HEAT CONDUCTION IN COMPOSITE MEDIA INVOLVING IMPERFECT CONTACT AND PERFECTLY CONDUCTIVE INCLUSIONS

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ABSTRACT. We study the thermal properties of a composite material made up of a medium hosting an ε-periodic array of perfect thermal conductors. The thermal potentials of the two phases are coupled across the interface through a non-standard imperfect contact transmission condition, involving the external thermal flux and a proportionality coefficient $D_0\varepsilon^\alpha$, where $\alpha \in \mathbb{R}$ is a scaling parameter and $D_0 > 0$ accounts for the imperfect contact. We perform the homogenization for all the scalings $\alpha \in \mathbb{R}$ and we compare the resulting models with the perfect contact transmission case addressed in [6, 7], letting $D_0 \to 0$, where this is meaningful.

KEYWORDS: Homogenization, time-periodic unfolding, total flux boundary conditions, imperfect contact conditions, parabolic problems.
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1. INTRODUCTION

We are interested in the study of the thermal properties of a composite material made up of a hosting medium in which a periodic array of thermal conductors is inserted. Since, usually, in the applications the hosting medium is a plastic material, we assume that the inclusions are made of perfect heat conductors (i.e. they have infinite thermal conductivity). This last assumption is in agreement with the fact that the heat conductivity of the inclusions is much larger than the one of the plastic hosting medium, as in the application we have in mind, i.e. the packaging of electronic devices (see [30, 31, 36, 37, 38, 40]). Other possible applications can be found in

heat diffusion, electric conduction, petroleum exploitation, wave equations or elastic properties of perforated materials (see, for instance, [2, 16, 17, 19, 26, 27, 32, 33, 34]).
From a mathematical point of view, the problem addressed in this paper reduces to a heat equation satisfied by the temperature $u_\varepsilon^{\mathrm{out}}$ in the hosting medium, while inside each inclusion the temperature $u_\varepsilon^{\mathrm{int}}$ depends only on time and satisfies an ordinary differential equation. Such an equation is not standard, since it involves a non-local condition. As a consequence, $u_\varepsilon^{\mathrm{int}}$ is a constant (possibly depending on the inclusion) with respect to the space variable. The two thermal potentials $u_\varepsilon^{\mathrm{int}}$ and $u_\varepsilon^{\mathrm{out}}$ are coupled through an imperfect contact transmission condition across the interface Γ^ε between the two conductive phases of the medium. More precisely, the jump of the temperature $[u_\varepsilon] = u_\varepsilon^{\mathrm{out}} - u_\varepsilon^{\mathrm{int}}$ satisfies the non-standard boundary condition (see (2.4))

$$[u_\varepsilon] = D_0 \varepsilon^\alpha \kappa_\varepsilon \nabla u_\varepsilon^{\mathrm{out}} \cdot \nu_\varepsilon \, , \tag{1.1}$$

where ν_ε is the normal vector to the interface Γ^ε pointing to the hosting medium, κ_ε is a matrix taking into account the diffusion properties of the hosting material, ε represents the characteristic length of the inclusions, $D_0 > 0$ is a factor taking into account the imperfect contact condition and $\alpha \in \mathbb{R}$ is a scaling parameter related to the interfacial heat exchange.
We mention that models like the one considered in the present paper have a wide area of possible applications, as recalled before, and they are known in the mathematical literature as "equivalued surface boundary value problems", this being justified by the fact that, on the boundary of the inclusions, the interior temperature field is assumed to be spatially constant (see, for instance, [16, 19, 26, 32, 33, 34] and the references therein).
In [6, 7], the authors studied a similar problem in which a perfect thermal contact between the two conductive phases was assumed. This implies that the temperature u_ε does not jump across the interface and, therefore, essentially allows one to restrict the problem to the outer domain.
The main novelty of our paper consists in the fact that we consider a more general model, in which we assume that the solution is no longer continuous across the interface between the two phases, but it has a jump, whose size is proportional to the outer heat flux through the coefficient $D_0\varepsilon^\alpha$, as shown in (1.1) above. Indeed, problems involving composite media with imperfect contacts are widely studied in various frameworks (see, among others, [5, 11, 12, 13, 14, 15, 18, 20, 21, 28, 29, 35] and the references therein).
Our goal in this paper is twofold. First of all, we want to determine the overall conduction properties of the composite medium, by means of a homogenization procedure based on the periodic unfolding technique. As a second goal, we want to recover the perfect contact condition and to compare the resulting models with the perfect transmission case addressed in [6, 7].
Therefore, we have two small parameters (the period ε of the geometry and the amplitude factor D_0) that we want to let tend to zero. For fixed $D_0 > 0$, we perform the homogenization process for each scaling $\alpha \in \mathbb{R}$, obtaining different macroscopic problems. More precisely, for $\alpha > -1$ we get three standard parabolic limit equations depending on different homogenized matrices (see (4.15) for $\alpha > 1$, (5.13) for $\alpha = 1$

and (6.9) for $\alpha \in (-1, 1)$). On the other hand, for $\alpha \leq -1$ we get two different systems: for $\alpha = -1$ we obtain a kind of standard bidomain problem consisting in the coupling of a partial differential equation with an ordinary differential one (see (7.5)), while for $\alpha < -1$ we obtain a completely decoupled system, consisting of a parabolic equation and a prescribed time-independent function (see (8.3) and (8.4)). We point out that, for $\alpha \neq \pm 1$, the homogenized limit problems do not keep memory of D_0. More precisely, in these cases, for $\alpha > 1$, we obtain the same parabolic equation as in [6, 7]; for $\alpha \in (-1, 1)$, the homogenized limit problem keeps a similar structure, but it depends on a different macroscopic diffusion matrix. Finally, for $\alpha < -1$, the decoupled system does not keep any memory of the physical properties of the inner phase; here, the equation for the relevant phase is different from the one obtained in [6, 7] even in structure and the macroscopic contribution of the other phase is given only by the limit of the original initial condition. On the contrary, for $\alpha = \pm 1$, the limit problem depends explicitly on the coefficient D_0. This justifies the need for performing our second step, i.e. to let $D_0 \to 0$. The consequence of such a procedure leads to two different situations: for $\alpha = 1$, we recover the same problem obtained for $\alpha > 1$ and, hence, the result in [6, 7] (see (5.31)), while for $\alpha = -1$, the bidomain structure disappears and we recover the same problem obtained for $\alpha \in (-1, 1)$ (see (7.10)). For more comments, see Remark 4.4, Remark 5.6, Remark 6.3, Remark 7.4 and Remark 8.3.

We notice that if we let $D_0 \to 0$ first, we can easily obtain from the energy estimate (2.9) that $[u_\varepsilon] \to 0$ in $L^2(\Gamma^\varepsilon \times (0, T))$, for any fixed $\alpha \in \mathbb{R}$. Therefore, passing to the limit in the weak formulation (2.7), we get the same problem, with perfect contact transmission conditions, studied in [6, 7], which is homogenized to the problem (4.15), found in Section 4.

On the contrary, when we take the two limits in the opposite order, the scaling α is relevant and determines the shape of the final limit problems. We stress that, at least for $\alpha < 1$, the two limits do not commute, while for $\alpha \geq 1$ they do commute.

Our proofs rely on the time-dependent periodic unfolding technique, which is the time-depending version of the one described in [25]. In the homogenization procedure, we need to consider non-standard test functions, inspired from the construction in [19, 26], for the elliptic case. Up to our knowledge, the problem we are addressing here is new in the literature, since it involves non-local conditions and imperfect contact between the hosting medium and the perfect conductive inclusions.

We mention that, in a forthcoming paper ([9]), we shall address the case of different inclusions with different imperfect contact conditions on the interface; in that paper, we will analyze all possible limit cases.

The paper is organized as follows: in Section 2, we state our problem and some preliminary results; in Subsections 2.1 and 2.2, we gather our general assumptions. In Section 3, we introduce the cell functions appearing in the homogenization procedure. In the remaining sections, we state and prove the homogenization results for the cases $\alpha > 1$ (Section 4), $\alpha = 1$ (Section 5), $-1 < \alpha < 1$ (Section 6), $\alpha = -1$ (Section 7) and $\alpha < -1$ (Section 8), respectively, and we compare the resulting models with the one arising for the perfect contact condition, letting $D_0 \to 0$, where this is meaningful.

2. Statement of the problem

2.1. **Geometrical setting.** Let Ω be an open smooth connected bounded subset of $\mathbb{R}^N$, with $N \geq 2$, and denote the unit cell by $Y = (0,1)^N$. For a given $T > 0$, we define $\Omega_T = \Omega \times (0,T)$ and, for $\varepsilon \in (0,1)$,

$$\Xi_\varepsilon = \left\{ \xi \in \mathbb{Z}^N , \quad \varepsilon(\xi + Y) \subset \Omega \right\} .$$

Next, we consider an open smooth subset $E \subset \mathbb{R}^N$, which we assume to be periodic in the sense that $E + z = E$ for all $z \in \mathbb{Z}^N$, setting also $E_{\text{int}} = E \cap Y$, $E_{\text{out}} = Y \setminus \overline{E}$, $\Gamma = \partial E \cap Y$. We assume E_{int} to be connected. We also assume $\overline{E_{\text{int}}} \subset Y$, implying that $\partial E_{\text{int}} = \Gamma$. For $\xi \in \Xi_\varepsilon$, we define

$$E^{\varepsilon,\xi}_{\text{int}} := \varepsilon(E_{\text{int}} + \xi) , \qquad \Gamma^\varepsilon_\xi := \partial E^{\varepsilon,\xi}_{\text{int}} .$$

We let

$$\Omega^\varepsilon_{\text{int}} = \bigcup_{\xi \in \Xi_\varepsilon} E^{\varepsilon,\xi}_{\text{int}} , \qquad \Gamma^\varepsilon = \partial \Omega^\varepsilon_{\text{int}} , \qquad \Omega^\varepsilon_{\text{out}} = \Omega \setminus \overline{\Omega^\varepsilon_{\text{int}}} .$$

We remark also that $\Omega^\varepsilon_{\text{out}}$ is connected, while $\Omega^\varepsilon_{\text{int}}$ is disconnected. From the physical point of view, they represent the hosting medium and the perfect conductive inclusions, respectively. Finally, let ν denote the normal unit vector to Γ pointing into E_{out}, extended by periodicity to the whole of $\mathbb{R}^N$, so that $\nu_\varepsilon(x) = \nu(x/\varepsilon)$ denotes the normal unit vector to Γ^ε pointing into $\Omega^\varepsilon_{\text{out}}$.
In the following, by γ we shall denote a strictly positive constant, independent of ε, which may vary from line to line.

Let us denote the unknown u_ε by the piecewise representation

$$u_\varepsilon = \begin{cases} u^{\text{int}}_\varepsilon , & \text{in } \Omega^\varepsilon_{\text{int}} \times (0,T), \\ u^{\text{out}}_\varepsilon , & \text{in } \Omega^\varepsilon_{\text{out}} \times (0,T). \end{cases} \tag{2.1}$$

We denote the jump of such a function as

$$[u_\varepsilon] = u^{\text{out}}_\varepsilon - u^{\text{int}}_\varepsilon , \qquad \text{on } \Gamma^\varepsilon \times (0,T).$$

The same notation will be employed for other functions, namely for test functions φ_ε which may exhibit jumps across $\Gamma^\varepsilon \times (0,T)$. The functions $u^{\text{int}}_\varepsilon$ and $u^{\text{out}}_\varepsilon$ represent the thermal potentials (or the temperatures) of the two phases.
We set

$$\kappa_\varepsilon(x) = \kappa\Big(x, \frac{x}{\varepsilon}\Big) ,$$

where $\kappa = (\kappa_{ij})$ is a Y-periodic symmetric matrix with $\kappa_{ij} \in L^\infty(\Omega \times Y)$ and such that there exists a constant $C \geq 1$ satisfying

$$C^{-1}|\zeta|^2 \leq \kappa(x,y)\zeta \cdot \zeta \leq C|\zeta|^2, \qquad \text{for a.e. } (x,y) \in \Omega \times Y \text{ and all } \zeta \in \mathbb{R}^N.$$

In addition, for a given positive constant λ, we set

$$a_\varepsilon(x) = \begin{cases} \frac{\lambda}{|E_{\text{int}}|} , & \text{in } \Omega^\varepsilon_{\text{int}}, \\ 1 , & \text{in } \Omega^\varepsilon_{\text{out}}. \end{cases}$$

From the physical point of view, the matrix κ_ε describes the diffusion features of the outer phase and a_ε accounts for the capacities of the two phases.

Then, we state, formally, our problem for u_ε:

$$\frac{\partial u_\varepsilon^{\mathrm{out}}}{\partial t} - \operatorname{div}(\kappa_\varepsilon \nabla u_\varepsilon^{\mathrm{out}}) = f\,, \qquad \text{in } \Omega_{\mathrm{out}}^\varepsilon \times (0,T); \tag{2.2}$$

$$\lambda \frac{\partial u_\varepsilon^{\mathrm{int}}}{\partial t} = \frac{1}{\varepsilon^N} \int_{\Gamma_\xi^\varepsilon} \kappa_\varepsilon \nabla u_\varepsilon^{\mathrm{out}} \cdot \nu_\varepsilon \, \mathrm{d}\sigma\,, \qquad \text{in } E_{\mathrm{int}}^{\varepsilon,\xi} \times (0,T),\ \xi \in \Xi_\varepsilon; \tag{2.3}$$

$$[u_\varepsilon] = D_0 \varepsilon^\alpha \kappa_\varepsilon \nabla u_\varepsilon^{\mathrm{out}} \cdot \nu_\varepsilon\,, \qquad \text{on } \Gamma^\varepsilon \times (0,T); \tag{2.4}$$

$$u_\varepsilon = 0\,, \qquad \text{on } \partial\Omega \times (0,T); \tag{2.5}$$

$$u_\varepsilon(x,0) = \overline{u_\varepsilon}(x)\,, \qquad \text{in } \Omega. \tag{2.6}$$

We assume $D_0 \in (0,+\infty)$, $f \in L^2(\Omega_T)$ and $\overline{u_\varepsilon} \in L^2(\Omega)$. We also require that $\overline{u_\varepsilon}$ is constant in each $E_{\mathrm{int}}^{\varepsilon,\xi}$, $\alpha \in \mathbb{R}$, and that $\overline{u_\varepsilon} \to \overline{u}$, as $\varepsilon \to 0$, strongly in $L^2(\Omega)$.
Note that, as a consequence of (2.3) and of the choice of the initial datum $\overline{u_\varepsilon}$, in each component $E_{\mathrm{int}}^{\varepsilon,\xi}$ of $\Omega_{\mathrm{int}}^\varepsilon$ the function $u_\varepsilon^{\mathrm{int}}$ depends only on (ξ,t), so that it is piecewise constant in $\Omega_{\mathrm{int}}^\varepsilon$. We mention that one can include in (2.3) a source term independent of the spatial variable, without additional difficulties.
By the usual process of formal integration by parts, we arrive at the following rigorous formulation of the problem.

Definition 2.1. Let the function u_ε be as in (2.1), with (abusing notation) $u_\varepsilon^{\mathrm{int}}(x,t) = u_\varepsilon^{\mathrm{int}}(\xi,t)$ in $E_{\mathrm{int}}^{\varepsilon,\xi}$ and $u_\varepsilon^{\mathrm{int}} \in L^2(\Omega_{\mathrm{int}}^\varepsilon \times (0,T))$, $u_\varepsilon^{\mathrm{out}} \in L^2(0,T;H^1(\Omega_{\mathrm{out}}^\varepsilon))$, $[u_\varepsilon] \in L^2(\Gamma^\varepsilon \times (0,T))$, $u_\varepsilon^{\mathrm{out}} = 0$ on $\partial\Omega$. Then, u_ε is a weak solution of problem (2.2)–(2.6) if

$$\int_0^T \int_\Omega \{-a_\varepsilon u_\varepsilon \varphi_{\varepsilon,t} + \chi_{\Omega_{\mathrm{out}}^\varepsilon} \kappa_\varepsilon \nabla u_\varepsilon^{\mathrm{out}} \cdot \nabla \varphi_\varepsilon\} \, \mathrm{d}x \, \mathrm{d}t + \frac{1}{D_0 \varepsilon^\alpha} \int_0^T \int_{\Gamma^\varepsilon} [u_\varepsilon][\varphi_\varepsilon] \, \mathrm{d}\sigma \, \mathrm{d}t = \int_0^T \int_{\Omega_{\mathrm{out}}^\varepsilon} f \varphi_\varepsilon \, \mathrm{d}x \, \mathrm{d}t + \int_\Omega a_\varepsilon \overline{u_\varepsilon} \varphi_\varepsilon(0) \, \mathrm{d}x\,, \tag{2.7}$$

for all test functions φ_ε such that $\varphi_\varepsilon^{\mathrm{int}}$ is constant in x in each $E_{\mathrm{int}}^{\varepsilon,\xi}$, $\varphi_\varepsilon^{\mathrm{int}} \in L^2(\Omega_{\mathrm{int}}^\varepsilon; H^1(0,T))$, $\varphi_\varepsilon^{\mathrm{out}} \in H^1(\Omega_{\mathrm{out}}^\varepsilon \times (0,T))$, $[\varphi_\varepsilon] \in L^2(\Gamma^\varepsilon \times (0,T))$, and $\varphi_\varepsilon(\cdot,T) = 0$, $\varphi_\varepsilon = 0$ on $\partial\Omega \times (0,T)$. □

Here, for $O \subseteq \mathbb{R}^N$, χ_O denotes the characteristic function of O. Moreover, by $\varphi_{\varepsilon,t}$ (or similar notation, in the following) we denote the time derivative of the function φ_ε.

Existence and uniqueness of solutions to (2.2)–(2.6), for each fixed $\varepsilon > 0$, can be proven by means of an approximation argument with strictly parabolic equations in the whole domain, for which a well-posedness result follows from the abstract parabolic theory as in [39].

We make use of the following energy inequality, which in turn can be proved by means of routine approximation procedures:

$$\sup_{0<t<T} \int_{\Omega} u_\varepsilon^2(t)\,\mathrm{d}x + \int_0^T \int_{\Omega_{\mathrm{out}}^\varepsilon} |\nabla u_\varepsilon^{\mathrm{out}}|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{D_0 \varepsilon^\alpha} \int_0^T \int_{\Gamma^\varepsilon} [u_\varepsilon]^2 \,\mathrm{d}\sigma\,\mathrm{d}t \\ \leq \gamma\big(\|f\|^2_{L^2(\Omega_T)} + \|\overline{u_\varepsilon}\|^2_{L^2(\Omega)}\big) . \quad (2.8)$$

For any $\alpha \in \mathbb{R}$, we infer at once

$$\sup_{0<t<T} \int_{\Omega} u_\varepsilon^2(t)\,\mathrm{d}x + \int_0^T \int_{\Omega_{\mathrm{out}}^\varepsilon} |\nabla u_\varepsilon^{\mathrm{out}}|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\varepsilon}{\varepsilon^{\alpha+1} D_0} \int_0^T \int_{\Gamma^\varepsilon} [u_\varepsilon]^2 \,\mathrm{d}\sigma\,\mathrm{d}t \leq \gamma\,, \qquad (2.9)$$

where γ does not depends on ε, since $\overline{u_\varepsilon}$ converges strongly.

2.2. **Time-depending unfolding operator.** In order to deal with our homogenization results, we need a space-time version of the unfolding operator studied in [22, 25] (see also [6, 7, 10, 8, 13] and, for a more general version, in which a time-microscale is actually present, see [3, 4]).
We recall, here, only the definitions and some of the properties needed in what follows (for other standard ones, we refer directly to [10, 25, 28]).
For $\xi \in \Xi_\varepsilon$, set

$$\widehat{\Omega}_\varepsilon = \text{interior}\left\{ \bigcup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \overline{Y}) \right\}, \qquad \Lambda_T^\varepsilon = \widehat{\Omega}_\varepsilon \times (0,T)\,.$$

Denoting by $[r]$ the integer part and by $\{r\}$ the fractional part of $r \in \mathbb{R}$, we define for $x \in \mathbb{R}^N$

$$\left[\frac{x}{\varepsilon}\right]_Y = \left(\left[\frac{x_1}{\varepsilon}\right], \dots, \left[\frac{x_N}{\varepsilon}\right] \right) \qquad \text{and} \qquad \left\{\frac{x}{\varepsilon}\right\}_Y = \left(\left\{\frac{x_1}{\varepsilon}\right\}, \dots, \left\{\frac{x_N}{\varepsilon}\right\} \right),$$

so that

$$x = \varepsilon \left(\left[\frac{x}{\varepsilon}\right]_Y + \left\{\frac{x}{\varepsilon}\right\}_Y \right) .$$

Then, let $Y_\varepsilon(x) = \varepsilon\Big(\left[\frac{x}{\varepsilon}\right]_Y + Y \Big)$ be the space cell containing x.

Definition 2.2. For w Lebesgue-measurable on Ω_T, the (time-depending) periodic unfolding operator $\mathcal{T}_\varepsilon$ is defined as

$$\mathcal{T}_\varepsilon(w)(x,t,y) = \begin{cases} w\left(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y, t\right), & (x,t,y) \in \Lambda_T^\varepsilon \times Y\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

For w Lebesgue-measurable on Γ_T^ε, the (time-depending) boundary unfolding operator $\mathcal{T}_\varepsilon^b$ is defined as

$$\mathcal{T}_\varepsilon^b(w)(x,t,y) = \begin{cases} w\left(\varepsilon \left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y, t\right), & (x,t,y) \in \Lambda_T^\varepsilon \times \Gamma\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

□

Definition 2.3. Let w be integrable in Ω_T. The local (time-depending) space average operator is defined by

$$\mathcal{M}_\varepsilon(w)(x,t) = \begin{cases} \dfrac{1}{\varepsilon^N} \displaystyle\int_{Y_\varepsilon(x)} w(\zeta,t)\,\mathrm{d}\zeta\,, & \text{if}\,(x,t) \in \Lambda_T^\varepsilon\,, \\ 0\,, & \text{otherwise.} \end{cases} \tag{2.10}$$

□

Notice that $\mathcal{M}_\varepsilon(w) = \mathcal{M}_Y(\mathcal{T}_\varepsilon(w))$, where, for a general set O, $\mathcal{M}_O(\cdot)$ denotes the integral average on O.

Proposition 2.4. *Let $w \in L^2(0,T;H^1(\Omega))$. Then,*

$$\frac{1}{\varepsilon}\left[\mathcal{T}_\varepsilon(w) - \mathcal{M}_\varepsilon(w)\right] \to y^c \cdot \nabla w\,, \quad \textit{strongly in}\, L^2(\Omega_T \times Y)\,, \tag{2.11}$$

where $y^c = (y^{c1},\dots,y^{cN}) = y - \mathcal{M}_Y(y)$.

Proposition 2.5. *Let $w_\varepsilon = (w_\varepsilon^{\mathrm{int}}, w_\varepsilon^{\mathrm{out}})$, with $w_\varepsilon^{\mathrm{int}} \in L^2(0,T;H^1(\Omega_{\mathrm{int}}^\varepsilon))$ and $w_\varepsilon^{\mathrm{out}} \in L^2(0,T;H^1(\Omega_{\mathrm{out}}^\varepsilon))$. Assume that there exists $\gamma > 0$ (independent of ε) such that*

$$\int_{\Omega_T} |w_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} |\nabla w_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t \le \gamma\,, \qquad \forall \varepsilon > 0. \tag{2.12}$$

Then, there exist $w^1 \in L^2(\Omega_T)$, $w^2 \in L^2((0,T);H^1(\Omega))$, $\overline{w}^{int} \in L^2(\Omega_T;H^1(E_{\mathrm{int}}))$ and $\widehat{w}^1 \in L^2(\Omega_T;H^1_\#(E_{\mathrm{out}}))$ such that, up to a subsequence,

$$\mathcal{T}_\varepsilon(\chi_{\Omega_{\mathrm{int}}^\varepsilon} w_\varepsilon) \rightharpoonup w^1, \qquad \textit{weakly in } L^2(\Omega_T \times E_{\mathrm{int}})\,; \tag{2.13}$$

$$\mathcal{T}_\varepsilon(\chi_{\Omega_{\mathrm{out}}^\varepsilon} w_\varepsilon) \rightharpoonup w^2, \qquad \textit{weakly in } L^2(\Omega_T \times E_{\mathrm{out}})\,; \tag{2.14}$$

$$\mathcal{T}_\varepsilon(\chi_{\Omega_{\mathrm{int}}^\varepsilon} \nabla w_\varepsilon) \rightharpoonup \nabla_y \overline{w}^{int}, \qquad \textit{weakly in } L^2(\Omega_T \times E_{\mathrm{int}})\,, \tag{2.15}$$

$$\mathcal{T}_\varepsilon(\chi_{\Omega_{\mathrm{out}}^\varepsilon} \nabla w_\varepsilon) \rightharpoonup \nabla w^2 + \nabla_y \widehat{w}^1, \qquad \textit{weakly in } L^2(\Omega_T \times E_{\mathrm{out}})\,, \tag{2.16}$$

for $\varepsilon \to 0$. Moreover, due to (2.12), we have

$$\varepsilon \int_{\Gamma_T^\varepsilon} [w_\varepsilon]^2\,\mathrm{d}\sigma\,\mathrm{d}t \le \gamma\,, \qquad \forall \varepsilon > 0\,, \tag{2.17}$$

with γ independent of ε, and then

$$\mathcal{T}_\varepsilon^b([w_\varepsilon]) \rightharpoonup w^2 - w^1\,, \qquad \textit{weakly in } L^2(\Omega_T \times \Gamma)\,. \tag{2.18}$$

Remark 2.6. We recall that, when $w_\varepsilon \to w$, strongly in $L^2(\Omega_T)$, then $\mathcal{T}_\varepsilon(w_\varepsilon) \to w$ strongly in $L^2(\Omega_T \times Y)$. However, the only classes for which the strong convergence of the unfolding $\mathcal{T}_\varepsilon(w_\varepsilon)$ is known to hold in $L^2(\Omega_T \times Y)$, without strong convergence of w_ε, are sums of the following cases: $w_\varepsilon(x,t) = f_1(x,t) f_2(\varepsilon^{-1}x)$ with f_1, f_2 suitable Lebesgue-measurable functions, $w_\varepsilon(x,t) = w(x,\varepsilon^{-1}x,t)$ with $w \in L^2(Y;\mathcal{C}(\Omega_T))$ or $w \in L^2(\Omega_T;\mathcal{C}(Y))$ (see [1, 23, 24] and [4, Remark 2.9]). □

In view of the previous remark, we will assume in the following that the matrix $\kappa \in \big(L^\infty(\Omega; \mathcal{C}_\#(\overline{Y}))\big)^{N\times N}$, even if this assumption can be weakened, requiring that there exists a Y-periodic matrix $K \in \big(L^\infty(\Omega \times Y)\big)^{N\times N}$ such that $\mathcal{T}_\varepsilon(\kappa_\varepsilon) \to K$, strongly in $L^2(\Omega \times Y)$.

2.3. **Compactness results.** Let u_ε be the unique solution of problem (2.2)–(2.6). We collect here some compactness results in all the cases of interest for the scaling parameter α. The convergences below are intended up to extracting subsequences.
By taking into account estimate (2.9) and Proposition 2.5, applied to our solution u_ε, it follows that there exist suitable functions $u_1 \in L^2(\Omega_T)$, $u_2 \in L^2(0,T; H^1_0(\Omega))$ and $\widehat{u}^1 \in L^2(\Omega_T; H^1_\#(E_{\text{out}}))$ such that

$$\mathcal{T}_\varepsilon(\chi_{\Omega^\varepsilon_{\text{int}}} u_\varepsilon) \rightharpoonup u_1\,, \qquad \text{weakly in } L^2(\Omega_T \times E_{\text{int}}); \tag{2.19}$$

$$\mathcal{T}_\varepsilon(\chi_{\Omega^\varepsilon_{\text{out}}} u_\varepsilon) \rightharpoonup u_2\,, \qquad \text{weakly in } L^2(\Omega_T \times E_{\text{out}}); \tag{2.20}$$

$$\mathcal{T}_\varepsilon(\chi_{\Omega^\varepsilon_{\text{out}}} \nabla u_\varepsilon) \rightharpoonup \nabla u_2 + \nabla_y \widehat{u}^1\,, \qquad \text{weakly in } L^2(\Omega_T \times E_{\text{out}}). \tag{2.21}$$

In addition, for every $\alpha \in \mathbb{R}$,

$$\int\limits_{\Omega_T\times\Gamma} \mathcal{T}^b_\varepsilon([u_\varepsilon])^2 \,\mathrm{d}x\,\mathrm{d}\sigma_y\,\mathrm{d}t \le \gamma\varepsilon^{\alpha+1}\,. \tag{2.22}$$

The last estimate clearly implies, for $\alpha > -1$,

$$\mathcal{T}^b_\varepsilon([u_\varepsilon]) \to 0\,, \qquad \text{strongly in } L^2(\Omega_T \times \Gamma), \tag{2.23}$$

and therefore, $u_1 = u_2 =: u \in L^2(0,T; H^1_0(\Omega))$ in virtue of (2.18). More precisely, in this case, we have

$$u_\varepsilon \rightharpoonup u\,, \qquad \text{weakly in } L^2(\Omega_T). \tag{2.24}$$

On the other hand, for $\alpha = -1$, we obtain

$$\mathcal{T}^b_\varepsilon([u_\varepsilon]) \to u_2 - u_1\,, \qquad \text{weakly in } L^2(\Omega_T \times \Gamma). \tag{2.25}$$

In the cases $\alpha > -1$ instead, we get:

$$\text{for } -1<\alpha<1\,, \quad \frac{1}{\varepsilon^\alpha}\mathcal{T}^b_\varepsilon([u_\varepsilon]) \to 0\,, \qquad \text{strongly in } L^2(\Omega_T \times \Gamma), \tag{2.26}$$

$$\text{for } \alpha>1\,, \quad \mathcal{T}^b_\varepsilon\Big(\frac{[u_\varepsilon]}{\varepsilon}\Big) \to 0\,, \qquad \text{strongly in } L^2(\Omega_T \times \Gamma), \tag{2.27}$$

$$\text{for } \alpha=1\,, \quad \mathcal{T}^b_\varepsilon\Big(\frac{[u_\varepsilon]}{\varepsilon}\Big) \rightharpoonup \widehat{u}^1 + y^c\cdot\nabla u + \zeta, \qquad \text{weakly in } L^2(\Omega_T \times \Gamma), \tag{2.28}$$

for a suitable function $\zeta \in L^2(\Omega_T)$. The convergences in (2.26) and (2.27) are consequences of (2.22), while for (2.28) we refer to [28].

Remark 2.7. We stress the fact that all the above convergences take place as $\varepsilon \to 0$ and D_0 is fixed. Later, after the homogenization procedure, we will discuss also the limit for $D_0 \to 0$. $\square$

3. Auxiliary results

We understand here all the general assumptions listed in Section 2. All the problems below are intended in the distributional sense.

Lemma 3.1. *For each $j = 1, \dots, N$, there exists a unique $\chi^j \in L^\infty(\Omega; H^1_\#(E_{\mathrm{out}}))$ which satisfies*

$$-\operatorname{div}_y(\kappa(x,y)\,\nabla_y(\chi^j(x,y) - y^j)) = 0\,, \qquad \textit{in } E_{\mathrm{out}}; \tag{3.1}$$

$$\int_\Gamma \kappa(x,y)\,\nabla_y(\chi^j(x,y) - y^j)\cdot\nu\,\mathrm{d}\sigma_y = 0\,, \tag{3.2}$$

$$\chi^j(x,y) - y^j \qquad \textit{is independent of } y \textit{ on } \Gamma, \tag{3.3}$$

$$\int_{E_{\mathrm{out}}} \chi^j(x,y)\,\mathrm{d}y = 0\,. \tag{3.4}$$

Proof. The result can be obtained following the same steps as in [7, Lemma 4.7]. □

Lemma 3.2. *For $j = 1, \dots, N$, let us consider the problem*

$$-\operatorname{div}_y(\kappa\nabla_y(\hat\chi^j(x,y) - y^{cj})) = 0\,, \qquad \textit{in } E_{\mathrm{out}}; \tag{3.5}$$

$$\kappa\nabla_y(\hat\chi^j(x,y) - y^{cj})\cdot\nu = \frac{1}{D_0}\Big(\hat\chi^j(x,y) - y^{cj} - \frac{1}{|\Gamma|}\int_\Gamma(\hat\chi^j(x,z) - z^{cj})\,\mathrm{d}\sigma_z\Big), \quad \textit{on } \Gamma; \tag{3.6}$$

$$\int_{E_{\mathrm{out}}} \hat\chi^j(x,y)\,\mathrm{d}y = 0\,. \tag{3.7}$$

Then, problem (3.5)–(3.7) admits a unique solution $\hat\chi^j \in L^\infty(\Omega; H^1_\#(E_{\mathrm{out}}))$.

Proof. For $j = 1, \dots, N$, let $\widetilde\chi^j \in H^1_\#(E_{\mathrm{out}})$ be the unique solution of the Robin problem

$$-\operatorname{div}_y(\kappa\nabla_y(\widetilde\chi^j - y^{cj})) = 0\,, \qquad \text{in } E_{\mathrm{out}}; \tag{3.8}$$

$$\kappa\nabla_y(\widetilde\chi^j - y^{cj})\cdot\nu - \frac{1}{D_0}(\widetilde\chi^j - y^{cj}) = 0\,, \qquad \text{on } \Gamma. \tag{3.9}$$

Set

$$\hat\zeta^j(x) = \frac{1}{|E_{\mathrm{out}}|}\int_{E_{\mathrm{out}}} \widetilde\chi^j(x,y)\,\mathrm{d}y \tag{3.10}$$

and define

$$\hat\chi^j(x,y) = \widetilde\chi^j(x,y) - \hat\zeta^j(x)\,. \tag{3.11}$$

By means of an integration by parts of the differential equation (3.8) and taking into account (3.9) and (3.11), it follows that

$$\hat\zeta^j(x) = -\frac{1}{|\Gamma|}\int_\Gamma(\hat\chi^j(x,y) - y^{cj})\,\mathrm{d}\sigma_y\,. \tag{3.12}$$

Moreover, replacing $\widetilde{\chi}^j$ with $\hat{\chi}^j + \hat{\zeta}^j$ in (3.8) and (3.9), we get

$$-\operatorname{div}_y(\kappa \nabla_y(\hat{\chi}^j(x,y) - y^{cj})) = 0\,, \qquad \text{in } E_{\text{out}}; \tag{3.13}$$

$$\kappa \nabla_y(\hat{\chi}^j(x,y) - y^{cj}) \cdot \nu = \frac{1}{D_0}(\hat{\chi}^j(x,y) - y^{cj} + \hat{\zeta}^j(x))\,, \qquad \text{on } \Gamma. \tag{3.14}$$

Therefore, taking into account (3.12)–(3.14), it is easily seen that $\hat{\chi}^j$ is the required unique solution of the problem (3.5)–(3.7). The uniqueness follows by standard energy estimates. □

Remark 3.3. Notice that (3.12) can be rewritten as

$$\int_\Gamma (\hat{\chi}^j(x,y) - y^{cj} + \hat{\zeta}^j(x))\,\mathrm{d}\sigma_y = 0\,. \tag{3.15}$$

Finally, we also recall the following well-known result.

Lemma 3.4. *For each $j = 1, \ldots, N$, there exists a unique $\bar{\chi}^j \in L^\infty(\Omega; H^1_\#(E_{\text{out}}))$ which satisfies*

$$-\operatorname{div}_y(\kappa(x,y)\,\nabla_y(\bar{\chi}^j(x,y) - y^j)) = 0\,, \quad \textit{in } E_{\text{out}}; \tag{3.16}$$

$$\kappa(x,y)\,\nabla_y(\bar{\chi}^j(x,y) - y^j)\cdot\nu = 0\,, \quad \textit{on } \Gamma; \tag{3.17}$$

$$\int_{E_{\text{out}}} \bar{\chi}^j(x,y)\,\mathrm{d}y = 0\,. \tag{3.18}$$

4. Homogenization of the case $\alpha > 1$

In this section, we study the homogenization for the case $\alpha > 1$ and we find, in the limit, a standard parabolic problem, in which the effective capacity is given by a weighted average of the original capacities of the outer and the inner phases, the effective diffusion matrix depends only on the properties of the outer phase, but no memory of the contact coefficient D_0 is kept (see Remark 4.4).

Theorem 4.1. *The limiting function u, appearing in* (2.24)*, is the unique solution of*

$$-(|E_{\text{out}}| + \lambda)\int_{\Omega_T} u\varphi_t\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} A_{\text{hom}}\nabla u \cdot \nabla\varphi\,\mathrm{d}x\,\mathrm{d}t = |E_{\text{out}}|\int_{\Omega_T} f\varphi\,\mathrm{d}x\,\mathrm{d}t + (|E_{\text{out}}| + \lambda)\int_\Omega \overline{u}\varphi(0)\,\mathrm{d}x\,, \tag{4.1}$$

for all $\varphi \in H^1(\Omega_T)$ with $\varphi = 0$ on $\partial\Omega \times (0,T)$ and for $t = T$. Here, the homogenized matrix A_{hom} is defined by

$$A^{ij}_{\text{hom}}(x) = -\int_{E_{\text{out}}} \kappa(x,y)\,\nabla_y(\chi^i - y^i)\cdot\nabla_y y^j\,\mathrm{d}y\,, \tag{4.2}$$

for $i, j = 1, \ldots, N$, where χ has been introduced in Lemma 3.1.

Proof. Following similar ideas as in [19, 26], we select as a test function

$$\varphi_\varepsilon(x,t) = \varepsilon z(t)\Big[\mathcal{M}_\varepsilon(w)(x)\psi\Big(\frac{x}{\varepsilon}\Big) + w(x)\phi\Big(\frac{x}{\varepsilon}\Big)\Big],$$

where $z \in \mathcal{C}^1([0,T])$, $z(T)=0$, $w \in \mathcal{C}_0^\infty(\Omega)$, $\psi \in \mathcal{C}_0^\infty(Y)$, $\phi \in \mathcal{C}_\#^\infty(\overline{Y})$; we also assume that ψ is constant and $\phi = 0$ in E_{int}. By unfolding the integrals appearing in the weak formulation (2.7), we arrive at

$$\begin{aligned}
&- \varepsilon \int\limits_{\Omega_T\times Y} \mathcal{T}_\varepsilon(a_\varepsilon u_\varepsilon)\mathcal{T}_\varepsilon(z')[\mathcal{M}_\varepsilon(w)\mathcal{T}_\varepsilon(\psi) + \mathcal{T}_\varepsilon(w)\mathcal{T}_\varepsilon(\phi)]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&+ \varepsilon \int\limits_{\Omega_T\times E_{\mathrm{out}}} \mathcal{T}_\varepsilon(z)\mathcal{T}_\varepsilon(\kappa_\varepsilon)\mathcal{T}_\varepsilon(\nabla u_\varepsilon)\cdot\mathcal{T}_\varepsilon(\nabla w)\mathcal{T}_\varepsilon(\phi)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&+ \int\limits_{\Omega_T\times E_{\mathrm{out}}} \mathcal{T}_\varepsilon(z)\mathcal{T}_\varepsilon(\kappa_\varepsilon)\mathcal{T}_\varepsilon(\nabla u_\varepsilon)\cdot[\mathcal{T}_\varepsilon(\nabla_y \psi)\mathcal{M}_\varepsilon(w) + \mathcal{T}_\varepsilon(w)\mathcal{T}_\varepsilon(\nabla_y \phi)]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&- \int\limits_{\Omega_T\times E_{\mathrm{out}}} \mathcal{T}_\varepsilon(f)\mathcal{T}_\varepsilon(\varphi_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t - \int\limits_{\Omega\times Y} \mathcal{T}_\varepsilon(a_\varepsilon\overline{u_\varepsilon})\mathcal{T}_\varepsilon(\varphi_\varepsilon(0))\,\mathrm{d}x\,\mathrm{d}y \to 0, \qquad \varepsilon\to 0.
\end{aligned}$$

However, it is easily seen that all the terms above, excepting the third integral, vanish in the limit independently of the previous relation, which therefore yields

$$\int\limits_{\Omega_T\times E_{\mathrm{out}}} z(t)w(x)\kappa(x,y)(\nabla u + \nabla_y \widehat{u}^1)\cdot\nabla_y(\psi+\phi)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 0. \tag{4.3}$$

By a standard density argument, this implies

$$\int\limits_{\Omega_T\times E_{\mathrm{out}}} z(t)\kappa(x,y)(\nabla u + \nabla_y \widehat{u}^1)\cdot\nabla_y \Phi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 0, \tag{4.4}$$

for all $\Phi \in \mathcal{C}^1(\Omega\times E_{\mathrm{out}})$, periodic in Y, vanishing on $\partial\Omega$ and constant on Γ.
Taking into account that, from Proposition 4.2 below, we have that $y^c\cdot\nabla u + \widehat{u}^1$ is independent of y on $\Omega_T\times\Gamma$, equality (4.4) turns out to be the weak formulation of problem

$$-\operatorname{div}_y(\kappa(\nabla u + \nabla_y \widehat{u}^1)) = 0, \qquad \text{in } \Omega_T\times E_{\mathrm{out}}; \tag{4.5}$$

$$\int\limits_{\Gamma} \kappa(\nabla u + \nabla_y \widehat{u}^1)\cdot\nu\,\mathrm{d}\sigma_y = 0, \tag{4.6}$$

$$y^c\cdot\nabla u + \widehat{u}^1 \qquad \text{is independent of } y \text{ on } \Omega_T\times\Gamma. \tag{4.7}$$

Next, we choose as a test function

$$\phi_\varepsilon(x,t) = z(t)\Big[\mathcal{M}_\varepsilon(w)\psi\Big(\frac{x}{\varepsilon}\Big) + w(x)\Big(1-\psi\Big(\frac{x}{\varepsilon}\Big)\Big)\Big],$$

where ψ is as above with, more specifically, $\psi = 1$ in E_{int}. Note that $\mathcal{T}_\varepsilon(\phi_\varepsilon) \to zw$ strongly in $L^2(\Omega \times Y)$ and that

$$
\begin{aligned}
&- \int\limits_{\Omega_T \times Y} \mathcal{T}_\varepsilon(a_\varepsilon u_\varepsilon)\mathcal{T}_\varepsilon(z'[\mathcal{M}_\varepsilon(w)\psi + w(1-\psi)])\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\quad + \int\limits_{\Omega_T \times E_{\text{out}}} \mathcal{T}_\varepsilon(\kappa_\varepsilon)\mathcal{T}_\varepsilon(\nabla u_\varepsilon)\mathcal{T}_\varepsilon\big(z(1-\psi)\nabla w + \varepsilon^{-1} z(\mathcal{M}_\varepsilon(w) - w)\nabla_y \psi\big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\quad - \int\limits_{\Omega_T \times E_{\text{out}}} \mathcal{T}_\varepsilon(f)\mathcal{T}_\varepsilon(\phi_\varepsilon)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t - \int\limits_{\Omega \times Y} \mathcal{T}_\varepsilon(a_\varepsilon \overline{u_\varepsilon})\mathcal{T}_\varepsilon(\phi_\varepsilon(0))\,\mathrm{d}x\,\mathrm{d}y \to 0\,, \qquad \varepsilon \to 0\,.
\end{aligned}
$$

Then we get, according to [26],

$$
\begin{aligned}
&- \int\limits_{\Omega_T \times E_{\text{out}}} u z' w\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t - \int\limits_{\Omega_T \times E_{\text{int}}} \frac{\lambda}{|E_{\text{int}}|} u z' w\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\quad + \int\limits_{\Omega_T \times E_{\text{out}}} z\kappa(\nabla u + \nabla_y \widehat{u}^1)\cdot[\nabla w - \nabla_y(\psi y^c \cdot \nabla w)]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\quad = \int\limits_{\Omega_T \times E_{\text{out}}} f w z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t + (|E_{\text{out}}| + \lambda)\int\limits_{\Omega} z(0) w \overline{u}\,\mathrm{d}x\,.
\end{aligned}
\tag{4.8}
$$

We add (4.8) to (4.4) and obtain

$$
\begin{aligned}
&- (|E_{\text{out}}| + \lambda)\int\limits_{\Omega_T} z' w u\,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\Omega_T \times E_{\text{out}}} z\kappa(\nabla u + \nabla_y \widehat{u}^1)\cdot(\nabla w + \nabla_y w^1)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\qquad = |E_{\text{out}}| \int\limits_{\Omega_T} z w f\,\mathrm{d}x\,\mathrm{d}t + (|E_{\text{out}}| + \lambda)\int\limits_{\Omega} z(0) w \overline{u}\,\mathrm{d}x\,,
\end{aligned}
\tag{4.9}
$$

where we set

$$
w^1(x,y) = \Phi(x,y) - \psi(y) y^c \cdot \nabla w - c(x)\,,
$$

with c such that $\mathcal{M}_{E_{\text{out}}}(w^1) = 0$. The choice to fix the mean value of w^1 in E_{out} is due to the fact that such a function acts only on E_{out}.
By appealing to (4.5)–(4.7), we expand

$$
\widehat{u}^1(x,y,t) = -\chi(x,y)\cdot\nabla u(x,t)\,, \qquad (x,y,t) \in \Omega \times E_{\text{out}} \times (0,T)\,,
\tag{4.10}
$$

where the vector χ has been introduced in Lemma 3.1. This implies that $\mathcal{M}_{E_{\text{out}}}(\widehat{u}^1) = 0$. Thus, we obtain from (4.9)

$$
\begin{aligned}
&- (|E_{\text{out}}| + \lambda)\int\limits_{\Omega_T} z' w u\,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\Omega_T} z A_{\text{hom}} \nabla u \cdot \nabla w\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\qquad = |E_{\text{out}}| \int\limits_{\Omega_T} z w f\,\mathrm{d}x\,\mathrm{d}t + (|E_{\text{out}}| + \lambda)\int\limits_{\Omega} z(0) w \overline{u}\,\mathrm{d}x\,,
\end{aligned}
\tag{4.11}
$$

which essentially is (4.1), up to a routine density argument. The uniqueness of the solution u of equation (4.1) is a standard result from the theory of parabolic equations, taking into account that the homogenized matrix A_{hom} is symmetric and positive definite (see Remark 4.3). Hence, all the above convergences hold true for the whole sequences. □

Proposition 4.2. *Let $\widehat{u}^1$ and u be the limiting functions appearing in* (2.21) *and* (2.24)*, respectively. Then, we have that*

$$y^c \cdot \nabla u + \widehat{u}^1 \qquad \text{is independent of } y \text{ on } \Omega_T \times \Gamma. \tag{4.12}$$

Proof. Let $u_\varepsilon = (u_\varepsilon^{\mathrm{int}}, u_\varepsilon^{\mathrm{out}})$ be the unique solution of problem (2.2)–(2.6). Following [28, Theorem 2.17], we have that, up to a subsequence, as $\varepsilon \to 0$,

$$\frac{1}{\varepsilon}\left[\mathcal{T}_\varepsilon(u_\varepsilon^{\mathrm{out}}) - \mathcal{M}_\Gamma(\mathcal{T}_\varepsilon^b(u_\varepsilon^{\mathrm{out}}))\right] \rightharpoonup y^c \cdot \nabla u + \widehat{u}^1\,, \quad \text{weakly in } L^2(\Omega_T; H^1(E_{\mathrm{out}}))\,,$$

up to an additive function independent of y. Therefore, by the linearity of the trace operator, as $\varepsilon \to 0$, we also obtain that

$$\frac{1}{\varepsilon}\left[\mathcal{T}_\varepsilon^b(u_\varepsilon^{\mathrm{out}}) - \mathcal{M}_\Gamma(\mathcal{T}_\varepsilon^b(u_\varepsilon^{\mathrm{out}}))\right] \rightharpoonup y^c \cdot \nabla u + \widehat{u}^1\,, \quad \text{weakly in } L^2(\Omega_T \times \Gamma)\,, \tag{4.13}$$

where, with abuse of notation, we denote by $\widehat{u}^1$ also the trace of $\widehat{u}^1$ on Γ. Clearly, the previous convergence is still true up to the same additive function independent of y, quoted above. Moreover, by (2.27) and (4.13), we have

$$\begin{aligned}\frac{1}{\varepsilon}\left(\mathcal{T}_\varepsilon^b(u_\varepsilon^{\mathrm{int}}) - \mathcal{M}_\Gamma(\mathcal{T}_\varepsilon^b(u_\varepsilon^{\mathrm{int}}))\right) &= -\mathcal{T}_\varepsilon^b\left(\frac{[u_\varepsilon]}{\varepsilon}\right) + \mathcal{M}_\Gamma\left(\frac{\mathcal{T}_\varepsilon^b([u_\varepsilon])}{\varepsilon}\right)\\ + \frac{1}{\varepsilon}\left(\mathcal{T}_\varepsilon^b(u_\varepsilon^{\mathrm{out}}) - \mathcal{M}_\Gamma(\mathcal{T}_\varepsilon^b(u_\varepsilon^{\mathrm{out}}))\right) &\rightharpoonup y^c \cdot \nabla u + \widehat{u}^1\,, \quad \text{weakly in } L^2(\Omega_T \times \Gamma)\,, \end{aligned} \tag{4.14}$$

still up to the same additive function independent of y quoted above. However, as already pointed out, by (2.3) and the choice of the initial datum $\overline{u_\varepsilon}$, it follows that $u_\varepsilon^{\mathrm{int}}$, in each component $E_{\mathrm{int}}^{\varepsilon,\xi}$ of $\Omega_{\mathrm{int}}^\varepsilon$, depends only on (ξ, t), i.e. $\mathcal{T}_\varepsilon^b(u_\varepsilon^{\mathrm{int}})$ is independent of y on Γ. Hence, also the limit of $\varepsilon^{-1}\left(\mathcal{T}_\varepsilon^b(u_\varepsilon^{\mathrm{int}}) - \mathcal{M}_\Gamma(\mathcal{T}_\varepsilon^b(u_\varepsilon^{\mathrm{int}}))\right)$ is independent of y on Γ, so that the thesis is achieved. □

Remark 4.3. We emphasize that the distributional formulation of the limit problem (4.1) is

$$\begin{aligned} &(|E_{\mathrm{out}}| + \lambda)\, u_t - \mathrm{div}(A_{\mathrm{hom}} \nabla u) = |E_{\mathrm{out}}| f\,, && \text{in } \Omega_T;\\ &u = 0\,, && \text{on } \partial\Omega \times (0,T);\\ &u(x,0) = \overline{u}\,, && \text{in } \Omega, \end{aligned} \tag{4.15}$$

which coincides with problem (50) in [6], since A_{hom} is the same matrix obtained in [6, formula (49)]. Indeed, from standard results, the matrix A_{hom} can be rewritten as

$$A_{\mathrm{hom}}^{ij}(x) = \int_{E_{\mathrm{out}}} \kappa(x,y)\, \nabla_y(\chi^i - y^i) \cdot \nabla_y(\chi^j - y^j)\, \mathrm{d}y\,, \tag{4.16}$$

which implies that it is symmetric and positive definite, ensuring, in this way, the uniqueness of the homogenized solution. □

Remark 4.4. The limiting homogenized problem in the case $\alpha > 1$ does not bear any memory of the contact coefficient D_0.
Indeed, the fact that we obtain the same result given by problem (50) of [6] (where, by assumption, $[u_\varepsilon] = 0$), can be viewed as a consequence of the high degeneracy of the case of power $\alpha > 1$, which we are dealing with. □

5. Homogenization in the case $\alpha = 1$

In this section, we study the homogenization for the case $\alpha = 1$ and we find, in the limit, a standard parabolic problem, in which the effective capacity is the same as in the previous case, while the effective diffusion matrix keeps memory of the contact coefficient D_0 (see Remark 5.3) and the geometrical properties of the interface. Such an explicit dependence on D_0 calls for a second limit procedure, in which we let $D_0 \to 0$, essentially arriving at the same situation discussed in Section 4 (see Remark 5.6).

Theorem 5.1. *The limiting function u, appearing in* (2.24)*, is the unique solution of*

$$- (|E_{\mathrm{out}}| + \lambda) \int_{\Omega_T} u\varphi_t \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} B_{\mathrm{hom}} \nabla u \cdot \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t = |E_{\mathrm{out}}| \int_{\Omega_T} f\varphi \,\mathrm{d}x\,\mathrm{d}t + (|E_{\mathrm{out}}| + \lambda) \int_{\Omega} \overline{u}\varphi(0) \,\mathrm{d}x \,, \quad (5.1)$$

for all $\varphi \in H^1(\Omega_T)$ with $\varphi = 0$ on $\partial\Omega \times (0,T)$ and for $t = T$. Here, the homogenized matrix B_{hom} is defined by

$$B^{ij}_{\mathrm{hom}}(x) = -\int_{E_{\mathrm{out}}} \kappa(x,y) \nabla_y(\hat{\chi}^j - y^{cj}) \cdot \nabla_y y^{ci} \,\mathrm{d}y - \frac{1}{D_0} \int_{\Gamma} (\hat{\chi}^j - y^{cj} + \hat{\zeta}^j) y^{ci} \,\mathrm{d}\sigma_y \,, \quad (5.2)$$

for $i,\ j = 1,\ \dots,\ N$, where $\hat{\chi} = (\hat{\chi}^1, \dots, \hat{\chi}^N)$ and $\hat{\zeta} = (\hat{\zeta}^1, \dots, \hat{\zeta}^N)$ have been introduced in Lemma 3.2.

Proof. We begin by recalling that $[u] = 0$, owing to (2.23), and that (2.28) is in force. Thus, we choose first the test function $\varphi_\varepsilon(x,t) = \varepsilon\varphi(x, x/\varepsilon, t)$, where

$$\varphi(x,y,t) = \begin{cases} \Psi(x,y)z(t)\,, & \text{in } \Omega_T \times E_{\mathrm{out}}, \\ 0\,, & \text{in } \Omega_T \times E_{\mathrm{int}}. \end{cases}$$

Here, $z \in \mathcal{C}^1([0,T])$, $z(T) = 0$, the function $\Psi \in \mathcal{C}^\infty(\overline{\Omega \times E_{\mathrm{out}}})$ vanishes near $\partial\Omega$ and is Y-periodic.

Reasoning as in Section 4, we arrive at

$$\int\limits_{\Omega_T\times E_{\rm out}} \mathcal{T}_\varepsilon(\kappa_\varepsilon z)\mathcal{T}_\varepsilon(\nabla u_\varepsilon^{\rm out})\cdot\mathcal{T}_\varepsilon(\nabla_y\Psi)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ +\frac{1}{D_0}\int\limits_{\Omega_T\times\Gamma}\mathcal{T}_\varepsilon^b\Big(\frac{[u_\varepsilon]}{\varepsilon}\Big)\mathcal{T}_\varepsilon^b(z\Psi)\,\mathrm{d}x\,\mathrm{d}\sigma_y\,\mathrm{d}t\to 0\,,\qquad \varepsilon\to 0\,. \tag{5.3}$$

Then, by a density argument and recalling (2.28), we get

$$\int\limits_{\Omega_T\times E_{\rm out}}\kappa(\nabla u+\nabla_y\widehat{u}^1)\cdot\nabla_y\phi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t+\frac{1}{D_0}\int\limits_{\Omega_T\times\Gamma}(\widehat{u}^1+y^c\cdot\nabla u+\zeta)\phi\,\mathrm{d}x\,\mathrm{d}\sigma_y\,\mathrm{d}t=0\,, \tag{5.4}$$

for all $\phi\in L^\infty(\Omega_T;H^1_\#(E_{\rm out}))$. The distributional formulation of (5.4) is given by

$$-\operatorname{div}_y(\kappa(\nabla u+\nabla_y\widehat{u}^1))=0\,,\qquad \text{in }\Omega_T\times E_{\rm out}; \tag{5.5}$$

$$\kappa(\nabla u+\nabla_y\widehat{u}^1)\cdot\nu=\frac{1}{D_0}(\widehat{u}^1+y^c\cdot\nabla u+\zeta)\,,\qquad \text{on }\Omega_T\times\Gamma. \tag{5.6}$$

Notice that (5.6) implies

$$\int\limits_{\Omega_T\times\Gamma}(\widehat{u}^1+y^c\cdot\nabla u+\zeta)\,\mathrm{d}x\,\mathrm{d}\sigma_y\,\mathrm{d}t=0\,. \tag{5.7}$$

Then, we will apply below the factorization

$$\widehat{u}^1(x,y,t)=-\hat{\chi}(x,y)\cdot\nabla u(x,t)-\hat{\zeta}(x)\cdot\nabla u(x,t)-\zeta(x,t)\,. \tag{5.8}$$

Next, we select the test function $\tilde{\varphi}_\varepsilon(x,x/\varepsilon,t)$, where

$$\tilde{\varphi}_\varepsilon(x,y,t)=\begin{cases} z(t)w(x)\,, & \text{in }\Omega_T\times E_{\rm out},\\ z(t)\mathcal{M}_\varepsilon(w)(x)\,, & \text{in }\Omega_T\times E_{\rm int},\end{cases}$$

with $z\in\mathcal{C}^1([0,T])$, $z(T)=0$, $w\in\mathcal{C}^\infty_0(\Omega)$. We get

$$-\int\limits_{\Omega_T\times E_{\rm out}}\mathcal{T}_\varepsilon(u_\varepsilon)\mathcal{T}_\varepsilon(z'w)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t-\frac{\lambda}{|E_{\rm int}|}\int\limits_{\Omega_T\times E_{\rm int}}\mathcal{T}_\varepsilon(u_\varepsilon)\mathcal{T}_\varepsilon(z')\mathcal{M}_\varepsilon(w)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ +\int\limits_{\Omega_T\times E_{\rm out}}\mathcal{T}_\varepsilon(z\kappa_\varepsilon)\mathcal{T}_\varepsilon(\nabla u_\varepsilon)\cdot\mathcal{T}_\varepsilon(\nabla w)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ +\frac{1}{D_0}\int\limits_{\Omega_T\times\Gamma}\mathcal{T}_\varepsilon^b(z)\mathcal{T}_\varepsilon^b\Big(\frac{[u_\varepsilon]}{\varepsilon}\Big)\mathcal{T}_\varepsilon^b\Big(\frac{w-\mathcal{M}_\varepsilon(w)}{\varepsilon}\Big)\,\mathrm{d}x\,\mathrm{d}\sigma_y\,\mathrm{d}t \\ -\int\limits_{\Omega_T\times E_{\rm out}}\mathcal{T}_\varepsilon(f)\mathcal{T}_\varepsilon(zw)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t-\int\limits_{\Omega}\Big(\chi_{E_{\rm out}}+\frac{\lambda}{|E_{\rm int}|}\chi_{E_{\rm int}}\Big)\mathcal{T}_\varepsilon(\overline{u_\varepsilon})\mathcal{T}_\varepsilon(\tilde{\varphi}_\varepsilon(0))\,\mathrm{d}x\,\mathrm{d}y\to 0\,,$$

as $\varepsilon \to 0$. Thus, in the limit, we obtain

$$
\begin{aligned}
& - |E_{\text{out}}| \int_{\Omega_T} u z' w \,\mathrm{d}x\,\mathrm{d}t - \lambda \int_{\Omega_T} u z' w \,\mathrm{d}x\,\mathrm{d}t \\
& + \int_{\Omega_T \times E_{\text{out}}} z \kappa (\nabla u + \nabla_y \widehat{u}^1) \cdot \nabla w \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
& + \frac{1}{D_0} \int_{\Omega_T \times \Gamma} z(\widehat{u}^1 + y^c \cdot \nabla u + \zeta) y^c \cdot \nabla w \,\mathrm{d}x\,\mathrm{d}\sigma_y\,\mathrm{d}t \\
& = |E_{\text{out}}| \int_{\Omega_T} f z w \,\mathrm{d}x\,\mathrm{d}t + (|E_{\text{out}}| + \lambda) \int_{\Omega} \overline{u} z(0) w \,\mathrm{d}x \,.
\end{aligned}
\tag{5.9}
$$

Equation (5.9) is, up to the usual density argument, the weak formulation of the limiting problem. Next, we insert into it the factorization for $\widehat{u}^1$ given in (5.8), obtaining (5.1). Since the homogenized matrix B_{hom} is symmetric and positive definite (see Proposition 5.2 below), by standard theory of parabolic equations, the solution of (5.1) is unique, and then all the above convergences hold true for the whole sequences. □

Proposition 5.2. *The matrix B_{hom} can be rewritten as*

$$
\begin{aligned}
B^{ij}_{\text{hom}}(x) = & \int_{E_{\text{out}}} \kappa(x,y) \nabla_y (\hat{\chi}^i - y^{ci}) \cdot \nabla_y (\hat{\chi}^j - y^{cj}) \,\mathrm{d}y \\
& + \frac{1}{D_0} \int_{\Gamma} (\hat{\chi}^i - y^{ci} + \hat{\zeta}^i)(\hat{\chi}^j - y^{cj} + \hat{\zeta}^j) \,\mathrm{d}\sigma_y \,,
\end{aligned}
\tag{5.10}
$$

for i, $j = 1, \ldots, N$. It follows that B_{hom} is symmetric and positive definite.

Proof. On using $\hat{\chi}^i$ in (3.5) as test function, we find

$$
\int_{E_{\text{out}}} \kappa \nabla_y (\hat{\chi}^j - y^{cj}) \cdot \nabla_y \hat{\chi}^i \,\mathrm{d}y + \frac{1}{D_0} \int_{\Gamma} (\hat{\chi}^j - y^{cj} + \hat{\zeta}^j) \hat{\chi}^i \,\mathrm{d}\sigma_y = 0 \,. \tag{5.11}
$$

On the other hand,

$$
\int_{\Gamma} (\hat{\chi}^j - y^{cj} + \hat{\zeta}^j) \hat{\zeta}^i \,\mathrm{d}\sigma_y = 0 \,, \tag{5.12}
$$

owing to (3.15) and recalling that $\hat{\zeta}^i$ is independent of y. By adding (5.11) and (5.12) to (5.2), we prove (5.10). The positive definiteness of B_{hom} is then standard. □

Remark 5.3. Notice that the distributional formulation of the problem (5.1) is

$$
\begin{aligned}
& (|E_{\text{out}}| + \lambda) u_t - \operatorname{div}(B_{\text{hom}} \nabla u) = |E_{\text{out}}| f \,, && \text{in } \Omega_T; \\
& u = 0 \,, && \text{on } \partial\Omega \times (0,T); \\
& u(x,0) = \overline{u} \,, && \text{in } \Omega.
\end{aligned}
\tag{5.13}
$$

The homogenized diffusion matrix depends on D_0 and this calls for a second limit $D_0 \to 0$. □

The final part of this section is devoted to perform the limit $D_0 \to 0$. In what follows, the dependence on D_0 of the involved functions is not denoted explicitly, but it is left understood.
For later use, let us define the functional space

$$\mathcal{X}^{\Gamma}_{\#}(E_{\text{out}}) := \{\psi \in H^1_{\#}(E_{\text{out}}) \,:\, \psi \text{ is independent of } y \text{ on } \Gamma\}\,. \tag{5.14}$$

Theorem 5.4. *For $D_0 \to 0$, we have that there exists $u_o \in L^2(0,T;H^1_0(\Omega))$ such that*

$$u \rightharpoonup u_o\,, \qquad \textit{weakly in } L^2(0,T;H^1_0(\Omega)),$$

where u_o is the unique solution of the problem

$$\begin{aligned} -\,(|E_{\text{out}}|+\lambda)\int_{\Omega_T} u_o\varphi_t \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} B_{o,hom}\nabla u_o\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \\ = |E_{\text{out}}|\int_{\Omega_T} f\varphi\,\mathrm{d}x\,\mathrm{d}t + (|E_{\text{out}}|+\lambda)\int_{\Omega}\overline{u}\varphi(0)\,\mathrm{d}x\,, \end{aligned} \tag{5.15}$$

for any $\varphi \in H^1(\Omega_T)$ with $\varphi = 0$ on $\partial\Omega$ and for $t = T$. Here, $B_{o,hom}$ is defined by

$$B^{ij}_{o,hom}(x) = -\int_{E_{\text{out}}} \kappa_{ik}(x,y)\partial_{y_k}(\hat{\chi}^j_o - y^{cj})\,\mathrm{d}y - \int_{\Gamma} \kappa\nabla_y(\hat{\chi}^j_o - y^{cj})\cdot\nu y^{ci}\,\mathrm{d}\sigma_y\,, \tag{5.16}$$

for i, $j = 1, \ldots, N$, where $\hat{\chi}_0 = (\hat{\chi}^1_o, \ldots, \hat{\chi}^N_o)$ is given in (5.26)–(5.29)*, below.*

Proof. First, we remark that the weak formulation of problem (3.5)–(3.6) is

$$\int_{E_{\text{out}}} \kappa(x,y)\nabla_y(\hat{\chi}^j - y^{cj})\cdot\nabla_y\varphi\,\mathrm{d}y + \frac{1}{D_0}\int_{\Gamma}(\hat{\chi}^j - y^{cj} + \hat{\zeta}^j)\varphi\,\mathrm{d}\sigma_y = 0\,, \tag{5.17}$$

for every $\varphi \in H^1_{\#}(E_{\text{out}})$. However, problem (3.5)–(3.6) can be written in a variational form. Following the ideas in [26], for $j = 1, \ldots, N$, we can introduce the function $U^j(x,y) = \hat{\chi}^j(x,y) - y^{cj}\Psi(y)$, where $\Psi \in \mathcal{C}^\infty(\overline{E_{\text{out}}})$ is a function such that $\Psi = 0$ on ∂Y, $\Psi \equiv 1$ on Γ and $\int_{E_{\text{out}}} y^{cj}\Psi\,\mathrm{d}y = 0$. Clearly, $U^j = \hat{\chi}^j - y^{cj} + y^{cj}(1-\Psi)$. Moreover, set

$$g^j_1 = -\operatorname{div}_y\big(\kappa\nabla_y(y^{cj}(1-\Psi))\big)\,, \qquad \text{in } E_{\text{out}}$$

and

$$g^j_2 = \kappa\nabla_y\big(y^{cj}(1-\Psi)\big)\cdot\nu\,, \qquad \text{on } \Gamma.$$

By (3.5)–(3.7), we obtain that $U^j \in L^\infty(\Omega; H^1_\#(E_{\mathrm{out}}))$ and it satisfies the following variational problem

$$-\operatorname{div}_y(\kappa \nabla_y U^j) = g_1^j\,, \qquad \text{in } E_{\mathrm{out}}; \tag{5.18}$$

$$\kappa \nabla_y U^j \cdot \nu = \frac{1}{D_0}(U^j + \hat{\zeta}^j) + g_2^j\,, \quad \text{on } \Gamma; \tag{5.19}$$

$$\int_{E_{\mathrm{out}}} U^j(x,y)\,\mathrm{d}y = 0\,. \tag{5.20}$$

Multiplying equation (5.18) by U^j (which is an admissible test function), integrating by parts and using (5.19), we arrive at

$$\begin{aligned}\int_{E_{\mathrm{out}}} \kappa \nabla_y U^j \cdot \nabla_y U^j\,\mathrm{d}y &+ \frac{1}{D_0}\int_{\Gamma} (U^j + \hat{\zeta}^j) U^j\,\mathrm{d}\sigma_y \\ &= \int_{E_{\mathrm{out}}} g_1^j U^j\,\mathrm{d}y - \int_{\Gamma} g_2^j U^j\,\mathrm{d}\sigma_y = \int_{E_{\mathrm{out}}} \kappa \nabla_y (y^{cj}(1-\Psi)) \cdot \nabla_y U^j\,\mathrm{d}y\,. \end{aligned} \tag{5.21}$$

Taking into account that, by construction, $U^j = \hat{\chi}^j - y^{cj}$ on Γ and that $\hat{\zeta}^j$ is independent of y on Γ, by (5.12), it follows

$$\int_{\Gamma} (U^j + \hat{\zeta}^j)\hat{\zeta}^j\,\mathrm{d}\sigma_y = \int_{\Gamma} (\hat{\chi}^j - y^{cj} + \hat{\zeta}^j)\hat{\zeta}^j\,\mathrm{d}\sigma_y = 0\,,$$

so that the equality (5.21) can be rewritten in the form

$$\int_{E_{\mathrm{out}}} \kappa \nabla_y U^j \cdot \nabla_y U^j\,\mathrm{d}y + \frac{1}{D_0}\int_{\Gamma} (U^j + \hat{\zeta}^j)(U^j + \hat{\zeta}^j)\,\mathrm{d}\sigma_y = \int_{E_{\mathrm{out}}} \kappa \nabla_y (y^{cj}(1-\Psi)) \cdot \nabla_y U^j\,\mathrm{d}y\,.$$

This leads to the energy estimate

$$\sup_{\Omega} \left(\int_{E_{\mathrm{out}}} |\nabla_y U^j|^2\,\mathrm{d}y + \frac{1}{D_0}\int_{\Gamma} (U^j + \hat{\zeta}^j)^2\,\mathrm{d}\sigma_y \right) \le \gamma\,, \tag{5.22}$$

where γ depends on C and $\|\Psi\|_{\mathcal{C}^1(E_{\mathrm{out}})}$, but it is independent of D_0. This implies that, for $j = 1, \dots, N$,

$$U^j + \hat{\zeta}^j = \hat{\chi}^j - y^{cj} + \hat{\zeta}^j \to 0\,, \qquad \text{strongly in } L^\infty(\Omega; L^2(\Gamma)), \tag{5.23}$$

when $D_0 \to 0$; moreover, there exists $\hat{\chi}_o^j \in L^\infty(\Omega; H^1_\#(E_{\mathrm{out}}))$ such that, up to a subsequence,

$$\hat{\chi}^j \rightharpoonup \hat{\chi}_o^j\,, \qquad \text{weakly in } L^\infty(\Omega; H^1_\#(E_{\mathrm{out}})), \tag{5.24}$$

when $D_0 \to 0$. In particular, from (5.23) and (5.24), it follows that $\hat{\chi}_o^j - y^{cj}$ is independent of y on Γ.

In order to pass to the limit in the weak formulation (5.17), let us take a test function $\varphi \in \mathcal{X}^{\Gamma}_{\#}(E_{\mathrm{out}})$. Then, for a.e. $x \in \Omega$, using again (5.12), we get

$$\int\limits_{E_{\mathrm{out}}} \kappa(x,y)\nabla_y(\hat{\chi}^j_o - y^{cj}) \cdot \nabla_y \varphi \, \mathrm{d}y = 0 \,, \qquad \forall \varphi \in \mathcal{X}^{\Gamma}_{\#}(E_{\mathrm{out}}) \,. \tag{5.25}$$

Taking into account that $\hat{\chi}^j_o - y^{cj}$ is independent of y on Γ, we can extend $\hat{\chi}^j_o$ to E_{int} in such a way that $\hat{\chi}^j_o - y^{cj}$ remains independent of y (still denoting this extension by $\hat{\chi}^j_o$). Hence, it is easy to see that such a $\hat{\chi}^j_o$ belongs to the space $L^\infty(\Omega; H^1_{\#}(Y))$ and, if we add to it a suitable function of x, it satisfies

$$-\operatorname{div}_y \big(\kappa \nabla_y(\hat{\chi}^j_o - y^{cj})\big) = 0 \qquad \text{in } E_{\mathrm{out}}; \tag{5.26}$$

$$\int\limits_{E_{\mathrm{out}}} \kappa(x,y)\nabla_y(\hat{\chi}^j_o - y^{cj}) \cdot \nu \, \mathrm{d}y = 0 \tag{5.27}$$

$$\hat{\chi}^j_o(x,y) - y^{cj} \qquad \text{is independent of } y \text{ in } E_{\mathrm{int}}; \tag{5.28}$$

$$\int\limits_{Y} \hat{\chi}^j_o(x,y) \, \mathrm{d}y = 0 \,, \tag{5.29}$$

for $j = 1, \dots, N$. Now, we have to pass to the limit in the homogenized matrix

$$B^{ij}_{hom}(x) = -\int\limits_{E_{\mathrm{out}}} \kappa_{ik}(x,y)\partial_{y_k}(\hat{\chi}^j - y^{cj}) \, \mathrm{d}y - \frac{1}{D_0}\int\limits_{\Gamma} (\hat{\chi}^j - y^{cj} + \hat{\zeta}^j) y^{ci} \, \mathrm{d}\sigma_y =: -I_1 - I_2 \,.$$

Clearly, when $D_0 \to 0$, for a.e. $x \in \Omega$, it follows

$$I_1 \to \int\limits_{E_{\mathrm{out}}} \kappa_{ik}(x,y)\partial_{y_k}(\hat{\chi}^j_o - y^{cj}) \, \mathrm{d}y \,.$$

On the other hand, using (3.14), for I_2 we have

$$\begin{aligned}
I_2 = \frac{1}{D_0}\int\limits_{\Gamma} (\hat{\chi}^j - y^{cj} + \hat{\zeta}^j) y^{ci} \, \mathrm{d}\sigma_y &= \frac{1}{D_0}\int\limits_{\Gamma} (\hat{\chi}^j - y^{cj} + \hat{\zeta}^j) y^{ci} \Psi \, \mathrm{d}\sigma_y \\
&= \int\limits_{\Gamma} \kappa \nabla_y(\hat{\chi}^j - y^{cj}) \cdot \nu y^{ci} \Psi \, \mathrm{d}\sigma_y \\
= -\int\limits_{E_{\mathrm{out}}} \kappa \nabla_y(\hat{\chi}^j - y^{cj}) \cdot \nabla_y(y^{ci}\Psi) \, \mathrm{d}y &\to -\int\limits_{E_{\mathrm{out}}} \kappa \nabla_y(\hat{\chi}^j_o - y^{cj}) \cdot \nabla_y(y^{ci}\Psi) \, \mathrm{d}y \\
&= \int\limits_{\Gamma} \kappa \nabla_y(\hat{\chi}^j_o - y^{cj}) \cdot \nu y^{ci} \Psi \, \mathrm{d}\sigma_y = \int\limits_{\Gamma} \kappa \nabla_y(\hat{\chi}^j_o - y^{cj}) \cdot \nu y^{ci} \, \mathrm{d}\sigma_y \,.
\end{aligned}$$

Thus, for a.e. $x \in \Omega$, $B^{ij}_{hom}(x) \to B^{ij}_{o,hom}(x)$, where the limit matrix $B^{ij}_{o,hom}(x)$ is given by (5.16). Moreover, by (5.22), it follows that the entries of the matrix B_{hom} are bounded by a constant in Ω and thus $B^{ij}_{hom} \to B^{ij}_{o,hom}$, strongly in $L^q(\Omega)$, for all $q \geq 1$.

By taking into account that the matrix B^{ij}_{hom} is symmetric and positive definite, independently of D_0, we get the standard energy estimate

$$\sup_{0<t<T} \int\limits_{\Omega} u^2(t)\,\mathrm{d}x + \int\limits_{\Omega_T} |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t \leq \gamma\,, \tag{5.30}$$

for a suitable $\gamma > 0$, independent of D_0. Then, up to a subsequence, we get that there exists $u_o \in L^2(0,T;H^1_0(\Omega))$ such that, when $D_0 \to 0$,

$$u \rightharpoonup u_o\,, \qquad \text{weakly in } L^2(0,T;H^1_0(\Omega)).$$

Thus, passing to the limit for $D_0 \to 0$ in the weak formulation (5.1), we arrive at (5.15). The uniqueness is a consequence of the symmetry and the positive definiteness of the matrix $B_{o,hom}$ (see Remark 5.5 below). □

Remark 5.5. The distributional formulation of the limit problem reads like

$$\begin{aligned} &(|E_{\mathrm{out}}| + \lambda)\, u_{o,t} - \mathrm{div}(B_{o,hom}\nabla u_o) = |E_{\mathrm{out}}| f\,, && \text{in } \Omega_T;\\ &u_o = 0\,, && \text{on } \partial\Omega \times (0,T);\\ &u_o(x,0) = \overline{u}\,, && \text{in } \Omega, \end{aligned} \tag{5.31}$$

which coincides with problem (50) in [6], once we have proven that the homogenized limit matrix $B_{o,hom}$ is the same as the matrix obtained in [6, formula (49)].
To this purpose, let us take $\varphi = \hat{\chi}^i_o - y^{ci}\Psi \in \mathcal{X}^{\Gamma}_{\#}(E_{\mathrm{out}})$, extended independently of y in E_{int}, as test function in (5.25) (with Ψ as above). It follows that

$$\int\limits_Y \kappa(x,y)\nabla_y(\hat{\chi}^j_o - y^{cj}) \cdot \nabla_y \hat{\chi}^i_o\,\mathrm{d}y - \int\limits_Y \kappa(x,y)\nabla_y(\hat{\chi}^j_o - y^{cj}) \cdot \nabla_y(y^{ci}\Psi)\,\mathrm{d}y = 0\,. \tag{5.32}$$

Hence,

$$B_{o,hom}^{ij}(x) = -\int\limits_{E_{\text{out}}} \kappa_{ik}(x,y)\partial_{y_k}(\hat{\chi}_o^j - y^{cj})\,\mathrm{d}y - \int\limits_{\Gamma} \kappa\nabla_y(\hat{\chi}_o^j - y^{cj})\cdot\nu y^{ci}\,\mathrm{d}\sigma_y$$

$$= -\int\limits_{Y} \kappa(x,y)\nabla_y(\hat{\chi}_o^j - y^{cj})\cdot\nabla_y y^{ci}\,\mathrm{d}y - \int\limits_{\Gamma} \kappa\nabla_y(\hat{\chi}_o^j - y^{cj})\cdot\nu y^{ci}\Psi\,\mathrm{d}\sigma_y$$

$$= \int\limits_{Y} \kappa(x,y)\nabla_y(\hat{\chi}_o^j - y^{cj})\cdot\nabla_y(\hat{\chi}_o^i - y^{ci})\,\mathrm{d}y$$

$$- \int\limits_{Y} \kappa(x,y)\nabla_y(\hat{\chi}_o^j - y^{cj})\cdot\nabla_y(y^{ci}\Psi)\,\mathrm{d}y - \int\limits_{\Gamma} \kappa\nabla_y(\hat{\chi}_o^j - y^{cj})\cdot\nu y^{ci}\Psi\,\mathrm{d}\sigma_y$$

$$= \int\limits_{Y} \kappa(x,y)\nabla_y(\hat{\chi}_o^j - y^{cj})\cdot\nabla_y(\hat{\chi}_o^i - y^{ci})\,\mathrm{d}y$$

$$- \int\limits_{E_{\text{out}}} \kappa(x,y)\nabla_y(\hat{\chi}_o^j - y^{cj})\cdot\nabla_y(y^{ci}\Psi)\,\mathrm{d}y - \int\limits_{\Gamma} \kappa\nabla_y(\hat{\chi}_o^j - y^{cj})\cdot\nu y^{ci}\Psi\,\mathrm{d}\sigma_y$$

$$= \int\limits_{Y} \kappa(x,y)\nabla_y(\hat{\chi}_o^j - y^{cj})\cdot\nabla_y(\hat{\chi}_o^i - y^{ci})\,\mathrm{d}y + \int\limits_{E_{\text{out}}} \operatorname{div}\big(\kappa(x,y)\nabla_y(\hat{\chi}_o^j - y^{cj})\big)y^{ci}\Psi\,\mathrm{d}y$$

$$= \int\limits_{Y} \kappa(x,y)\nabla_y(\hat{\chi}_o^j - y^{cj})\cdot\nabla_y(\hat{\chi}_o^i - y^{ci})\,\mathrm{d}y\,,$$

where, in the third and fourth lines, we added (5.32) and, in the last line, we used (5.26). Therefore, we get that $B_{o,hom}$ is the same symmetric and positive definite matrix appearing in [6, formula (49)], once we notice that (5.26)–(5.29) coincides with problem (44)–(46) in Lemma 4.4 of [6]. Hence, the previous limit problem (5.31) is exactly problem (50) in [6] (recall that y^{cj} and y^j coincide up to an additive constant). □

Remark 5.6. Notice that (5.26)–(5.29) are essentially the same problem defining the cell function χ in Lemma 3.1, which is involved in the homogenization of the case $\alpha > 1$; indeed, the two cell functions coincide up to a quantity independent of y. This implies that $B_{o,hom} = A_{\text{hom}}$. Therefore, after the two limit procedures, we arrive at the same problem given in (4.15), for $\alpha > 1$. This fact seems to be connected to the high degeneracy of the ε-problem in this case. □

6. Homogenization of the case $-1 < \alpha < 1$

In this section, we study the homogenization for the case $-1 < \alpha < 1$ and we find, in the limit, a standard parabolic problem, in which the effective capacity is given by a weighted average of the original capacities of the outer and the inner phases, the effective diffusion matrix depends only on the properties of the outer phase, but no memory of the contact coefficient D_0 is kept. However, the effective diffusion in this case is different from the one obtained in Section 4 (see Remark 6.3).

Theorem 6.1. *The limiting function u, appearing in (2.24), is the unique solution of*

$$-(|E_{\mathrm{out}}|+\lambda)\int_{\Omega_T} u\varphi_t \,\mathrm{d}x\,\mathrm{d}t+\int_{\Omega_T} C_{\mathrm{hom}}\nabla u\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \\ =|E_{\mathrm{out}}|\int_{\Omega_T} f\varphi\,\mathrm{d}x\,\mathrm{d}t+(|E_{\mathrm{out}}|+\lambda)\int_{\Omega}\overline{u}\varphi(0)\,\mathrm{d}x\,, \quad (6.1)$$

for all $\varphi\in H^1(\Omega_T)$ with $\varphi=0$ on $\partial\Omega\times(0,T)$ and for $t=T$. Here, the homogenized matrix C_{hom} is defined by

$$C^{ij}_{\mathrm{hom}}(x)=-\int_{E_{\mathrm{out}}}\kappa(x,y)\,\nabla_y(\bar{\chi}^j-y^j)\cdot\nabla_y\, y^i\,\mathrm{d}y\,, \quad (6.2)$$

for i, $j=1$, $\dots$, N, where $\bar{\chi}$ has been introduced in Lemma 3.4.

Proof. We proceed as in the proof of Theorem 5.1, with the same test function φ_ε, arriving at

$$\int_{\Omega_T\times E_{\mathrm{out}}}\mathcal{T}_\varepsilon(\kappa_\varepsilon z)\mathcal{T}_\varepsilon(\nabla u^{\mathrm{out}}_\varepsilon)\cdot\mathcal{T}_\varepsilon(\nabla_y\Psi)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ +\frac{\varepsilon^{\frac{1-\alpha}{2}}}{D_0}\int_{\Omega_T\times\Gamma}\mathcal{T}^b_\varepsilon\Big(\frac{[u_\varepsilon]}{\varepsilon^{\frac{1+\alpha}{2}}}\Big)\mathcal{T}^b_\varepsilon(z\Psi)\,\mathrm{d}x\,\mathrm{d}\sigma_y\,\mathrm{d}t\to 0\,, \qquad \varepsilon\to 0\,. \quad (6.3)$$

When we recall (2.22), we get by density

$$\int_{\Omega_T\times E_{\mathrm{out}}}\kappa(\nabla u+\nabla_y\widehat{u}^1)\cdot\nabla_y\phi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t=0\,, \quad (6.4)$$

for all $\phi\in L^\infty(\Omega_T;H^1_\#(E_{\mathrm{out}}))$. The distributional formulation of (6.4) is

$$-\mathrm{div}_y(\kappa(\nabla u+\nabla_y\widehat{u}^1))=0\,, \qquad \text{in } \Omega_T\times E_{\mathrm{out}}; \quad (6.5)$$

$$\kappa(\nabla u+\nabla_y\widehat{u}^1)\cdot\nu=0\,, \qquad \text{on } \Omega_T\times\Gamma. \quad (6.6)$$

Then, we will apply below the factorization

$$\widehat{u}^1(x,y,t)=-\bar{\chi}(x,y)\cdot\nabla u(x,t)\,, \quad (6.7)$$

where $\bar{\chi}=(\bar{\chi}^1,\dots,\bar{\chi}^N)$ is introduced in Lemma 3.4.
We go on reproducing the argument in the proof of Theorem 5.1, that is we select the test function $\tilde{\varphi}_\varepsilon(x,x/\varepsilon,t)$, as there. However, owing to the different scaling, we

obtain here

$$
\begin{aligned}
&-\int\limits_{\Omega_T\times E_{\mathrm{out}}} \mathcal{T}_\varepsilon(u_\varepsilon)\mathcal{T}_\varepsilon(z'w)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t - \frac{\lambda}{|E_{\mathrm{int}}|}\int\limits_{\Omega_T\times E_{\mathrm{int}}} \mathcal{T}_\varepsilon(u_\varepsilon)\mathcal{T}_\varepsilon(z')\mathcal{M}_\varepsilon(w)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\quad+\int\limits_{\Omega_T\times E_{\mathrm{out}}} \mathcal{T}_\varepsilon(z\kappa_\varepsilon)\mathcal{T}_\varepsilon(\nabla u_\varepsilon)\cdot\mathcal{T}_\varepsilon(\nabla w)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\quad+\frac{\varepsilon^{\frac{1-\alpha}{2}}}{D_0}\int\limits_{\Omega_T\times\Gamma} \mathcal{T}^b_\varepsilon(z)\mathcal{T}^b_\varepsilon\Big(\frac{[u_\varepsilon]}{\varepsilon^{\frac{1+\alpha}{2}}}\Big)\frac{\mathcal{T}^b_\varepsilon(w-\mathcal{M}_\varepsilon(w))}{\varepsilon}\,\mathrm{d}x\,\mathrm{d}\sigma_y\,\mathrm{d}t \\
&\quad-\int\limits_{\Omega_T\times E_{\mathrm{out}}} \mathcal{T}_\varepsilon(f)\mathcal{T}_\varepsilon(zw)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t - \int\limits_{\Omega}\Big(\chi_{E_{\mathrm{out}}}+\frac{\lambda}{|E_{\mathrm{int}}|}\chi_{E_{\mathrm{int}}}\Big)\mathcal{T}_\varepsilon(\overline{u_\varepsilon})\mathcal{T}_\varepsilon(\varphi_\varepsilon(0))\,\mathrm{d}x\,\mathrm{d}y \to 0\,,
\end{aligned}
$$

as $\varepsilon\to 0$. In the limit, on invoking again (2.22), we obtain

$$
\begin{aligned}
&-|E_{\mathrm{out}}|\int\limits_{\Omega_T} uz'w\,\mathrm{d}x\,\mathrm{d}t - \lambda\int\limits_{\Omega_T} uz'w\,\mathrm{d}x\,\mathrm{d}t \\
&\quad+\int\limits_{\Omega_T\times E_{\mathrm{out}}} z\kappa(\nabla u+\nabla_y\widehat{u}^1)\cdot\nabla w\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\quad=|E_{\mathrm{out}}|\int\limits_{\Omega_T} fzw\,\mathrm{d}x\,\mathrm{d}t + (|E_{\mathrm{out}}|+\lambda)\int\limits_{\Omega}\overline{u}z(0)w\,\mathrm{d}x\,,
\end{aligned}
\tag{6.8}
$$

which, as above, yields the weak formulation of the limiting problem. Next, we insert into it the factorization for $\widehat{u}^1$ given in (6.7), obtaining (6.1). Due to Remark 6.2 below, we obtain uniqueness of the solution of (6.1) and, therefore, all the above convergences hold true for the whole sequences. □

Remark 6.2. The distributional formulation of the problem (6.1) reads as follows:

$$
\begin{aligned}
&(|E_{\mathrm{out}}|+\lambda)\,u_t-\operatorname{div}(C_{\mathrm{hom}}\nabla u)=|E_{\mathrm{out}}|f\,, && \text{in } \Omega_T;\\
&u=0\,, && \text{on } \partial\Omega\times(0,T);\\
&u(x,0)=\overline{u}\,, && \text{in } \Omega.
\end{aligned}
\tag{6.9}
$$

Moreover, by standard results, the matrix C_{hom} can be rewritten as

$$
C^{ij}_{\mathrm{hom}}(x)=\int\limits_{E_{\mathrm{out}}}\kappa(x,y)\,\nabla_y(\bar{\chi}^i(x,y)-y^i)\cdot\nabla_y(\bar{\chi}^j(x,y)-y^j)\,\mathrm{d}y\,,
\tag{6.10}
$$

which implies that it is symmetric and positive definite. Therefore, problem (6.1) is a standard parabolic problem, for which existence and uniqueness are well-known results.

□

Remark 6.3. Note that the limiting problem (6.9) does not keep any memory of the contact constant D_0. Moreover, we point out that, despite the fact that problem (6.9) presents the same form as the problem (4.15), it describes a different macroscopic material, since the corresponding homogenized matrices are different, as they involve different cell functions. $\square$

7. Homogenization of the case $\alpha = -1$

In this section, we study the case $\alpha = -1$, which leads to a completely different structure of the macroscopic problem. More precisely, we arrive at a bidomain problem, where the homogenized solution splits into two components satisfying a system coupled through an exchange term involving D_0 (see (7.5)). As $D_0 \to 0$, the bidomain structure is lost and we recover a standard parabolic problem.

Theorem 7.1. *The limiting functions u_1 and u_2, appearing in* (2.19)–(2.20)*, give the unique solution (u_1, u_2) of the system*

$$
\begin{aligned}
&-\lambda \int_{\Omega_T} u_1 \varphi_{1,t} \,\mathrm{d}x\,\mathrm{d}t - |E_{\mathrm{out}}| \int_{\Omega_T} u_2 \varphi_{2,t} \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} C_{\mathrm{hom}} \nabla u_2 \cdot \nabla \varphi_2 \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \frac{|\Gamma|}{D_0} \int_{\Omega_T} (u_2 - u_1)(\varphi_2 - \varphi_1) \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = |E_{\mathrm{out}}| \int_{\Omega_T} f \varphi_2 \,\mathrm{d}x\,\mathrm{d}t + \lambda \int_{\Omega} \overline{u} \varphi_1(0) \,\mathrm{d}x + |E_{\mathrm{out}}| \int_{\Omega} \overline{u} \varphi_2(0) \,\mathrm{d}x \,, \qquad (7.1)
\end{aligned}
$$

for all $\varphi_1, \varphi_2 \in H^1(\Omega_T)$ with $\varphi_1, \varphi_2 = 0$ on $\partial\Omega \times (0,T)$ and for $t = T$. Here, C_{hom} is the same matrix defined in (6.2).

Proof. We proceed as in the proof of Theorem 5.1, with the same test function φ_ε, arriving at

$$
\begin{aligned}
&\int_{\Omega_T \times E_{\mathrm{out}}} \mathcal{T}_\varepsilon(\kappa_\varepsilon z) \mathcal{T}_\varepsilon(\nabla u_\varepsilon^{\mathrm{out}}) \cdot \mathcal{T}_\varepsilon(\nabla_y \Psi) \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\qquad + \frac{\varepsilon}{D_0} \int_{\Omega_T \times \Gamma} \mathcal{T}_\varepsilon^b([u_\varepsilon]) \mathcal{T}_\varepsilon^b(z\Psi) \,\mathrm{d}x\,\mathrm{d}\sigma_y\,\mathrm{d}t \to 0 \,, \qquad \varepsilon \to 0 \,. \qquad (7.2)
\end{aligned}
$$

Recalling (2.25), we get that (6.4)–(6.7) are still in force.
Next, we select the test function $\tilde{\varphi}_\varepsilon(x,t) = \tilde{\varphi}(x, x/\varepsilon, t)$, with $\tilde{\varphi}$ given by

$$
\tilde{\varphi}(x,y,t) = \begin{cases} z_1(t) w_1(x) \,, & \text{in } \Omega_T \times E_{\mathrm{int}}, \\ z_2(t) w_2(x) \,, & \text{in } \Omega_T \times E_{\mathrm{out}}, \end{cases}
$$

with $z_1, z_2 \in \mathcal{C}^1([0,T])$, $z_1(T) = 0 = z_2(T)$, $w_1, w_2 \in \mathcal{C}_0^\infty(\Omega)$. Owing to the present scaling, we obtain

$$
\begin{aligned}
-\int\limits_{\Omega_T \times E_{\mathrm{out}}} \mathcal{T}_\varepsilon(u_\varepsilon)\mathcal{T}_\varepsilon(z_2' w_2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t - \frac{\lambda}{|E_{\mathrm{int}}|}\int\limits_{\Omega_T \times E_{\mathrm{int}}} \mathcal{T}_\varepsilon(u_\varepsilon)\mathcal{T}_\varepsilon(z_1' w_1)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
+ \int\limits_{\Omega_T \times E_{\mathrm{out}}} \mathcal{T}_\varepsilon(z_2)\mathcal{T}_\varepsilon(\kappa_\varepsilon)\mathcal{T}_\varepsilon(\nabla u_\varepsilon)\cdot\mathcal{T}_\varepsilon(\nabla w_2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
+ \frac{1}{D_0}\int\limits_{\Omega_T \times \Gamma} \mathcal{T}_\varepsilon^b([u_\varepsilon])\mathcal{T}_\varepsilon^b(z_2 w_2 - z_1 w_1)\,\mathrm{d}x\,\mathrm{d}\sigma_y\,\mathrm{d}t - \int\limits_{\Omega_T \times E_{\mathrm{out}}} \mathcal{T}_\varepsilon(f)\mathcal{T}_\varepsilon(z_2 w_2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
- \int\limits_{\Omega \times E_{\mathrm{out}}} \mathcal{T}_\varepsilon(\overline{u_\varepsilon})\mathcal{T}_\varepsilon(z_2(0) w_2)\,\mathrm{d}x\,\mathrm{d}y - \int\limits_{\Omega \times E_{\mathrm{int}}} \frac{\lambda}{|E_{\mathrm{int}}|}\mathcal{T}_\varepsilon(\overline{u_\varepsilon})\mathcal{T}_\varepsilon(z_1(0) w_1)\,\mathrm{d}x\,\mathrm{d}y \to 0\,,
\end{aligned}
\tag{7.3}
$$

as $\varepsilon \to 0$. In the limit, on invoking again (2.25), we obtain

$$
\begin{aligned}
-|E_{\mathrm{out}}|\int\limits_{\Omega_T} u_2 z_2' w_2\,\mathrm{d}x\,\mathrm{d}t - \lambda\int\limits_{\Omega_T} u_1 z_1' w_1\,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\Omega_T \times E_{\mathrm{out}}} z_2 \kappa(\nabla u_2 + \nabla_y \widehat{u}^1)\cdot\nabla w_2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
+ \frac{1}{D_0}\int\limits_{\Omega_T \times \Gamma} (u_2 - u_1)(z_2 w_2 - z_1 w_1)\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}t \\
= |E_{\mathrm{out}}|\int\limits_{\Omega_T} f z_2 w_2\,\mathrm{d}x\,\mathrm{d}t + |E_{\mathrm{out}}|\int\limits_{\Omega} \overline{u} z_2(0) w_2\,\mathrm{d}x\,\mathrm{d}y + \lambda\int\limits_{\Omega} \overline{u} z_1(0) w_1\,\mathrm{d}x\,\mathrm{d}y\,,
\end{aligned}
\tag{7.4}
$$

which, as above, by a density argument yields the weak formulation of the limiting problem. Next, we insert into it the factorization for $\widehat{u}^1$ given in (6.7), obtaining (7.1). Due to Remark 6.2 above and to the energy estimate (7.8) below, the problem (7.1) has uniqueness. Therefore, all the above convergences hold true for the whole sequences. □

Remark 7.2. The distributional formulation of (7.1) leads to the homogenized bidomain system given by

$$
\begin{aligned}
&|E_{\mathrm{out}}| u_{2,t} - \operatorname{div}(C_{\mathrm{hom}}\nabla u_2) + \frac{|\Gamma|}{D_0}(u_2 - u_1) = |E_{\mathrm{out}}| f\,, && \text{in } \Omega_T; \\
&\lambda u_{1,t} - \frac{|\Gamma|}{D_0}(u_2 - u_1) = 0\,, && \text{in } \Omega_T; \\
&u_1(0) = u_2(0) = \overline{u}\,, && \text{in } \Omega.
\end{aligned}
\tag{7.5}
$$

We stress the fact that the matrix C_{hom}, which is the same matrix appearing in Section 6, does not depend on D_0. □

Theorem 7.3. *For $D_0 \to 0$, we have that there exists $u_o \in L^2(0,T;H^1_0(\Omega))$ such that*

$$\begin{aligned} &u_1 \rightharpoonup u_o\,, && \textit{weakly in } L^2(\Omega_T);\\ &u_2 \rightharpoonup u_o\,, && \textit{weakly in } L^2(0,T;H^1_0(\Omega));\\ &u_2 - u_1 \to 0\,, && \textit{strongly in } L^2(\Omega_T), \end{aligned} \tag{7.6}$$

where u_o is the unique solution of the problem

$$\begin{aligned} -(|E_{\text{out}}|+\lambda)\int_{\Omega_T} u_o\varphi_t\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} C_{\text{hom}}\nabla u_o\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \\ = |E_{\text{out}}|\int_{\Omega_T} f\varphi\,\mathrm{d}x\,\mathrm{d}t + (|E_{\text{out}}|+\lambda)\int_{\Omega}\overline{u}\varphi(0)\,\mathrm{d}x\,, \end{aligned} \tag{7.7}$$

for any $\varphi \in H^1(\Omega_T)$ with $\varphi = 0$ on $\partial\Omega\times(0,T)$ and for $t = T$. Here, C_{hom} is the same matrix defined in (6.2).

Proof. Multiplying the first equation in (7.5) by u_2 and the second one by u_1, summing and integrating by parts, we get

$$\begin{aligned} \frac{\lambda}{2}\int_{\Omega} u_1^2\,\mathrm{d}x + \frac{|E_{\text{out}}|}{2}\int_{\Omega} u_2^2\,\mathrm{d}x + \int_{\Omega_T} C_{\text{hom}}\nabla u_2\cdot\nabla u_2\,\mathrm{d}x\,\mathrm{d}t + \frac{|\Gamma|}{D_0}\int_{\Omega_T}(u_2-u_1)^2\,\mathrm{d}x\,\mathrm{d}t \\ = |E_{\text{out}}|\int_{\Omega_T} f u_2\,\mathrm{d}x\,\mathrm{d}t + \left(\frac{\lambda+|E_{\text{out}}|}{2}\right)\int_{\Omega}\overline{u}^2\,\mathrm{d}x\,, \end{aligned}$$

which, by using Poincaré's inequality on u_2, leads to the following energy estimate

$$\sup_{0<t<T}\left(\int_{\Omega} u_1^2\,\mathrm{d}x + \int_{\Omega} u_2^2\,\mathrm{d}x\right) + \int_{\Omega_T}|\nabla u_2|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{D_0}\int_{\Omega_T}(u_2-u_1)^2\,\mathrm{d}x\,\mathrm{d}t \le \gamma\,, \tag{7.8}$$

for a suitable positive constant independent of D_0.
The energy estimate (7.8) implies that there exist two functions $u_{o1} \in L^2(\Omega_T)$ and $u_{o2} \in L^2(0,T;H^1_0(\Omega))$, such that, up to a subsequence, we have

$$\begin{aligned} &u_1 \rightharpoonup u_{o1}\,, && \text{weakly in } L^2(\Omega_T);\\ &u_2 \rightharpoonup u_{o2}\,, && \text{weakly in } L^2(0,T;H^1_0(\Omega));\\ &u_2 - u_1 \to 0\,, && \text{strongly in } L^2(\Omega_T), \end{aligned} \tag{7.9}$$

for $D_0 \to 0$. Since in the limit, $u_{01} = u_{02} =: u_o \in L^2(0,T;H^1_0(\Omega))$, in the weak formulation of problem (7.5) we can take, in both the equations, the same test function $\varphi \in H^1(\Omega_T)$, with $\varphi = 0$ on $\partial\Omega\times(0,T)$ and $\varphi(T) = 0$ in Ω. Then, passing to the limit for $D_0 \to 0$, we obtain

$$|E_{\rm out}| \int\limits_{\Omega_T} f\varphi \,\mathrm{d}x\,\mathrm{d}t + (\lambda + |E_{\rm out}|) \int\limits_{\Omega} \overline{u}\varphi(0)\,\mathrm{d}x$$

$$= -\lambda \int\limits_{\Omega_T} u_1\varphi_t \,\mathrm{d}x\,\mathrm{d}t - |E_{\rm out}| \int\limits_{\Omega_T} u_2\varphi_t \,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\Omega_T} C_{\rm hom}\nabla u_2 \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t$$

$$\to -\lambda \int\limits_{\Omega_T} u_o\varphi_t \,\mathrm{d}x\,\mathrm{d}t - |E_{\rm out}| \int\limits_{\Omega_T} u_o\varphi_t \,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\Omega_T} C_{\rm hom}\nabla u_o \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t\,.$$

This proves (7.7). We can conclude that the limit problem has no longer a bidomain structure; indeed, it is a standard parabolic problem, for which existence and uniqueness are well-known, proving the statement. □

Remark 7.4. The distributional formulation of the problem (7.7) is given by

$$\begin{aligned} &(|E_{\rm out}| + \lambda)\, u_{o,t} - \mathrm{div}(C_{\rm hom}\nabla u_o) = |E_{\rm out}| f\,, && \text{in } \Omega_T; \\ &u_o = 0\,, && \text{on } \partial\Omega \times (0,T); \\ &u_o(x,0) = \overline{u}\,, && \text{in } \Omega. \end{aligned} \tag{7.10}$$

Notice that (7.10) coincides with problem (50) in [6] only formally, since the present matrix is different from the one appearing there. On the other hand, the problem (7.10) coincides with (6.9) and this is still due to the degeneracy of the problem in this case. □

8. Homogenization of the case $\alpha < -1$

In this section, we study the case $\alpha < -1$, which leads to a completely different problem (see Remarks 8.2 and 8.3), given by a kind of "bidomain" system, in which the equation governing the outer phase coincides with the homogenized equation of a standard homogenous Neumann problem in a periodically perforated domain. In turn, the inner phase coincides with the limit of the initial datum.

Theorem 8.1. *The limiting functions u_1 and u_2, appearing in* (2.19)–(2.20)*, give the unique solution (u_1, u_2) of the system*

$$- \lambda \int\limits_{\Omega_T} u_1\varphi_{1,t} \,\mathrm{d}x\,\mathrm{d}t - |E_{\rm out}| \int\limits_{\Omega_T} u_2\varphi_{2,t} \,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\Omega_T} C_{\rm hom}\,\nabla\, u_2 \cdot \nabla\, \varphi_2 \,\mathrm{d}x\,\mathrm{d}t$$

$$= |E_{\rm out}| \int\limits_{\Omega_T} f\varphi_2 \,\mathrm{d}x\,\mathrm{d}t + \lambda \int\limits_{\Omega} \overline{u}\varphi_1(0)\,\mathrm{d}x + |E_{\rm out}| \int\limits_{\Omega} \overline{u}\varphi_2(0)\,\mathrm{d}x\,, \tag{8.1}$$

for all $\varphi_1, \varphi_2 \in H^1(\Omega_T)$ with $\varphi_1, \varphi_2 = 0$ on $\partial\Omega \times (0,T)$ and for $t = T$. Here, $C_{\rm hom}$ is the same matrix defined in (6.2).

Proof. We proceed as in the proof of Theorem 7.1, with the same test functions. Here, we only highlight the changes with respect to the previous case. More precisely, concerning the computations for the microscopic part, we still arrive at (7.2) where, recalling (2.22), the second integral is replaced by

$$\begin{aligned}
&\frac{1}{D_0\varepsilon^{\alpha}}\int\limits_{\Omega_T\times\Gamma}\mathcal{T}^b_\varepsilon([u_\varepsilon])\mathcal{T}^b_\varepsilon(z\Psi)\,\mathrm{d}x\,\mathrm{d}\sigma_y\,\mathrm{d}t \\
&\qquad=\frac{\varepsilon}{D_0\varepsilon^{(\alpha+1)/2}}\int\limits_{\Omega_T\times\Gamma}\mathcal{T}^b_\varepsilon\left(\frac{[u_\varepsilon]}{\varepsilon^{(\alpha+1)/2}}\right)\mathcal{T}^b_\varepsilon(z\Psi)\,\mathrm{d}x\,\mathrm{d}\sigma_y\,\mathrm{d}t \\
&\qquad=\frac{\varepsilon^{(1-\alpha)/2}}{D_0}\int\limits_{\Omega_T\times\Gamma}\mathcal{T}^b_\varepsilon\left(\frac{[u_\varepsilon]}{\varepsilon^{(\alpha+1)/2}}\right)\mathcal{T}^b_\varepsilon(z\Psi)\,\mathrm{d}x\,\mathrm{d}\sigma_y\,\mathrm{d}t\le\gamma\varepsilon^{(1-\alpha)/2}\to 0\,,
\end{aligned}$$

since $\alpha < -1$. So, we have that (6.4)–(6.7) are still in force. On the other hand, concerning the computations for the macroscopic part, we still arrive at (7.3) where, similarly as above, the fourth integral is replaced by

$$\begin{aligned}
&\frac{1}{D_0\varepsilon^{\alpha+1}}\int\limits_{\Omega_T\times\Gamma}\mathcal{T}^b_\varepsilon([u_\varepsilon])\mathcal{T}^b_\varepsilon(z_2w_2-z_1w_1)\,\mathrm{d}x\,\mathrm{d}\sigma_y\,\mathrm{d}t \\
&\qquad=\frac{\varepsilon^{-(\alpha+1)/2}}{D_0}\int\limits_{\Omega_T\times\Gamma}\mathcal{T}^b_\varepsilon\left(\frac{[u_\varepsilon]}{\varepsilon^{(\alpha+1)/2}}\right)\mathcal{T}^b_\varepsilon(z_2w_2-z_1w_1)\,\mathrm{d}x\,\mathrm{d}\sigma_y\,\mathrm{d}t\to 0
\end{aligned}$$

as $\varepsilon \to 0$, as a consequence of (2.22) and of the fact that $\alpha < -1$ (which implies $-(\alpha+1)/2 > 0$). Thus, in the limit, we obtain

$$\begin{aligned}
&-|E_{\mathrm{out}}|\int\limits_{\Omega_T}u_2z_2'w_2\,\mathrm{d}x\,\mathrm{d}t-\lambda\int\limits_{\Omega_T}u_1z_1'w_1\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad+\int\limits_{\Omega_T\times E_{\mathrm{out}}}z_2\kappa(\nabla u_2+\nabla_y\widehat{u}^1)\cdot\nabla w_2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&=|E_{\mathrm{out}}|\int\limits_{\Omega_T}fz_2w_2\,\mathrm{d}x\,\mathrm{d}t+|E_{\mathrm{out}}|\int\limits_{\Omega}\overline{u}z_2(0)w_2\,\mathrm{d}x\,\mathrm{d}y+\lambda\int\limits_{\Omega}\overline{u}z_1(0)w_1\,\mathrm{d}x\,\mathrm{d}y\,, \qquad (8.2)
\end{aligned}$$

which coincides with (7.4), apart form the lack of the integral containing D_0, appearing there. Therefore, the weak formulation of the limiting problem follows by the same density argument. Next, we insert into it the factorization for $\widehat{u}^1$ given in (6.7), obtaining (8.1). Due to Remark 6.2, the problem (8.1) clearly has uniqueness and, therefore, all the above convergences hold true for the whole sequence. □

Remark 8.2. The distributional formulation of the homogenized system (8.1) is given by

$$
\begin{aligned}
&|E_{\text{out}}|u_{2,t} - \operatorname{div}(C_{\text{hom}}\nabla u_2) = |E_{\text{out}}|f\,, && \text{in } \Omega_T; \\
&\lambda u_{1,t} = 0\,, && \text{in } \Omega_T; \\
&u_1(0) = u_2(0) = \overline{u}\,, && \text{in } \Omega,
\end{aligned}
\tag{8.3}
$$

where no dependence on D_0 appears. Moreover, we easily obtain

$$
u_1(x,t) = \overline{u}(x)\,, \qquad \text{a.e. in } \Omega_T. \tag{8.4}
$$

It is worthwhile to notice that the problem (8.3) can be considered as being a kind of "bidomain" system, in which the two overlapping phases are completely decoupled. □

Remark 8.3. We stress that, in the present case, the equation governing the relevant phase in the "bidomain" limit system (8.3) corresponds neither to the problem (4.15) in Section 4 (which coincides with [6, problem (50)]) nor to the problem (6.9) in Section 6 (which recovers, at least formally, [6, problem (50)]). Indeed, here both the capacity and the diffusion matrix are different from the ones appearing in [6, problem (50)], leading to a completely new equation which does not keep any memory of the physical properties of the inner phase. We notice that such a memory is lost also by u_1, which, as it can be seen in (8.4), depends only on the limit of the initial condition. □

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