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Article

Approximation of Space-Time Fractional Equations

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Abstract: The aim of this paper is to provide approximation results for space-time non-local equations with general non-local (and fractional) operators in space and time. We consider a general Markov process time changed with general subordinators or inverses to general subordinators. Our analysis is based on Bernstein symbols and Dirichlet forms, where the symbols characterize the time changes, and the Dirichlet forms characterize the Markov processes.

Keywords: space–time fractional equations; Dirichlet forms; asymptotics

MSC: 26A33; 60B10; 60H30; 31C25

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1. Introduction

We consider space–time fractional equations with general fractional operators in space and time. More precisely, we deal with a very general fractional space operator that covers a large class of non-local operators, such as fractional Laplacians.

This non-local space-operator can be related to time-changed processes where the time change is given by a subordinator (for definitions, examples and applications, see ref. [1]) characterized by a symbol, which is a Bernstein function (see [2]).

Additionally, the non-local time operator is very general, and it includes a huge class of convolution-type operators, such as the Caputo fractional derivatives. This non-local time-operator can be related to time-changed processes where the time change is given by an inverse to a subordinator characterized by a symbol, which is again a Bernstein function.

The literature on space–time fractional equations and their applications is very extensive. We mention here only some basic works [3–7] and the references therein. Connections with Sturm–Liouville problems were investigated in [8], whereas for the the higher-order counterpart, we refer to [9,10]. For the fractional Cauchy problem on manifolds, we reference the work in [11]. Recently, the authors also obtained results on irregular domains in the case of randomly varying fractals [12,13].

The aim of this paper is to relate the asymptotic analysis of space–time fractional equations to the convergence of corresponding symbols (see Theorems 5 and 6). Our result extends Theorem 7 in [9], where asymptotic properties for time-changed processes were investigated for pseudo-processes. The symbol of a subordinator may be approximated by the symbols of a continuous-time random walk (see [9] (Theorem 5)). The base process is Markovian, but it is driven by a signed measure; that is, the governing equation of the base process is a higher-order equation.

We highlight that the results of the present paper provide a useful tool for studying the approximation of space–time fractional equations in several contexts since the theory of Dirichlet forms allows us to describe many structures in an appropriate functional environment.

For example, we can approximate space–time fractional equations related to relativistic 2α-stable processes, spherical symmetric 2α-stable processes, and gamma processes with suitable sequences of relativistic 2α-stable processes in $\mathbb{R}^d$. Moreover, we can use the results of the present paper for "denoising" variance gamma processes, that is, Brownian motions time changed using a gamma subordinator as a random time. Such a kind of "denoising" can be carried out by considering the asymptotic limits of the parameters characterizing the symbol of the subordinator (see Section 7).

The paper is set out as follows. In Section 2, we recall some basic facts about processes associated with Dirichlet forms. In Section 3, we introduce symbols corresponding to Bernstein functions associated with subordinate processes. In Section 4, we consider space fractional equations, and we recall asymptotic results via the convergence of symbols obtained in [14]. In Section 5, we introduce the time fractional equations associated with inverse processes. In Section 6, we consider space–time fractional equations, and we prove asymptotic results via the convergence of symbols. Finally, in the last section, we provide some examples and applications.

2. Processes Associated with Dirichlet Forms

We now recall some basic facts about processes associated with Dirichlet forms (see [15]).

We consider an m-symmetric right process X on a Lusin space E. Without loss of generality, X can be assumed as an m-symmetric Hunt process associated with a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on a locally compact separable metric space E, where m is a Radon measure with full support on E (by using quasi-homeomorphism, see [16]). The L^2-infinitesimal generator $\mathcal{A}$ is a non-positive definite self-adjoint operator, and it has the following spectral representation:

$$-\mathcal{A} = \int_0^\infty \lambda dE_\lambda$$

with domain

$$Dom(\mathcal{A}) = \left\{ u \in L^2(E,m) : \int_0^\infty \lambda^2 d(E_\lambda u, u) < \infty \right\}.$$

Here, $\{E_\lambda, \lambda \geq 0\}$ is the spectral family of $-\mathcal{A}$: it is a right continuous increasing sequence of orthogonal projections in $L^2(E,m)$ with $E_0 = 0$, and $E_\infty = I$ the identity operator. The corresponding Dirichlet form $(\mathcal{E}, \mathcal{F})$ associated with X is defined as follows:

$$\mathcal{E}(u,v) = (\sqrt{-\mathcal{A}}u, \sqrt{-\mathcal{A}}v)_{L^2(E,m)}$$

for $u, v \in \mathcal{F}$, where

$$\mathcal{F} = Dom(\sqrt{-\mathcal{A}}).$$

We highlight that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ can be described by using spectral representation in the following way:

$$\mathcal{E}(u,v) = \int_0^\infty \lambda d(E_\lambda u, v)$$

for $u, v \in \mathcal{F}$ where

$$\mathcal{F} = \left\{ u \in L^2(E,m) : \int_0^\infty \lambda d(E_\lambda u, u) < \infty \right\},$$

We now recall the definition of Mosco convergence (see [17]). We consider a sequence of forms $\{\mathcal{E}^n\}$ with domain $D(\mathcal{E}^n)$ and a form $\{\mathcal{E}\}$ with domain $D(\mathcal{E})$. The forms $\mathcal{E}$, $\mathcal{E}^n$ can be defined in the whole of $L^2(E,m)$ by setting the following:

$$\mathcal{E}(u,u) = +\infty \quad \forall\, u \in L^2(E,m) \setminus D(\mathcal{E}).$$

$$\mathcal{E}^n(u,u) = +\infty \quad \forall\, u \in L^2(E,m) \setminus D(\mathcal{E}^n).$$

Definition 1. *A sequence of forms $\{\mathcal{E}^n\}$ M-converges to a form $\mathcal{E}$ in $L^2(E,m)$ if*

- *For every v_n converging weakly to u in $L^2(E,m)$*

$$\liminf \mathcal{E}^n(v_n, v_n) \geq \mathcal{E}(u,u), \quad as \quad n \to +\infty.$$

- *For every $u \in L^2(E,m)$, there exists u_n converging strongly to u in $L^2(E,m)$, such that*

$$\limsup \mathcal{E}^n(u_n, u_n) \leq \mathcal{E}(u,u), \quad as \quad n \to +\infty.$$

The M-convergence of forms can be characterized in terms of convergence of the resolvent operators and semigroup operators.

Theorem 1 (see [18]). *$\{\mathcal{E}^n\}$ M-converges to $\mathcal{E}$ in $L^2(E,m)$ if and only if the sequence of the resolvent operators $\{G^n_\lambda : \lambda > 0\}$ converges to the resolvent operator G_λ in the strong operator topology of $L^2(E,m)$.*

Theorem 2 (see [18]). *$\{\mathcal{E}^n\}$ M-converges to $\mathcal{E}$ in $L^2(E,m)$ if and only if for every $t > 0$ the sequence $\{T^n_t\}$ of the semigroup operators converges to the semigroup operator T_t associated with the strong operator topology of $L^2(E,m)$ uniformly on every interval $0 < t \leq t_1$.*

3. Symbols and Associated Subordinators

Here, we focus on subordination, which is a time change given by a subordinator. The corresponding semigroup is termed a subordinated semigroup.

We consider the following symbols corresponding to Bernstein functions

$$\Phi(\lambda) = \int_0^\infty \left(1 - e^{-\lambda z}\right)\Pi(dz), \quad \lambda \geq 0 \tag{1}$$

where Π is a Lévy measure on $(0,\infty)$ with $\int_0^\infty (1 \wedge z)\Pi(dz) < \infty$. We also recall that

$$\frac{\Phi(\lambda)}{\lambda} = \int_0^\infty e^{-\lambda z}\overline{\Pi}(z)dz, \qquad \overline{\Pi}(z) = \Pi((z,\infty)) \tag{2}$$

and $\overline{\Pi}$ is the so called tail of the Lévy measure (see [2]). We highlight that the symbol Φ can be related to the Laplace exponent of a subordinator H—that is, a one-dimensional almost surely increasing Lévy process—as follows:

$$\mathbf{E}_0[\exp(-\lambda H_t)] = \exp(-t\Phi(\lambda))$$

(see [1]).

Typical examples are the following:

- $\Phi(\lambda) = \lambda$;
- $\Phi(\lambda) = \lambda^\beta$, $\beta \in (0,1)$, associated with stable subordinator;
- $\Phi(\lambda) = (\lambda + \eta)^\alpha - \eta^\alpha$ with $\eta > 0$ and $\alpha \in (0,1)$ associated with generalized stable subordinator;
- $\Phi(\lambda) = \sigma^{-2}\left(\sqrt{2\lambda\sigma^2 + \mu^2} - \mu\right)$ with $\sigma \neq 0$ associated with inverse Gaussian subordinator;
- $\Phi(\lambda) = a\ln(1 + \lambda/b)$ with $ab > 0$ associated with gamma subordinator.

By using spectral representation, we have

$$\Phi(-\mathcal{A}) = \int_0^\infty \Phi(\lambda)dE_\lambda.$$

For example, if H is a stable subordinator with symbol $\Phi(\lambda) = \lambda^\alpha$ and X is a Brownian motion, then $-\Phi(-\mathcal{A})$ is the fractional Laplacian.

For the process X with generator $(\mathcal{A}, D(\mathcal{A}))$ and the independent subordinator H, we define the time-changed process as follows:

$$X_t^H = X_{H_t} = X \circ H_t$$

for $t \geq 0$.

The process $X_t^H, t \geq 0$ can be considered in order to solve the following equation:

$$\frac{\partial u}{\partial t} = -\Phi(-\mathcal{A})u, \quad u_0 = f \in D(\Phi(-\mathcal{A})) \subset D(\mathcal{A}). \tag{3}$$

as the probabilistic representation of the solutions to (3) is given by

$$u = P_t^H f(x) := \mathbf{E}_x[f(X_t^H)] = \mathbf{E}_x[f({}^*X_t^H), t < \zeta^H]$$

where ζ^H is the lifetime of X^H, which is the part process of ${}^*X^H$ on E.

4. Space Fractional Equations via Convergence of Symbols

We consider a subordinator H with Laplace exponent Φ and subordinators H_n with Laplace exponent Φ_n.

We suppose that the process X is independent of H and H_n, and we define the subordinate processes

$$X^\Phi := X \circ H$$

and

$$X^{\Phi_n} := X \circ H_n.$$

We denote by $(\mathcal{E}^\Phi, \mathcal{F}^\Phi)$ the corresponding Dirichlet form associated with X^Φ and by $(\mathcal{E}^{\Phi_n}, \mathcal{F}^{\Phi_n})$ the Dirichlet form associated with X^{Φ_n}.

By spectral representation, we have

$$\mathcal{F}^\Phi = \left\{ u \in L^2(E, m) : \int_0^\infty \Phi(\lambda) d(E_\lambda u, u) < \infty \right\}$$

and

$$\mathcal{E}^\Phi(u, v) = \int_0^\infty \Phi_n(\lambda) d(E_\lambda u, v)$$

for $u, v \in \mathcal{F}^\Phi$.

In a similar way,

$$\mathcal{F}^{\Phi_n} = \left\{ u \in L^2(E, m) : \int_0^\infty \Phi_n(\lambda) d(E_\lambda u, u) < \infty \right\}$$

and

$$\mathcal{E}^{\Phi_n}(u, v) = \int_0^\infty \Phi_n(\lambda) d(E_\lambda u, v)$$

for $u, v \in \mathcal{F}^{\Phi_n}$.

Thus, the generator of X^Φ is $\mathcal{L}$, where

$$-\mathcal{L} = \int_0^\infty \Phi(\lambda) dE_\lambda$$

with domain

$$Dom(\mathcal{L}) = \left\{ u \in L^2(E, m) : \int_0^\infty \Phi(\lambda)^2 d(E_\lambda u, u) < \infty \right\}$$

and the generator of X^{Φ_n} is $\mathcal{L}_n$, where

$$-\mathcal{L}_n = \int_0^\infty \Phi_n(\lambda) dE_\lambda$$

with domain

$$Dom(\mathcal{L}_n) = \left\{ u \in L^2(E,m) : \int_0^\infty \Phi_n(\lambda)^2 d(E_\lambda u, u) < \infty \right\}.$$

We recall the following results.

Lemma 1 (Lemma 4.2 of [14])**.** *Assume that*

$$\lim_{n\to\infty} \Phi_n(\lambda) = \Phi(\lambda) \quad \text{for every} \quad \lambda \geq 0.$$

Then, the Dirichlet form $(\mathcal{E}^{\Phi_n}, \mathcal{F}^{\Phi_n})$ *M-converges to* $(\mathcal{E}^{\Phi}, \mathcal{F}^{\Phi})$.

For an open subset D of E, we denote by $X^{\Phi_n,D}$ the part process of X^{Φ_n} on D and by $X^{\Phi,D}$ the part process of X^{Φ} on D. $(\mathcal{E}^{\Phi_n}, \mathcal{F}^{\Phi_n,D})$ is the corresponding Dirichlet form associated with the part process $X^{\Phi_n,D}$, and $(\mathcal{E}^{\Phi}, \mathcal{F}^{\Phi,D})$ is the corresponding Dirichlet form associated with part process $X^{\Phi,D}$.

Theorem 3 (Theorem 4.3 of [14])**.** *Assume that*

$$\lim_{n\to\infty} \Phi_n(\lambda) = \Phi(\lambda) \quad \text{for every} \quad \lambda \geq 0.$$

Then, the Dirichlet form $(\mathcal{E}^{\Phi_n}, \mathcal{F}^{\Phi_n,D})$ *M-converges to* $(\mathcal{E}^{\Phi}, \mathcal{F}^{\Phi;D})$.

5. Inverse of Subordinators and Time Fractional Derivatives

We introduce the inverse process L

$$L_t = \inf\{s \geq 0 \,:\, H_s > t\}$$

and define (for $t \geq 0$) the time-changed process

$$X_t^L := X_{L_t} = X \circ L_t.$$

This process is strictly related to the following time fractional equation:

$$\mathfrak{D}_t^\Phi u = \mathcal{A}u, \qquad u_0 = f \in D(\mathcal{A}). \tag{4}$$

Here, the fractional time operator $\mathfrak{D}_t^\Phi$ is defined in the following way. For $M > 0$ and $w \geq 0$, we consider the set $\mathcal{M}_w$ of (piecewise) continuous function on $[0,\infty)$ of exponential order w, such that $|u(t)| \leq Me^{wt}$. We define the operator $\mathfrak{D}_t^\Phi : \mathcal{M}_w \mapsto \mathcal{M}_w$, such that

$$\int_0^\infty e^{-\lambda t} \mathfrak{D}_t^\Phi u(t)\, dt = \Phi(\lambda)\tilde{u}(\lambda) - \frac{\Phi(\lambda)}{\lambda} u(0), \quad \lambda > w$$

where $\tilde{u}$ is the Laplace transform of u.

By using $\overline{\Pi}$ (the tail of the Lévy measure), we can also write the following:

$$\mathfrak{D}_t^\Phi u(t) = \int_0^t u'(s)\overline{\Pi}(t-s)ds.$$

We highlight that operator $\mathfrak{D}_t^\Phi$ was previously considered in [19] (Remark 4.8) as the generalized Caputo derivative. In particular, we observe the following:

- if $\Phi(\lambda) = \lambda$, the operator $\mathfrak{D}_t^\Phi$ becomes the ordinary derivative;

- if $\Phi(\lambda) = \lambda^\beta$, the operator $\mathfrak{D}_t^\Phi$ becomes the Caputo fractional derivative

$$\mathfrak{D}_t^\Phi u(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{u'(s)}{(t-s)^\beta} ds$$

with $u'(s) = du/ds$;
- if $\Phi(\lambda) = (\lambda + \eta)^\alpha - \eta^\alpha$, the operator $\mathfrak{D}_t^\Phi$ becomes tempered fractional derivative;
- if $\Phi(\lambda) = \lambda^{2\beta} + \lambda^\beta$ for $\beta \in (0, 1/2)$, the operator $\mathfrak{D}_t^\Phi$ becomes the telegraph fractional operator.

The probabilistic representation of the solution of time fractional Equation (4) is given by the following:

$$u(t, x) = \mathbf{E}_x[f(X_t^L)] = \mathbf{E}_x[f({}^*X_t^L), t < \zeta^L] \tag{5}$$

where ζ^L is the lifetime of X^L, the part process of ${}^*X^L$ on E. In particular, the following theorem states the existence and uniqueness of a strong solution in $L^2(E, m)$ to (4) (see [12,20,21]).

Theorem 4 ([12] (Theorem 5.2))**.** *The function* (5) *is the unique strong solution in* $L^2(E, m)$ *to* (4) *in the sense that:*

1. $\varphi : t \mapsto u(t, \cdot)$ *is such that* $\varphi \in C([0, \infty), \mathbb{R}_+)$ *and* $\varphi' \in \mathcal{M}_0$;
2. $\vartheta : x \mapsto u(\cdot, x)$ *is such that* $\vartheta, \mathcal{A}\vartheta \in D(\mathcal{A})$;
3. $\forall\, t > 0, \mathfrak{D}_t^\Phi u(t, x) = \mathcal{A}u(t, x)$ *holds m-a.e in E;*
4. $\forall\, x \in E, u(t, x) \to f(x)$ *as* $t \downarrow 0$.

6. Time–Space Fractional Equations via Convergence of Symbols

As in Section 4, we consider symbols Φ and Φ_n and their corresponding subordinators, H and H_n. We assume that the process X is independent of H and H_n and consider the subordinate processes

$$X^{\Phi_n} := X \circ H_n$$

and

$$X^\Phi := X \circ H.$$

Moreover, we consider a symbol Ψ and the corresponding inverse of its associated subordinator denoted again by L. We examine the following time–space fractional equations

$$\mathfrak{D}_t^\Psi u = -\Phi_n(-\mathcal{A})u, \quad u_0 = f \in D(\Phi_n(-\mathcal{A})) \subset D(\mathcal{A}). \tag{6}$$

The probabilistic representation of the solution can be written in terms of the time-changed process $(X^{\Phi_n})^L$, that is,

$$u_n(t, x) = \mathbf{E}_x[f((X_t^{\Phi_n})^L)]. \tag{7}$$

Similarly, the probabilistic representation of the solution to

$$\mathfrak{D}_t^\Psi u = -\Phi(-\mathcal{A})u, \quad u_0 = f \in D(\Phi(-\mathcal{A})) \subset D(\mathcal{A}), \tag{8}$$

can be written in terms of the time-changed process $(X^\Phi)^L$, that is,

$$u(t, x) = \mathbf{E}_x[f((X_t^\Phi)^L)]. \tag{9}$$

$\mathbb{D}$ is the set of continuous functions from $[0, \infty)$ to $E_\partial = E \cup \partial$, which are right continuous on $[0, \infty)$ with left limits on $(0, \infty)$, where ∂ is the cemetery point. In the following Theorem 5, we prove the asymptotic results for space–time fractional equations via the convergence of symbols.

Theorem 5. *Assume that*

$$\lim_{n\to\infty} \Phi_n(\lambda) = \Phi(\lambda) \qquad \text{for every} \qquad \lambda \geq 0.$$

Then

$$(X_t^{\Phi_n})^L \to (X_t^{\Phi})^L \quad \text{in distribution in} \quad \mathbb{D}.$$

Proof. As

$$\lim_{n\to\infty} \Phi_n(\lambda) = \Phi(\lambda) \qquad \text{for every} \qquad \lambda \geq 0$$

by Lemma 1, we observed that $(\mathcal{E}^{\Phi_n}, \mathcal{F}^{\Phi_n})$ M-converges to $(\mathcal{E}^{\Phi}, \mathcal{F}^{\Phi})$.

By using the results of a recent paper [12], we found the convergence of the time-changed processes from the M-convergence of the forms $\mathcal{E}^n(\cdot,\cdot)$. More precisely, from the M-convergence of the forms, $\mathcal{E}^n(\cdot,\cdot)$ we found the strong convergence of the corresponding semigroups. Then, by using Theorem 17.25 in [22] by means of which we know that the strong convergence of semigroups (Feller semigroups) is equivalent to the weak convergence of measures if $X_0^{\Phi_n} \to X_0^{\Phi}$ in distribution, we found that $X_t^{\Phi_n} \to X_t^{\Phi}$ in distribution in $\mathbb{D}$.

By using results in [23], we found that as $n \to \infty$

$$(X_t^{\Phi_n})^L \to (X_t^{\Phi})^L$$

in distribution in $\mathbb{D}$. □

As in Section 4, D is an open subset of E. We use $X^{\Phi_n,D}$ to denote the part process of X^{Φ_n} on D and $X^{\Phi,D}$ to denote the part process of X^{Φ} on D.

Theorem 6. *Assume that*

$$\lim_{n\to\infty} \Phi_n(\lambda) = \Phi(\lambda) \qquad \text{for every} \qquad \lambda \geq 0.$$

Then

$$(X_t^{\Phi_n,D})^L \to (X_t^{\Phi,D})^L \quad \text{in distribution in} \quad \mathbb{D}.$$

Proof. By Theorem 3, following the same tools of previous proof, we obtained the result. □

Remark 1. *We remark that a probabilistic interpretation in terms of the mean lifetime of the base and time-changed processes was given recently in [24].*

7. Examples and Applications

In this section, we present some examples to illustrate the main results of this paper. First, we consider the case where X is a Brownian motion in $\mathbb{R}^d$ running twice as fast as the standard Brownian motion. We consider the following relativistic $(2\alpha_n)$-stable process in $\mathbb{R}^d$ with $\alpha_n \in (0,1]$

$$\Phi_n(\lambda) = \frac{(\lambda + \gamma_n)^{\alpha_n} - {\gamma_n}^{\alpha_n}}{\alpha_n}$$

where (α_n, γ_n) is a sequence in $(0,1] \times [0,\infty)$. When $\gamma_n > 0$, the time changed process X^{Φ_n} is a relativistic 2α-stable process in $\mathbb{R}^d$.

If (α_n, γ_n) converges to some (α, γ) in $(0,1] \times (0,\infty)$ as $n \to \infty$ we find that $\Phi_n(\lambda)$ tends to a relativistic 2α-stable process in $\mathbb{R}^d$.

If (α_n, γ_n) converges to some $(\alpha, 0)$ with $\alpha \in (0,1]$ as $n \to \infty$ we find that $\Phi_n(\lambda)$ tends to a spherical symmetric 2α-stable process in $\mathbb{R}^d$.

If (α_n, γ_n) converges to some $(0, \gamma)$ with $\gamma \in (0, \infty)$ as $n \to \infty$ we find that $\Phi_n(\lambda)$ tends to

$$\Phi(\lambda) = \ln\left(\frac{\lambda}{\gamma} + 1\right)$$

which is related to a gamma process.

Then, by using the results of the previous sections, we can approximate space–time fractional equations related to relativistic 2α-stable processes, spherical symmetric 2α-stable processes, and gamma processes with suitable sequences of relativistic 2α-stable processes in $\mathbb{R}^d$.

We highlight that other examples can be given in a similar way by replacing X with another kind of symmetric process, such as a spherically symmetric α-stable process, symmetric Lévy process, and symmetric diffusions with infinitesimal generators of divergence form.

Another interesting example is the following. Consider the sequences of symbols

$$\Phi_n(\lambda) = \beta_n \ln\left(\frac{\lambda}{\delta_n} + 1\right)$$

that are related to a gamma processes with parameters $\beta_n \to \infty$ and $\delta_n \to \infty$. The parameters β_n, δ_n can be related to the plot of the observed data that may fit the path of a realization of the Laplace motion (also termed variance gamma processes, that is, the Brownian motion time-changed by a gamma subordinator).

If the parameters β_n, δ_n characterizing the phenomenon satisfy

$$\frac{\beta_n}{\delta_n} \to c$$

as $n \to \infty$, we find that $\Phi_n(\lambda)$ tends to

$$\Phi(\lambda) = c\lambda$$

as $n \to \infty$; that is we have a sort of "denoising", and the underline (base) process appears.

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