf{A}(t_j, \mathbf{u}(t_j,\omega)), \mathbf{z}(t_j,\omega) - \mathbf{u}(t_j,\omega)\rangle_{d,\mathbf{m}(t_j)} \geq 0.$$

Since $\mathbf{A}(t,\cdot)$ is strongly pseudo-monotone with degree $\alpha > 1$, we obtain the estimate

$$\begin{aligned}
&\eta \|\mathbf{z}(t_j,\omega) - \mathbf{u}(t_j,\omega)\|^{\alpha}_{d,\mathbf{p}(t_j),\mathbf{m}(t_j)} \\
&\quad \leq \langle \mathbf{A}(t_j, \mathbf{z}(t_j,\omega)), \mathbf{z}(t_j,\omega) - \mathbf{u}(t_j,\omega)\rangle_{d,\mathbf{m}(t_j)} \\
&\quad \leq \|\mathbf{A}(t_j,\mathbf{z}(t_j,\omega))\|_{d,\mathbf{p}^{-1}(t_j),\mathbf{m}(t_j)} \|\mathbf{z}(t_j,\omega) - \mathbf{u}(t_j,\omega)\|_{d,\mathbf{p}(t_j),\mathbf{m}(t_j)}
\end{aligned}$$

from which we deduce that

$$\|\mathbf{z}(t_j,\omega) - \mathbf{u}(t_j,\omega)\|_{d,\mathbf{p}(t_j),\mathbf{m}(t_j)} \leq \eta^{\frac{1}{1-\alpha}} \|\mathbf{A}(t_j,\mathbf{z}(t_j,\omega))\|^{\frac{1}{\alpha-1}}_{d,\mathbf{p}^{-1}(t_j),\mathbf{m}(t_j)}.$$

The further estimate

$$\begin{aligned}
&\|\mathbf{u}(t_j,\omega)\|_{d,\mathbf{p}(t_j),\mathbf{m}(t_j)} \\
&\quad \le \|\mathbf{u}(t_j,\omega)-\mathbf{z}(t_j,\omega)\|_{d,\mathbf{p}(t_j),\mathbf{m}(t_j)} + \|\mathbf{z}(t_j,\omega)\|_{d,\mathbf{p}(t_j),\mathbf{m}(t_j)} \\
&\quad \le \eta^{\frac{1}{1-\alpha}} \|\mathbf{A}(t_j,\mathbf{z}(t_j,\omega))\|^{\frac{1}{\alpha-1}}_{d,\mathbf{p}^{-1}(t_j),\mathbf{m}(t_j)} + \|\mathbf{z}(t_j,\omega)\|_{d,\mathbf{p}(t_j),\mathbf{m}(t_j)},
\end{aligned}$$

shows that $\{\mathbf{u}(t_j,\omega)\}_{j\in\mathbb{N}}$ is a bounded sequence. Thus, there exist $\mathbf{z}\in\mathbb{R}^d$ and a subsequence not relabeled and still denoted by $\{\mathbf{u}(t_j,\omega)\}_{j\in\mathbb{N}}$ such that $\mathbf{u}(t_j,\omega)\in\mathbf{M}(t_j,\omega)$, $\forall j\in\mathbb{N}$ and $\mathbf{u}(t_j,\omega)\to\mathbf{z}$. The Kuratowski's set convergence assumption ensures that $\mathbf{z}\in\mathbf{M}(t,\omega)$.

Now, we are left to prove that $\mathbf{z}=\mathbf{u}(t,\omega)$. A further application of the generalized version of Minty-Browder's Lemma to any $\mathbf{u}(t_j,\omega)$ gives

$$\langle \mathbf{A}(t_j,\mathbf{v}(t_j,\omega)), \mathbf{v}(t_j,\omega)-\mathbf{u}(t_j,\omega)\rangle_{d,\mathbf{m}(t_j)} \ge 0, \quad \forall \mathbf{v}(t_j,\omega)\in\mathbf{M}(t_j,\omega).$$

Making use again of the set convergence in Kuratowski's sense, for any $\mathbf{v}(t,\omega)\in\mathbf{M}(t,\omega)$, we can obtain $\{\mathbf{v}(t_j,\omega)\}_{j\in\mathbb{N}}$ such that $\mathbf{v}(t_j,\omega)\in\mathbf{M}(t_j,\omega)$ for j large enough and $\mathbf{v}(t_j,\omega)\to\mathbf{v}(t,\omega)$. We have

$$\begin{aligned}
&\langle \mathbf{A}(t_j,\mathbf{v}(t_j,\omega)), \mathbf{v}(t_j,\omega)-\mathbf{u}(t_j,\omega)\rangle_{d,\mathbf{m}(t_j)} \\
&\quad = \langle \mathbf{m}(t_j)\mathbf{A}(t_j,\mathbf{v}(t_j,\omega)), \mathbf{v}(t_j,\omega)-\mathbf{u}(t_j,\omega)\rangle_{d} \ge 0, \quad \forall \mathbf{v}(t_j,\omega)\in\mathbf{M}(t_j,\omega),
\end{aligned}$$

where $\langle\cdot,\cdot\rangle_d$ is the inner product of $\mathbb{R}^d$. Let $j\to+\infty$, we obtain

$$\langle \mathbf{A}(t,\mathbf{v}(t,\omega)), \mathbf{v}(t,\omega)-\mathbf{z}\rangle_{d,\mathbf{m}(t)} \ge 0, \quad \forall \mathbf{v}(t,\omega)\in\mathbf{M}(t,\omega).$$

Taking into account the generalized version of Minty-Browder's Lemma, it results

$$\langle \mathbf{A}(t,\mathbf{z}), \mathbf{v}(t,\omega)-\mathbf{z}\rangle_{d,\mathbf{m}(t)} \ge 0, \quad \forall \mathbf{v}(t,\omega)\in\mathbf{M}(t,\omega).$$

Finally, the uniqueness of the solution to (9) ensures that $\mathbf{z}=\mathbf{u}(t,\omega)$ and $\mathbf{u}(t_j,\omega)\to\mathbf{u}(t,\omega)$. □

5. The random weighted traffic equilibrium model

This section is devoted to propose a more general transportation model. In particular we consider an extended version of the traffic models described in

[6, 15, 16] where all the relevant quantities, like as path flows and costs, will be random variables, being dependent both on time and probability. As already remarked in [6] for the deterministic weighted model, the non-pivot Hilbert space setting permits to treat some "congested" traffic problems that otherwise would have no solution in L^2. Moreover, the "weighted" bilinear form allows us to formulate the results developed by Carlo Ratti at the SENSEable City Laboratory. More precisely, we are looking for the optimal distribution of flows taking into account the traffic density obtained by means of a system of wireless communications. However, the user's choices (and, as a consequence, the equilibrium distribution) are often variable over time in a non-regular and unpredictable manner. Such an uncertainty can be caused by several factors such as the particular hour of the day but also by a sudden accident or maintenance works. The mathematical model, taking into account these remarks, can be expressed by the weights on the routes and the cost function. Indeed, if we analyze a term of the bilinear form which explains the model, namely

$$\sum_{k=1}^{d} \mathbb{E}\Big(\int_0^T A_k(\xi,\omega)\sqrt{p_k(\xi)}\sqrt{m_k(\xi)}B_k(\xi,\omega)\frac{1}{\sqrt{p_k(\xi)}}\sqrt{m_k(\xi)}d\xi \Big),$$

we note that if $(\sqrt{p_k(t)})^{-1}$ is the weight associated to the route $B_k(t,\omega)$, then the weight $\sqrt{p_k(t)}$ acts on the cost function. As a consequence, if $p_k^{-1}(t)$ is very large then the cost has to be very small (almost surely); if $p_k(t)$ is very small, then the cost has to be very large (almost surely). Moreover, the weight $\mathbf{p}$ influences the traffic density. In particular the network user discards the path in which the associated weight is greater than others for a fixed path cost.

For the reader's convenience we introduce in detail the model of random weighted traffic equilibrium problem (see [5, 10] for the deterministic and the random static cases).

A traffic network consists of a triple (N, A, W), where $N = \{N_1, \dots, N_p\}$ is the set of nodes, $A = \{A_1, \dots, A_n)$ represents the set of the directed arcs connecting couples of nodes and $W = \{w_1, \dots, w_l\} \subset N \times N$ is the set of the origin-destination (O/D) pairs. Let $T > 0$ and $\Omega \subset \mathbb{R}$. The planning horizon of our problem is $]0,T[$. The flow on the arc A_k is denoted by f_k and the uncertainty which affects the knowledge of f_k is given by the dependence of f_k also on ω, namely $f_k = f_k(t,\omega)$, where $t \in]0,T[$, $\omega \in \Omega$. We will set $\mathbf{f}(t,\omega) = (f_1(t,\omega),\dots,f_n(t,\omega))$. We call a set of a finite number of consecutive arcs a path and assume that each O/D pair w_j is connected by $r_j \geq 1$ paths, whose set is denoted by $\mathcal{R}_j, j = 1,\dots,l$. All the paths in the network are grouped into a vector $(R_1,\dots,R_d)$. We can describe the arc structure of the path by using the arc-path incidence matrix $\Delta = (\delta_{ir})$, $k = 1,\dots,n$, $r = 1,\dots,d$, whose entries take the value 1 if $A_k \in R_r$ and 0 if $A_k \notin R_r$.

Let $\mathbf{p} = (p_1,\dots,p_d)$ and $\mathbf{p^{-1}} = (p_1^{-1},\dots,p_d^{-1})$ be two families of weights such that for every $1 \leq r \leq d$, $p_r \in \mathcal{C}(]0,T[,\mathbb{R}^+\backslash\{0\})$. Let us introduce also the real time density $\mathbf{m} = (m_1,\dots,m_d)$ such that for every $1 \leq r \leq d$, $m_r \in \mathcal{C}(]0,T[,\mathbb{R}^+\backslash\{0\})$. Let us associate the components p_r and m_r of the weights $\mathbf{p}$ and $\mathbf{m}$, respectively, to every path R_r, $r = 1,2,\dots,d$. Correspondingly, we define the weighted spaces X and X^*, as presented in Section 2.

To each path R_r there corresponds a flow $F_r(t,\omega) \in L^2(\Omega_T,\mathbb{R},p_r,m_r)$, and the path flows are grouped into a vector $\mathbf{F}(t,\omega) =$ $(F_1(t,\omega),\dots,F_d(t,\omega))$, which is called the path flow vector. The flow f_k on the arc A_k is equal to the sum of the flows on the paths which contains A_k so that $\mathbf{f}(t,\omega) = \Delta\mathbf{F}(t,\omega)$, $(t,\omega) \in \Omega_T$. Let us now introduce the unit cost of going through A_k as a function $c_k(t,\mathbf{f}(t,\omega)) \geq 0$ of the flows on the network, so that $\mathbf{c}(t,\mathbf{f}(t,\omega)) = (c_1(t,\mathbf{f}(t,\omega)),\dots,c_n(t,\mathbf{f}(t,\omega)))$ denotes the arc cost on the network. Analogously $\mathbf{C}(t,\mathbf{F}(t,\omega)) = (C_1(t,\mathbf{F}(t,\omega)),\dots,C_n(t,\mathbf{F}(t,\omega)))$ will denote the cost on the paths. Usually $C_r(t,\mathbf{F}(t,\omega))$ is given by the sum of the costs on the arcs building the path: $C_r(t,\mathbf{F}(t,\omega)) = \displaystyle\sum_{k=1}^{n} \delta_{kr} c_k(t,\mathbf{f}(t,\omega))$ or, in vector notation, $\mathbf{C}(t,\mathbf{F}(t,\omega)) = \Delta^T \mathbf{c}(\Delta\mathbf{F}(t,\omega))$.

We suppose that there exist two random capacity constraints $\boldsymbol{\lambda},\boldsymbol{\mu} \in L^2(\Omega_T,\mathbb{R}^d,\mathbf{p},\mathbf{m})$, $\boldsymbol{\lambda} \leq \boldsymbol{\mu}$, such that for all $r = 1,\dots,d$

$$0 \leq \lambda_r(t,\omega) \leq F_r(t,\omega) \leq \mu_r(t,\omega) \quad \text{a.e. in }]0,T[,\ \mathcal{P}\text{-a.s.}$$

Let Φ be the pair-incidence matrix, whose element ϕ_{jr} is equal to 1, if the path

R_r connects the pair w_j and equal 0 otherwise. Let a_j the family of indices r such that $\phi_{jr} = 1$, for $j = 1, \ldots, l$, let $\alpha^j = |a_j|$, for $j = 1, \ldots, l$, let $p^*_j = \max(p_{(a_j)_1}, \ldots, p_{(a_j)_{\alpha^j}})$[1], for $j = 1, \ldots, l$, and let $m^*_j = \max(m_{(a_j)_1}, \ldots, m_{(a_j)_{d^j}})$, $j = 1, \ldots, l$. Let us group the previous components in $\mathbf{p}^* = (p^*_1, \ldots, p^*_l)$ and $\mathbf{m}^* = (m^*_1, \ldots, m^*_l)$. Let $\rho_j \in L^2(\Omega_T, \mathbb{R}^l, p^*_j, m^*_j)$, $j = 1, \ldots, l$, be the travel demand associated with the users moving between the O/D pair w_j and let $\rho = (\rho_1, \ldots, \rho_l)^T \in L^2(\Omega_T, \mathbb{R}^l, \mathbf{p}^*, \mathbf{m}^*) = \prod_{j=1}^l L^2(\Omega_T, \mathbb{R}, p^*_j, m^*_j)$ be the total demand function. We require that the traffic conservation law is fulfilled, namely

$$\sum_{r=1}^{d} \varphi_{jr} F_r(t,\omega) = \rho_j(t,\omega) \quad j = 1, \ldots, l \quad \text{a.e. in }]0,T[,\ \mathcal{P}\text{-a.s.}$$

that can be written also as

$$\Phi \mathbf{F}(t,\omega) = \boldsymbol{\rho}(t,\omega), \text{ a.e. in }]0,T[,\ \mathcal{P}\text{-a.s.}.$$

Hence, the set of feasible flows $\mathbb{K}$ is

$$\begin{aligned} \mathbb{K} = \{\mathbf{F} \in L^2(\Omega_T, \mathbb{R}^d, \mathbf{p}, \mathbf{m}) : \quad & \boldsymbol{\lambda}(t,\omega) \le \mathbf{F}(t,\omega) \le \boldsymbol{\mu}(t,\omega), \\ & \Phi \mathbf{F}(t,\omega) = \boldsymbol{\rho}(t,\omega), \text{ a.e. in }]0,T[,\ \mathcal{P}\text{-a.s.}\}. \end{aligned}$$

It can be shown that $\mathbb{K}$ is a nonempty, closed and convex subset of space $L^2(\Omega_T, \mathbb{R}^d, \mathbf{p}, \mathbf{m})$. Moreover, $\mathbb{K}$ satisfies the Kuratowski's set convergence property, which we present without proof (see for instance [17]).

Proposition 5.1. *Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in L^2(\Omega_T, \mathbb{R}^d, \mathbf{p}, \mathbf{m}) \cap C(\Omega_T, \mathbb{R}^d_+)$, let $\boldsymbol{\rho} \in L^2(\Omega_T, \mathbb{R}^l, \mathbf{p}^*, \mathbf{m}^*) \cap C(\Omega_T, \mathbb{R}^l_+)$ and let $\{t_j\}_{j\in\mathbb{N}} \subseteq]0,T[$ be a sequence such that $t_j \to t \in]0,T[$, as $j \to +\infty$. Then, for any fixed $\omega \in \Omega$, the sequence of sets*

$$\mathbb{K}(t_j,\omega) = \Big\{\mathbf{F}(t_j,\omega) \in \mathbb{R}^d : \boldsymbol{\lambda}(t_j,\omega) \le \mathbf{F}(t_j,\omega) \le \boldsymbol{\mu}(t_j,\omega),\ \Phi\mathbf{F}(t_j,\omega) = \boldsymbol{\rho}(t_j,\omega)\Big\},$$

$\forall j \in \mathbb{N}$, converges to

$$\mathbb{K}(t,\omega) = \Big\{\mathbf{F}(t,\omega) \in \mathbb{R}^d : \boldsymbol{\lambda}(t,\omega) \le \mathbf{F}(t,\omega) \le \boldsymbol{\mu}(t,\omega),\ \Phi\mathbf{F}(t,\omega) = \boldsymbol{\rho}(t,\omega)\Big\},$$

[1]Where $p_{(a_j)_k}$ is the k-th element of the family a_j, for $j = 1, \ldots, l$ and $k = 1, \ldots, l$.

as $j \to +\infty$, in Kuratowski's sense.

Now we introduce the equilibrium condition, the so-called random weighted Wardrop equilibrium principle.

Definition 5.2. A distribution $\mathbf{H} \in \mathbb{K}$ is an equilibrium distribution from the user's point of view iff $\forall w_j \in W$, $\forall R_p, R_s \in \mathcal{R}_j$, a.e. in $]0,T[$, $\mathcal{P}$-a.s. there holds

$$\begin{aligned} &m_p(t)C_p(t,\mathbf{H}(t,\omega)) < m_s(t)C_s(t,\mathbf{H}(t,\omega)) \\ &\Longrightarrow H_p(t,\omega) = \mu_p(t,\omega) \text{ or } H_s(t,\omega) = \lambda_s(t,\omega). \end{aligned} \tag{11}$$

The meaning of Definition 5.2 is that the network's users discard the more expensive paths with respect to the weighted cost. In the following, we apply the theoretical results for stochastic weighted variational inequalities obtained in the previous sections to the traffic problem. More precisely, we show the equivalence between the random weighted equilibrium flow distribution and the solution to a stochastic weighted variational inequality.

Theorem 5.3. *The stochastic process $\mathbf{H} \in \mathbb{K}$ is an equilibrium distribution according to Definition 5.2 iff it solves the stochastic weighted variational inequality:*

$$\begin{aligned} &\langle\langle \mathbf{C}(\mathbf{H}), \mathbf{F}-\mathbf{H}\rangle\rangle_{\mathbf{m}} = \\ &\sum_{k=1}^{d} \mathbb{E}\Big(\int_0^T C_k(\xi,\mathbf{H}(\xi,\omega))(F_k(\xi,\omega)-H_k(\xi,\omega))m_k(\xi)\,d\xi\Big) \geq 0, \quad \forall \mathbf{F} \in \mathbb{K}. \end{aligned} \tag{12}$$

Proof. First, we prove that if $\mathbf{H} \in \mathbb{K}$ is an equilibrium distribution as in (11), then $\mathbf{H}$ is a solution to (12). It is enough to prove that a.e. in $]0,T[$ and $\mathcal{P}$-a.s.

$$\sum_{j=1}^{l} \sum_{R_r \in \mathcal{R}_j} C_r(t,\mathbf{H}(t,\omega))(F_r(t,\omega)-H_r(t,\omega))m_r(t,\omega) \geq 0, \quad \forall \mathbf{F}(t,\omega) \in \mathbb{K}(t,\omega), \tag{13}$$

since the assertion (12) follows by integrating on Ω_T.

Let $w_j \in W$ be an arbitrary O/D pair. We denote by

$$A_j := \{R_q \in \mathcal{R}_j : \, H_q(t,\omega) < \mu_q(t,\omega) \text{ a.e. in }]0,T[, \, \mathcal{P}\text{-a.s.}\},$$

and

$$B_j := \{R_k \in \mathcal{R}_j : \, H_k(t,\omega) > \lambda_k(t,\omega) \text{ a.e. in }]0,T[, \, \mathcal{P}\text{-a.s.}\}.$$

From equilibrium condition (11) we obtain

$$m_q(t)C_q(t,\mathbf{H}(t,\omega)) \geq m_k(t)C_k(t,\mathbf{H}(t,\omega)), \forall R_q \in A_j, \, \forall R_k \in B_j, \text{ a.e. in }]0,T[, \, \mathcal{P}-\text{a.s.}$$

As a consequence, there exists a function $\gamma_j : \Omega_T \to \mathbb{R}$ such that

$$\inf_{R_q \in A_j} m_q(t)C_q(t,\mathbf{H}(t,\omega)) \geq \gamma_j(t,\omega) \geq \sup_{R_k \in B_j} m_k(t)C_k(t,\mathbf{H}(t,\omega)),$$

a.e. in $]0,T[$, $\mathcal{P}$-a.s.

Let $\mathbf{F} \in \mathbb{K}$ be fixed. Then, for every $R_r \in \mathcal{R}_j$, the inequality

$$m_r(t)C_r(t,\mathbf{H}(t,\omega)) < \gamma_j(t,\omega), \text{ a.e. in }]0,T[, \, \mathcal{P}\text{-a.s.},$$

implies $R_r \notin A_j$, from which we deduce that $H_r(t,\omega) = \mu_r(t,\omega)$, a.e. in $]0,T[$, $\mathcal{P}$-a.s. and then $F_r(t,\omega) - H_r(t,\omega) \leq 0$, a.e. in $]0,T[$, $\mathcal{P}$-a.s. As a consequence, we have

$$(C_r(t,\mathbf{H}(t,\omega)) - \gamma_j(t,\omega))(F_r(t,\omega) - H_r(t,\omega))m_r(t) \geq 0.$$

Likewise, $m_r(t)C_r(t,\mathbf{H}(t,\omega)) > \gamma_j(t,\omega)$, a.e. in $]0,T[$, $\mathcal{P}$-a.s., implies $(m_r(t)C_r(t,\mathbf{H}(t,\omega)) - \gamma_j(t,\omega))(F_r(t,\omega) - H_r(t,\omega)) \geq 0$, a.e. in $]0,T[$, $\mathcal{P}$-a.s..

Thus, we have

$$\begin{aligned} &\sum_{R_r \in \mathcal{R}_j} m_r(t)C_r(t,\mathbf{H}(t,\omega))(F_r(t,\omega) - H_r(t,\omega)) \\ &\qquad \geq \gamma_j(t,\omega) \sum_{R_r \in \mathcal{R}_j} (F_r(t,\omega) - H_r(t,\omega)) = 0, \end{aligned}$$

from which (12) immediately follows.

Now we prove that any solution to (12) satisfies condition (11). We argue by contradiction and we assume that there exist $w_j \in W$, $R_q, R_k \in \mathcal{R}_j$ and a set

$E \in \mathcal{A}$ with $\nu(E) > 0$ such that for a.e. $(t, \omega) \in E$ one has $m_q(t) C_q(t, \mathbf{H}(t, \omega)) < m_k(t) C_k(t, \mathbf{H}(t, \omega))$, but $H_q(t, \omega) < \mu_q(t, \omega)$ and $H_k(t, \omega) > \lambda_k(t, \omega)$.

We denote by $\delta(t, \omega) := \min\{\mu_q(t, \omega) - H_q(t, \omega), H_k(t, \omega) - \lambda_k(t, \omega)\}$, $(t, \omega) \in E$, satisfying $\delta(t, \omega) > 0$ a.e. in E, and consider the flow $\mathbf{F}^*(t, \omega)$ defined as:

$$\mathbf{F}^*(t, \omega) = \mathbf{H}(t, \omega) \text{ in } \Omega_T \setminus E$$

$$F_r^*(t, \omega) = \begin{cases} H_r(t, \omega) & \text{if } r \neq q, k \\ H_q(t, \omega) + \delta(t, \omega) & \text{if } r = q \\ H_k(t, \omega) - \delta(t, \omega) & \text{if } r = k \end{cases} \quad \text{a.e. in } E.$$

It is easy to verify that $\mathbf{F}^* \in \mathbb{K}$, then we can valuate the stochastic variational inequality (12) in $\mathbf{F}^*$:

$$\sum_{k=1}^{d} \mathbb{E}\Big(\int_0^T C_k(\xi, \mathbf{H}(\xi, \omega))(F_k^*(\xi, \omega) - H_k(\xi, \omega)) m_k(\xi)\, d\xi \Big)$$
$$= \mathbb{E}\Big(\int_{E_\omega} [m_q(\xi) C_q(\xi, \mathbf{H}(\xi, \omega)) - m_k(\xi) C_k(\xi, \mathbf{H}(\xi, \omega))] \delta(\xi, \omega)\, d\xi \Big) < 0$$

which is in contradiction with (12).

$\square$

As remarked previously, $\mathbb{K}$ is a nonempty, closed, convex and bounded subset of X. Thus, taking into account Theorem 3.3 and Theorem 3.4, we obtain the existence of a random weighted traffic equilibrium solution, either with or without monotonicity assumptions on the cost function. Moreover, assuming that the data are continuous, Proposition 5.1 and Theorem 4.1 guarantee that the equilibrium flow is a stochastically continuous process. As already remarked in [6], the continuity is a fundamental tool to develop numerical schemes to compute the random weighted traffic equilibrium solution. These topics will be subject of future investigations.

6. A simple numerical example

Now we apply the theory developed in the previous sections to a simple traffic network model and we compute explicitly the vector stochastic process describing the equilibrium flow.

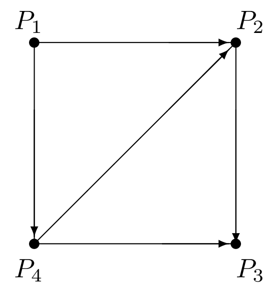

Figure 1: A simple network model.

We analyze a traffic network as in Fig.1, where the set of nodes is given by $N = (P_1, P_2, P_3, P_4)$ and the set of links by

$$A = \{(P_1, P_2), (P_1, P_4), (P_2, P_3), (P_4, P_2), (P_4, P_3)\}.$$

The set of O/D pairs is $W = \{(P_1, P_3)\}$, and consequently the routes are

$$\begin{aligned}
R_1 &= (P_1, P_2) \cup (P_2, P_3), \\
R_2 &= (P_1, P_4) \cup (P_4, P_3), \\
R_3 &= (P_1, P_4) \cup (P_4, P_2) \cup (P_2, P_3).
\end{aligned}$$

The planning horizon of the traffic problem is $]0, 1[$. Let $\mathbf{p}(t) = (p_1(t), p_2(t), p_3(t)) = (1, 1, \sqrt{t})$ be the weight and let $\mathbf{m}(t) = (m_1(t), m_2(t), m_3(t)) = (3t, t^2, 2t\sqrt{t})$ be the real time density. We suppose that the path cost vector-function $\mathbf{C}(t, \mathbf{F}(t, \omega))$ has components

$$\begin{aligned}
C_1(t, \mathbf{F}(t, \omega)) &= F_1(t, \omega), \\
C_2(t, \mathbf{F}(t, \omega)) &= \frac{1}{t}(F_1(t, \omega) + F_2(t, \omega)), \\
C_3(t, \mathbf{F}(t, \omega)) &= \frac{1}{\sqrt{t}} F_3(t, \omega).
\end{aligned}$$

Let us introduce $\rho(t, \omega)$ the random traffic density. Let us assume that $\rho(t, \omega)$ is normally distributed with mean 0 and variance 1 under the constraint $\rho \in [0, +\infty)$, a.e. in $]0, 1[$, $\mathcal{P}$-a.s.

Hence, the set of feasible flows is

$$\mathbb{K} = \Big\{ \mathbf{F} \in L^2(]0,1[\times\Omega, \mathbb{R}^3, \mathbf{p}, \mathbf{m}) : F_k(t,\omega) \geq 0, \forall k = 1,2,3 \text{ a.e. in }]0,1[, \mathcal{P}\text{-a.s.}, \\ \sum_{k=1}^{3} F_k(t,\omega) = \rho(t,\omega), \text{ a.e. in }]0,1[, \mathcal{P}\text{-a.s.} \Big\}.$$

The stochastic weighted variational inequality, which governs the problem, is

$$\sum_{k=1}^{3} \mathbb{E}\Big(\int_0^1 C_k(\xi, \mathbf{H}(\xi,\omega))(F_k(\xi,\omega) - H_k(\xi,\omega)) m_k(\xi)\, d\xi \Big) \geq 0, \quad \forall\, \mathbf{F} \in \mathbb{K}. \quad (14)$$

In order to compute the equilibrium flow, we can apply the direct method introduced in [18]. Firstly, we remark that

$$F_3(t,\omega) = \rho(t,\omega) - F_1(t,\omega) - F_2(t,\omega), \text{ a.e. in }]0,1[, \mathcal{P}\text{-a.s.} \quad (15)$$

and correspondingly we can consider

$$\widetilde{\mathbb{K}} = \Big\{ \widetilde{\mathbf{F}} \in L^2(]0,1[\times\Omega, \mathbb{R}^2, (1,1), (t,t^2)) : F_k(t,\omega) \geq 0, k = 1,2 \text{ a.e. in }]0,1[, \mathcal{P}\text{-a.s.}, \\ \sum_{k=1}^{2} F_k(t,\omega) \leq \rho(t,\omega), \text{ a.e. in }]0,1[, \mathcal{P}\text{-a.s.} \Big\}.$$

We define

$$\begin{aligned} \Gamma_1(t, \widetilde{\mathbf{F}}(t,\omega)) &= m_1(t) C_1(t, \widetilde{\mathbf{F}}(t,\omega)) - m_3(t) C_3(t, \widetilde{\mathbf{F}}(t,\omega)) \\ &= 5t F_1(t,\omega) + 2t F_2(t,\omega) - 2t\rho(t,\omega), \end{aligned}$$

$$\begin{aligned} \Gamma_2(t, \widetilde{\mathbf{F}}(t,\omega)) &= m_2(t) C_2(t, \widetilde{\mathbf{F}}(t,\omega)) - m_3(t) C_3(t, \widetilde{\mathbf{F}}(t,\omega)) \\ &= 3t F_1(t,\omega) + 3t F_2(t,\omega) - 2t\rho(t,\omega). \end{aligned}$$

Thus, problem (14) may be written as

$$\sum_{k=1}^{2} \mathbb{E}\Big(\int_0^1 \Gamma_k(\xi, \widetilde{\mathbf{H}}(\xi,\omega))(F_k(\xi,\omega) - H_k(\xi,\omega)) m_k(\xi)\, d\xi \Big) \geq 0, \quad \forall \widetilde{\mathbf{F}} \in \widetilde{\mathbb{K}}. \quad (16)$$

It can be shown that if $\widetilde{\mathbf{H}}$ is a solution to the system

$$\begin{cases} \Gamma_1(t, \widetilde{\mathbf{H}}(t,\omega)) = 0, \\ \Gamma_2(t, \widetilde{\mathbf{H}}(t,\omega)) = 0, \\ \widetilde{\mathbf{H}} \in \widetilde{\mathbb{K}}, \end{cases}$$

then it solves (16). We obtain, a.e. in $]0,1[$, $\mathcal{P}$-a.s.

$$H_1(t,\omega) = \frac{2}{9}\rho(t,\omega),$$
$$H_2(t,\omega) = \frac{4}{9}\rho(t,\omega),$$

under the constraints that

$$H_1(t,\omega), H_2(t,\omega) \geq 0, \ \text{ a.e. in }]0,1[, \ \mathcal{P}\text{-a.s.},$$
$$H_1(t,\omega) + H_2(t,\omega) \leq \rho(t,\omega), \ \text{ a.e. in }]0,1[, \ \mathcal{P}\text{-a.s.}$$

Finally, from (15) we can compute $H_3(t,\omega)$ and the vector stochastic process describing the equilibrium flow a.e. in $]0,1[$ is given by

$$\mathbf{H}(t,\omega) = \Big(\frac{2}{9}\rho(t,\omega), \frac{4}{9}\rho(t,\omega), \frac{1}{3}\rho(t,\omega)\Big).$$

Let us check that $\mathbf{H} \in \mathbb{K}$. It is well known (see e.g. [11]) that the probability density functions of H_1, H_2 and H_3 can be obtained from p_ρ by a scaling argument. More precisely, we have (see Fig. 2)

$$p_{H_1}(x) = \frac{9}{2}p_\rho\left(\frac{9}{2}x\right) = \frac{9}{\sqrt{2\pi}}e^{-\frac{81}{8}x^2}, \ x \geq 0,$$
$$p_{H_2}(x) = \frac{9}{4}p_\rho\left(\frac{9}{4}x\right) = \frac{9}{2\sqrt{2\pi}}e^{-\frac{81}{32}x^2}, \ x \geq 0,$$
$$p_{H_3}(x) = 3p_\rho\left(3x\right) = \frac{6}{\sqrt{2\pi}}e^{-\frac{9}{2}x^2}, \ x \geq 0,$$

from which a straightforward computation gives

$$\mathcal{P}(\{H_k \geq 0\}) = \int_0^{+\infty} p_{H_k}(z)\,dz = 1, \ \forall k = 1,2,3, \ \text{ a.e. in }]0,1[.$$

Moreover, since $\mathbb{E}(c\rho) = c\mathbb{E}(\rho)$ and $\text{Var}(c\rho) = c^2 Var(\rho)$, $c \in \mathbb{R}$, we have

$$\mathbb{E}(H_1) = \frac{2\sqrt{2}}{9\sqrt{\pi}}, \quad \text{Var}(H_1) = \frac{4\pi - 8}{81\pi},$$
$$\mathbb{E}(H_2) = \frac{4\sqrt{2}}{9\sqrt{\pi}}, \quad \text{Var}(H_2) = \frac{16\pi - 32}{81\pi},$$
$$\mathbb{E}(H_3) = \frac{\sqrt{2}}{3\sqrt{\pi}}, \quad \text{Var}(H_3) = \frac{\pi - 2}{9\pi}.$$

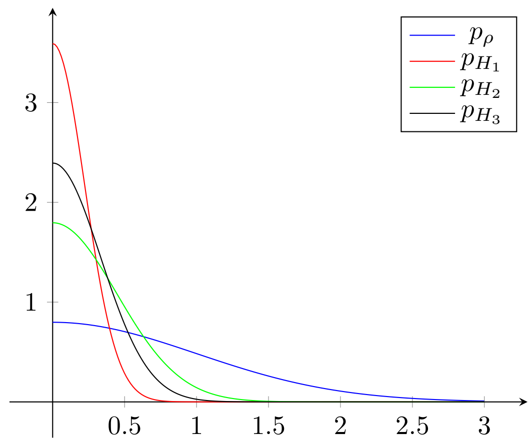

Figure 2: Probability density functions of traffic demand and equilibrium flow.

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