

Quantitative ergodicity for the symmetric exclusion process with stationary initial data

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Abstract

We consider the symmetric exclusion process on the d -dimensional lattice with initial data invariant with respect to space shifts and ergodic. It is then known that as t diverges the distribution of the process at time t converges to a Bernoulli product measure. Assuming a summable decay of correlations of the initial data, we prove a quantitative version of this convergence by obtaining an explicit bound on the Ornstein \bar{d} -distance. The proof is based on the analysis of a two species exclusion process with annihilation.

Keywords: exclusion process; Ornstein distance; speed of convergence to equilibrium.

MSC2020 subject classifications: 60K35; 82C20.

Submitted to ECP on May 31, 2021, final version accepted on August 7, 2021.

Supersedes arXiv:2101.02487.

1 Introduction

The analysis of the speed of the convergence to equilibrium for Markov processes is a major topic in probability theory. Referring to [15] for a general overview, we focus the discussion to the case of reversible stochastic lattice gases, i.e. conservative interacting particles systems satisfying the detailed balance condition with respect to a Gibbs measure. If these processes are considered on a bounded subset Λ of the d -dimensional lattice they are ergodic when restricted to the configurations with fixed number of particles and the corresponding reversible measure is the finite volume canonical Gibbs measure. In the high temperature regime, in [7, 8, 9, 10, 19] it has been shown that both the inverse of the spectral gap and the logarithmic Sobolev constant grow as the square of the diameter of Λ . On the infinite lattice, stochastic lattice gases are reversible with respect to the (infinite volume) canonical Gibbs measures, see [11, Thm. 2.14]. In the high temperature regime, by [11, Thm. 5.14], the extremal elements of the set of the canonical Gibbs measures consist in the one parameter family $\{\pi_\rho\}$ where π_ρ is the grand-canonical Gibbs measure with density, i.e. expected number of particles per site, given by ρ . Moreover, as follows from [11, Thm. 1.72], the semigroup P_t , $t \geq 0$ associated to a reversible stochastic lattice gases is ergodic in $L_2(\pi_\rho)$ namely,

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$\|P_t f - \pi_\rho(f)\|_{L_2(\pi_\rho)} \rightarrow 0$ for each $f \in L_2(\pi_\rho)$. A quantitative version of this statement can be obtained when the function f is local, i.e. it depends on the particles configuration only through its value on finitely many sites. For this class of functions it has been shown, for the exclusion and the zero range processes, that $\|P_t f - \pi_\rho(f)\|_{L_2(\pi_\rho)}^2 \leq C t^{-d/2}$ for some constant $C = C(f)$ [3, 12]. The case in which the reversible probability is a grand-canonical Gibbs measure in the high temperature regime is discussed in [4, 6, 14] where a slightly worse bound is proven.

We here consider the simple symmetric exclusion process. It corresponds to the infinite temperature case and the probability measure π_ρ is the product Bernoulli measure with parameter $\rho \in [0, 1]$. If the probability μ is a suitable local perturbation of π_ρ it has been proven in [3] that $\text{Ent}(\mu P_t | \pi_\rho) \leq C t^{-d/2}$ for some constant $C = C(\mu)$, here Ent denotes the relative entropy. See also [16] for further details on this issue. In general, it appears to be quite difficult to characterize the probabilities μ on the configuration space such that μP_t converges to π_ρ as $t \rightarrow \infty$. However, as proven in [18, Thm. VIII.1.47], such convergence holds when μ is stationary, i.e. invariant with respect to space shifts, and ergodic with density ρ . Our purpose is to provide a quantitative version of this statement. More precisely, denoting by \bar{d} the Ornstein distance on the set of stationary probabilities [22, § I.9.b], we prove here that if μ is stationary, ergodic with density ρ , and has absolutely summable correlations, then $\bar{d}(\mu P_t, \pi_\rho) \leq C t^{-\gamma(d)}$ for some constant $C = C(\mu)$ and $\gamma(d) = d/4$ for $d < 4$ and $\gamma(d) = 1$ for $d \geq 4$; see Theorem 2.1 below. The proof is achieved by combining a simple coupling argument with the analysis on the decay of the density for the two-species symmetric exclusion process with annihilation [1, 2], that relies on an analogous result for the two-species independent random walks [5].

Referring to [5] for more details, we next explain heuristically the power law decay of the Ornstein \bar{d} distance. By [22, Thm. I.9.7] the \bar{d} distance between μP_t and π_ρ can be bounded using a coupling between μP_t and π_ρ invariant with respect to space shifts: $\bar{d}(\mu P_t, \pi_\rho) \leq \mathbb{P}(\eta_0(t) \neq \zeta_0(t))$. Here $(\eta_0(t), \zeta_0(t))$ are the occupation numbers at the origin at time $t \geq 0$ of a two-species annihilating exclusion process with equal density. Namely, two species of particles that evolve on \mathbb{Z}^d according to exclusion processes and annihilating when they meet. Let $\rho(t) = \mathbb{P}(\eta_0(t) \neq \zeta_0(t))$ be the probability that the origin is occupied by either species of particles. In the mean field approximation, $\rho(t)$ decays to zero according to $\dot{\rho}(t) = -\rho(t)^2$ which would imply $\bar{d}(\mu P_t, \pi_\rho) \approx t^{-1}$. This approximation yields the correct behavior when $d \geq 4$ while for $d \leq 3$ the Gaussian fluctuations of the initial data become relevant and, due to the underlying particle's diffusion, the decay becomes $\bar{d}(\mu P_t, \pi_\rho) \approx t^{-d/4}$.

To our knowledge, the present analysis of the symmetric exclusion process is the first example in which the quantitative ergodicity for a stochastic lattice gas with stationary initial data has been achieved. The arguments here developed cover directly the case of independent random walks. We conclude by discussing possible extensions to other models. As mentioned before, the crucial ingredient in the proof is the quantitative decay of the density for the two-species exclusion process with annihilation. This decay might be proven for other attractive stochastic lattice gases such as the zero range process with increasing rates, see e.g. [13, Thm. 2.5.2], or the special class of reversible stochastic lattice gases in [17, § 4.1]. Another simple model for which the quantitative ergodicity could be investigated is the inclusion process (SIP), indeed this model is self-dual and a coupling with independent random walks has been constructed in [21]. For the generic case of reversible stochastic lattice gases where the invariant measure is not a product measure, it seems however difficult that coupling arguments suffice to establish the quantitative ergodicity, cfr. the corresponding problem of the decay to equilibrium for local functions in [4, 6, 12, 14]. We remark that another possible setting

to discuss the quantitative ergodicity for stochastic lattice gases with stationary initial data μ is the decay rate of the relative entropy per site of μP_t with respect to π_ρ . In view of [20], such decay would imply a quantitative decay on the \bar{d} distance between μP_t and π_ρ .

2 Notation and results

Let \mathbb{Z}^d be the d -dimensional lattice. We write $\Lambda \subset\subset \mathbb{Z}^d$ when Λ is a finite subset of \mathbb{Z}^d . Set $\Omega := \{0, 1\}^{\mathbb{Z}^d}$ that it is considered endowed with the product topology and the corresponding Borel σ -algebra. Elements of \mathbb{Z}^d will be called sites while elements of Ω configurations. For $\eta \in \Omega$ the value of the configuration η at the site x , denoted by $\eta_x \in \{0, 1\}$, is interpreted as the absence/presence of a particle at x and called occupation number. In particular, η_0 is the occupation number at the origin of \mathbb{Z}^d .

The simple symmetric exclusion process (SEP) is the Markov process on the state space Ω whose generator acts on local functions $f: \Omega \rightarrow \mathbb{R}$, i.e. functions depending on η only through the values $\{\eta_x\}$ for finitely many sites x , as

$$Lf(\eta) = \sum_{\langle x,y \rangle} [f(\eta^{x,y}) - f(\eta)]. \tag{2.1}$$

The sum is carried out over the unordered edges of \mathbb{Z}^d and $\eta^{x,y}$ is the configuration obtained from η by exchanging the occupation numbers at x and y ,

$$\eta_z^{x,y} := \begin{cases} \eta_y & \text{if } z = x, \\ \eta_x & \text{if } z = y, \\ \eta_z & \text{otherwise.} \end{cases} \tag{2.2}$$

We denote by $P_t, t \geq 0$, the semigroup generated by L that acts on the Banach space $\mathcal{C}(\Omega)$, the family of continuous function on Ω endowed with the uniform norm. We refer to [18, Ch. VIII] for the construction of this process and its properties. In Theorem 1.44 there, it is proven in particular that a probability μ on Ω is invariant for SEP if and only if μ is exchangeable, equivalently μ is a mixture, i.e. a possibly infinite convex combination, of i.i.d. Bernoulli measures.

Let $\mathcal{P}_\tau(\Omega)$ be the set of *stationary* probabilities on Ω , i.e. the probabilities on Ω invariant with respect to the space shifts on Ω . Observe that $\mathcal{P}_\tau(\Omega)$ is a convex set and the set of its extremal points, denoted by $\mathcal{P}_{\tau,e}(\Omega)$, consists of the ergodic probabilities. For $\rho \in [0, 1]$ let $\pi_\rho \in \mathcal{P}_\tau(\Omega)$ the Bernoulli product probability with parameter ρ . In [18, Thm. VIII.1.47] it is proven that if $\mu \in \mathcal{P}_{\tau,e}(\Omega)$ and $\mu(\eta_0) = \rho$ then μP_t weakly converges to π_ρ as $t \rightarrow +\infty$. The purpose of the present analysis is to provide a quantitative version of this statement with an explicit control on the rate of convergence. This will be achieved when the probability μ has absolutely summable correlations.

To formulate the quantitative ergodicity we need a distance on $\mathcal{P}_\tau(\Omega)$. We shall use the so-called Ornstein \bar{d} distance. Given $\Lambda \subset\subset \mathbb{Z}^d$ let d_Λ be the distance on $\Omega_\Lambda := \{0, 1\}^\Lambda$ defined by

$$d_\Lambda(\eta, \zeta) := \sum_{x \in \Lambda} |\eta_x - \zeta_x|.$$

Denoting by $\mathcal{P}(\Omega_\Lambda)$ the set of probabilities on Ω_Λ , let W_Λ be the 1-Wasserstein distance on $\mathcal{P}(\Omega_\Lambda)$ associated to d_Λ , i.e.

$$W_\Lambda(\mu, \nu) := \inf_Q \int Q(d\eta, d\zeta) d_\Lambda(\eta, \zeta),$$

where the infimum is carried out over all the *couplings* Q of μ and ν , i.e. the set of probabilities on $\Omega \times \Omega$ with marginals μ and ν . For $\mu \in \mathcal{P}_\tau(\Omega)$ and $\Lambda \subset \mathbb{Z}^d$ let μ_Λ be the marginal of μ on Ω_Λ . By a standard super-additive argument, if $\mu, \nu \in \mathcal{P}_\tau(\Omega)$ then

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} W_\Lambda(\mu_\Lambda, \nu_\Lambda) = \sup_{\Lambda \subset \mathbb{Z}^d} \frac{1}{|\Lambda|} W_\Lambda(\mu_\Lambda, \nu_\Lambda) =: \bar{d}(\mu, \nu). \tag{2.3}$$

Moreover, see e.g. [22, Thm. I.9.7], \bar{d} defines a distance on $\mathcal{P}_\tau(\Omega)$ that can be represented as

$$\bar{d}(\mu, \nu) = \inf_Q \int Q(d\eta, d\zeta) |\eta_0 - \zeta_0| \tag{2.4}$$

where we recall that η_0, ζ_0 are the occupation numbers at the origin and the infimum is carried out over all the *stationary couplings* Q of μ and ν , i.e. the set of couplings of μ and ν that are invariant with respect to space shifts on $\Omega \times \Omega$. By (2.3), the topology induced by \bar{d} on $\mathcal{P}_\tau(\Omega)$ is finer than the topology induced by the weak convergence. Denoting by $\text{ent}(\mu|\nu)$ the relative entropy *per unit of volume* of μ with respect to ν , we finally mention two remarkable properties of the Ornstein \bar{d} distance, see e.g. [22, Thm. I.9.15 and I.9.16]: (i) $\mathcal{P}_{\tau,e}(\Omega)$ is \bar{d} closed, (ii) for each $\rho \in [0, 1]$ the map $\mathcal{P}_{\tau,e}(\Omega) \ni \mu \mapsto \text{ent}(\mu|\pi_\rho)$ is \bar{d} continuous.

Given two functions f, g and a probability μ we let $\mu(f; g) := \mu(fg) - \mu(f)\mu(g)$ be the μ -covariance of f and g . For $\mu \in \mathcal{P}_{\tau,e}(\Omega)$ we set

$$A(\mu) := \sum_{x \in \mathbb{Z}^d} |\mu(\eta_0; \eta_x)|. \tag{2.5}$$

The quantitative ergodicity for SEP with stationary initial data is then stated as follows.

Theorem 2.1. *For each d there exists a constant C such that for any $t > 0, \rho \in [0, 1]$, and $\mu \in \mathcal{P}_{\tau,e}(\Omega)$ satisfying $\mu(\eta_0) = \rho$,*

$$\bar{d}(\mu P_t, \pi_\rho) \leq C \frac{\sqrt{A(\mu)}}{t^{\gamma(d)}} \tag{2.6}$$

where $\gamma(d) = d/4$ if $d \leq 4$ and $\gamma(d) = 1$ for $d > 4$.

The above results implies that if $\mu \in \mathcal{P}_{\tau,e}(\Omega)$, $\mu(\eta_0) = \rho$, and $A(\mu) < +\infty$ then μP_t converges to π_ρ in the topology induced by the \bar{d} distance. In particular, by remark (ii) above, $\text{ent}(\mu P_t|\pi_\rho) \rightarrow 0$. It is unclear to us whether this statement holds without the condition $A(\mu) < +\infty$.

As we next argue, there exist probabilities $\mu \in \mathcal{P}_{\tau,e}(\Omega)$ such that $\bar{d}(\mu P_t, \pi_\rho)$ decays to zero arbitrarily slow as $t \rightarrow +\infty$. Fix two sites $x \neq y$. By (2.3) it is enough to exhibit $\mu \in \mathcal{P}_{\tau,e}(\Omega)$ with $\mu(\eta_0) = \rho$ such that $(\mu P_t)(\eta_x = 1, \eta_y = 1)$ converges to ρ^2 arbitrarily slow. By the so-called self-duality of SEP, see [18, Cor. VIII.1.3],

$$(\mu P_t)(\eta_x = 1, \eta_y = 1) = \mathbf{E}_{(x,y)} \mu(\eta_{X(t)} = 1, \eta_{Y(t)} = 1),$$

where $(X(t), Y(t)), t \geq 0$, are two particles in exclusion starting from (x, y) . Since correlations of μ can decay arbitrarily slow, we deduce that as $t \rightarrow +\infty$ $(\mu P_t)(\eta_x = 1, \eta_y = 1)$ converges to ρ^2 arbitrarily slow.

3 Reduction to the two species SEP with annihilation

In view of (2.4), an upper bound for $\bar{d}(\mu P_t, \pi_\rho)$ can be obtained by exhibiting a stationary coupling between μP_t and π_ρ . Starting a time $t = 0$ by a stationary coupling of μ and π_ρ and coupling the corresponding two SEP we obtain, at time $t > 0$, a stationary

coupling between μP_t and π_ρ good enough to produce the bound (2.6). In words, the coupling between the two SEP can be described as follows. Particles of the two processes that are at the same site jump together while particles alone jump independently. In formulae, we consider the Markov process whose generator is the closure of the operator \tilde{L} that acts on local functions $F: \Omega \times \Omega \rightarrow \mathbb{R}$ as

$$\begin{aligned} \tilde{L}F(\eta, \zeta) = & \sum_{\langle x,y \rangle} \left\{ (1 - \mathbf{1}_{\eta_x \neq \zeta_x, \eta_y \neq \zeta_y}) [F(\eta^{x,y}, \zeta^{x,y}) - F(\eta, \zeta)] \right. \\ & \left. + \mathbf{1}_{\eta_x \neq \zeta_x, \eta_y \neq \zeta_y} [F(\eta^{x,y}, \zeta) + F(\eta, \zeta^{x,y}) - 2F(\eta, \zeta)] \right\}. \end{aligned} \tag{3.1}$$

The corresponding semigroup is denoted by $\tilde{P}_t, t \geq 0$. Note that, even if not apparent from the notation, \tilde{L} is the operator used in [18, § VIII.2] to prove the attractiveness of the exclusion process. The next statement can be checked by a direct computation that is omitted.

Lemma 3.1. *Let $(\eta(t), \zeta(t)), t \geq 0$ be the Markov process generated by \tilde{L} . Then $\xi(t) := \{\eta_x(t) - \zeta_x(t), x \in \mathbb{Z}^d\}, t \geq 0$ is a Markov process with state space $\mathcal{S} = \{-1, 0, 1\}^{\mathbb{Z}^d}$ and generator \mathcal{L} that acts on the local functions $f: \mathcal{S} \rightarrow \mathbb{R}$ as*

$$\mathcal{L}f(\xi) = \sum_{\langle x,y \rangle} \left\{ \mathbf{1}_{\xi_x \xi_y \neq -1} [f(\xi^{x,y}) - f(\xi)] + 2 \mathbf{1}_{\xi_x \xi_y = -1} [f(\xi^{x,y;\dagger}) - f(\xi)] \right\} \tag{3.2}$$

where $\xi^{x,y}$ has been defined in (2.2) and

$$\xi_z^{x,y;\dagger} := \begin{cases} 0 & \text{if } z \in \{x, y\}, \\ \xi_z & \text{otherwise.} \end{cases}$$

The process generated by (3.2), that will be referred to as the *two species SEP with annihilation*, can be described by visualizing the sites x where $\xi_x = -1$ as occupied by anti-particles and the sites x where $\xi_x = 1$ occupied by particles. Particles and anti-particles evolve following two independent SEP and when a particle jumps over a anti-particle, or conversely a anti-particle jumps over a particle, the two particles are annihilated. It can be therefore seen as kinetic model corresponding to the reaction anti-particle + particle $\mapsto \emptyset$. As we show in the next section, an analysis of this dynamics yields an upper bound for the Ornstein distance between two SEP with different initial data.

4 Long time behavior of the two species SEP with annihilation

In this section we consider the two species SEP with annihilation obtaining – for suitable stationary initial data – an upper bound for the probability that at time $t > 0$ the origin is occupied by either types of particles. Given a probability \wp on \mathcal{S} the law of the two species SEP with annihilation, i.e. the process generated by (3.2), and initial datum \wp is denoted by \mathbb{P}_\wp , the corresponding expectation by \mathbb{E}_\wp . For $\wp \in \mathcal{P}_{\tau,e}(\mathcal{S})$, the set of stationary and ergodic probabilities on \mathcal{S} , we set

$$B(\wp) := \sum_{x \in \mathbb{Z}^d} \sum_{\alpha, \beta \in \{-1, 1\}} |\wp(\xi_0 = \alpha; \xi_x = \beta)| \tag{4.1}$$

where $\wp(\xi_0 = \alpha; \xi_x = \beta) := \wp(\xi_0 = \alpha, \xi_x = \beta) - \wp(\xi_0 = \alpha)\wp(\xi_x = \beta)$.

Theorem 4.1. *For each d there exists a constant C such that for any $t > 0$ and any $\wp \in \mathcal{P}_{\tau,e}(\mathcal{S})$ satisfying $\wp(\xi_0 = -1) = \wp(\xi_0 = 1)$*

$$\mathbb{E}_\wp |\xi_0(t)| \leq C \frac{\sqrt{B(\wp)}}{t^{\gamma(d)}}, \tag{4.2}$$

where $\gamma(d) = d/4$ if $d \leq 4$ and $\gamma(d) = 1$ for $d > 4$.

The analogous statement for two species annihilating independent random walks and stationary product initial condition has been proven in [5]. Relying on the arguments there, the bound (4.2) is proven in [1, 2] when the initial datum φ is a product measure. This assumption on the initial datum is used only in [1, Lemma 2.1]; however, as we show in Lemma 4.2 below, it can be relaxed to the condition $B(\varphi) < +\infty$. The rest of the arguments in [1, 2] carries out unchanged to the present setting and yields the statement in Theorem 4.1. Assuming it, we first conclude the proof of the quantitative ergodicity for the SEP with stationary initial data.

Proof of Theorem 2.1. Given $\mu \in \mathcal{P}_{\tau,e}(\Omega)$ with $\mu(\eta_0) = \rho$, let $\varphi \in \mathcal{P}_{\tau}(\mathcal{S})$ be the law of $\eta - \zeta$ where η and ζ are independently sampled from μ and π_{ρ} . By (2.5) and (4.1), a direct computation yields

$$\begin{aligned} B(\varphi) &\leq \sum_{\alpha \in \{-1,1\}} [1 - \rho(1 - \rho)] \left| \mu(\eta_0 = \frac{1+\alpha}{2}; \eta_0 = \frac{1+\alpha}{2}) \right| \\ &\quad + \sum_{\alpha \neq \beta \in \{-1,1\}} \rho(1 - \rho) \left| \mu(\eta_0 = \frac{1+\alpha}{2}; \eta_0 = \frac{1+\beta}{2}) \right| \\ &\quad + \sum_{x \neq 0} \sum_{\alpha, \beta \in \{-1,1\}} \pi_{\rho}(\zeta_0 = \frac{1-\alpha}{2}, \zeta_x = \frac{1-\beta}{2}) \left| \mu(\eta_0 = \frac{1+\alpha}{2}; \eta_x = \frac{1+\beta}{2}) \right| \\ &\leq A(\mu). \end{aligned}$$

Since the process $(\eta(t), \zeta(t))$, $t \geq 0$ couples two SEP, the probability $(\mu \otimes \pi_{\rho})\tilde{P}_t$ is a coupling of μP_t and $\pi_{\rho} P_t = \pi_{\rho}$. Here P_t and \tilde{P}_t , $t \geq 0$, are the semigroups associated to SEP and (3.1), respectively. Moreover, as the probability $\mu \otimes \pi_{\rho}$ on $\Omega \times \Omega$ is invariant with respect to space shifts, $(\mu \otimes \pi_{\rho})\tilde{P}_t$ is a stationary coupling of μP_t and π_{ρ} . According to Lemma 3.1, $\xi(t) = \eta(t) - \zeta(t)$ is distributed as the two-species SEP with annihilation, i.e. the process generated by (3.2), whose law is denoted by \mathbb{E}_{φ} . Hence, by (2.4) and Theorem 4.1,

$$\bar{d}(\mu P_t, \pi_{\rho}) \leq \mathbb{E}_{\varphi} |\xi_0(t)| \leq C \frac{\sqrt{A(\mu)}}{t^{\gamma(d)}}$$

for some constant C depending only on d . □

In order to extend the result in [1, 2] to non-product initial data, we need to realize the two species SEP with annihilation on the probability space associated to the so-called stirring process. This construction is achieved in two steps: from two independent stirring processes we first obtain the two species SEP without annihilation then, by a thinning procedure, we construct the the two species SEP with annihilation.

We start by recalling the graphical construction of the stirring process. To each site $x \in \mathbb{Z}^d$ attach a copy of the positive half-axis \mathbb{R}_+ . For each edge $\langle x, y \rangle$ draw a set of double-arrows sampled according to independent Poisson point processes with intensity one. The stirring process $W = \{W_x(t), x \in \mathbb{Z}^d, t \in \mathbb{R}_+\}$ is defined as follows: the value $W_x(t) \in \mathbb{Z}^d$ is obtained by placing a marker at time $t = 0$ at the point x and letting it evolve following the path dictated by the arrows. Given $\zeta \in \{0, 1\}^{\mathbb{Z}^d}$ the SEP with initial datum ζ can be realized as $\eta_x(t) = \sum_{y \in \mathbb{Z}^d} \zeta_y \mathbf{1}_{\{x\}}(W_y(t))$, $x \in \mathbb{Z}^d$. Let finally $\mathbf{W} = (W^-, W^+)$ be two independent copies of the stirring process.

The *two species SEP without annihilation* can be described as follows. Each site can be: empty, occupied by a particle, occupied by a anti-particle, or occupied by both a particle and an anti-particle. The anti-particles evolve according to the stirring process W^- while particles according to W^+ . Setting $\tilde{\mathcal{S}} := \{0, -1, +1, \pm\}^{\mathbb{Z}^d}$, the two species SEP

without annihilation is thus the process on the state space $\tilde{\mathcal{S}}$ defined by

$$\tilde{\xi}_x(t) = \begin{cases} -1 & \text{if } \exists y \in \mathbb{Z}^d: \tilde{\zeta}_y \in \{-1, \pm\} \text{ and } W_y^-(t) = x, \\ & \quad \nexists z \in \mathbb{Z}^d: \tilde{\zeta}_z \in \{1, \pm\} \text{ and } W_z^+(t) = x, \\ +1 & \text{if } \nexists y \in \mathbb{Z}^d: \tilde{\zeta}_y \in \{-1, \pm\} \text{ and } W_y^-(t) = x, \\ & \quad \exists z \in \mathbb{Z}^d: \tilde{\zeta}_z \in \{1, \pm\} \text{ and } W_z^+(t) = x, \\ \pm & \text{if } \exists y \in \mathbb{Z}^d: \tilde{\zeta}_y \in \{-1, \pm\} \text{ and } W_y^-(t) = x, \\ & \quad \exists z \in \mathbb{Z}^d: \tilde{\zeta}_z \in \{1, \pm\} \text{ and } W_z^+(t) = x, \\ 0 & \text{otherwise.} \end{cases} \quad (4.3)$$

Then $\tilde{\xi}(t)$, $t \geq 0$, is the Markov process whose generator $\tilde{\mathcal{L}}$ acts on local functions $f: \tilde{\mathcal{S}} \rightarrow \mathbb{R}$ as

$$\begin{aligned} \tilde{\mathcal{L}}f(\tilde{\xi}) = & \sum_{(x,y)} \left\{ \sum_{\alpha \in \{-1, +1\}} \sum_{\beta \in \{0, \pm\}} \mathbf{1}_{\{\alpha\}}(\tilde{\xi}_x) \mathbf{1}_{\{\beta\}}(\tilde{\xi}_y) [f(\tilde{\xi}^{x,y}) - f(\tilde{\xi})] \right. \\ & + \mathbf{1}_{\{-1\}}(\tilde{\xi}_x) \mathbf{1}_{\{+1\}}(\tilde{\xi}_y) [f(\tilde{\xi}^{x,y;\pm,0}) + f(\tilde{\xi}^{x,y;0,\pm}) - 2f(\tilde{\xi})] \\ & \left. + \mathbf{1}_{\{0\}}(\tilde{\xi}_x) \mathbf{1}_{\{\pm\}}(\tilde{\xi}_y) [f(\tilde{\xi}^{x,y;+1,-1}) + f(\tilde{\xi}^{x,y;-1,+1}) - 2f(\tilde{\xi})] \right\}, \end{aligned} \quad (4.4)$$

where the leftmost sum is carried out over the set of *oriented* edges of \mathbb{Z}^d , $\tilde{\xi}^{x,y}$ has been defined in (2.2) and, given $\alpha, \beta \in \{0, -1, +1, \pm\}$,

$$(\tilde{\xi}^{x,y;\alpha,\beta})_z := \begin{cases} \alpha & \text{if } z = x, \\ \beta & \text{if } z = y, \\ \tilde{\xi}_z & \text{otherwise.} \end{cases}$$

Given a probability $\tilde{\varphi}$ on $\tilde{\mathcal{S}}$ we denote by $\tilde{\mathbb{P}}_{\tilde{\varphi}}$ the law of this process with initial condition $\tilde{\varphi}$ and by $\tilde{\mathbb{E}}_{\tilde{\varphi}}$ the corresponding expectation.

The two species SEP with annihilation $\xi(t)$, $t \geq 0$, can be finally realized by a *thinning* of two species SEP without annihilation by recursively removing pairs of particles of different species that occupy the same site. This thinning procedure provides a coupling of the processes $\xi(t)$ and $\tilde{\xi}(t)$ such that for any $t \geq 0$ and $\alpha \in \{-1, 1\}$ we have $\{x \in \mathbb{Z}^d: \xi_t(x) = \alpha\} \subset \{x \in \mathbb{Z}^d: \tilde{\xi}_t(x) = \alpha\}$ with probability one.

Given $\Lambda \subset \subset \mathbb{Z}^d$ and $\alpha \in \{-1, 1\}$ we set $N_{\Lambda,\alpha}(\xi) := \sum_{x \in \Lambda} \mathbf{1}_{\{\alpha\}}(\xi_x)$. Namely, $N_{\Lambda,-1}$ and $N_{\Lambda,1}$ are respectively the number of anti-particles and the number particles in Λ . The same notation is used for $\tilde{\xi} \in \tilde{\mathcal{S}}$. In the next statement we regard $\varphi \in \mathcal{P}_{\tau}(\mathcal{S})$ as a stationary probability on \mathcal{S} .

Lemma 4.2. *Let B as defined in (4.1). Then for each $\Lambda \subset \subset \mathbb{Z}^d$, $t \geq 0$, and $\varphi \in \mathcal{P}_{\tau}(\mathcal{S})$ such that $\varphi(\xi_0 = -1) = \varphi(\xi_0 = 1)$,*

$$\tilde{\mathbb{E}}_{\varphi} \left(N_{\Lambda,1}(\tilde{\xi}(t)) - N_{\Lambda,-1}(\tilde{\xi}(t)) \right)^2 \leq 2|\Lambda| B(\varphi).$$

Proof. We write $N_{\Lambda,1}(\tilde{\xi}(t)) - N_{\Lambda,-1}(\tilde{\xi}(t)) = \sum_{x \in \Lambda} [\mathbf{1}_{\{+1,\pm\}}(\tilde{\xi}_x(t)) - \mathbf{1}_{\{-1,\pm\}}(\tilde{\xi}_x(t))]$ and observe that its expectation with respect to $\tilde{\mathbb{P}}_{\varphi}$ vanishes. Thus

$$\begin{aligned} \tilde{\mathbb{E}}_{\varphi} \left(N_{\Lambda,1}(\tilde{\xi}(t)) - N_{\Lambda,-1}(\tilde{\xi}(t)) \right)^2 = & \sum_{x \in \Lambda} \sum_{\alpha \in \{-1, 1\}} \tilde{\mathbb{P}}_{\varphi}(\tilde{\xi}_x(t) \in \{\alpha, \pm\}) \\ & + \sum_{\substack{x,y \in \Lambda \\ x \neq y}} \sum_{\alpha, \beta \in \{-1, 1\}} (-1)^{\frac{\alpha+\beta}{2}+1} \tilde{\mathbb{P}}_{\varphi}(\tilde{\xi}_x(t) \in \{\alpha, \pm\}; \tilde{\xi}_y(t) \in \{\beta, \pm\}). \end{aligned} \quad (4.5)$$

Let $p_t(x, y) := \mathbf{P}(W_x^\alpha(t) = y)$, i.e. the transition probability of the standard continuous time simple symmetric random walk on \mathbb{Z}^d . By (4.3), the diagonal term in (4.5) is given by

$$\sum_{x \in \Lambda} \sum_{\alpha \in \{-1, 1\}} \sum_{y \in \mathbb{Z}^d} \wp(\xi_y = \alpha) p_t(x, y) = 2\sigma |\Lambda|$$

where $\sigma := \wp(\xi_0 = -1) = \wp(\xi_0 = 1)$.

By (4.3), for $x \neq y$ and $\alpha, \beta \in \{-1, +1\}$

$$\begin{aligned} & \tilde{\mathbf{P}}_\wp(\tilde{\xi}_x(t) \in \{\alpha, \pm\}; \tilde{\xi}_y(t) \in \{\beta, \pm\}) \\ &= \sum_{x', y'} p_t^{\alpha, \beta}(x', y', x, y) \wp(\tilde{\xi}_{x'} = \alpha, \tilde{\xi}_{y'} = \beta) \\ & \quad - \sum_{x'} p_t(x', x) \wp(\tilde{\xi}_{x'} = \alpha) \sum_{y'} p_t(y', y) \wp(\tilde{\xi}_{y'} = \beta) \end{aligned}$$

where $p_t^{\alpha, \beta}(x', y', x, y) := \mathbf{P}(W_{x'}^\alpha(t) = x, W_{y'}^\beta(t) = y)$. Observe that if $\alpha \neq \beta$ then $p_t^{\alpha, \beta}(x', y', x, y) = p_t(x', x)p_t(y', y)$ while, by the Liggett's inequality [18, Prop. VIII.1.7], for $\alpha = \beta$ we have $p_t^{\alpha, \beta}(x', y', x, y) \leq p_t(x', x)p_t(y', y)$. By the invariance of \wp with respect to space shifts we deduce that the off diagonal term in (4.5) can be bounded by

$$\begin{aligned} & \sum_{\substack{x, y \in \Lambda \\ x \neq y}} \sum_{x', y'} \sum_{\alpha, \beta \in \{-1, 1\}} p_t(x', x) p_t(y', y) |\wp(\tilde{\xi}_0 = \alpha; \tilde{\xi}_{y'-x'} = \beta)| \\ & \leq \sum_{x \in \Lambda} \sum_z \sum_{x', y'} p_t(x', x) p_t(y', x+z) |\wp(\tilde{\xi}_0 = \alpha; \tilde{\xi}_{y'-x'} = \beta)| \\ & \leq |\Lambda| \sum_z \sum_{z'} q_t(z', z) |\wp(\tilde{\xi}_0 = \alpha; \tilde{\xi}_{z'} = \beta)| \leq |\Lambda| B(\wp) \end{aligned}$$

where $q_t(z', z)$ is the transition probability for a rate two symmetric random walk on \mathbb{Z}^d , i.e. the difference of two i.i.d. rate one symmetric random walks on \mathbb{Z}^d . Since $2\sigma = \sum_{\alpha, \beta \in \{-1, 1\}} |\wp(\xi_0 = \alpha; \xi_0 = \beta)|$ the statement follows. \square

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Acknowledgments. We thank L. Danella and D. Gabrielli for introducing us to the Ornstein distance.

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