ORIGINAL PAPER

Diffusion in inhomogeneous media with periodic microstructures

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Summary

Diffusion in inhomogeneous materials can be described by both the Fick and Fokker–Planck diffusion equations. Here, we study a mixed Fick and Fokker–Planck diffusion problem with coefficients rapidly oscillating both in space and time. We obtain macroscopic models performing the homogenization limit by means of the unfolding technique.

KEYWORDS:

Diffusion, Fick's law, Fokker–Planck diffusion law, homogenization

1 | INTRODUCTION

The study of the motion of particles diffusing in a confined region is relevant in many different fields (see, for instance, the recent papers[1,2,3,4,5,6] and the references therein). In several studies, it has been shown that the interaction of particles with the walls results into a diffusive coefficient depending on the space coordinates[7,8,9]. A rather natural microscopic counterpart is represented by the random walk models, with hopping probabilities depending on the site coordinates. Such kind of models have been, for instance, introduced in the study of wetting phenomena, in which the effect of competition between long range attraction and reflection at the wall is modeled[10]. We also mention that space dependent diffusion is also considered in some biological ionic channel models, to justify the selection of ionic species[11,12].

In the context of diffusion motion in inhomogeneous materials, due to the space dependence of the diffusion coefficient, the derivation of the macroscopic equation is not straightforward. Indeed, assuming that the flux is given either by $-B\nabla u$ or $-\nabla(cu)$, where B and c represent the diffusion coefficient and u the density field, gives rise to two different diffusion equations, known in the literature as the Fick and the Fokker–Planck diffusion laws[13,14,15,16,17], respectively. In the recent paper[18], relying on a hydrodynamic limit computation, it has been proved that the two different choices mentioned above for the flux are connected to the microscopic structure of the inhomogeneity. Indeed, for local isotropic space inhomogeneities, the Fokker–Planck version of the flux is found, whereas when the space inhomogeneity is exclusively due to local anisotropy, the Fick expression is recovered. In mixed situations, the general flux structure $-B\nabla(cu)$ is found and the corresponding general diffusion law $u_t = \nabla \cdot [B\nabla(cu)]$ is obtained.

Here, we study such a mixed Fick and Fokker–Planck diffusion problem for inhomogeneous materials, whose diffusion properties are described by means of rapidly oscillating coefficients with respect to both space and time (see the initial–boundary value problem (4)–(6) below). We assume that such a material has an underlying periodic microstructure, whose characteristic length is of order ε^{α} (ε and α being strictly positive real parameters), while its time oscillation has a period of order ε^{β}, β being another strictly positive parameter.

As usual in this kind of very fast oscillating problems, the main purpose is to obtain a macroscopic model, overcoming the difficulties due to the intricate original geometry and appearing, for instance, in the numerical approach. To this purpose, we are led to let $\varepsilon \to 0$, thus performing a homogenization limit. The resulting equation models the effective behavior of the medium in the macroscopic setting, keeping memory, in general, of the underlying periodic structure. However, the homogenization of the problem (4)–(6) seems to be a too ambitious goal, without some further structural assumptions on the coefficients. For this

reason, we shall confine our investigation to a particular case introduced in Section 2.2, where the capacitive coefficient in front of the time-derivative and the Fokker coefficient inside the spatial gradient are assumed to have a separate dependence on the time and space oscillating variables (similarly to the classical Fick case, treated in [19]), but admitting that the Fokker coefficient can be perturbed by a non–product additional coefficient of amplitude ε. We refer to this case as the *weakly non–product case*. However, as we will see in the sequel, when we consider the pure Fokker–Planck model (i.e., the diffusion matrix in front of the gradient term is the identity), with unit capacity, the perturbation does not play any role in the limit equation and disappears from the expression of the effective coefficients (see Subsections 2.2.2 and 2.2.3).

More precisely, in Subsection 2.2.2, by using the well–known two–scale expansion technique, introduced in [20], we formally show that the non–product perturbation does not affect the upscaled equation, when its amplitude ε is of the same order of the spatial oscillation period ε^α (i.e. $\alpha = 1$), as long as we assume the diffusion matrix $B = I$. This result is also rigorously proven in Section 4. It is rather natural to ask what would happen if such a perturbation were more intense with respect to the microscopic oscillation scale. This case will be considered in Subsection 2.2.3. However, since in the formal expansions we are obliged to deal only with integer powers, we cannot consider an exponent smaller than one for the ε amplitude of the non–product perturbation. Hence, we accelerate the microscopic spatial oscillations choosing the smaller oscillation period ε^2. We will show that also in this case the small non–product perturbation does not affect the upscaled equation. However, we do not propose this as a general conclusion, since it could depend on the special choice of the diffusion matrix and the capacity coefficients.

In Section 4, we will rigorously prove that the same property holds also in the general mixed Fick and Fokker-Planck case, if the amplitude of the non–product perturbation in the Fokker coefficient is strictly smaller than the spatial oscillation period, i.e. $\alpha < 1$.

At our knowledge, diffusion problems governed by Fick and/or Fokker–Planck laws depending on capacitive, diffusive and Fokker coefficients, highly oscillating with respect to time and space simultaneously are not considered in an extensive body of mathematical literature. Among the few results, we recall [19,11,21,22,23,24,25,26].

In particular, in [19] the authors have considered a homogenization problem in the framework of the standard heat equation, which is very close to the case analyzed in the present research. The main novelty of that paper, with respect to former literature, is not only the presence of a capacitive term, oscillating both in space and time, but also the fact that the homogenization problem has been solved under completely general assumptions on the space and time microscopic oscillation periods, i.e. the oscillation periods τ and ε, respectively, for time and space variables, are completely independent.

In this respect, those authors had to distinguish between two cases, namely, when the space period is smaller than or larger than the square of the time period. These cases were called *fast* and *slow* oscillations, respectively.

In the present paper, we shall have to distinguish between these two situations, as well; however, we will confine our investigation only to the case where time and space oscillation periods are powers of the common small parameter ε which, as recalled above, represents the perturbation size.

The approach we follow here is essentially the same as the one adopted in [19] and it is based on the periodic unfolding homogenization technique, first introduced in [27,28]. Part of our results are consequences of some properties already proven in [19], but the novelty of the present research relies on the new structure of the equation under consideration, which cannot be reduced to the classical Fick case considered in [19]. Moreover, the presence of the non–product perturbation in the Fokker coefficient represents a further non trivial feature of the problem.

It is worthwhile also to point out that the resulting homogenized equation has a non–standard structure, since it remains in an integral form with respect to the microvariables and, moreover, the capacity, the diffusivity and the Fokker coefficients mix in the limit (see Subsection 4.4), similarly as in the pure Fick case studied in [19], where the capacity and the diffusion coefficients appear in a mixed form in the upscaled equation (see (22)).

Only when the diffusion matrix is the identity (i.e., in the pure Fokker–Planck case) and the capacity is constant, the limit equations assume the standard form analogous to the starting one (see equation (45)), and in this case the memory of the periodic microstructure remains in the limit only as an average of the coefficients.

The paper is organized as follows: in Section 2.1, we present the general problem. In Section 2.2 we introduce the weaker version that we shall be able to treat rigorously, state our main results, and discuss some heuristics based on formal expansions. Some preliminary statements are proven in Section 3 and, finally, in Section 4 we state and prove our rigorous results.

2 | THE PROBLEM

Let Ω be an open connected bounded set in $\mathbb{R}^n$ with Lipschitz boundary, $T > 0$, and set $\Omega_T = \Omega \times (0, T)$. Let $\mathcal{Y} = (0, 1)^n$, $S = (0, 1)$, and call $\mathcal{Q} = \mathcal{Y} \times S$ the *microscopic cell* or, simply, the *cell*.

Given a function $w \in L^2(\Omega)$ (or $w \in L^2(\Omega_T)$), we will denote by $\|w\|_2$ its $L^2(\Omega)$-norm (or $L^2(\Omega_T)$-norm, respectively). Finally, γ will denote a strictly positive constant, which may vary from line to line.

2.1 | The general problem

Consider the real functions $a(x, t, y, \tau)$, $c(x, t, y, \tau)$, and the $n \times n$–matrix function $B(x, t, y, \tau)$ with $(x, t) \in \Omega_T$ and $\mathcal{Q}$–periodic in (y, τ). We assume that $B \in L^\infty(\Omega_T \times \mathcal{Q}; \mathbb{R}^{n \times n})$ is symmetric and satisfies the bounds

$$C^{-1}|\xi|^2 \le B_{ij}(x, t, y, \tau)\xi_i \xi_j \le C|\xi|^2, \tag{1}$$

for every $\xi \in \mathbb{R}^n$ and almost every $(x, t, y, \tau) \in \Omega_T \times \mathcal{Q}$. We assume, also, that $a, c \in L^\infty(\Omega_T \times \mathcal{Q})$ satisfy the bounds

$$C^{-1} \le a(x, t, y, \tau), c(x, t, y, \tau) \le C, \tag{2}$$

for almost every $(x, t, y, \tau) \in \Omega_T \times \mathcal{Q}$. Moreover, we assume that a, c, B_{ij} are Lipschitz–continuous on $\overline{\Omega}_T \times \overline{\mathcal{Q}}$.

Let $\alpha, \beta > 0$ and set

$$a^\varepsilon(x, t) = a\left(x, t, \frac{x}{\varepsilon^\alpha}, \frac{t}{\varepsilon^\beta}\right), \quad c^\varepsilon(x, t) = c\left(x, t, \frac{x}{\varepsilon^\alpha}, \frac{t}{\varepsilon^\beta}\right), \quad B^\varepsilon(x, t) = B\left(x, t, \frac{x}{\varepsilon^\alpha}, \frac{t}{\varepsilon^\beta}\right). \tag{3}$$

Given $f \in L^2(\Omega_T)$ and $\bar{u} \in H_0^1(\Omega)$, we are interested in studying the family of mixed Fick and Fokker–Planck problems with oscillating coefficients

$$a^\varepsilon \frac{\partial u_\varepsilon}{\partial t} - \text{div}(B^\varepsilon \nabla(c^\varepsilon u_\varepsilon)) = f, \quad \text{in } \Omega_T = \Omega \times (0, T); \tag{4}$$

$$u_\varepsilon(x, t) = 0, \quad \text{on } \partial\Omega \times (0, T); \tag{5}$$

$$u_\varepsilon(x, 0) = \bar{u}(x), \quad \text{in } \Omega. \tag{6}$$

Note that, in the case $c = 1$, the pure Fick problem is recovered, while in the case $B = I$, we obtain the pure Fokker-Planck problem. The terms a, B, c, and f will be respectively called *capacity coefficient*, *diffusion matrix*, *Fokker coefficient*, and *source term*.

If we let $v_\varepsilon = c^\varepsilon u_\varepsilon$, the above problem can be rewritten as the following Fick problem with linear lower order terms

$$\frac{\partial v_\varepsilon}{\partial t} - \text{div}\left(\frac{c^\varepsilon}{a^\varepsilon} B^\varepsilon \nabla v_\varepsilon\right) + B^\varepsilon \nabla\left(\frac{c^\varepsilon}{a^\varepsilon}\right) \cdot \nabla v_\varepsilon - \frac{1}{c^\varepsilon}\frac{\partial c^\varepsilon}{\partial t} v_\varepsilon = \frac{c^\varepsilon}{a^\varepsilon} f, \quad \text{in } \Omega_T = \Omega \times (0, T); \tag{7}$$

$$v_\varepsilon(x, t) = 0, \quad \text{on } \partial\Omega \times (0, T); \tag{8}$$

$$u_\varepsilon(x, 0) = c^\varepsilon(x, 0)\bar{u}(x), \quad \text{in } \Omega. \tag{9}$$

We remark that, by [29] Chapter 4, Theorem 9.1, for every $\varepsilon > 0$ fixed, the problem (7)–(9) admits a unique solution $v_\varepsilon \in L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega))$. Clearly this implies existence and uniqueness of the solution $u_\varepsilon \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$ of the problem (4)–(6).

As we pointed out above, the homogenization of the previous problem in its full generality provides some very hard technical difficulties. For this reason, we shall treat only the special *weakly non–product* case described in the following Section 2.2.

2.2 | The weakly non–product problem

Here we consider a special case in which the coefficients a and c of the problem (4)–(6) are factored in one term depending on (x, t, y) and another on (x, t, τ); namely, the dependence on the micro–variables is separated. However, for the coefficient c, typical of the Fokker–Planck equation, we can admit a small general perturbation of the product part.

Consider the real functions $a_1(x, t, y)$, $a_2(x, t, \tau)$, $b_1(x, t, y)$, $b_2(x, t, \tau)$, and $b(x, t, y, \tau)$, with $(x, t) \in \Omega_T$ and $\mathcal{Q}$–periodic in (y, τ). We assume, also, that $a_1, b_1 \in L^\infty(\Omega_T \times \mathcal{Y})$, $a_2, b_2 \in L^\infty(\Omega_T \times S)$, and $b \in L^\infty(\Omega_T \times \mathcal{Q})$ satisfy the bounds

$$C^{-1} \le a_1(x, t, y), a_2(x, t, \tau), b_1(x, t, y), b_2(x, t, \tau), b(x, t, y, \tau) \le C, \tag{10}$$

for almost every $(x, t, y, \tau) \in \Omega_T \times Q$. Moreover, we assume that a_1, b_1 are Lipschitz–continuous in $\overline{\Omega}_T \times \overline{\mathcal{Y}}$, a_2, b_2 in $\overline{\Omega}_T \times \overline{\mathcal{S}}$, and b in $\overline{\Omega}_T \times \overline{Q}$. We keep the same assumptions as in Subsection 2.1 for B, f, and $\bar{u}$.

Similarly as above, for $(x, t) \in \Omega_T$, we set

$$a_1^\varepsilon(x, t) = a_1\left(x, t, \frac{x}{\varepsilon^\alpha}\right), \quad a_2^\varepsilon(x, t) = a_2\left(x, t, \frac{t}{\varepsilon^\beta}\right), \tag{11}$$

$$b_1^\varepsilon(x, t) = b_1\left(x, t, \frac{x}{\varepsilon^\alpha}\right), \quad b_2^\varepsilon(x, t) = b_2\left(x, t, \frac{t}{\varepsilon^\beta}\right), \tag{12}$$

and

$$b^\varepsilon(x, t) = b\left(x, t, \frac{x}{\varepsilon^\alpha}, \frac{t}{\varepsilon^\beta}\right). \tag{13}$$

For $\varepsilon > 0$, we will study the family of problems

$$a_1^\varepsilon a_2^\varepsilon \frac{\partial u_\varepsilon}{\partial t} - \mathrm{div}(B^\varepsilon \nabla((b_1^\varepsilon b_2^\varepsilon + \varepsilon b^\varepsilon) u_\varepsilon)) = f, \quad \text{in } \Omega_T; \tag{14}$$

$$u_\varepsilon(x, t) = 0, \quad \text{on } \partial\Omega \times (0, T); \tag{15}$$

$$u_\varepsilon(x, 0) = \bar{u}(x), \quad \text{in } \Omega, \tag{16}$$

where $\varepsilon > 0$. Note that in the case $b = 0$ and $b_1 = b_2 = 1$, we recover the pure Fick problem discussed in[19] Section 3.2. We shall also consider the following auxiliary problem for the function $v_\varepsilon(x, t) = b_1^\varepsilon(x, t) u_\varepsilon(x, t)$:

$$a_1^\varepsilon a_2^\varepsilon \frac{\partial}{\partial t}\left(\frac{v_\varepsilon}{b_1^\varepsilon}\right) - \mathrm{div}\left(B^\varepsilon \nabla\left(\left(b_2^\varepsilon + \varepsilon \frac{b^\varepsilon}{b_1^\varepsilon}\right) v_\varepsilon\right)\right) = f, \quad \text{in } \Omega_T; \tag{17}$$

$$v_\varepsilon(x, t) = 0, \quad \text{on } \partial\Omega \times (0, T); \tag{18}$$

$$v_\varepsilon(x, 0) = \bar{v}_\varepsilon(x), \quad \text{in } \Omega, \tag{19}$$

where $\bar{v}_\varepsilon(x) = b_1^\varepsilon(x, 0) \bar{u}(x)$. Note that, since $b_1(\cdot, 0, \cdot) \in L^\infty(\Omega \times \mathcal{Y})$, it follows

$$\|\bar{v}_\varepsilon\|_2 \leq C. \tag{20}$$

On the other hand $\|\nabla \bar{v}_\varepsilon\|_2 = O(\varepsilon^{-\alpha})$.

Note that the weak formulation of problem (17)–(19) can be written as follows:

$$-\int_0^T \int_\Omega \frac{v_\varepsilon}{b_1^\varepsilon} \frac{\partial}{\partial t}(a_1^\varepsilon a_2^\varepsilon \phi) \, dx \, dt + \int_0^T \int_\Omega B^\varepsilon \nabla\left(\left(b_2^\varepsilon + \varepsilon \frac{b^\varepsilon}{b_1^\varepsilon}\right) v_\varepsilon\right) \cdot \nabla\phi \, dx \, dt$$

$$= \int_0^T \int_\Omega f\phi \, dx \, dt + \int_\Omega \frac{\bar{v}_\varepsilon(x)}{b_1^\varepsilon(x, 0)} a_1^\varepsilon(x, 0) a_2^\varepsilon(x, 0) \phi(x, 0) \, dx, \tag{21}$$

for any $\phi \in H^1(\Omega_T)$ such that $\phi = 0$ on $\partial\Omega \times (0, T)$ and $\phi(x, T) = 0$ a.e. in Ω.

For later use, we set

$$\omega_{\alpha,1} = \begin{cases} 1, & \alpha = 1, \\ 0, & 0 < \alpha < 1. \end{cases}$$

The main result of the paper is the following homogenization theorem, whose proof can be found in Subsection 4.3.

Theorem 1. *Assume $0 < \alpha \leq 1$ and $\beta > 0$. For any $\varepsilon > 0$, let u_ε be the unique solution of problem (14)–(16). Then, when $\varepsilon \to 0$, $u_\varepsilon \rightharpoonup u$ weakly in $L^2(\Omega_T)$, where $u \in L^2(0, T; H_0^1(\Omega))$ is the unique weak solution of the homogenized problem*

$$\int_Q \left[\frac{a_1}{\int_S a_2^{-1} \, d\tau} \frac{\partial}{\partial t}\left(\frac{u}{b_1 \int_\mathcal{Y} b_1^{-1} \, dy}\right) - \frac{1}{a_2 \int_S a_2^{-1} \, d\tau} \mathrm{div}\left(B_{\mathit{eff}} \nabla\left(b_2 \frac{u}{\int_\mathcal{Y} b_1^{-1} \, dy}\right)\right)\right.$$

$$\left. - \frac{1}{a_2 \int_S a_2^{-1} \, d\tau} \mathrm{div}\left(B\nabla_y\left(\omega_{\alpha,1} \frac{b}{b_1} - b_2\zeta + \chi \cdot \nabla b_2\right) \frac{u}{\int_\mathcal{Y} b_1^{-1} \, dy}\right)\right] \, dy \, d\tau = f, \quad \text{in } \Omega_T; \tag{22}$$

and

$$\frac{u(x, 0)}{\int_\mathcal{Y} b_1^{-1} \, dy} = \bar{u}(x)\left(\int_\mathcal{Y} a_1(x, 0, y) \, dy\right)\left(\int_\mathcal{Y} \frac{a_1(x, 0, y)}{b_1(x, 0, y)} \, dy\right)^{-1}, \quad \text{in } \Omega. \tag{23}$$

Here, the matrix B_{eff} is given by

$$B_{eff} = B\nabla_y(y - \chi) \tag{24}$$

and

1) if $\beta = 2\alpha$, χ and ζ are the solutions of (123) and (124), respectively;

2) if $\beta > 2\alpha$, χ and ζ are the solutions of (135) and (136), respectively;

3) if $\beta < 2\alpha$, χ and ζ are the solutions of (144) and (145), respectively.

2.2.1 | Formal expansions for the weakly non–product problem

In Section 4, we will prove rigorously the macroscopic equations for problem (14)–(16) in the case $0 < \alpha \leq 1$ and $\beta > 0$. Namely, we will be able to homogenize the system in the case in which the spatial oscillations are not too fast with respect to the amplitude of the non–product perturbation in the Fokker coefficient appearing in the Fokker–Planck equation (14).

However, before the rigorous approach, we first set up some formal expansions. More precisely, in Subsection 2.2.2, we consider, as an example, the case $\alpha = 1$ and $\beta = 2$, which is indeed rigorously covered by Theorem 1. Moreover, in Subsection 2.2.3 we formally approach also some cases (with integer exponents) not covered by the theory developed in Section 4, that is to say, when the spatial oscillations are faster than the amplitude of the non–product perturbation in the Fokker coefficient (i.e., $\alpha > 1$). In details, we consider the case $\alpha = 2$ and $\beta = 1, 2, 4$, in which time oscillations are respectively slower, as fast as, and faster than spatial ones. Note that the case $\alpha = 2$ and $\beta = 4$ corresponds to the natural parabolic scaling. In our formal expansions arguments, we assume that the diffusion matrix is $B = \text{Id}$, the capacity coefficients are $a_1 = a_2 = 1$, and the source term is $f = 0$. Note that this last assumption could be easily removed.

2.2.2 | Space oscillations as fast as the perturbation amplitude

As mentioned above in this section, we formally study the case $\alpha = 1$ and $\beta = 2$, which is covered by Theorem 1.

We let $y = x/\varepsilon$ and $\tau = t/\varepsilon^2$ and, by abusing the notation, we write the differential rules

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \frac{1}{\varepsilon^2}\frac{\partial}{\partial \tau}, \quad \nabla_x = \nabla_x + \frac{1}{\varepsilon}\nabla_y, \text{ and } \quad \Delta_x = \Delta_x + \frac{1}{\varepsilon}(\nabla_y \cdot \nabla_x + \nabla_x \cdot \nabla_y) + \frac{1}{\varepsilon^2}\Delta_y. \tag{25}$$

We then look for a solution of (14), using the formal expansion

$$u_\varepsilon(x, t) = u_0(x, t, y, \tau) + \varepsilon u_1(x, t, y, \tau) + \varepsilon^2 u_2(x, t, y, \tau) + \cdots, \tag{26}$$

with u_k a Q-periodic function with respect to (y, τ). By replacing (26) in (14), we get

$$\frac{\partial u_\varepsilon}{\partial t} = \frac{\partial u_0}{\partial t} + \frac{1}{\varepsilon^2}\frac{\partial u_0}{\partial \tau} + \varepsilon\frac{\partial u_1}{\partial t} + \frac{1}{\varepsilon}\frac{\partial u_1}{\partial \tau} + \varepsilon^2\frac{\partial u_2}{\partial t} + \frac{\partial u_2}{\partial \tau} + \varepsilon\frac{\partial u_3}{\partial \tau} + \varepsilon^2\frac{\partial u_4}{\partial \tau} + o(\varepsilon^2) \tag{27}$$

and

$$\begin{aligned}
\Delta[(b_1^\varepsilon b_2^\varepsilon + \varepsilon b^\varepsilon)u_\varepsilon] = &\Delta_x[(b_1 b_2 + \varepsilon b)u_0] + \frac{1}{\varepsilon}\nabla_y \cdot \nabla_x[(b_1 b_2 + \varepsilon b)u_0] \\
&+ \frac{1}{\varepsilon}\nabla_x \cdot \nabla_y[(b_1 b_2 + \varepsilon b)u_0] + \frac{1}{\varepsilon^2}\Delta_y[(b_1 b_2 + \varepsilon b)u_0] \\
&+ \varepsilon\Delta_x[(b_1 b_2 + \varepsilon b)u_1] + \nabla_y \cdot \nabla_x[(b_1 b_2 + \varepsilon b)u_1] \\
&+ \nabla_x \cdot \nabla_y[(b_1 b_2 + \varepsilon b)u_1] + \frac{1}{\varepsilon}\Delta_y[(b_1 b_2 + \varepsilon b)u_1] \\
&+ \varepsilon^2\Delta_x[(b_1 b_2 + \varepsilon b)u_2] + \varepsilon\nabla_y \cdot \nabla_x[(b_1 b_2 + \varepsilon b)u_2] \\
&+ \varepsilon\nabla_x \cdot \nabla_y[(b_1 b_2 + \varepsilon b)u_2] + \Delta_y[(b_1 b_2 + \varepsilon b)u_2] \\
&+ \varepsilon^2\nabla_y \cdot \nabla_x[b_1 b_2 u_3] + \varepsilon^2\nabla_x \cdot \nabla_y[b_1 b_2 u_3] \\
&+ \varepsilon\Delta_y[(b_1 b_2 + \varepsilon b)u_3] + \varepsilon^2\Delta_y[b_1 b_2 u_4] + o(\varepsilon^2).
\end{aligned} \tag{28}$$

Thus, at order $1/\varepsilon^2$, we find the equation

$$\frac{\partial u_0}{\partial \tau} - \Delta_y(b_1 b_2 u_0) = 0. \tag{29}$$

Recalling that b_1 does not depend on τ, see (12), we let $v_0(x, t, y, \tau) = b_1(x, t, y)u_0(x, t, y, \tau)$ and find for v_0 the equation

$$\frac{1}{b_1}\frac{\partial v_0}{\partial \tau} - \Delta_y(b_2 v_0) = 0, \tag{30}$$

which must be solved assuming that v_0 is $\mathcal{Q}$-periodic in (y, τ). We prove, indeed, that v_0 does not depend on the microscopic variables: we first multiply (30) times v_0 and integrate on the microscopic cell

$$
\begin{aligned}
0 &= \int_{\mathcal{Q}} \frac{1}{b_1} \frac{\partial v_0}{\partial \tau} v_0 \, dy \, d\tau - \int_{\mathcal{Q}} [\Delta_y(b_2 v_0)] v_0 \, dy \, d\tau \\
&= \int_{\mathcal{Y}} \frac{1}{2b_1} \left(\int_S \frac{\partial v_0^2}{\partial \tau} \, d\tau \right) dy + \int_{\mathcal{Q}} b_2 \nabla_y v_0 \cdot \nabla_y v_0 \, dy \, d\tau - \int_S b_2 \left(\int_{\partial \mathcal{Y}} v_0 \nabla_y v_0 \cdot \nu \, d\sigma \right) d\tau .
\end{aligned}
\tag{31}
$$

By periodicity, the first and the third integral vanish, hence

$$
0 = \int_{\mathcal{Q}} b_2 |\nabla_y v_0|^2 \, dy \, d\tau \geq C^{-1} \int_{\mathcal{Q}} |\nabla_y v_0|^2 \, dy \, d\tau ,
\tag{32}
$$

where we used (10). This implies that v_0 is constant with respect to y.

On the other hand, since both b_2 and v_0 do not depend on y, from (30) we immediately get that v_0 does not depend on τ as well. Note that, since $v_0(x, t) = b_1(x, t, y) u_0(x, t, y, \tau)$, we have that u_0 does not depend on τ.

We now consider the $1/\varepsilon$ order equation. From (14), (27), and (28), we have

$$
\frac{\partial u_1}{\partial \tau} - \nabla_y \cdot \nabla_x(b_1 b_2 u_0) - \nabla_x \cdot \nabla_y(b_1 b_2 u_0) - \Delta_y(b u_0) - \Delta_y(b_1 b_2 u_1) = 0.
\tag{33}
$$

Since b_2 and $v_0 = b_1 u_0$ do not depend on y, (33) simplifies to

$$
\frac{\partial u_1}{\partial \tau} - \Delta_y(b u_0) - \Delta_y(b_1 b_2 u_1) = 0.
\tag{34}
$$

We now let $v_1(x, t, y, \tau) = b_1(x, t, y) u_1(x, t, y, \tau)$ and, from (34), we get

$$
\frac{1}{b_1} \frac{\partial v_1}{\partial \tau} - b_2 \Delta_y v_1 = \Delta_y\left(\frac{b}{b_1} \right) v_0.
\tag{35}
$$

We now look for a solution of the above equation in the factored form

$$
v_1(x, t, y, \tau) = -\zeta(x, t, y, \tau) v_0(x, t),
\tag{36}
$$

with ζ a $\mathcal{Q}$-periodic function with respect to (y, τ). By plugging (36) into (35), we get that ζ has to solve the equation

$$
\frac{1}{b_1} \frac{\partial \zeta}{\partial \tau} - b_2 \Delta_y \zeta = -\Delta_y\left(\frac{b}{b_1} \right).
\tag{37}
$$

We, finally, consider the ε^0 order equation, which will yield a compatibility condition providing an equation for u_0. From (14), (27), and (28), we have

$$
\begin{aligned}
\frac{\partial u_0}{\partial t} + \frac{\partial u_2}{\partial \tau} &- \Delta_x(b_1 b_2 u_0) - \nabla_y \cdot \nabla_x(b u_0) - \nabla_x \cdot \nabla_y(b u_0) - \nabla_y \cdot \nabla_x(b_1 b_2 u_1) \\
&- \nabla_x \cdot \nabla_y(b_1 b_2 u_1) - \Delta_y(b u_1) - \Delta_y(b_1 b_2 u_2) = 0,
\end{aligned}
\tag{38}
$$

which can be seen as an equation for u_2. Hence, as usual, we introduce the function $v_2(x, t, y, \tau) = b_1(x, t, y) u_2(x, t, y, \tau)$ and rewrite (38) as

$$
\begin{aligned}
\frac{1}{b_1} \frac{\partial v_2}{\partial \tau} - b_2 \Delta_y v_2 &= \Delta_x(b_2 v_0) + \nabla_y \cdot \nabla_x(b u_0) + \nabla_x \cdot \nabla_y(b u_0) + \nabla_y \cdot \nabla_x(b_2 v_1) \\
&\quad + \nabla_x \cdot \nabla_y(b_2 v_1) + \Delta_y(b u_1) - \frac{\partial u_0}{\partial t} ,
\end{aligned}
\tag{39}
$$

for v_2 a $\mathcal{Q}$-periodic function with respect to (y, τ).

Now, if we integrate (39) on $\mathcal{Q}$, since b_1 does not depend on τ and b_2 does not depend on y, on the left hand side we find zero. Hence, we have the compatibility condition

$$
\begin{aligned}
\int_{\mathcal{Q}} \Big[\frac{\partial u_0}{\partial t} &- \Big(\Delta_x(b_2 v_0) + \nabla_y \cdot \nabla_x(b u_0) + \nabla_x \cdot \nabla_y(b u_0) + \nabla_y \cdot \nabla_x(b_2 v_1) \\
&+ \nabla_x \cdot \nabla_y(b_2 v_1) + \Delta_y(b u_1) \Big) \Big] \, dy \, d\tau = 0 .
\end{aligned}
\tag{40}
$$

By the periodicity on Q and Gauss-Green formulas, we also have

$$\int_Q \left[\frac{\partial u_0}{\partial t} - \left(\Delta_x(b_2 v_0) + \nabla_x \cdot \nabla_y(bu_0) + \nabla_x \cdot \nabla_y(b_2 v_1) \right) \right] dy \, d\tau = 0. \tag{41}$$

Since, again by periodicity,

$$\int_Q \left[\nabla_x \cdot \nabla_y(bu_0) + \nabla_x \cdot \nabla_y(b_2 v_1) \right] dy \, d\tau = \int_S \left[\nabla_x \cdot \int_y \nabla_y(bu_0) \, dy + \nabla_x \cdot \int_y \nabla_y(b_2 v_1) \, dy \right] d\tau = 0, \tag{42}$$

we have

$$\frac{\partial}{\partial t} \left[\left(\int_y \frac{1}{b_1} \, dy \right) v_0 \right] - \Delta_x \left[\left(\int_S b_2 \, d\tau \right) v_0 \right] = 0, \tag{43}$$

where we have used that v_0 does not depend on y and τ.

We have, finally, found an equation for v_0. Indeed, we can deduce the equation that must be satisfied by the mean value of u_0 on the microscopic cell. If we let

$$u(x,t) = \int_Q u_0(x,t,y,\tau) \, dy \, d\tau = \int_Q \frac{v_0(x,t)}{b_1(x,t,y)} \, dy \, d\tau = v_0(x,t) \int_y \frac{dy}{b_1(x,t,y)}, \tag{44}$$

we can rewrite (43) as an equation for u, finding

$$\frac{\partial u}{\partial t} - \Delta_x \left[\left(\int_S b_2 \, d\tau \right) \left(\int_y \frac{dy}{b_1} \right)^{-1} u \right] = 0. \tag{45}$$

It is interesting to note that the non–product small correction $\varepsilon b^\varepsilon$ in (14) does not play any role in the upscaled equation.

Note that equation (43) (resp. (45)) coincides with the rigorous equations obtained in (125) (resp. (22)), once we have taken into account that, under the present assumptions, we have: i) the cell functions χ^j in Theorem 3 are identically equal to zero; ii) the cell function ζ in Theorem 3, which is equal to the function ζ introduced in the equation (37) above, and the term $\nabla_y(b/b_1)$ disappear from the expressions of P_{eff} in (128) and z_{eff} in (129), due to the periodicity.

2.2.3 | Space oscillations faster than the perturbation amplitude

In this section, we formally study the homogenization for the equation (14) in some cases not covered by the rigorous theory developed in Section 4. As mentioned above, we shall consider situations in which the spatial oscillation is faster than the amplitude of the non–product perturbation present in the Fokker coefficient.

We remark that, as we shall prove in Section 4 (for $\alpha < 1$ or $\alpha = 1$ and $B = I$) and as we found in Subsection 2.2.2 (for $\alpha = 1$), the non–product perturbation $\varepsilon b^\varepsilon$ appearing in (14) does not affect the upscaled equation. In the three cases discussed below, we shall see that this property is preserved in the case $\alpha = 2$, namely, even when the spatial oscillation is fast, which is expected to reinforce the effect of the perturbation. We cannot conclude that this is a general result for the scaling $\alpha > 1$; indeed, it might depend on our peculiar choice of the diffusion matrix and the capacity coefficients in the formal computation.

We first consider the problem (14)–(16) for $\alpha = \beta = 2$. Indeed, from the point of view of computations, such a case seems to be the most delicate among those discussed in this section. We then let $y = x/\varepsilon^2$ and $\tau = t/\varepsilon^2$ and, by abusing the notation, we write the differential rules

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \frac{1}{\varepsilon^2} \frac{\partial}{\partial \tau}, \quad \nabla_x = \nabla_x + \frac{1}{\varepsilon^2} \nabla_y, \quad \text{and} \quad \Delta_x = \Delta_x + \frac{1}{\varepsilon^2}(\nabla_y \cdot \nabla_x + \nabla_x \cdot \nabla_y) + \frac{1}{\varepsilon^4} \Delta_y \tag{46}$$

and look for a solution of (14), using the formal expansion (26). Differentiating in time, we are led again to (27), while differentiation in space yields

$$
\begin{aligned}
\nabla \cdot \nabla[(b_1^\epsilon b_2^\epsilon + \epsilon b^\epsilon) u_\epsilon] = {}& \Delta_x[b_1 b_2 u_0] + \frac{1}{\epsilon^2} \nabla_y \cdot \nabla_x[(b_1 b_2 + \epsilon b) u_0] \\
&+ \frac{1}{\epsilon^2} \nabla_x \cdot \nabla_y[(b_1 b_2 + \epsilon b) u_0] + \frac{1}{\epsilon^4} \Delta_y[(b_1 b_2 + \epsilon b) u_0] \\
&+ \frac{1}{\epsilon} \nabla_y \cdot \nabla_x[(b_1 b_2 + \epsilon b) u_1] + \frac{1}{\epsilon} \nabla_x \cdot \nabla_y[(b_1 b_2 + \epsilon b) u_1] \\
&+ \frac{1}{\epsilon^3} \Delta_y[(b_1 b_2 + \epsilon b) u_1] + \nabla_y \cdot \nabla_x[b_1 b_2 u_2] \\
&+ \nabla_x \cdot \nabla_y[b_1 b_2 u_2] + \frac{1}{\epsilon^2} \Delta_y[(b_1 b_2 + \epsilon b) u_2] \\
&+ \frac{1}{\epsilon} \Delta_y[(b_1 b_2 + \epsilon b) u_3] + \Delta_y[b_1 b_2 u_4] + o(1),
\end{aligned}
\tag{47}
$$

where we took into account the powers of ϵ up to the order ϵ^0.

Thus, at order $1/\epsilon^4$, we find the equation

$$
\Delta_y(b_1 b_2 u_0) = 0.
\tag{48}
$$

Recalling that b_2 does not depend on y (see (12)), from (48) we have that $b_1(x, t, y) u_0(x, t, y, \tau)$ does not depend on y, thus we set $v_0(x, t, \tau) = b_1(x, t, y) u_0(x, t, y, \tau)$.

We now consider the $1/\epsilon^3$ order equation. From (14), (27), and (47), we have

$$
\Delta_y(b u_0) + \Delta_y(b_1 b_2 u_1) = 0,
\tag{49}
$$

which, provided we let $v_1(x, t, y, \tau) = b_1(x, t, y) u_1(x, t, y, \tau)$, can be rewritten as

$$
b_2 \Delta_y(v_1) = -v_0 \Delta_y\left(\frac{b}{b_1}\right),
\tag{50}
$$

where we have used that b_2 and v_0 do not depend on y. We now look for a solution of the above equation in the factored form

$$
v_1(x, t, y, \tau) = \chi_1(x, t, y, \tau) v_0(x, t, \tau).
\tag{51}
$$

By plugging (51) into (50) and using again that v_0 does not depend on y, we get that χ_1 has to solve the equation

$$
b_2 \Delta_y(\chi_1) = -\Delta_y\left(\frac{b}{b_1}\right).
\tag{52}
$$

We now consider the $1/\epsilon^2$ order equation. From (14), (27), and (47), we have

$$
\frac{\partial u_0}{\partial \tau} - [\nabla_y \cdot \nabla_x(b_1 b_2 u_0) + \nabla_x \cdot \nabla_y(b_1 b_2 u_0) + \Delta_y(b u_1) + \Delta_y(b_1 b_2 u_2)] = 0.
\tag{53}
$$

Since b_2 and $v_0 = b_1 u_0$ do not depend on y, we get

$$
\frac{1}{b_1} \frac{\partial v_0}{\partial \tau} - \left[\Delta_y\left(\frac{b}{b_1} v_1\right) + \Delta_y(b_2 v_2)\right] = 0,
\tag{54}
$$

where we set $v_2(x, t, y, \tau) = b_1(x, t, y) u_2(x, t, y, \tau)$. Since the last two terms above integrate to zero on $\mathcal{Y}$ and v_0 does not depend on y, we have the compatibility condition

$$
\left(\int_y \frac{dy}{b_1}\right) \frac{\partial v_0}{\partial \tau} = 0,
\tag{55}
$$

which implies that v_0 does not depend on τ. Hence, $v_0 = v_0(x, t)$ and, since $v_0 = b_1 u_0$, also u_0 does not depend on τ, namely $u_0 = u_0(x, t, y)$. Inserting, now, (51) in (54), we get the following equation for v_2:

$$
\Delta_y(b_2 v_2) = -v_0 \Delta_y\left(\frac{b}{b_1} \chi_1\right).
\tag{56}
$$

We will look for a solution of the above equation in the factored form

$$
v_2(x, t, y, \tau) = \chi_2(x, t, y, \tau) v_0(x, t).
\tag{57}
$$

This leads to the equation

$$
\Delta_y(b_2 \chi_2) = -\Delta_y\left(\frac{b}{b_1} \chi_1\right)
\tag{58}
$$

for the unknown function χ_2.

Next we consider the $1/\varepsilon$ order equation. From (14), (27), and (47) we have

$$\frac{\partial u_1}{\partial \tau} - [(\nabla_x \cdot \nabla_y + \nabla_y \cdot \nabla_x)(bu_0 + b_1 b_2 u_1) + \Delta_y(bu_2) + \Delta_y(b_1 b_2 u_3)] = 0. \tag{59}$$

Since all the terms above but the first one integrate to zero on $\mathcal{Y}$, we have the compatibility condition

$$\int_{\mathcal{Y}} \frac{\partial u_1}{\partial \tau} \, dy = 0, \tag{60}$$

that, recalling the definition of v_1 given below (49) and (51), yields the condition

$$\int_{\mathcal{Y}} \frac{\partial}{\partial \tau}\left(\frac{\chi_1}{b_1}\right) dy = 0, \tag{61}$$

which completes the definition of χ_1 as solution of the equation (52). Setting, now, $v_3(x,t,y,\tau) = b_1(x,t,y)u_3(x,t,y,\tau)$, we can rewrite (59) as an equation for v_3; indeed, we find

$$\Delta_y(b_2 v_3) = \frac{v_0}{b_1}\frac{\partial \chi_1}{\partial \tau} - \left[(\nabla_x \cdot \nabla_y + \nabla_y \cdot \nabla_x)\left(\frac{b}{b_1}v_0 + b_2\chi_1 v_0\right) + v_0\Delta_y\left(\frac{b}{b_1}\chi_2\right)\right]. \tag{62}$$

Then, we turn to the ε^0 order equation. From (14), (27), and (47), we have

$$\frac{\partial u_0}{\partial t} + \frac{\partial u_2}{\partial \tau} - \left[\Delta_x(b_1 b_2 u_0) + (\nabla_y \cdot \nabla_x + \nabla_x \cdot \nabla_y)(bu_1 + b_1 b_2 u_2) + \Delta_y(bu_3) + \Delta_y(b_1 b_2 u_4)\right] = 0. \tag{63}$$

Since all the terms above but the first three on the left integrate to zero on $\mathcal{Y}$, we have the compatibility condition

$$\frac{\partial}{\partial t}\left(v_0 \int_{\mathcal{Y}} \frac{dy}{b_1}\right) + \int_{\mathcal{Y}} \frac{\partial u_2}{\partial \tau} \, dy - \Delta_x(b_2 v_0) = 0, \tag{64}$$

where we have used that $v_0 = b_1 u_0$ and that v_0 and b_2 do not depend on y. Finally, by integrating over $\mathcal{S}$, using the $\mathcal{Q}$-periodicity of u_2 in (y, τ) and the fact that both b_1 and v_0 do not depend on τ, we get for v_0 the equation

$$\frac{\partial}{\partial t}\left[\left(\int_{\mathcal{Y}} \frac{dy}{b_1}\right)v_0\right] - \Delta_x\left[\left(\int_{\mathcal{S}} b_2 \, d\tau\right)v_0\right] = 0, \tag{65}$$

which coincides with the equation (43), found in Section 2.2.2. Hence, also in this case, equation (45) is still in force.

The second case we consider here is the problem (14)–(16) for $\alpha = 2$ and $\beta = 4$. We then let $y = x/\varepsilon^2$ and $\tau = t/\varepsilon^4$ and, by abusing the notation, we write the differential rules

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \frac{1}{\varepsilon^4}\frac{\partial}{\partial \tau}, \quad \nabla_x = \nabla_x + \frac{1}{\varepsilon^2}\nabla_y, \quad \text{and} \quad \Delta_x = \Delta_x + \frac{1}{\varepsilon^2}(\nabla_y \cdot \nabla_x + \nabla_x \cdot \nabla_y) + \frac{1}{\varepsilon^4}\Delta_y \tag{66}$$

and look for a solution of (14), using the formal expansion (26). By substituting (26) in (14), we get

$$\frac{\partial u_\varepsilon}{\partial t} = \frac{\partial u_0}{\partial t} + \frac{1}{\varepsilon^4}\frac{\partial u_0}{\partial \tau} + \varepsilon\frac{\partial u_1}{\partial t} + \frac{1}{\varepsilon^3}\frac{\partial u_1}{\partial \tau} + \varepsilon^2\frac{\partial u_2}{\partial t} + \frac{1}{\varepsilon^2}\frac{\partial u_2}{\partial \tau} + \frac{1}{\varepsilon}\frac{\partial u_3}{\partial \tau} + \frac{\partial u_4}{\partial \tau} + \varepsilon\frac{\partial u_5}{\partial \tau} + \varepsilon^2\frac{\partial u_6}{\partial \tau} + o(\varepsilon^2) \tag{67}$$

and (47). Thus, at order $1/\varepsilon^4$, we find the equation

$$\frac{\partial u_0}{\partial \tau} - \Delta_y(b_1 b_2 u_0) = 0. \tag{68}$$

Recalling that b_2 does not depend on y, from (68) we have that $b_1(x,t,y)u_0(x,t,y,\tau)$ does not depend on y and τ, thus we set $v_0(x,t) = b_1(x,t,y)u_0(x,t,y,\tau)$. Indeed, we write (68) as an equation for v_0 (which, clearly, has uniqueness) and note that v_0, constant with respect to τ and y, solves such an equation.

Now, we pass directly to the ε^0 order equation. From (14), (67), and (47), we have

$$\frac{\partial u_0}{\partial t} + \frac{\partial u_4}{\partial \tau} - \left[\Delta_x(b_1 b_2 u_0) + (\nabla_y \cdot \nabla_x + \nabla_x \cdot \nabla_y)(bu_1 + b_1 b_2 u_2) + \Delta_y(bu_3) + \Delta_y(b_1 b_2 u_4)\right] = 0. \tag{69}$$

Integrating on $\mathcal{Q}$, we find again (65) and, with the same arguments as those used above, we derive (45).

The last situation we discuss in this subsection is the problem (14)–(16) for $\alpha = 2$ and $\beta = 1$. We, then, let $y = x/\varepsilon^2$ and $\tau = t/\varepsilon$ and, by abusing the notation, we write the differential rules

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \frac{1}{\varepsilon}\frac{\partial}{\partial \tau}, \quad \nabla_x = \nabla_x + \frac{1}{\varepsilon^2}\nabla_y, \quad \text{and} \quad \Delta_x = \Delta_x + \frac{1}{\varepsilon^2}(\nabla_y \cdot \nabla_x + \nabla_x \cdot \nabla_y) + \frac{1}{\varepsilon^4}\Delta_y, \tag{70}$$

and we look for a solution of (14), using the formal expansion (26). By substituting (26) in (14), we get

$$\frac{\partial u_\varepsilon}{\partial t} = \frac{\partial u_0}{\partial t} + \frac{1}{\varepsilon}\frac{\partial u_0}{\partial \tau} + \varepsilon\frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial \tau} + \varepsilon^2\frac{\partial u_2}{\partial t} + \varepsilon\frac{\partial u_2}{\partial \tau} + \varepsilon^2\frac{\partial u_3}{\partial \tau} + o(\varepsilon^2) \tag{71}$$

and (47).

Thus, at order $1/\varepsilon^4$, we find again the equation (48), which leads, as above, to $v_0(x, t, \tau) = b_1(x, t, y)u_0(x, t, y, \tau)$.

We now consider the $1/\varepsilon$ order equation. From (14), (71), and (47), we have

$$\frac{\partial u_0}{\partial \tau} - [(\nabla_x \cdot \nabla_y + \nabla_y \cdot \nabla_x)(bu_0 + b_1 b_2 u_1) + \Delta_y(bu_2) + \Delta_y(b_1 b_2 u_3)] = 0. \tag{72}$$

By integrating on $\mathcal{Y}$ and using that v_0 does not depend on y and b_1 does not depend on τ, we arrive again to the compatibility condition (55), which implies that $v_0 = v_0(x, t)$.

We finally consider the ε^0 order equation. From (14), (71), and (47), we have

$$\frac{\partial u_0}{\partial t} + \frac{\partial u_1}{\partial \tau} - \left[\Delta_x(b_1 b_2 u_0) + (\nabla_y \cdot \nabla_x + \nabla_x \cdot \nabla_y)(bu_1 + b_1 b_2 u_2) + \Delta_y(bu_3) + \Delta_y(b_1 b_2 u_4)\right] = 0. \tag{73}$$

Integrating on $\mathcal{Q}$, we get once again (65) and, with the same arguments as those used above, we derive (45).

3 | PRELIMINARY RESULTS

In this Section, we always assume that $\alpha \le 1$ and v_ε is the solution to (21), under the assumptions listed in Subsection 2.2.

3.1 | Estimates

We collect here some estimates that will be used in the sequel.

Lemma 1. *There exists $\gamma > 0$, depending on T, $\|f\|_2$, $\|\bar{v}_\varepsilon\|_2$ and the structural constants of the problem, but independent of ε, such that*

$$\sup_{t\in[0,T]} \int_\Omega v_\varepsilon^2 \, dx + \int_0^T \int_\Omega |\nabla v_\varepsilon|^2 \, dx \, dt \le \gamma. \tag{74}$$

Proof. Multiplying (17) by $v_\varepsilon/a_2^\varepsilon$ and integrating by parts, we obtain

$$\begin{aligned}
\frac{1}{2}\int_\Omega v_\varepsilon^2 \frac{a_1^\varepsilon}{b_1^\varepsilon}dx &+ \int_0^t\int_\Omega \frac{1}{a_2^\varepsilon}\left(b_2^\varepsilon + \varepsilon\frac{b^\varepsilon}{b_1^\varepsilon}\right)B^\varepsilon\nabla v_\varepsilon \cdot \nabla v_\varepsilon \, dx \, ds \\
&= \int_0^t\int_\Omega \left[f\frac{v_\varepsilon}{a_2^\varepsilon} + \frac{1}{2}v_\varepsilon^2\left\{\frac{1}{b_1^\varepsilon}\frac{\partial a_1^\varepsilon}{\partial s} - a_1^\varepsilon\frac{\partial}{\partial s}\frac{1}{b_1^\varepsilon}\right\}\right]dx \, ds \\
&\quad - \int_0^t\int_\Omega v_\varepsilon\left[\left(b_2^\varepsilon + \varepsilon\frac{b^\varepsilon}{b_1^\varepsilon}\right)B^\varepsilon\nabla v_\varepsilon \cdot \nabla\frac{1}{a_2^\varepsilon} + \frac{1}{a_2^\varepsilon}B^\varepsilon\nabla v_\varepsilon \cdot \nabla\left(b_2^\varepsilon + \varepsilon\frac{b^\varepsilon}{b_1^\varepsilon}\right)\right]dx \, ds \\
&\quad - \int_0^t\int_\Omega v_\varepsilon^2 B^\varepsilon\nabla\frac{1}{a_2^\varepsilon} \cdot \nabla\left(b_2^\varepsilon + \varepsilon\frac{b^\varepsilon}{b_1^\varepsilon}\right)dx \, ds + \frac{1}{2}\int_\Omega\left[v_\varepsilon^2\frac{a_1^\varepsilon}{b_1^\varepsilon}\right]_{s=0}dx.
\end{aligned} \tag{75}$$

Under our assumptions on the sign of the coefficients, the left hand side of (75) can be bounded from below by the left hand side of (74). Again appealing to our assumptions and, in particular, to $\alpha \le 1$, we see that all the functions appearing in the integrals on the right hand side of (75) are bounded by an absolute constant, with the exception of f, v^ε, and ∇v^ε.

Then, by Young inequality, the right hand side of (75) can be bounded from above by

$$\gamma\left(\|f\|_2^2 + \delta\int_0^t\int_\Omega |\nabla v_\varepsilon|^2 \, dx \, ds + \frac{1}{\delta}\int_0^t\int_\Omega v_\varepsilon^2 \, dx \, ds + \|\bar{v}_\varepsilon\|_2^2\right), \tag{76}$$

where γ is independent of ε and $\delta > 0$ can be chosen so that the gradient term can be absorbed into the left hand side. Finally, the result follows from the application of Gronwall lemma. $\qquad\square$

Taking into account that the initial datum $\bar{u}_\varepsilon$ (and, therefore, $\bar{v}_\varepsilon$) belongs not only to the space $L^2(\Omega)$ (as needed in the previous estimate), but it is, indeed, in $H_0^1(\Omega_T)$, we can obtain also some estimates for the time-derivative of the solution v_ε, as stated in the next two lemmas.

Lemma 2. *There exists $\gamma > 0$, depending on T, $\|f\|_2$, $\|\bar{v}_\varepsilon\|_2$ and the structural constants of the problem, but independent of ε, such that*

$$\int_0^T \int_\Omega \left(\frac{\partial v_\varepsilon}{\partial t}\right)^2 dx\, dt + \sup_{t\in[0,T]} \int_\Omega |\nabla v_\varepsilon|^2 \, dx \leq \frac{\gamma}{\varepsilon^\beta} + \gamma \int_\Omega |\nabla \bar{v}_\varepsilon|^2 \, dx. \tag{77}$$

Proof. Let us multiply (17) times $\partial v_\varepsilon/\partial t$ and integrate by parts to obtain

$$
\begin{aligned}
\int_0^t \int_\Omega & \frac{a_1^\varepsilon a_2^\varepsilon}{b_1^\varepsilon}\left(\frac{\partial v_\varepsilon}{\partial s}\right)^2 dx\, ds + \frac{1}{2}\int_\Omega \left(b_2^\varepsilon + \varepsilon\frac{b^\varepsilon}{b_1^\varepsilon}\right)B^\varepsilon \nabla v_\varepsilon \cdot \nabla v_\varepsilon \, dx \\
&= -\int_0^t \int_\Omega a_1^\varepsilon a_2^\varepsilon v_\varepsilon \left(\frac{\partial v_\varepsilon}{\partial s}\right)\frac{\partial}{\partial s}\frac{1}{b_1^\varepsilon}\, dx\, ds + \frac{1}{2}\int_0^t\int_\Omega \frac{\partial}{\partial s}\left[\left(b_2^\varepsilon + \varepsilon\frac{b^\varepsilon}{b_1^\varepsilon}\right)B^\varepsilon\right]\nabla v_\varepsilon \cdot \nabla v_\varepsilon \, dx\, ds \\
&\quad -\int_0^t \int_\Omega v_\varepsilon B^\varepsilon \nabla\left(b_2^\varepsilon + \varepsilon\frac{b^\varepsilon}{b_1^\varepsilon}\right)\cdot \nabla\frac{\partial v_\varepsilon}{\partial s}\, dx\, ds + \int_0^t\int_\Omega f\frac{\partial v_\varepsilon}{\partial s}\, dx\, ds \\
&\quad + \frac{1}{2}\int_\Omega \left[\left(b_2^\varepsilon + \varepsilon\frac{b^\varepsilon}{b_1^\varepsilon}\right)B^\varepsilon \nabla v_\varepsilon \cdot \nabla v_\varepsilon\right]_{s=0}\, dx \\
&= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
\tag{78}
$$

Under our assumptions on the sign of the coefficients, the left hand side of (78) can be bounded from below by the left hand side of (77). Next, we give estimates for each term I_i. By Young inequality, we get

$$|I_1| \leq \delta \int_0^T \int_\Omega \left(\frac{\partial v_\varepsilon}{\partial t}\right)^2 dx\, dt + \frac{\gamma}{\delta}\int_0^T\int_\Omega v_\varepsilon^2 \, dx\, dt \tag{79}$$

$$|I_2| \leq \frac{\gamma}{\varepsilon^\beta}\int_0^T \int_\Omega |\nabla v_\varepsilon|^2 \, dx\, dt \tag{80}$$

$$|I_4| \leq \delta \int_0^T \int_\Omega \left(\frac{\partial v_\varepsilon}{\partial t}\right)^2 dx\, dt + \frac{\gamma}{\delta}\int_0^T\int_\Omega f^2 \, dx\, dt \tag{81}$$

$$|I_5| \leq \gamma \int_\Omega |\nabla \bar{v}_\varepsilon|^2 \, dx. \tag{82}$$

Moreover, we calculate

$$I_3 = -\int_\Omega \left[v_\varepsilon B^\varepsilon \nabla\left(b_2^\varepsilon + \varepsilon\frac{b^\varepsilon}{b_1^\varepsilon}\right)\cdot \nabla v_\varepsilon\right]_0^t dx + \int_0^t\int_\Omega \frac{\partial}{\partial s}\left[v_\varepsilon B^\varepsilon \nabla\left(b_2^\varepsilon + \varepsilon\frac{b^\varepsilon}{b_1^\varepsilon}\right)\right]\cdot \nabla v_\varepsilon \, dx\, ds \tag{83}$$

and thus, recalling that $\alpha \leq 1$, we obtain

$$|I_3| \leq \gamma \int_\Omega |\nabla v_\varepsilon||v_\varepsilon|\, dx + \gamma \int_\Omega |\nabla \bar{v}_\varepsilon||\bar{v}_\varepsilon|\, dx + \frac{\gamma}{\varepsilon^\beta}\int_0^T\int_\Omega |\nabla v_\varepsilon||v_\varepsilon|\, dx\, dt + \gamma \int_0^T\int_\Omega \left|\nabla v_\varepsilon\right|\left|\frac{\partial v_\varepsilon}{\partial t}\right|\, dx\, dt. \tag{84}$$

Again, an application of Young inequality gives

$$|I_3| \leq \delta \int_\Omega |\nabla v_\varepsilon|^2 \, dx + \frac{\gamma}{\delta} \int_\Omega v_\varepsilon^2 \, dx + \gamma \int_\Omega (\bar{v}_\varepsilon^2 + |\nabla \bar{v}_\varepsilon|^2) \, dx$$

$$+ \frac{\gamma}{\varepsilon^\beta} \int_0^T \int_\Omega (v_\varepsilon^2 + |\nabla v_\varepsilon|^2) \, dx \, dt + \delta \int_0^T \int_\Omega \left(\frac{\partial v_\varepsilon}{\partial t}\right)^2 \, dx \, dt + \frac{\gamma}{\delta} \int_0^T \int_\Omega |\nabla v_\varepsilon|^2 \, dx \, dt \,. \tag{85}$$

For δ suitably small, we can absorb the terms in (85) multiplied by δ into the left hand side of (78). Then, the claim follows by applying Lemma 1. $\qquad\square$

Lemma 3. *Let $T_1 \in (0, T)$. Then, there exists $\gamma > 0$, depending on $T_1, T, \|f\|_2, \|\bar{v}_\varepsilon\|_2$ and the structural constants of the problem, but independent of ε, such that*

$$\int_{T_1}^T \int_\Omega \left(\frac{\partial v_\varepsilon}{\partial t}\right)^2 dx \, dt + \sup_{t \in [T_1, T]} \int_\Omega |\nabla v_\varepsilon|^2 \, dx \leq \frac{\gamma}{\varepsilon^\beta} \,. \tag{86}$$

Proof. Let us multiply (17) times $\phi(t)\partial v_\varepsilon/\partial t$, where $\phi(t) \in C^\infty(\mathbb{R})$ such that $\phi(t) = 0$ for $t \leq T_1/2$, $\phi(t) = 1$ for $t > T_1$, and $0 \leq \phi'(t) \leq 4/T_1$, and integrate by parts to obtain

$$\int_0^t \int_\Omega \phi \frac{a_1^\varepsilon a_2^\varepsilon}{b_1^\varepsilon} \left(\frac{\partial v_\varepsilon}{\partial s}\right)^2 dx \, ds + \frac{1}{2} \int_\Omega \phi \left(b_2^\varepsilon + \varepsilon \frac{b^\varepsilon}{b_1^\varepsilon}\right) B^\varepsilon \nabla v_\varepsilon \cdot \nabla v_\varepsilon \, dx$$

$$= -\int_0^t \int_\Omega \phi a_1^\varepsilon a_2^\varepsilon v_\varepsilon \left(\frac{\partial v_\varepsilon}{\partial s}\right) \frac{\partial}{\partial s} \frac{1}{b_1^\varepsilon} \, dx \, ds + \frac{1}{2} \int_0^t \int_\Omega \frac{\partial}{\partial s} \left[\phi \left(b_2^\varepsilon + \varepsilon \frac{b^\varepsilon}{b_1^\varepsilon}\right) B^\varepsilon\right] \nabla v_\varepsilon \cdot \nabla v_\varepsilon \, dx \, ds$$

$$- \int_0^t \int_\Omega \phi v_\varepsilon B^\varepsilon \nabla \left(b_2^\varepsilon + \varepsilon \frac{b^\varepsilon}{b_1^\varepsilon}\right) \cdot \nabla \frac{\partial v_\varepsilon}{\partial s} \, dx \, ds + \int_0^t \int_\Omega \phi f \frac{\partial v_\varepsilon}{\partial s} \, dx \, ds$$

$$= I_1 + I_2 + I_3 + I_4 \,.$$

Now, the terms I_1 and I_4 are treated as in the proof of Lemma 2. For I_2, we write

$$|I_2| \leq \frac{\gamma(1 + T_1^{-1})}{\varepsilon^\beta} \int_0^T \int_\Omega |\nabla v_\varepsilon|^2 \, dx \, dt \,. \tag{87}$$

Moreover, we calculate

$$I_3 = -\int_\Omega \left[\phi v_\varepsilon B^\varepsilon \nabla \left(b_2^\varepsilon + \varepsilon \frac{b^\varepsilon}{b_1^\varepsilon}\right) \cdot \nabla v_\varepsilon\right]_0^t dx + \int_0^t \int_\Omega \frac{\partial}{\partial s} \left[\phi v_\varepsilon B^\varepsilon \nabla \left(b_2^\varepsilon + \varepsilon \frac{b^\varepsilon}{b_1^\varepsilon}\right)\right] \cdot \nabla v_\varepsilon \, dx \, ds \tag{88}$$

and thus, recalling that $\alpha \leq 1$, we obtain

$$|I_3| \leq \gamma \int_\Omega |\nabla v_\varepsilon| |v_\varepsilon| \, dx + \frac{\gamma(1 + T_1^{-1})}{\varepsilon^\beta} \int_0^T \int_\Omega |\nabla v_\varepsilon| |v_\varepsilon| \, dx \, dt + \gamma \int_0^T \int_\Omega \phi |\nabla v_\varepsilon| \left|\frac{\partial v_\varepsilon}{\partial t}\right| \, dx \, dt \,. \tag{89}$$

As in the proof of Lemma 2, a final application of the Young inequality yields (86). $\qquad\square$

Proposition 1. *For any $0 < \delta < T/2$, there exists $\gamma > 0$ (depending on $T, \|f\|_2, \|\bar{v}_\varepsilon\|_2$, the structural constants of the problem and δ), such that*

$$\int_\delta^{T-\delta} \int_\Omega |v_\varepsilon(x, t+h) - v_\varepsilon(x, t)|^2 \, dx \, dt \leq \gamma \sqrt{h}, \tag{90}$$

for any $0 < h < \delta/2$.

Proof. We select, as a test function in the integral formulation (21), the function

$$\phi(x,t) = \frac{\varphi(x,t)}{a_2^\epsilon(x,t)}, \qquad \text{with } \varphi \in H_0^1(\Omega_T).$$

We obtain

$$\iint_{\Omega_T} \left\{ -\frac{v_\epsilon}{b_1^\epsilon} \frac{\partial}{\partial t}(a_1^\epsilon \varphi) + B^\epsilon \nabla \left[\left(b_2^\epsilon + \epsilon \frac{b^\epsilon}{b_1^\epsilon} \right) v_\epsilon \right] \cdot \nabla \left(\frac{\varphi}{a_2^\epsilon} \right) \right\} \mathrm{d}x\, \mathrm{d}t = \iint_{\Omega_T} f \frac{\varphi}{a_2^\epsilon} \mathrm{d}x\, \mathrm{d}t. \tag{91}$$

Here, for any $F = F(x,t)$, we denote by $\widetilde{F}(x,t) = F(x, t+h)$ its time shift. Let $\delta \in (0, T/2)$, $0 < h < \delta/2$, and assume that $\varphi(x,t) = 0$ for $t < \delta/2$ and for $t > T - \delta/2$. Using the formula (91) with $\varphi(x,t)$ replaced with $\varphi(x, t-h)$, and then changing variables to $(x, t+h)$, but still keeping the old variable names, we obtain

$$\iint_{\Omega_T} \left\{ -\frac{\widetilde{v}_\epsilon}{\widetilde{b}_1^\epsilon} \frac{\partial}{\partial t}(\widetilde{a}_1^\epsilon \varphi) + \widetilde{B}^\epsilon \nabla \left[\left(\widetilde{b}_2^\epsilon + \epsilon \frac{\widetilde{b}^\epsilon}{\widetilde{b}_1^\epsilon} \right) \widetilde{v}_\epsilon \right] \cdot \nabla \left(\frac{\varphi}{\widetilde{a}_2^\epsilon} \right) \right\} \mathrm{d}x\, \mathrm{d}t = \iint_{\Omega_T} \widetilde{f} \frac{\varphi}{\widetilde{a}_2^\epsilon} \mathrm{d}x\, \mathrm{d}t. \tag{92}$$

Next, in (91)–(92), we select $\varphi = \varphi_h$ where

$$\varphi_h(x,t) = -\zeta(t) \int_t^{t+h} v_\epsilon(x,s)\, \mathrm{d}s,$$

where $\zeta \in C_0^1(\delta/2, T - \delta/2)$ is a nonnegative function such that $\zeta = 1$ in $(\delta, T - \delta)$ and $|\zeta'| \leq \gamma/\delta$.
On subtracting the two integral formulations (91) and (92), we obtain

$$\iint_{\Omega_T} \left\{ -\left[\frac{\widetilde{v}_\epsilon}{\widetilde{b}_1^\epsilon} \widetilde{a}_1^\epsilon - \frac{v_\epsilon}{b_1^\epsilon} a_1^\epsilon \right] \right\} \frac{\partial \varphi_h}{\partial t} \mathrm{d}x\, \mathrm{d}t + \iint_{\Omega_T} \left\{ -\left[\frac{\widetilde{v}_\epsilon}{\widetilde{b}_1^\epsilon} \frac{\partial \widetilde{a}_1^\epsilon}{\partial t} - \frac{v_\epsilon}{b_1^\epsilon} \frac{\partial a_1^\epsilon}{\partial t} \right] \right\} \varphi_h \mathrm{d}x\, \mathrm{d}t$$

$$+ \iint_{\Omega_T} \left\{ \frac{1}{\widetilde{a}_2^\epsilon} \widetilde{B}^\epsilon \nabla \left[\left(\widetilde{b}_2^\epsilon + \epsilon \frac{\widetilde{b}^\epsilon}{\widetilde{b}_1^\epsilon} \right) \widetilde{v}_\epsilon \right] - \frac{1}{a_2^\epsilon} B^\epsilon \nabla \left[\left(b_2^\epsilon + \epsilon \frac{b^\epsilon}{b_1^\epsilon} \right) v_\epsilon \right] \right\} \cdot \nabla \varphi_h \mathrm{d}x\, \mathrm{d}t$$

$$+ \iint_{\Omega_T} \left\{ \widetilde{B}^\epsilon \nabla \left[\left(\widetilde{b}_2^\epsilon + \epsilon \frac{\widetilde{b}^\epsilon}{\widetilde{b}_1^\epsilon} \right) \widetilde{v}_\epsilon \right] \cdot \nabla \frac{1}{\widetilde{a}_2^\epsilon} - B^\epsilon \nabla \left[\left(b_2^\epsilon + \epsilon \frac{b^\epsilon}{b_1^\epsilon} \right) v_\epsilon \right] \cdot \nabla \frac{1}{a_2^\epsilon} \right\} \varphi_h \mathrm{d}x\, \mathrm{d}t \tag{93}$$

$$= \iint_{\Omega_T} \left\{ \frac{\widetilde{f}}{\widetilde{a}_2^\epsilon} - \frac{f}{a_2^\epsilon} \right\} \varphi_h \mathrm{d}x\, \mathrm{d}t.$$

For the sake of notational simplicity, we denote each integral with a different symbol, thereby rewriting (93) as

$$I_1 + I_2 + I_3 + I_4 = I_5,$$

where, actually, only the estimation of I_1 requires a detailed calculation. Indeed,

$$I_1 = \iint_{\Omega_T} \left[\frac{\widetilde{v}_\epsilon}{\widetilde{b}_1^\epsilon} \widetilde{a}_1^\epsilon - \frac{v_\epsilon}{b_1^\epsilon} a_1^\epsilon \right] \left\{ \zeta[\widetilde{v}_\epsilon - v_\epsilon] + \zeta' \int_t^{t+h} v_\epsilon(x,s)\, \mathrm{d}s \right\} \mathrm{d}x\, \mathrm{d}t$$

$$= \iint_{\Omega_T} [\widetilde{v}_\epsilon - v_\epsilon]^2 \frac{\zeta a_1^\epsilon}{b_1^\epsilon} \mathrm{d}x\, \mathrm{d}t + \iint_{\Omega_T} \widetilde{v}_\epsilon [\widetilde{v}_\epsilon - v_\epsilon] \left[\frac{\widetilde{a}_1^\epsilon}{\widetilde{b}_1^\epsilon} - \frac{a_1^\epsilon}{b_1^\epsilon} \right] \zeta \mathrm{d}x\, \mathrm{d}t$$

$$+ \iint_{\Omega_T} \left[\left(\frac{\widetilde{v}_\epsilon}{\widetilde{b}_1^\epsilon} \widetilde{a}_1^\epsilon - \frac{v_\epsilon}{b_1^\epsilon} a_1^\epsilon \right) \zeta' \int_t^{t+h} v_\epsilon(x,s)\, \mathrm{d}s \right] \mathrm{d}x\, \mathrm{d}t = I_{11} + I_{12} + I_{13}. \tag{94}$$

The term I_{11} essentially equals the one estimated in the statement. The term I_{12} is estimated, invoking the time regularity of a_1, b_1, by

$$|I_{12}| \leq \gamma \int_\Omega \int_{\delta/2}^{T-\delta/2} |\widetilde{v}_\epsilon| \left(|\widetilde{v}_\epsilon| + |v_\epsilon| \right) \left[\left| \widetilde{a}_1^\epsilon - a_1^\epsilon \right| + \left| \widetilde{b}_1^\epsilon - b_1^\epsilon \right| \right] \mathrm{d}x\, \mathrm{d}t \leq \gamma \|v_\epsilon\|_2^2 h. \tag{95}$$

The integral I_{13} can be bounded by means of the Hölder inequality as follows

$$|I_{13}| \le \gamma \, \|\zeta'\|_\infty \left(\int\limits_\Omega \int\limits_{\delta/2}^{T-\delta/2} |\widetilde{v}_\varepsilon|^2 + |v_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}t \right)^{\frac{1}{2}} \left(\int\limits_\Omega \int\limits_{\delta/2}^{T-\delta/2} \left| \int\limits_t^{t+h} v_\varepsilon(x,s) \, \mathrm{d}s \right|^2 \mathrm{d}x \, \mathrm{d}t \right)^{\frac{1}{2}} \le \frac{\gamma}{\delta} \, \|v_\varepsilon\|_2^2 \, \sqrt{h} \,. \tag{96}$$

Clearly, the integrals I_2, I_3, I_4 and I_5 can be estimated by means of a similar device, once we remark that, owing to the assumed regularity in space of b_2, b_1, b, we get

$$\left| \nabla \left(b_2^\varepsilon + \varepsilon \frac{b^\varepsilon}{b_1^\varepsilon} \right) \right| \le |\nabla b_2^\varepsilon| + \varepsilon \left| \nabla \frac{b^\varepsilon}{b_1^\varepsilon} \right| \le \gamma \,. \tag{97}$$

For example, the integral I_3 can be estimated by

$$|I_3| \le \gamma \int\limits_\Omega \int\limits_{\delta/2}^{T-\delta/2} \left(|v_\varepsilon| + |\nabla v_\varepsilon| + |\widetilde{v}_\varepsilon| + |\nabla \widetilde{v}_\varepsilon| \right) \left| \int\limits_t^{t+h} \nabla v_\varepsilon(x,s) \, \mathrm{d}s \right| \mathrm{d}x \, \mathrm{d}t \le \gamma \left(\|v_\varepsilon\|_2^2 + \|\nabla v_\varepsilon\|_2^2 \right) \sqrt{h} \,. \tag{98}$$

Finally, on collecting all the estimates above, we get (90). $\qquad\qquad\Box$

3.2 | Unfolding

In the sequel, we denote by $[r]$ the integer part of $r \in \mathbb{R}$ and, for $x \in \mathbb{R}^n$, we define the vector with integer components $[x] = ([x_1], \dots, [x_n])$.

Let us consider the tiling of $\mathbb{R}^n$ given by the boxes $\varepsilon^\alpha(\xi + \mathcal{Y})$, with $\xi \in \mathbb{Z}^n$. Following[19], we set

$$\Xi_\varepsilon = \{ \xi \in \mathbb{Z}^n \, : \, \varepsilon^\alpha(\xi + \mathcal{Y}) \subset \Omega \}, \quad \hat{\Omega}_\varepsilon = \text{interior} \left\{ \bigcup_{\xi \in \Xi_\varepsilon} \varepsilon^\alpha(\xi + \overline{\mathcal{Y}}) \right\}, \tag{99}$$

and

$$\hat{T}_\varepsilon = \left\{ t \in (0,T) \, : \, \varepsilon^\beta \left(\left[\frac{t}{\varepsilon^\beta} \right] + 1 \right) \le T \right\}, \quad \Lambda_\varepsilon = \hat{\Omega}_\varepsilon \times \hat{T}_\varepsilon \,. \tag{100}$$

We introduce also the space-time cell containing the point (x,t) as

$$Q_\varepsilon(x,t) = \varepsilon^\alpha \left(\left[\frac{x}{\varepsilon^\alpha} \right] + \mathcal{Y} \right) \times \varepsilon^\beta \left(\left[\frac{t}{\varepsilon^\beta} \right] + S \right) \,.$$

Definition 1. The time–periodic unfolding operator $\mathcal{T}_\varepsilon$ of a Lebesgue measurable function w defined on Ω_T is given by

$$\mathcal{T}_\varepsilon(w)(x,t,y,\tau) = \begin{cases} w \left(\varepsilon^\alpha \left[\frac{x}{\varepsilon^\alpha} \right] + \varepsilon^\alpha y, \varepsilon^\beta \left[\frac{t}{\varepsilon^\beta} \right] + \varepsilon^\beta \tau \right), & (x,t,y,\tau) \in \Lambda_\varepsilon \times Q, \\ 0, & \text{otherwise.} \end{cases} \tag{101}$$

Note that, by definition, it easily follows that

$$\mathcal{T}_\varepsilon(w_1 w_2) = \mathcal{T}_\varepsilon(w_1) \mathcal{T}_\varepsilon(w_2) \,. \tag{102}$$

Definition 2. The space–time average operator $\mathcal{M}_\varepsilon$ of a Lebesgue integrable function w defined on Ω_T is given by

$$\mathcal{M}_\varepsilon(w)(x,t) = \begin{cases} \dfrac{1}{\varepsilon^{N\alpha+\beta}} \displaystyle\int\limits_{Q_\varepsilon(x,t)} w(\zeta,s) \, \mathrm{d}\zeta \, \mathrm{d}s, & (x,t) \in \Lambda_\varepsilon, \\ 0, & \text{otherwise.} \end{cases} \tag{103}$$

Moreover, the space–time oscillation operator is defined as

$$\mathcal{Z}_\varepsilon(w)(x,t,y,\tau) = \mathcal{T}_\varepsilon(w)(x,t,y,\tau) - \mathcal{M}_\varepsilon(w)(x,t) \,. \tag{104}$$

Notice that, by a simple change of variables, it easily follows that

$$\mathcal{M}_\varepsilon(w)(x,t) = \int\limits_Q \mathcal{T}_\varepsilon(w)(x,t,y,\tau) \, \mathrm{d}y \, \mathrm{d}\tau \,. \tag{105}$$

Finally, we denote by $\mathcal{M}_S$ the microscopic time average of an integrable function $\phi(x, t, y, \tau)$, i.e.

$$\mathcal{M}_S(\phi)(x, t, y) = \int_S \phi(x, t, y, \tau) \, d\tau \,. \tag{106}$$

We conclude this section, recalling the following result (see[19] Remark 2.9).

Proposition 2. *For $\phi \in L^2(Q; C(\overline{\Omega}_T))$ or $\phi \in L^2(\Omega_T; C(\overline{Q}))$, denote again by ϕ its extension by Q–periodicity to $\Omega_T \times \mathbb{R}^{n+1}$ and set $\phi_\varepsilon(x, t) = \phi(x, t, \varepsilon^{-\alpha}x, \varepsilon^{-\beta}t)$. Then, $\mathcal{T}_\varepsilon(\phi_\varepsilon) \to \phi$ strongly in $L^2(\Omega_T \times Q)$.*

For later use, we define the functional spaces

$$\begin{aligned} H^1_\#(\mathcal{Y}) &= \{v \in H^1_{loc}(\mathbb{R}^n) \,:\, v \text{ is } \mathcal{Y}\text{-periodic}\}, \\ H^1_\#(Q) &= \{v \in H^1_{loc}(\mathbb{R}^{n+1}) \,:\, v \text{ is } Q\text{-periodic}\}. \end{aligned} \tag{107}$$

4 | HOMOGENIZATION

In this section, u_ε and v_ε are the solutions of problem (14)–(16) and (17)–(19) in Subsection 2.2, and we assume all the hypotheses listed there. As in Section 3, we always assume $\alpha \leq 1$.

We remark that, in all the cases we deal with, the final structure of the macroscopic homogenized equation will be the same, though the coefficients in it have to be defined case–by–case. Results are presented in two subsections: Section 4.1 is devoted to the case $\beta \geq 2\alpha$ (fast oscillations), while in Section 4.2 the case $\beta < 2\alpha$ (slow oscillations) is studied.

In each case we prove two theorems, the first states the homogenization result and gives the limit two–scale system, while the second one introduces the corrector factorization and the resulting single scale equation. For technical reasons, the uniqueness of the solutions of the two limit problems is dealt in the corollaries following the theorems.

4.1 | Fast oscillations

Here, we treat the cases where $\beta \geq 2\alpha$, distinguishing between $\beta = 2\alpha$ and $\beta > 2\alpha$.

Theorem 2. *Let $\beta = 2\alpha$. Then, there exist $v \in L^2(0, T; H^1_0(\Omega))$ and $v_1 \in L^2(\Omega_T; H^1_\#(Q))$, with $\int_Q v_1 \, dy \, d\tau = 0$, such that*

$$\begin{aligned} v_\varepsilon &\rightharpoonup v\,, & \text{weakly in } L^2(\Omega_T); & \tag{108} \\ v_\varepsilon &\rightharpoonup v\,, & \text{weakly in } L^2(0, T; H^1_0(\Omega)); & \tag{109} \\ \mathcal{T}_\varepsilon(v_\varepsilon) &\rightharpoonup v\,, & \text{weakly in } L^2(\Omega_T; H^1(Q)); & \tag{110} \\ \mathcal{T}_\varepsilon(\nabla v_\varepsilon) &\rightharpoonup \nabla v + \nabla_y v_1\,, & \text{weakly in } L^2(\Omega_T \times Q); & \tag{111} \\ \varepsilon^\alpha \mathcal{T}_\varepsilon\left(\frac{\partial v_\varepsilon}{\partial t}\right) &\rightharpoonup \frac{\partial v_1}{\partial \tau}\,, & \text{weakly in } L^2(\Omega_T \times Q). & \tag{112} \end{aligned}$$

Moreover, the pair (v, v_1) is a weak solution of the two–scale problem

$$\int_Q \left[a_1 \frac{\partial}{\partial t}\left(\frac{v}{b_1}\right) \right.$$
$$\left. - \frac{1}{a_2} div\left(B\left(b_2(\nabla v + \nabla_y v_1) + v\nabla b_2 + \omega_{\alpha,1} v \nabla_y\left(\frac{b}{b_1}\right) \right) \right) \right] dy \, d\tau = f \int_S \frac{d\tau}{a_2}\,, \quad \text{in } \Omega_T\,; \tag{113}$$

$$\frac{a_1}{b_1} \frac{\partial v_1}{\partial \tau} - \frac{1}{a_2} div_y\left[B\left(b_2(\nabla v + \nabla_y v_1) + v\nabla b_2 + \omega_{\alpha,1} v \nabla_y\left(\frac{b}{b_1}\right) \right) \right] = 0\,, \quad \text{in } \Omega_T \times Q\,; \tag{114}$$

$$v = 0\,, \qquad \text{on } \partial\Omega \times (0, T)\,; \tag{115}$$

$$v(x, 0) = \bar{u}(x)\left(\int_{\mathcal{Y}} a_1(x, 0, y) \, dy \right)\left(\int_{\mathcal{Y}} \frac{a_1(x, 0, y)}{b_1(x, 0, y)} \, dy \right)^{-1}\,, \qquad \text{in } \Omega\,. \tag{116}$$

Proof The convergence results in (108) and (109) follow from the energy estimate (74); (110) follows from (77) and[19] Proposition 2.12 with $m = 1/2$ and by replacing τ with $\varepsilon^{2\alpha}$. Finally, (111) and (112) follow from (77) and[19] Theorem 2.18 with $m = 1/2$ by replacing τ with $\varepsilon^{2\alpha}$ and ε with ε^α.

Now, we choose, as test function in (21), $\phi_\varepsilon(x,t) = \varphi(x,t)/a_2(x,t,t/\varepsilon^{2\alpha})$, where $\varphi \in C^\infty(\overline{\Omega}_T)$ with $\varphi(x,T) =$ in $\overline{\Omega}$ and $\varphi = 0$ on $\partial\Omega \times [0,T]$, and we unfold the resulting equation. We obtain

$$-\int_0^T \int_\Omega \int_Q \mathcal{T}_\varepsilon(v_\varepsilon)\mathcal{T}_\varepsilon\left(\frac{1}{b_1^\varepsilon}\right)\mathcal{T}_\varepsilon\left(a_1^\varepsilon \frac{\partial\varphi}{\partial t} + \varphi\frac{\partial a_1^\varepsilon}{\partial t}\right) dy\,d\tau\,dx\,dt$$

$$+\int_0^T \int_\Omega \int_Q \mathcal{T}_\varepsilon(B^\varepsilon)\mathcal{T}_\varepsilon\left(\nabla\left(\left(b_2^\varepsilon + \varepsilon\frac{b^\varepsilon}{b_1^\varepsilon}\right)v_\varepsilon\right)\right) \cdot \mathcal{T}_\varepsilon\left(\nabla\left(\frac{\varphi}{a_2^\varepsilon}\right)\right) dy\,d\tau\,dx\,dt$$

$$= \int_0^T \int_\Omega f\frac{\varphi}{a_2^\varepsilon}\,dx\,dt + \int_\Omega \bar{u}a_1^\varepsilon(x,0)\varphi(x,0)\,dx + R_\varepsilon\,, \quad (117)$$

where $R_\varepsilon \to 0$ for $\varepsilon \to 0$.

We first note that

$$\mathcal{T}_\varepsilon\left(\nabla\left(\left(b_2^\varepsilon + \varepsilon\frac{b^\varepsilon}{b_1^\varepsilon}\right)v_\varepsilon\right)\right) = \mathcal{T}_\varepsilon\left(\left(b_2^\varepsilon + \varepsilon\frac{b^\varepsilon}{b_1^\varepsilon}\right)\nabla v_\varepsilon\right) + \mathcal{T}_\varepsilon\left(v_\varepsilon\left(\nabla b_2^\varepsilon + \varepsilon\nabla_x\left(\frac{b^\varepsilon}{b_1^\varepsilon}\right) + \varepsilon^{1-\alpha}\nabla_y\left(\frac{b^\varepsilon}{b_1^\varepsilon}\right)\right)\right). \quad (118)$$

Therefore, passing to the limit $\varepsilon \to 0$ in (117), and taking into account (110) and (111), we get

$$-\int_0^T \int_\Omega \int_Q \frac{v}{b_1}\frac{\partial}{\partial t}(a_1\varphi)\,dy\,d\tau\,dx\,dt + \int_0^T \int_\Omega \int_Q B\left(b_2(\nabla v + \nabla_y v_1) + v\left(\nabla b_2 + \omega_{\alpha,1}\nabla_y\left(\frac{b}{b_1}\right)\right)\right) \cdot \nabla\left(\frac{\varphi}{a_2}\right) dy\,d\tau\,dx\,dt$$

$$= \int_0^T \int_\Omega f\varphi\left(\int_S \frac{d\tau}{a_2(x,t,\tau)}\right) dx\,dt + \int_\Omega \int_\mathcal{Y} \bar{u}a_1(x,0,y)\varphi(x,0)\,dy\,dx\,, \quad (119)$$

i.e., the weak formulation of (113) and (116).

Next, we choose $\phi_\varepsilon(x,t) = \varepsilon^\alpha(\varphi(x,t)/a_2(x,t,t/\varepsilon^{2\alpha}))\psi(x/\varepsilon^\alpha, t/\varepsilon^{2\alpha})$, where $\varphi \in C^\infty(\overline{\Omega}_T)$ with $\varphi = 0$ on $\partial\Omega \times [0,T]$, and $\psi \in H^1_\#(Q)$, as test function in (21) (where we do not integrate by parts in time) and we unfold the resulting equation. We obtain

$$\varepsilon^\alpha \int_0^T \int_\Omega \int_Q \mathcal{T}_\varepsilon(a_1^\varepsilon)\left(\mathcal{T}_\varepsilon\left(\frac{\partial v_\varepsilon}{\partial t}\right)\mathcal{T}_\varepsilon\left(\frac{1}{b_1^\varepsilon}\right) + \mathcal{T}_\varepsilon(v_\varepsilon)\mathcal{T}_\varepsilon\left(\frac{\partial}{\partial t}\frac{1}{b_1^\varepsilon}\right)\right)\mathcal{T}_\varepsilon(\varphi\psi)\,dy\,d\tau\,dx\,dt$$

$$+\varepsilon^\alpha \int_0^T \int_\Omega \int_Q \mathcal{T}_\varepsilon(B^\varepsilon)\left(\mathcal{T}_\varepsilon(v_\varepsilon)\mathcal{T}_\varepsilon\left(\nabla_x b_2^\varepsilon + \varepsilon\nabla_x\left(\frac{b^\varepsilon}{b_1^\varepsilon}\right) + \varepsilon^{1-\alpha}\nabla_y\left(\frac{b^\varepsilon}{b_1^\varepsilon}\right)\right) + \mathcal{T}_\varepsilon\left(b_2^\varepsilon + \varepsilon\frac{b^\varepsilon}{b_1^\varepsilon}\right)\mathcal{T}_\varepsilon(\nabla v_\varepsilon)\right)$$

$$\cdot \left(\mathcal{T}_\varepsilon\left(\nabla\left(\frac{\varphi}{a_2}\right)\right)\mathcal{T}_\varepsilon(\psi) + \mathcal{T}_\varepsilon\left(\frac{\varphi}{a_2}\right)\mathcal{T}_\varepsilon(\nabla_x\psi) + \frac{1}{\varepsilon^\alpha}\mathcal{T}_\varepsilon\left(\frac{\varphi}{a_2}\right)\mathcal{T}_\varepsilon(\nabla_y\psi)\right) dy\,d\tau\,dx\,dt$$

$$= \varepsilon^\alpha \int_0^T \int_\Omega f\frac{\varphi}{a_2}\psi\,dx\,dt + R_\varepsilon\,, \quad (120)$$

where $R_\varepsilon \to 0$ for $\varepsilon \to 0$. Now, passing to the limit $\varepsilon \to 0$ and taking into account (111) and (112), we get

$$\int_0^T \int_\Omega \int_Q \frac{a_1}{b_1}\frac{\partial v_1}{\partial \tau}\varphi\psi\,dy\,d\tau\,dx\,dt + \int_0^T \int_\Omega \int_Q \frac{1}{a_2}B\left(v\left(\nabla b_2 + \omega_{\alpha,1}\nabla_y\left(\frac{b}{b_1}\right)\right) + b_2(\nabla v + \nabla_y v_1)\right) \cdot \varphi\nabla_y\psi\,dy\,d\tau\,dx\,dt = 0\,, \quad (121)$$

which is the weak formulation of (114). $\qquad\square$

Corollary 1. *Given* $v \in L^2(0,T;H^1_0(\Omega))$, *the equation* (114) *admits a unique solution* $v_1 \in L^2(\Omega_T;H^1_\#(Q))$ *with* $\int_Q v_1\,dy\,d\tau = 0$.

Proof. The proof of the uniqueness follows by standard energy estimates and Young inequality, taking into account the linearity of the problem and the periodicity of v_1. □

Theorem 3. *In the same hypotheses of Theorem 2, the corrector v_1 can be written in the factored form*

$$v_1(x,t,y,\tau) = -\chi^j(x,t,y,\tau)\frac{\partial v}{\partial x_j}(x,t) - \zeta(x,t,y,\tau)v(x,t), \tag{122}$$

where the cell functions χ^j, $j = 1, \ldots, n$, and ζ are Q–periodic, with null mean average over Q, and are the unique solutions of

$$\frac{a_1}{b_1}\frac{\partial \chi^j}{\partial \tau} - \frac{1}{a_2}div_y\big(b_2 B\nabla_y(\chi^j - y_j)\big) = 0 \tag{123}$$

and

$$\frac{a_1}{b_1}\frac{\partial \zeta}{\partial \tau} - \frac{1}{a_2}div_y\big(b_2 B\nabla_y\zeta\big) + \frac{1}{a_2}div_y\Big(B\Big(\nabla b_2 + \omega_{\alpha,1}\nabla_y\frac{b}{b_1}\Big)\Big) = 0. \tag{124}$$

Moreover, the system (113) and (114) can be written as the single scale equation

$$q_{eff}\frac{\partial v}{\partial t} - div(B_{hom}\nabla v) + P_{eff}\cdot\nabla v + z_{eff}v = f\int_S \frac{d\tau}{a_2}, \tag{125}$$

where

$$q_{eff} = \int_{\mathcal{Y}} \frac{a_1}{b_1}\,\mathrm{d}y, \tag{126}$$

$$B_{hom}^{ij} = \int_Q \frac{b_2}{a_2}B^{ik}\partial_k(y^j - \chi^j)\,\mathrm{d}y\,\mathrm{d}\tau = \int_Q \frac{b_2}{a_2}B^{\ell k}\partial_k(y^j - \chi^j)\partial_\ell(y^i - \chi^i)\,\mathrm{d}y\,\mathrm{d}\tau + \int_Q \frac{a_1}{b_1}\frac{\partial \chi^j}{\partial \tau}\chi^i\,\mathrm{d}y\,\mathrm{d}\tau, \tag{127}$$

$$P_{eff} = \int_Q \Big[\frac{b_2}{a_2}B\nabla_y\zeta + b_2(B\nabla_y(y - \chi))^\dagger\nabla\frac{1}{a_2} - \frac{1}{a_2}B\nabla b_2 - \omega_{\alpha,1}B\nabla_y\Big(\frac{b}{a_2 b_1}\Big)\Big]\,\mathrm{d}y\,\mathrm{d}\tau, \tag{128}$$

$$z_{eff} = \int_{\mathcal{Y}} a_1\frac{\partial}{\partial t}\frac{1}{b_1}\,\mathrm{d}y + \int_Q \Big[div\Big(\frac{Bb_2}{a_2}\nabla_y\zeta\Big) - \frac{1}{a_2}div\Big(B\Big(\nabla b_2 + \omega_{\alpha,1}\nabla_y\frac{b}{b_1}\Big)\Big) - b_2 B\nabla_y\zeta\cdot\nabla\frac{1}{a_2}\Big]\,\mathrm{d}y\,\mathrm{d}\tau. \tag{129}$$

Proof We first note that, by classical results (see, i.e.,[20] Chapter 1, Section 2.2), equations (123) and (124) admit a unique Q–periodic solution with null mean average. Then, a standard computation shows that v_1 defined in (122) satisfies (114). Finally, inserting (122) into (113) and performing some algebraic computations we get equation (125).

In particular, the second equality in (127) can be obtained as follows. We first note that

$$B_{hom}^{ij} = -\int_Q \frac{b_2}{a_2}B^{\ell k}\partial_k(\chi^j - y^j)\partial_\ell y^i\,\mathrm{d}y\,\mathrm{d}\tau\ .$$

Moreover, from (123), we have that

$$\int_Q \frac{b_2}{a_2}B^{\ell k}\partial_k(\chi^j - y^j)\partial_\ell \chi^i\,\mathrm{d}y\,\mathrm{d}\tau + \int_Q \frac{a_1}{b_1}\frac{\partial \chi^j}{\partial \tau}\chi^i\,\mathrm{d}y\,\mathrm{d}\tau = 0\ .$$

By summing the two equations above we get (127). □

Corollary 2. *In the same hypotheses of Theorem 2, equation (125), complemented with the boundary and initial conditions (115) and (116), and the two–scale problem (113)–(116) admit a unique solution.*

Proof. First we note that the matrix B_{hom} in (127) is made of two parts, the first one is symmetric and by standard calculations it is also positive definite. On the other hand, the second part, which is due to the presence of the derivative with respect to the microscopic time τ in the parabolic equation (123) for the cell functions χ^j, is antisymmetric.

However, the uniqueness for equation (125), complemented with (115) and (116), still follows by standard energy estimates, Gronwall and Young inequalities, taking into account that the antisymmetric part of the homogenized matrix B_{hom} disappears in the energy estimate. Indeed, it is multiplied by the symmetric matrix $(v - \tilde{v})_{x_i}(v - \tilde{v})_{x_j}$, where v and $\tilde{v}$ are two different solutions of (125). Thus, the estimation can be performed as usual.

To prove uniqueness for the problem (113)–(116), we assume that there exist two solutions (v, v_1) and $(\tilde{v}, \tilde{v}_1)$. >From Corollary 1 and Theorem 3 it follows that v_1 and $\tilde{v}_1$ are given as in (122) for v and $\tilde{v}$, respectively. By substituting these two

representations of v_1 and $\tilde{v}_1$ in (113), it follows that both v and $\tilde{v}$ satisfy (125). Thus, by uniqueness of the solution of (125), we have that $v = \tilde{v}$ and, therefore, we also have $v_1 = \tilde{v}_1$. $\qquad\square$

Remark 1. Notice that the antisymmetric term disappears in the homogenized matrix (127) under some additional assumptions. For instance, when the matrix B and the coefficients a_1, a_2, b_1, and b_2 do not depend on the macroscopic space variable x [20].

Theorem 4. *Let $\beta > 2\alpha$. Then, there exist $v \in L^2(0,T; H_0^1(\Omega))$ and $v_1 \in L^2(\Omega_T; H_\#^1(Q))$, with $\int_Q v_1 \, dy \, d\tau = 0$, such that (108)–(111) hold and*

$$\frac{\partial v_1}{\partial \tau} = 0. \tag{130}$$

Moreover, the pair (v, v_1) is a weak solution of the two–scale problem (113), (115), (116), complemented with the microscale equation

$$div_y\left[\mathcal{M}_S\left(\frac{Bb_2}{a_2}\right)(\nabla v + \nabla_y v_1) + v\mathcal{M}_S\left(\frac{B\nabla b_2}{a_2}\right) + \omega_{\alpha,1}v\mathcal{M}_S\left(\frac{B}{a_2}\nabla_y\left(\frac{b}{b_1}\right)\right)\right] = 0, \qquad in \; \Omega_T \times \mathcal{Y}. \tag{131}$$

Proof The convergence results in (108) and (109) still follow from (74); (110) follows from (77) and [19] Proposition 2.12, with $m = 1/2$ and by replacing τ with ε^β. Finally, (111) and (130) follow from (77) and [19] Theorem 2.16, with $m = 1/2$, by replacing τ with ε^β, ε with ε^α and taking into account that $\beta > 2\alpha$.

The proof of (113) and (116) is exactly the same as in the case $\beta = 2\alpha$.

In order to prove (131), we take into account (130) and choose the test function $\phi_\varepsilon(x,t) = \varepsilon^\alpha(\varphi(x,t)/a_2(x,t,t/\varepsilon^\beta))\psi(x/\varepsilon^\alpha)$, where $\varphi \in C^\infty(\overline{\Omega}_T)$ with $\varphi = 0$ on $\partial\Omega \times [0,T]$, and $\psi \in H_\#^1(\mathcal{Y})$, as test function in (21). We unfold the resulting equation and obtain

$$-\varepsilon^\alpha \int_0^T \int_\Omega \int_Q \mathcal{T}_\varepsilon(v_\varepsilon)\mathcal{T}_\varepsilon\left(\frac{1}{b_1^\varepsilon}\right)\left(\mathcal{T}_\varepsilon\left(\frac{\partial a_1^\varepsilon}{\partial t}\right)\mathcal{T}_\varepsilon(\varphi\psi) + \mathcal{T}_\varepsilon(a_1^\varepsilon)\mathcal{T}_\varepsilon(\psi)\mathcal{T}_\varepsilon\left(\frac{\partial\varphi}{\partial t}\right)\right) dy \, d\tau \, dx \, dt$$

$$+ \varepsilon^\alpha \int_0^T \int_\Omega \int_Q \mathcal{T}_\varepsilon(B^\varepsilon)\left(\mathcal{T}_\varepsilon(v_\varepsilon)\mathcal{T}_\varepsilon\left(\nabla_x b_2^\varepsilon + \varepsilon\nabla_x\left(\frac{b^\varepsilon}{b_1^\varepsilon}\right) + \varepsilon^{1-\alpha}\nabla_y\left(\frac{b^\varepsilon}{b_1^\varepsilon}\right)\right) + \mathcal{T}_\varepsilon\left(b_2^\varepsilon + \varepsilon\frac{b^\varepsilon}{b_1^\varepsilon}\right)\mathcal{T}_\varepsilon(\nabla v_\varepsilon)\right)$$

$$\cdot \left(\mathcal{T}_\varepsilon\left(\nabla\left(\frac{\varphi}{a_2}\right)\right)\mathcal{T}_\varepsilon(\psi) + \mathcal{T}_\varepsilon\left(\frac{\varphi}{a_2}\right)\mathcal{T}_\varepsilon(\nabla_x\psi) + \frac{1}{\varepsilon^\alpha}\mathcal{T}_\varepsilon\left(\frac{\varphi}{a_2}\right)\mathcal{T}_\varepsilon(\nabla_y\psi)\right) dy \, d\tau \, dx \, dt$$

$$= \varepsilon^\alpha \int_0^T \int_\Omega f\frac{\varphi}{a_2}\psi \, dx \, dt + \varepsilon^\alpha \int_\Omega \frac{\bar{v}_\varepsilon(x)}{b_1^\varepsilon(x,0)}a_1^\varepsilon(x,0)\varphi(x,0)\psi\left(\frac{x}{\varepsilon^\alpha}\right) dx + R_\varepsilon, \quad (132)$$

where $R_\varepsilon \to 0$ for $\varepsilon \to 0$. Now, passing to the limit $\varepsilon \to 0$ and taking into account (111), we get

$$\int_0^T \int_\Omega \int_Q \frac{1}{a_2}B\left(v\left(\nabla b_2 + \omega_{\alpha,1}\nabla_y\left(\frac{b}{b_1}\right)\right) + b_2(\nabla v + \nabla_y v_1)\right) \cdot \varphi\nabla_y\psi \, dy \, d\tau \, dx \, dt = 0, \tag{133}$$

which is the weak formulation of (131). $\qquad\square$

Notice that, by (130), actually $v_1 \in H_\#^1(\mathcal{Y})$ with $\int_\mathcal{Y} v_1 \, dy = 0$. Moreover, similarly to the case $\beta = 2\alpha$ discussed above, we have the following corollary.

Corollary 3. *Given $v \in L^2(0,T; H_0^1(\Omega))$, equation (131) admits a unique solution $v_1 \in L^2(\Omega_T; H_\#^1(\mathcal{Y}))$ with $\int_\mathcal{Y} v_1 \, dy = 0$.*

Theorem 5. *In the same hypotheses of Theorem 4, the corrector v_1 can be written in the factored form*

$$v_1(x,t,y) = -\chi^j(x,t,y)\frac{\partial v}{\partial x_j}(x,t) - \zeta(x,t,y)v(x,t) \tag{134}$$

where the cell functions χ^j, $j = 1, \dots, n$, and ζ are $\mathcal{Y}$–periodic, with null mean average over $\mathcal{Y}$, and are the unique solutions of

$$div_y\left(\mathcal{M}_S\left(\frac{b_2 B}{a_2}\right)\nabla_y(\chi^j - y_j)\right) = 0 \tag{135}$$

and

$$-div_y\Big(\mathcal{M}_S\Big(\frac{b_2 B}{a_2}\Big)\nabla_y\zeta\Big) + div_y\mathcal{M}_S\Big(\frac{B}{a_2}\Big(\nabla b_2 + \omega_{\alpha,1}\nabla_y\frac{b}{b_1}\Big)\Big) = 0. \tag{136}$$

Moreover, the system (113) *and* (131) *can be written as the single scale equation* (125), *where* q_{eff}, P_{eff}, *and* z_{eff} *are formally defined as in Theorem 3, and*

$$B_{hom}^{ij} = \int_Q \frac{b_2}{a_2}B^{ik}\partial_k(y^j - \chi^j)\,\mathrm{d}y\,\mathrm{d}\tau = \int_Q \frac{b_2}{a_2}B^{\ell k}\partial_k(y^j - \chi^j)\partial_\ell(y^i - \chi^i)\,\mathrm{d}y\,\mathrm{d}\tau, \tag{137}$$

with χ *and* ζ *being the solutions of* (135) *and* (136).

Proof We first note that, by classical results (see, i.e.,[20] Chapter 1, Section 2.2), equations (135) and (136) admit a unique $\mathcal{Y}$–periodic solution with null mean average. Then, a standard computation shows that v_1 defined in (134) satisfies (131). Finally, inserting (134) into (113) and performing some algebraic computations, we get equation (125). In particular, the second equality (137) is obtained as done for (127) in Theorem 3, by using, now, (135). □

Corollary 4. *In the same hypotheses of Theorem 4, equation* (125), *with the homogenized matrix* B_{hom} *given in* (137) *and complemented with the boundary and initial conditions* (115) *and* (116), *admits a unique solution. Moreover, the two–scale problem* (113), (131), (115), *and* (116) *admits a unique solution.*

Proof. First we note that the matrix B_{hom} in (137) is symmetric and, by standard calculations, it is also positive definite. Thus, the uniqueness for equation (125), complemented with (115) and (116), as usual follows by standard energy estimates, Gronwall and Young inequalities.

The second part of the corollary can be proven as we did for Corollary 2. □

4.2 | Slow oscillations

In this section, we consider the remaining case $\beta < 2\alpha$.

Theorem 6. *Let* $\beta < 2\alpha$. *Then, there exist* $v \in L^2(0,T;H_0^1(\Omega))$ *and* $v_1 \in L^2(\Omega_T \times S; H_\#^1(\mathcal{Y}))$, *with* $\int_{\mathcal{Y}} v_1\,dy = 0$, *such that* (108), (109) *and* (111) *hold, as well as*

$$\mathcal{T}_\varepsilon(v_\varepsilon) \rightharpoonup v, \qquad \text{weakly in } L^2(\Omega_T \times Q). \tag{138}$$

Moreover, the pair (v, v_1) *is a weak solution of the two–scale problem* (113), (115), (116), *complemented with the microscale equation*

$$\frac{1}{a_2}div_y\Big[B\Big(b_2(\nabla v + \nabla_y v_1) + v\nabla b_2 + \omega_{\alpha,1}v\nabla_y\Big(\frac{b}{b_1}\Big)\Big)\Big] = 0, \qquad \text{in } \Omega_T \times Q. \tag{139}$$

Proof The convergence results in (108) and (109) follow from (74); (111) follows from (109) and[19] Theorem 2.11. In order to prove (138), we proceed as follows. As, for instance, in[19] Proposition 2.8, for a suitable $\widetilde{v}(x,t,y,\tau)$, we have

$$\mathcal{T}_\varepsilon(v_\varepsilon) \rightharpoonup \widetilde{v}, \qquad \text{weakly in } L^2(\Omega_T \times Q),$$

as a consequence of (109). On the other hand, by (86) and[19] Proposition 2.12, with $m = 1/2$ and $\tau = \varepsilon^\beta$, we get that

$$\mathcal{T}_\varepsilon(v_\varepsilon) \rightharpoonup v, \qquad \text{weakly in } L^2(\Omega \times (T_1,T); H^1(Q)).$$

By testing with compactly supported functions in $\Omega_T \times Q$ we conclude that $\widetilde{v} = v$.

The proof of (113) and (116) is the same as in the case $\beta = 2\alpha$. In order to prove (139), we choose $\phi_\varepsilon(x,t) = \varepsilon^\alpha(\varphi(x,t)/a_2(x,t,t/\varepsilon^\beta))\psi(x/\varepsilon^\alpha,t/\varepsilon^\beta)$, where $\varphi \in C^\infty(\overline{\Omega}_T)$ with $\varphi(x,T) = 0$ in $\overline{\Omega}$ and $\varphi = 0$ on $\partial\Omega \times [0,T]$, and $\psi \in H_\#^1(Q)$

with $\psi(y,0) = \psi(y,1) = 0$ in $\mathcal{Y}$, as test function in (21). We unfold the resulting equation and obtain

$$-\varepsilon^{\alpha} \int_0^T \int_\Omega \int_Q \mathcal{T}_\varepsilon(v_\varepsilon)\mathcal{T}_\varepsilon\left(\frac{1}{b_1^\varepsilon}\right)\left(\mathcal{T}_\varepsilon\left(\frac{\partial a_1^\varepsilon}{\partial t}\right)\mathcal{T}_\varepsilon(\varphi\psi) + \mathcal{T}_\varepsilon(a_1^\varepsilon)\mathcal{T}_\varepsilon(\psi)\mathcal{T}_\varepsilon\left(\frac{\partial\varphi}{\partial t}\right)\right) dy\,d\tau\,dx\,dt$$

$$-\varepsilon^{\alpha-\beta} \int_0^T \int_\Omega \int_Q \mathcal{T}_\varepsilon(v_\varepsilon)\mathcal{T}_\varepsilon\left(\frac{1}{b_1^\varepsilon}\right)\mathcal{T}_\varepsilon(a_1^\varepsilon)\mathcal{T}_\varepsilon(\varphi)\mathcal{T}_\varepsilon\left(\frac{\partial\psi}{\partial\tau}\right) dy\,d\tau\,dx\,dt$$

$$+\varepsilon^{\alpha} \int_0^T \int_\Omega \int_Q \mathcal{T}_\varepsilon(B^\varepsilon)\left(\mathcal{T}_\varepsilon(v_\varepsilon)\mathcal{T}_\varepsilon\left(\nabla_x b_2^\varepsilon + \varepsilon\nabla_x\left(\frac{b^\varepsilon}{b_1^\varepsilon}\right) + \varepsilon^{1-\alpha}\nabla_y\left(\frac{b^\varepsilon}{b_1^\varepsilon}\right)\right) + \mathcal{T}_\varepsilon\left(b_2^\varepsilon + \varepsilon\frac{b^\varepsilon}{b_1^\varepsilon}\right)\mathcal{T}_\varepsilon(\nabla v_\varepsilon)\right)$$

$$\cdot \left(\mathcal{T}_\varepsilon\left(\nabla\left(\frac{\varphi}{a_2}\right)\right)\mathcal{T}_\varepsilon(\psi) + \mathcal{T}_\varepsilon\left(\frac{\varphi}{a_2}\right)\mathcal{T}_\varepsilon(\nabla_x\psi) + \frac{1}{\varepsilon^\alpha}\mathcal{T}_\varepsilon\left(\frac{\varphi}{a_2}\right)\mathcal{T}_\varepsilon(\nabla_y\psi)\right) dy\,d\tau\,dx\,dt$$

$$= \varepsilon^{\alpha} \int_0^T \int_\Omega f\frac{\varphi}{a_2}\psi\,dx\,dt + R_\varepsilon\,, \quad (140)$$

where $R_\varepsilon \to 0$ for $\varepsilon \to 0$.

Using that, as we show below, the second term in (140) tends to zero in the limit $\varepsilon \to 0$, we get the weak formulation of (139), similarly as we did for (133) in the proof of Theorem 4.

Indeed, the second term in (140) can be written as

$$\varepsilon^{2\alpha-\beta} \int_0^T \int_\Omega \int_Q \frac{1}{\varepsilon^\alpha}\mathcal{Z}_\varepsilon(v_\varepsilon)\mathcal{T}_\varepsilon\left(\frac{1}{b_1^\varepsilon}\right)\mathcal{T}_\varepsilon(a_1^\varepsilon)\mathcal{T}_\varepsilon(\varphi)\mathcal{T}_\varepsilon\left(\frac{\partial\psi}{\partial\tau}\right) dy\,d\tau\,dx\,dt$$

$$+\varepsilon^{\alpha-\beta} \int_0^T \int_\Omega \int_Q \mathcal{M}_\varepsilon(v_\varepsilon)\mathcal{T}_\varepsilon\left(\frac{1}{b_1^\varepsilon}\right)\mathcal{T}_\varepsilon(a_1^\varepsilon)\mathcal{T}_\varepsilon(\varphi)\mathcal{T}_\varepsilon\left(\frac{\partial\psi}{\partial\tau}\right) dy\,d\tau\,dx\,dt = J_1^\varepsilon + J_2^\varepsilon\,. \quad (141)$$

Recalling[19] Proposition 2.22 (with, in the notation there, $m = r = 1/2$, $\alpha = 1$, ε replaced by ε^α, and τ replaced by ε^β), it follows that $J_1^\varepsilon \to 0$, for $\varepsilon \to 0$. Moreover, we write

$$J_2^\varepsilon = \varepsilon^{2\alpha-\beta} \int_0^T \int_\Omega \int_Q \mathcal{M}_\varepsilon(v_\varepsilon)\mathcal{T}_\varepsilon\left(\frac{1}{b_1^\varepsilon}\right)\mathcal{T}_\varepsilon(a_1^\varepsilon)\frac{1}{\varepsilon^\alpha}\mathcal{Z}_\varepsilon(\varphi)\mathcal{T}_\varepsilon\left(\frac{\partial\psi}{\partial\tau}\right) dy\,d\tau\,dx\,dt$$

$$+\varepsilon^{\alpha-\beta} \int_0^T \int_\Omega \int_Q \mathcal{M}_\varepsilon(v_\varepsilon)\mathcal{T}_\varepsilon\left(\frac{1}{b_1^\varepsilon}\right)\mathcal{T}_\varepsilon(a_1^\varepsilon)\mathcal{M}_\varepsilon(\varphi)\mathcal{T}_\varepsilon\left(\frac{\partial\psi}{\partial\tau}\right) dy\,d\tau\,dx\,dt \quad (142)$$

and note that, using[19] Remark 2.23 (with, in the notation there, $m = r = 1/2$, $\alpha = 1$, ε replaced by ε^α, and τ replaced by ε^β), the first term tends to zero, in the limit $\varepsilon \to 0$.

Now, we integrate the second term in (142) with respect to τ taking into account that

$$\mathcal{T}_\varepsilon\left(\frac{\partial\psi}{\partial\tau}\right) = \frac{\partial}{\partial\tau}\mathcal{T}_\varepsilon(\psi)$$

and we get that it is equal to

$$-\varepsilon^{\alpha} \int_0^T \int_\Omega \int_Q \mathcal{M}_\varepsilon(v_\varepsilon)\frac{1}{\varepsilon^\beta}\frac{\partial}{\partial\tau}\left(\mathcal{T}_\varepsilon\left(\frac{a_1^\varepsilon}{b_1^\varepsilon}\right)\right)\mathcal{M}_\varepsilon(\varphi)\mathcal{T}_\varepsilon(\psi)\,dy\,d\tau\,dx\,dt$$

$$= -\varepsilon^{\alpha} \int_0^T \int_\Omega \int_Q \mathcal{M}_\varepsilon(v_\varepsilon)\mathcal{T}_\varepsilon\left(\frac{\partial}{\partial t}\left(\frac{a_1^\varepsilon}{b_1^\varepsilon}\right)\right)\mathcal{M}_\varepsilon(\varphi)\mathcal{T}_\varepsilon(\psi)\,dy\,d\tau\,dx\,dt \to 0\,, \quad (143)$$

where we used that

$$\frac{1}{\varepsilon^\beta}\frac{\partial}{\partial\tau}\left(\mathcal{T}_\varepsilon\left(\frac{a_1^\varepsilon}{b_1^\varepsilon}\right)\right) = \mathcal{T}_\varepsilon\left(\frac{\partial}{\partial t}\left(\frac{a_1^\varepsilon}{b_1^\varepsilon}\right)\right).$$

□

Similarly to the cases $\beta \geq 2\alpha$ discussed above, we have the following corollary.

Corollary 5. *Given* $v \in L^2(0, T; H_0^1(\Omega))$, *equation* (139) *admits a unique solution* $v_1 \in L^2(\Omega_T \times S; H_\#^1(\mathcal{Y}))$ *with* $\int_\mathcal{Y} v_1 \, dy = 0$.

Theorem 7. *In the same hypotheses of Theorem 6, the corrector* v_1 *can be written in the factored form* (122), *where the cell functions* χ^j, $j = 1, \ldots, n$ *and* ζ *are* $\mathcal{Y}$*–periodic, with null mean average over* $\mathcal{Y}$, *and are the unique solutions of*

$$\frac{1}{a_2}div_y\left(b_2 B\nabla_y(\chi^j - y_j)\right) = 0 \tag{144}$$

and

$$\frac{1}{a_2}div_y\left(b_2 B\nabla_y\zeta\right) + \frac{1}{a_2}div_y\left(B\left(\nabla b_2 + \omega_{\alpha,1}\nabla_y\frac{b}{b_1}\right)\right) = 0. \tag{145}$$

Moreover, the system (113) *and* (139) *can be written as the single scale equation* (125), *where* q_{eff}, P_{eff}, *and* z_{eff} *are formally defined as in Theorem 3, and* B_{hom} *is defined as in* (137), *with* χ *and* ζ *being the solutions of* (144) *and* (145).

Proof As above, by classical results[20] Chapter 1, Section 2.2, equations (144) and (145) admit a unique $\mathcal{Y}$–periodic solution, with null mean average. Then, a standard computation shows that v_1 defined in (122) satisfies (131). Finally, inserting (122) into (113) and performing some algebraic computations, we get equation (125). □

Similarly to the case $\beta > 2\alpha$ discussed above, we have the following corollary.

Corollary 6. *In the same hypotheses of Theorem 6, equation* (125), *with the homogenized matrix* B_{hom} *given in* (137), *where* χ *and* ζ *are the solutions of* (144) *and* (145), *and complemented with the boundary and initial conditions* (115) *and* (116), *admits a unique solution. Moreover, the two–scale problem* (113), (139), (115), *and* (116) *admits a unique solution.*

Notice that, in the present case, i.e. $\beta < 2\alpha$, the dependence of the cell functions χ and ζ on the microtime τ is only parametric (as well as on (x, t)), via the coefficients of the corresponding equations.

4.3 | Proof of Theorem 1.

In the case $\beta = 2\alpha$, by Theorems 2 and 3, we obtain that $v_\varepsilon \rightharpoonup v$ weakly in $L^2(\Omega_T)$, where v is the solution of (125). By using (74) and (90), it follows that the convergence is, indeed, strong in $L^2(\Omega_T)$. Moreover, by the assumptions on b_1, it follows that $1/b_1^\varepsilon \rightharpoonup \int_\mathcal{Y} dy/b_1$ weakly* in $L^\infty(\Omega_T)$. Therefore,

$$u_\varepsilon = \frac{v_\varepsilon}{b_1^\varepsilon} \rightharpoonup v\int_\mathcal{Y}\frac{dy}{b_1} =: u. \tag{146}$$

By replacing $v = u\left(\int_\mathcal{Y} dy/b_1\right)^{-1}$ in (125), we eventually get (22) and (23).

The uniqueness of the solution u of (22)–(23) follows by the uniqueness for equation (125), complemented with the boundary and the initial conditions (115) and (116).

The cases $\beta \neq 2\alpha$ are treated in the same way, of course by appealing to Theorems 4 and 3, for $\beta > 2\alpha$ and Theorems 6 and 7, for $\beta < 2\alpha$, respectively.

□

4.4 | Some particular cases

Here we discuss some very special cases in which the upscaled equations take specific forms.

4.4.1 | Pure product case

In the case $b = 0$, we can fix $\alpha = 1$ without loss of generality, since no other scaling, excepted x/ε^α and t/ε^β, is present in the equation.

The homogenized equations for the limit function u, appearing in Theorem 1, take the form

$$\int_Q \left[\frac{a_1}{\int_S a_2^{-1}\,d\tau} \frac{\partial}{\partial t}\left(\frac{u}{b_1 \int_y b_1^{-1}\,dy}\right) - \frac{1}{a_2 \int_S a_2^{-1}\,d\tau} \mathrm{div}\left(B_{\mathrm{eff}}\nabla\left(b_2 \frac{u}{\int_y b_1^{-1}\,dy}\right)\right) \right.$$
$$\left. - \frac{1}{a_2 \int_S a_2^{-1}\,d\tau} \mathrm{div}\left(B\nabla_y\left(-b_2\zeta + \chi \cdot \nabla b_2\right)\frac{u}{\int_y b_1^{-1}\,dy}\right) \right] dy\,d\tau = f, \qquad \text{in } \Omega_T, \quad (147)$$

where $B_{eff} = B\nabla_y(y - \chi)$ and the cell functions χ and ζ satisfy (123) and (124) with $b = 0$ for $\beta = 2$, (135) and (136) with $b = 0$ for $\beta > 2$, (144) and (145) with $b = 0$ for $\beta < 2$.

Incidentally, this is also the case when $\alpha < 1$ even if $b \neq 0$, which means that the space oscillation is greater than the non–product perturbation.

4.4.2 | Pure Fick case

In the case $b_1 = b_2 = 1$ and $b = 0$, as above we can fix $\alpha = 1$ without loss of generality, In such a case, our problem is a particular case of the one studied in [19], with the time oscillation being a power of the space oscillation.

Indeed, it is easy to prove that the cell function ζ is equal to zero for any β and the cell function χ satisfies [19] equation (7.1) for $\beta = 2$, [19] equation (7.2) for $\beta > 2$, and [19] equation (7.3) for $\beta < 2$, respectively.

Moreover, the homogenized equation (22) reduces to [19] equation (7.4), for any choice of β. In particular, when the capacity is independent of the macrovariables, the resulting equation turns to be the pure Fick equation

$$\frac{\int_y a_1\,dy}{\int_S a_2^{-1}\,d\tau} u_t - \mathrm{div}\left(\frac{\int_Q (B_{\mathrm{eff}}/a_2)\,dy\,d\tau}{\int_S a_2^{-1}\,d\tau}\nabla u\right) = f, \qquad (148)$$

where the capacity and the diffusion matrix appear mixed in the upscaled diffusion coefficient.

4.4.3 | Pure Fokker–Planck case

If B is the identity matrix, it follows that χ is always identically zero, so that $B_{\mathrm{eff}} = B$, and by periodicity

$$\int_y \nabla_y\left(\omega_{\alpha,1}\frac{b}{b_1} - b_2\zeta\right)dy = 0. \qquad (149)$$

Thus, the limit equation reduces to

$$\int_Q \left[\frac{a_1}{\int_S a_2^{-1}\,d\tau}\frac{\partial}{\partial t}\left(\frac{u}{b_1 \int_y b_1^{-1}\,dy}\right) - \frac{1}{a_2 \int_S a_2^{-1}\,d\tau}\Delta\left(b_2 \frac{u}{\int_y b_1^{-1}\,dy}\right)\right] dy\,d\tau = f, \qquad \text{in } \Omega_T, \qquad (150)$$

which does not depend on the non–product perturbation b. We remark that this is also valid under the milder hypothesis that B does not depend on y.

If in addition $\alpha < 1$ or $b = 0$, then also the cell function $\zeta = 0$ and therefore (149) is trivially satisfied.

The limit equation in the pure Fokker–Planck case has been written in the from (150) to make it as close as possible to the starting Fokker–Planck problem. However, it is possible to formally reduce it to a standard parabolic equation with lower order terms, in which the coefficients are expressed in terms of the mean value on Q of the coefficients of the original equation, i.e., a_1, a_2, b_1, and b_2.

Finally, we remark that in the very particular case in which the coefficients a_1, a_2, b_1, and b_2, do not depend on the macroscopic variables, the equation (150) becomes

$$\frac{\int_y (a_1/b_1)\,dy}{\int_S a_2^{-1}\,d\tau \int_y b_1^{-1}\,dy} u_t - \Delta\left(\frac{\int_S (b_2/a_2)\,d\tau}{\int_y b_1^{-1}\,dy \int_S a_2^{-1}\,d\tau} u\right) = f, \qquad \text{in } \Omega_T, \qquad (151)$$

which shows that, even in such a special case, the capacity and the Fokker coefficient are mixed in the upscaled equation.

Conflict of interest

The authors declare no potential conflict of interests.

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Text for the TABLE OF CONTENTS:

Diffusion in an inhomogeneous environment may lead to Fick or to Fokker-Planck models. We investigate the upscaling of problems where both types of variability coexist in a finely mixed and periodic microstructure. We show that for equations with a factored structure, allowing for a general perturbation, the two kinds of inhomogeneity behave differently in the homogenization limit.