Uniformly Positive Correlations in the Dimer Model and Macroscopic Interacting Self-Avoiding Walk in $\mathbb{Z}^d$, $d \geq 3$

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Abstract

Our first main result is that correlations between monomers in the dimer model in $\mathbb{Z}^d$ do not decay to 0 when $d > 2$. This is the first rigorous result about correlations in the dimer model in dimensions greater than 2 and shows that the model behaves drastically differently than in two dimensions, in which case it is integrable and correlations are known to decay to zero polynomially. Such a result is implied by our more general, second main result, which states the occurrence of a phase transition in the model of lattice permutations, which is related to the quantum Bose gas. More precisely, we consider a self-avoiding walk interacting with lattice permutations and we prove that, in the regime of fully packed loops, such a walk is 'long' and the distance between its endpoints grows linearly with the diameter of the box. These results follow from the derivation of a version of the infrared bound from a new general probabilistic settings, with coloured loops and walks interacting at sites and walks entering into the system from some 'virtual' vertices. © 2021 The Authors. *Communications on Pure and Applied Mathematics* published by Wiley Periodicals LLC.

1 Introduction

This paper considers two models related to each other, the dimer model and lattice permutations.

The dimer model is a classical statistical mechanics model whose configurations are perfect matchings of a graph, namely subsets of edges which cover every vertex precisely once. The model attracts interest from a wide range of perspectives, which include combinatorics, statistical mechanics, and algorithm complexity studies. Its rigorous mathematical study achieved a breakthrough with the works of Kasteleyn, Temperley and Fisher, [24, 38, 49] in 1961, who showed that on planar graphs the dimer problem is *exactly solvable*. By then, various aspects of the dimer model have been explored: For example its close relation to the Ising model [2, 38], a characterisation of the model's correlations [25], the arctic circle phenomenon [15], their continuous limits and the emergence of conformal symmetry [32, 39, 40].

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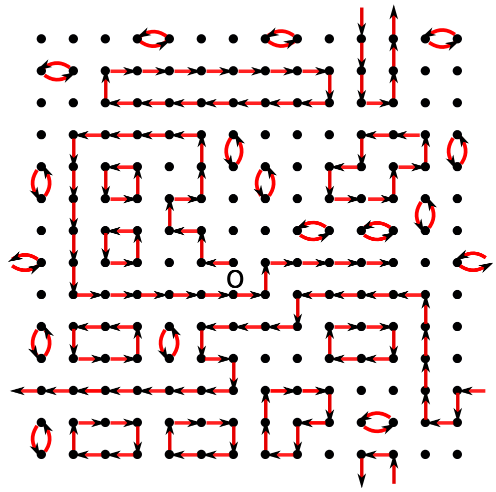

FIGURE 0.1. Lattice permutation with a self-avoiding walk.

Despite much progress on planar graphs, the rigorous mathematical understanding of the dimer model in higher dimensional graphs is still very poor. Indeed, as it was formalised by Hammersley *et al.* [35], the method of Kasteleyn, Temperley and Fisher, which consists of reducing the problem of enumerating the number of dimer covers to the problem of computing the Pfaffian of the so-called Kasteleyn matrix, cannot be naturally extended to $\mathbb{Z}^d$, $d > 2$, in which case it was shown [37] that the dimer model is computationally intractable.

This paper presents the first result about correlations in the dimer model in $\mathbb{Z}^d$, when $d > 2$. More precisely, we consider the monomer-monomer correlation, i.e., the ratio between the number of dimer covers with two monomers and the number of dimer covers with no monomers, which is a central quantity in the study of this model. In dimensions $d = 2$, it was shown that it decays to zero *polynomially* with the distance between the two monomers [16, 25]. Our first main result, Theorem 2.1 below, states that such a function *does not decay to zero* with the distance when $d > 2$. This is in agreement with physicists predictions [36] based on heuristic arguments. As a by-product of our technique we also deduce that, in the infinite volume limit, the correlation between monomers along the Cartesian axis equals $\frac{1}{2d}$ up to nonpositive corrections term of order $O(\frac{1}{d^2})$, which are uniform with respect to the distance between such monomers.

Our first main result is implied by our more general main result about the model of *lattice permutations*, which, in the form as we define it, can be viewed as a generalisation of the *double dimer model* [17, 41]. The configuration space of the model can be viewed as the set of directed multigraphs whose vertex set are the vertices of a box in $\mathbb{Z}^d$ and such that any connected component is either a 'monomer' (a single vertex with no edges which are incident to it), a 'double edge' (a connected component consisting of two vertices and two parallel edges pointing opposite directions), or a directed self-avoiding loop. A measure which assigns to each such graph a weight which depends on two parameters, $\rho \in [0, \infty)$, the *monomer activity*, and $N \in [0, \infty)$, the *number of colours*, is introduced. The parameter ρ rewards the number of monomers, while the parameter N rewards the number of loops and double edges.

The study of lattice permutations has been proposed in [4, 11, 31, 34] in view of their connections to *Bose-Einstein condensation* [23], which is an important unsolved statistical mechanics problem. Contrary to these papers, where jumps of arbitrary length are allowed and penalised according to a Gaussian weight (and no multiplicity factor for the number of loops and double edges is considered), here we only allow jumps of length 1 or 0; this feature gives the model a combinatorial flavour and allows the connection with the dimer model. The relevance of lattice permutations for the study of Bose-Einstein condensation is that, contrary to other spatial random permutation models that were studied before (for example, [1, 5, 7–9, 12, 21]) and similarly to the *interacting* quantum Bose gas, a spatial interaction that depends on the mutual distance of the loops takes place (loops interact by mutual exclusion). This feature makes the techniques that have been employed in such previous works ineffective for the rigorous analysis of lattice permutations and the model interesting and challenging. The central question for the quantum Bose gas is whether Bose-Einstein condensation takes place. In [51] it is shown that, in a random loop model which is related to lattice permutations, the two-point function, namely the ratio of the partition functions of a system with a forced 'open' cycle and one without, can be used to detect Bose-Einstein condensation: If this ratio stays positive uniformly in the volume and in the spatial separation of the two endpoints of the forced cycle, this is equivalent to the presence of off-diagonal long-range order [47], which itself is equivalent to Bose-Einstein condensation. This paper (our Theorem 2.3 below) provides a rigorous proof of this fact in the model of lattice permutations.

The relevance of lattice permutations goes even beyond their connection to the dimer model and Bose-Einstein condensation, which holds when $N = 2$. Indeed, they are an intriguing mathematical object for any value of $N \in [0, \infty)$ and can be viewed as a version of the *loop $O(N)$ model*, which is in turn related to spin systems with continuous symmetry for integer values of N (see [46] for an overview). The difference between our setting and the model considered in [46] is that we also allow double edges and that the loop containing the origin is 'open', namely it is a self-avoiding walk that starts from the origin and ends at an arbitrary vertex of the box. One of the most important questions for this class of models is the identification of regions of the phase diagram where the loop length *does not* admit exponential decay. This was recently accomplished for the loop $O(N)$ model on the hexagonal lattice using various techniques, for example parafermionic observables, planar spin representations, and Russo-Welsh estimates [19, 33]; see also further references in [46]. Although very powerful, these techniques are specific for planar graphs and cannot be naturally extended to $\mathbb{Z}^d$, $d > 2$, in which case only results stating exponential decay have been derived [14, 48] and techniques are missing. Our Theorem 2.2 below states that, in any dimension $d > 2$, in the regime of fully packed loops, the length of the self-avoiding walk in lattice permutations grows unboundedly with the size of the box, and the distance between its two endpoints is of the same order of magnitude as the diameter of the box. Hence, not only do we rule out exponential decay in any dimension $d > 2$, but we also identify the correct scaling of the distance between the endpoints of the self-avoiding walk.

Our proof technique is of independent interest and is a reformulation of the famous approach of Fröhlich, Simon, and Spencer [30] in the space of paths. In [30], the property of *reflection positivity* of a system of spins with continuous symmetry, the spin $O(N)$ model, was employed for the derivation of the so-called *infrared bound*, which implies that correlations do not decay in such a spin system. Such an approach was further developed in [27, 28] and implemented in several other research works in the framework of quantum and classical spin systems. Here we implement such an approach in a completely different setting that does not involve spins, but a general probabilistic model of interacting coloured loops and walks. Our approach generalises [30]; indeed, our general framework *includes* not only lattice permutations and the dimer model, for which no spin representation exists or is easy to derive in dimension $d > 2$, but also the (loop representation of) the spin $O(N)$ model (see also Remark 3.2 below).

2 Definitions and Main Results

We now provide a precise definition of the dimer model and of lattice permutations, and we state our main results formally. This section is divided into three subsections with each subsection stating a main theorem. Our third theorem, Theorem 2.3 below, involves lattice permutations, and it can be viewed as a reformulation of our Theorem 2.2 and as a generalisation of Theorem 2.1, which involves the dimer model.

2.1 The Dimer model

A *dimer cover* of the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a subgraph of $\mathcal{G}$ whose vertex set is $\mathcal{V}$ and such that every vertex has degree 1. Let $(\mathbb{T}_L, \mathbb{E}_L)$ be a graph with vertex set $\mathbb{T}_L := \big\{(x_1, \dots, x_d) \in \mathbb{Z}^d : \ x_i \in (-\frac{L}{2}, \frac{L}{2}]\big\}$ and edges connecting nearest-neighbour vertices and boundary vertices so that $(\mathbb{T}_L, \mathbb{E}_L)$ can be identified with the torus $\mathbb{Z}^d / L\mathbb{Z}^d$, where $L \in \mathbb{N}_{>0}$. For any set of sites $M \subset \mathbb{T}_L$, let $\mathcal{D}(M)$ be the (possibly empty) set of dimer covers of the graph which is obtained from $(\mathbb{T}_L, \mathbb{E}_L)$ by removing all the sites that are in M, and from $\mathbb{E}_L$ all the edges which are incident to at least one vertex in M. The *monomer-monomer correlation* is a fundamental quantity for the analysis of the dimer model, and it corresponds to the ratio between the number of dimer covers with two monomers and the number of

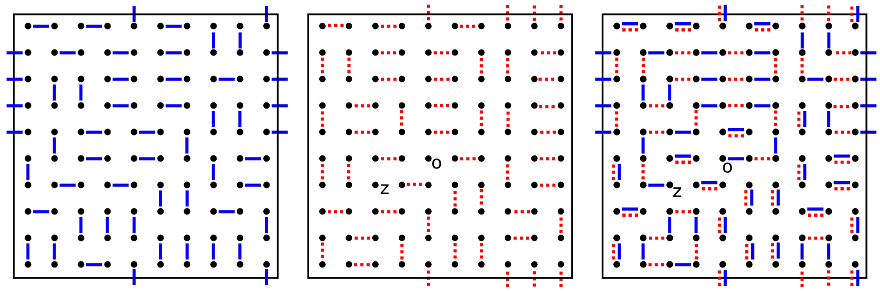

FIGURE 2.1. Left: A dimer cover in $\mathcal{D}(\varnothing)$. Centre: A dimer cover in $\mathcal{D}(\{o,z\})$. Right: superposition of the dimer cover on the left and the dimer cover in the centre.

dimer comers with no monomer,

$$\forall x \in \mathbb{T}_L \quad \Xi_L(x) := \frac{|\mathcal{D}(\{o,x\})|}{|\mathcal{D}(\varnothing)|}, \tag{2.1}$$

where o is used to denote the origin, $o = (0,\dots,0) \in \mathbb{T}_L$. See also Figure 2.1. This function equals 0 if $L \in 2\mathbb{N}$ and x belongs to the even sublattice of $\mathbb{T}_L^e \subset \mathbb{T}_L$, which is now defined together with the odd sublattice,

$$\begin{aligned} \mathbb{T}_L^e &:= \{x \in \mathbb{T}_L : d(o,x) \in 2\mathbb{N}\}, \\ \mathbb{T}_L^o &:= \{x \in \mathbb{T}_L : d(o,x) \in 2\mathbb{N}+1\}, \end{aligned} \tag{2.2}$$

where $d(o,x)$ is the graph distance in $(\mathbb{T}_L, \mathbb{E}_L)$. Let $N_+ = \sum_{n>0} \mathbb{1}\{S_n = o\}$ be the number of returns to the origin of a simple random walk starting from the origin, S_n, in $\mathbb{Z}^d$, whose probability measure and expectation are denoted by P^d and E^d, respectively, define $r_d := E^d(N_+)$, the expected number of returns to the origin. We use $\boldsymbol{e}_i \in \mathbb{R}^d$ to denote the Cartesian vectors, where $i \in \{1,\dots,d\}$.

THEOREM 2.1. *Suppose that $d > 2$. Then,*

$$\liminf_{\substack{L\to\infty \\ L \text{ even}}} \frac{1}{|\mathbb{T}_L^o|} \sum_{x \in \mathbb{T}_L^o} \Xi_L(x) \geq \frac{1}{2d}\left(1 - \frac{r_d}{2}\right). \tag{2.3}$$

Moreover, for any $\varphi \in (0, \frac{1}{2d}(1 - \frac{r_d}{2}))$, there exists an (explicit) constant $c_1 = c_1(\varphi, d) \in (0, \frac{1}{2})$ such that for any large enough $L \in 2\mathbb{N}$ and any odd integer $n \in (0, c_1 L)$,

$$\Xi_L(n\,\boldsymbol{e}_1) \geq \varphi. \tag{2.4}$$

Note that the monomer-monomer correlation equals 0 at even sites, that is, $\Xi_L(z) = 0$ for any $L \in 2\mathbb{N}$ and $z \in \mathbb{T}_L^e$ (as we prove in Lemma 5.4 below); hence the restriction of (2.4) to odd integers is necessary. An exact computation made by Watson [53] shows that $0.51 < r_d < 0.52$ when $d = 3$, and from the Rayleigh monotonicity principle [45] we deduce that r_d is nonincreasing with d. Thus, the Cesáro sum in (2.3) is bounded away from 0 uniformly for large L for any $d > 2$. Contrary to this, when $d = 2$ such a sum converges to 0 with the system size L [16]. From the general site-monotonicity properties that were derived in [43, remark 2.5] we deduce that

$$\forall L \in 2\mathbb{N},\ \forall x \in \mathbb{T}_L, \quad \Xi_L(x) \leq \frac{1}{2d}. \tag{2.5}$$

Since $r_d = O(\frac{1}{d})$, our lower bound in (2.3) gets closer to the pointwise upper bound (2.5) as the dimension increases. Hence, the larger the dimension, the more uniform is the correlation between monomers across the odd sites of the torus.

For $x \in \mathbb{Z}^d$, define now $\Xi(x) := \liminf_{L\to\infty} \Xi_{2L}(x)$. Our bound (2.4) and the pointwise upper bound (2.5) imply that, when $d > 2$, for any integer $n \in 2\mathbb{Z}+1$,

$$\frac{1}{2d}\left(1 - \frac{r_d}{2}\right) \leq \Xi(n\boldsymbol{e}_i) \leq \frac{1}{2d}, \tag{2.6}$$

where $\boldsymbol{e}_i$ is any Cartesian vector. Contrary to (2.6), $\Xi(n\boldsymbol{e}_i)$ was conjectured by Fisher and Stephenson [25] and proved by [16] to decay like $n^{-1/2}$ when $d = 2$. From (2.6) we deduce the asymptotic behaviour of the monomer-monomer correlation in the limit of large dimensions, i.e., for any odd integer n,

$$\Xi(n\boldsymbol{e}_i) = \frac{1}{2d} + O\left(\frac{1}{d^2}\right),$$

where the error term in the right-hand side is uniform in n.

2.2 Lattice permutations

We now introduce the model of lattice permutations. Recall that $(\mathbb{T}_L, \mathbb{E}_L)$ denotes the torus, with edges connecting nearest-neighbour vertices. To begin, for any pair of sites $x, y \in \mathbb{T}_L$ such that $x \neq y$, let $\Omega_{x,y}$ be the set of directed multigraphs $\pi = (\mathbb{T}_L, E_\pi)$ such that: (i) the edges of E_π connect nearest-neighbour vertices in the torus; (ii) the in- and the out-degree of every vertex in $\mathbb{T}_L \setminus \{x, y\}$ are equal and their value is either 0 or 1; (iii) the out-degree of x is 1 and its in-degree is 0, the out-degree of y is 0 and its in-degree is 1. This implies that the connected component of the graph $(\mathbb{T}_L, E_\pi)$ that contains x is a walk which starts at x and ends at y and that any other connected component is either a monomer, a double edge, or a loop, which we now define: a *walk* is a subgraph that is isomorphic to a simple open curve in $\mathbb{R}^d$ and is directed (and self-avoiding); a *monomer* is a connected component consisting of a single vertex with no edges incident to it; a *double edge* is a connected component corresponding to a pair of nearest-neighbour vertices, $z, w \in \mathbb{T}_L$, with an edge directed from z to w and an edge directed from w to z; and a *loop* is a subgraph that is isomorphic to a simple closed curve in $\mathbb{R}^d$ and is directed (and self-avoiding). See also Figure 0.1 for an example.

When $x = y$, we define $\Omega_{x,y}$ as the set of directed multigraphs $\pi = (\mathbb{T}_L, E_\pi)$ such that: (i) the edges of E_π connect nearest-neighbour vertices in the torus; (ii) the in- and the out-degree of every vertex in $\mathbb{T}_L \setminus \{x\}$ are equal and their value is either 0 or 1; (iii) the vertex $x = y$ is a monomer (i.e., the walk is 'degenerate', namely, it consists of just one vertex and no edges). We define the *configuration space* $\Omega := \bigcup_{x \in \mathbb{T}_L} \Omega_{o,x}$. Each such $\pi \in \Omega$ can be viewed as a system of monomers, loops, and double edges with a walk that starts from the origin and ends at one unspecified vertex of the torus, and all these objects are mutually disjoint. For any $\pi \in \Omega$, let $\mathcal{M}(\pi)$ be the number of monomers of π. Furthermore, for any $\pi \in \Omega$, let $\mathcal{L}(\pi)$ be the *number of loops and double edges* in π. We introduce the probability measure $\mathbb{P}_{L,N,\rho}$ in Ω, which depends on two parameters $\rho \in [0, \infty)$, the *monomer activity*, and $N \in [0, \infty)$, the *number of colours*, as follows:

$$\forall \pi \in \Omega \quad \mathbb{P}_{L,N,\rho}\big(\pi\big) := \frac{\rho^{\mathcal{M}(\pi)} \, (\frac{N}{2})^{\mathcal{L}(\pi)}}{Z_{L,N,\rho}}, \tag{2.7}$$

where $Z_{L,N,\rho}$ is a normalisation constant. Let $X : \Omega \to \mathbb{T}_L$ be the endpoint of the walk, which we call the *target point.* More precisely, for any $\pi \in \Omega$, we define $X(\pi) \in \mathbb{T}_L$ as the unique vertex such that $\pi \in \Omega_{o,X(\pi)}$. It is known that, if the monomer activity is large enough, the length of the walk admits uniformly bounded exponential moments [6, 48]. This implies that the distance between the target point and the origin does not grow unboundedly with the size of the system. Our Theorem 2.2 below states that, contrary to the case of high monomer activity, when the monomer activity is zero, the distance between the target point and the origin grows with the size of the system and scales linearly with the diameter of the box. In other words, a phase transition takes place at a finite, possibly 0 value of the monomer activity. Recall that r_d is the expected number of returns of a simple random walk in $\mathbb{Z}^d$, and recall also the properties of r_d, which were stated above.

THEOREM 2.2. *Suppose that $d > 2$ and that N is an integer in $(0, \frac{4}{r_d})$. There exists an (explicit) constant $c_2 = c_2(N, d) \in (0, \infty)$ such that for any large enough $L \in 2\mathbb{N}$,*

$$\forall A \subset \mathbb{T}_L \quad \mathbb{P}_{L,N,0}(X \in A) \leq c_2 \frac{|A|}{L^d}. \tag{2.8}$$

For example, by choosing $A = \mathbb{T}_{\lfloor \epsilon L \rfloor}$ for a small enough ϵ, we see that with uniformly positive probability the target point is at a distance at least ϵL from the origin. The restriction of our result to not-too-large values of N is not a limitation of our technique: It was shown by Chayes, Pryadko, and Shtengel [14] that, in any dimension $d \geq 2$, if N is a large enough integer, the loop length admits uniformly bounded exponential moments for any value $\rho \in [0, \infty)$.[1] Hence, not only do we prove the occurrence of a phase transition with respect to the variation of ρ for integer values $N \in (0, \frac{4}{r_d})$, but we also prove the occurrence of a phase transition with respect to the variation of N when we fix $\rho = 0$.

[1] In [14] a different setting than ours is considered, with only loops, which are allowed to overlap a bounded number of times, and no walk; the proof of [14] can be adapted to our setting implying that the length of the self-avoiding walk does not grow unboundedly with the size of the system and admits uniformly bounded exponential moments.

2.3 Uniform positivity

Our third main theorem, Theorem 2.3 below, can be viewed as a generalisation of Theorem 2.1 and states that the two-point function of lattice permutations is bounded away from 0 pointwise when the points lie along the same Cartesian axis and 'on average' across all points, uniformly with respect to the system size. To define the two-point function we need to introduce the set of multigraphs Ω^{ℓ}, whose connected components are loops, double edges, or monomers and no walk is present. Thus, let Ω^{ℓ} be the set of directed multigraphs $\pi = (\mathbb{T}_L, E_\pi)$ such that: (i) the edges connect nearest-neighbour vertices in the torus and, (ii) the in- and the out-degree of every vertex in $\mathbb{T}_L$ are equal and their value is either 0 or 1. It follows from this definition that every connected component of the graph $\pi \in \Omega^{\ell}$ is either a monomer, a loop or a double edge, which we defined before. We extend the definition of the number of monomers, $\mathcal{M}(\pi)$, and of the number or loops and double edges, $\mathcal{L}(\pi)$, which were provided before, to the graphs $\pi \in \Omega^{\ell}$. For any $L \in \mathbb{N}$, $\rho, N \in [0, \infty)$, we define the *loop partition function,*

$$\mathbb{Z}^{\ell}_{L,N,\rho} := \sum_{\pi \in \Omega^{\ell}} \rho^{\mathcal{M}(\pi)} \left(\frac{N}{2}\right)^{\mathcal{L}(\pi)}, \tag{2.9}$$

and, for any $x, y \in \mathbb{T}_L$, we define the *directed partition function,*

$$\mathbb{Z}_{L,N,\rho}(x, y) := \sum_{\pi \in \Omega_{x,y}} \rho^{\mathcal{M}(\pi)} \left(\frac{N}{2}\right)^{\mathcal{L}(\pi)}, \tag{2.10}$$

Finally, we define the *two-point function*,

$$\mathbb{G}_{L,N,\rho}(x, y) := \frac{\mathbb{Z}_{L,N,\rho}(x, y)}{\mathbb{Z}^{\ell}_{L,N,\rho}}, \tag{2.11}$$

and we define $\mathbb{G}_{L,N,\rho}(x) := \mathbb{G}_{L,N,\rho}(o, x)$. In the special case of $N = 2$ and $\rho = 0$, the two-point function of lattice permutations corresponds to the monomer-monomer correlation function of the dimer model,

$$\forall x \in \mathbb{T}_L \quad \mathbb{G}_{L,2,0}(x) = \Xi_L(x). \tag{2.12}$$

Indeed, as we prove in (3.11) below, the set of configurations that are obtained by superimposing two independent dimer covers, as in Figure 2.1, are in a one-to-one correspondence with the set of fully packed lattice permutations and this leads to (2.12). In light of (2.12), our Theorem 2.3 below, which holds for arbitrary (not necessarily equal to 2) integers N, can be viewed as a generalisation of Theorem 2.1.

THEOREM 2.3. *Suppose that $d > 2$ and that N is an integer in $(0, \frac{4}{r_d})$. Then*

$$\liminf_{\substack{L \to \infty \\ L \text{ even}}} \frac{1}{|\mathbb{T}^o_L|} \sum_{x \in \mathbb{T}^o_L} \mathbb{G}_{L,N,0}(x) \geq \frac{1}{2d}\left(\frac{2}{N} - \frac{r_d}{2}\right). \tag{2.13}$$

Moreover, for any $\varphi \in (0, \frac{1}{2d}(\frac{2}{N} - \frac{r_d}{2}))$, *there exists an (explicit) constant* $c_3 = c_3(\varphi, d, N) \in (0, \frac{1}{2})$ *such that for any large enough* $L \in 2\mathbb{N}$ *and any odd integer* $n \in (\, 0,\ c_3\, L\,)$,

$$\mathbb{G}_{L,N,0}(\, n\, \boldsymbol{e}_1\,) \geq \varphi. \tag{2.14}$$

Similarly to the case of the dimer model, from the site-monotonicity properties that were derived in [43], we deduce that, for any integer $N \in \mathbb{N}_{>0}$ and any $\rho \in [0, \infty)$,

$$\forall x \in \mathbb{T}_L \quad \mathbb{G}_{L,N,\rho}(x) \leq \frac{1}{dN}. \tag{2.15}$$

Since $r_d = O(\frac{1}{d})$, our uniform lower bound on the average (2.13) and the uniform pointwise upper bound (2.15) on the two-point function get closer to each other as d gets larger. From this we deduce that the larger the dimension, the more uniform is the distribution of the target point across the sites of the torus.

3 Proof Description

Most of the paper is devoted to the proof of (2.13), from which all our main results follow. The proof of (2.13) is divided into two main parts. The first part is devoted to the derivation of the key inequality, Theorem 3.1 below, from the analysis of a general soup of loops and walks, to which we refer as *random path model.* The random path model was introduced in [43], and it is a generalisation of the random wire model [3], which in turn can be viewed as a reformulation of the random walk representation of the spin $O(N)$ model [13]. In [43] it was shown that the random path model satisfies the important property of reflection positivity (which will be stated later). The property of reflection positivity for random loop models was also used in [14, 44, 52].

However, in such works the additional structures that allow the derivation of the key inequality *directly* from the space of loops and walks (i.e., without employing any spin representation) have not been introduced. The most important technical novelty of this paper is the introduction of such structures. This allows the extension of the method of [28, 28, 30] to random loop models for which no spin representation exists or is easy to derive, for example, lattice permutations (and, consequently, the dimer model). More precisely, our analysis involves the study of the random path model with appropriate weights in an 'extended' graph, which is obtained from the original torus by adding 'virtual' vertices 'on the top' of each vertex of the original torus; such virtual vertices serve as a source for the walks, and the walks get a different weight depending on where they start from; the whole setting is designed in such a way that the reflection positivity property, which was proved to hold true in the torus [43], is preserved. Such virtual vertices play the same role of the external magnetic field in spin systems (see [42] for a recent further application of this setting). More precisely, the presence of multiple virtual vertices, with one virtual vertex on the top of each original vertex, represents the action of an external field that might have a different intensity on each vertex.

The second part is devoted to the derivation of a version of the so-called infrared bound from the key inequality. Here we use Fourier transforms similarly to the case of spin systems with continuous symmetry [27,28,30], in which case the two-point function corresponds to the correlation between two spins. Our analysis differs from the classical case for some nontrivial aspects. The most important difference is that, in our case, the two-point function vanishes at any even site as $\rho \to 0$. In other words, the model exhibits a sort of antiferromagnetic ordering, similarly to [22]. This introduces some difficulties that are overcome by exploiting the different symmetry properties of the odd and even Fourier two-point functions (which will be introduced later) with respect to appropriate translations in the (Fourier) dual torus.

We now describe the two parts of the proof in greater detail and state Theorem 3.1 and Lemma 3.3. In the third and last subsection, we present the (short) proof of Theorem 2.1 given Theorem 2.3.

3.1 Description of part I: Derivation of the key inequality

The first part of the proof, which is presented in Section 4, is devoted to the derivation of Theorem 3.1, which is stated below. For an arbitrary vector of real numbers, $\boldsymbol{v} = (v_z)_{z \in \mathbb{T}_L}$, define the *discrete Laplacian* of $\boldsymbol{v}$,

$$\forall x \in \mathbb{T}_L \quad (\Delta v)_x := \sum_{\substack{y \in \mathbb{T}_L: \\ x \sim y}} (v_y - v_x). \tag{3.1}$$

THEOREM 3.1 (**Key Inequality**). *For any $N \in \mathbb{N}_{>0}$, $\rho \in \mathbb{R}_{\geq 0}$, $L \in 2\mathbb{N}_{>0}$, and any real-valued vector $\boldsymbol{v} = (v_x)_{x \in \mathbb{T}_L}$, we have that*

$$\sum_{x,y \in \mathbb{T}_L} \mathbb{G}_{L,N,\rho}(x,y)(\Delta v)_x (\Delta v)_y \leq \sum_{\{x,y\} \in \mathbb{E}_L} (v_y - v_x)^2. \tag{3.2}$$

The proof of Theorem 3.1 uses several ingredients that we now describe informally. We deal with the random path model, namely a probabilistic model of coloured closed and open paths, which interact at sites through a weight function, which will be denoted by U. The model depends on an edge-parameter $\lambda \in [0, \infty)$ that, informally, has the effect of increasing the typical length of the paths as λ gets larger.

We introduce a new setting that is reminiscent of the random current representation of the Ising model [18]. It involves the random path model on a graph $(\mathcal{T}_L, \mathcal{E}_L)$, which is obtained from the torus $(\mathbb{T}_L, \mathbb{E}_L)$ by adding a new vertex (which will be referred to as *virtual*) on the top of each vertex in $\mathbb{T}_L$ and by connecting this new vertex to the one that is below it by an edge, as in Figure 3.1. We refer to such a new graph $(\mathcal{T}_L, \mathcal{E}_L)$ as an *extended torus* and to the graph $(\mathbb{T}_L, \mathbb{E}_L) \subset (\mathcal{T}_L, \mathcal{E}_L)$, which was defined previously, as the *original torus*. Virtual vertices play the role of sources for open paths, and closed paths are not allowed to 'touch' any virtual vertex.

This setting is designed in such a way that the measure associated to the random path model on the extended graph satisfies two fundamental properties *at the same time*. The first fundamental property is *reflection positivity*. The second fundamental property involves a central quantity, $\mathcal{Z}_{L,N,\lambda,U}(\boldsymbol{v})$, where $\boldsymbol{v} = (v_z)_{z\in\mathbb{T}_L}$ is a vector of real numbers indexed by the vertices of the original torus. The quantity $\mathcal{Z}_{L,N,\lambda,H}(\boldsymbol{v})$ is defined as the average of a function which assigns a multiplicative weight v_z every time that a walk starts (or ends) at a vertex of the original torus $z \in \mathbb{T}_L$, and a multiplicative weight $-2dv_z$ every time that a walk starts (or ends) at the virtual vertex which is 'on the top' of $z \in \mathbb{T}_L$. This fundamental property is stated in (3.3) below and involves the infinitesimal variation of the function $\mathcal{Z}_{L,N,\lambda,U}(\boldsymbol{v})$ around the point $\boldsymbol{v} = 0$ when a specific choice of the weight function, $U = H$, is made. When $\boldsymbol{v} = 0$, $\mathcal{Z}_{L,N,\lambda,U}(\boldsymbol{v})$ equals the partition function of the model with all loops and no walk. More precisely, for an arbitrary choice of $\boldsymbol{v} \in \mathbb{R}^{\mathbb{T}_L}$ and $\varphi \in \mathbb{R}$, in the limit as $\varphi \to 0$

$$\begin{aligned} &\mathcal{Z}_{L,N,\lambda,H}(\varphi\boldsymbol{v}) \\ &\quad = \lambda^{|\mathbb{T}_L|}\mathbb{Z}^{\ell}_{L,N,\frac{1}{\lambda}} - \varphi^2\,\frac{\lambda N}{2}\,\lambda^{|\mathbb{T}_L|}\mathbb{Z}^{\ell}_{L,N,\frac{1}{\lambda}} \sum_{\{x,y\}\in\mathbb{E}_L} \big(v_y - v_x\big)^2 \\ &\qquad + \varphi^2\,\frac{\lambda N}{2}\,\lambda^{|\mathbb{T}_L|} \sum_{x,y\in\mathbb{T}_L} \mathbb{Z}_{L,N,\frac{1}{\lambda}}(x,y)(\triangle v)_x(\triangle v)_y + o(\varphi^2). \end{aligned} \tag{3.3}$$

To derive (3.3) we introduce a map that maps configurations of the random path model to configurations of lattice permutations and compare their weights. Here we use in an essential way the structure of the extended torus: the walks that enter into the original torus from a virtual vertex are weighted differently than the walks that start from a vertex of the original torus, and the weights are chosen appropriately so that we get the discrete Laplacians and the sum involving factors $(v_y - v_x)^2$ in (3.3). Also, the properties of the random path model and of the weight function H, which forces the walk to be vertex-self-avoiding at every vertex except for its endpoints, are used in an essential way. We refer to this central step of the proof as *polynomial expansion.* The reason why the expansion (3.3) is so important is that it is possible to deduce the key inequality by showing that, for any vector $\boldsymbol{v} \in \mathbb{R}^{\mathbb{T}_L}$, the term of order $O(\varphi^2)$ in (3.3) is nonpositive. Indeed, the reader can verify that, from the nonpositivity of the term of order $O(\varphi^2)$ and from the definition of the two-point function, (2.11), Theorem 3.1 follows immediately after dividing the whole expression by $\frac{\lambda N}{2}\lambda^{|\mathbb{T}_L|}\mathbb{Z}^{\ell}_{L,N,1/\lambda}$.

It is for the proof of such a concavity property of the function $\mathcal{Z}_{L,N,\lambda,H}(\boldsymbol{v})$ that we use reflection positivity. More precisely, such a concavity property follows from an iterative use of reflections, which leads by reflection positivity to the

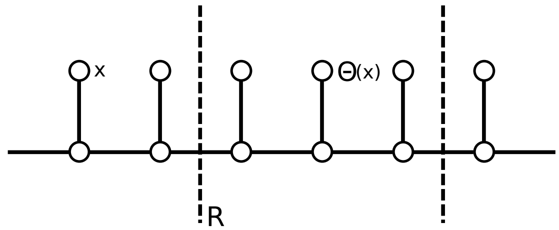

FIGURE 3.1. An *extended torus* when $d = 1$ and $L = 6$. The leftmost and the rightmost horizontal edges are identified. The leftmost vertical dashed line represents a reflection plane, R, which, for example, maps the vertex x to $\Theta(x)$. In the figure x is a virtual vertex, while the one that is 'below it' is original.

chessboard estimate,

$$\mathcal{Z}_{L,N,\lambda,U}(\boldsymbol{v}) \leq \Big(\prod_{x \in \mathbb{T}_L} \mathcal{Z}_{L,N,\lambda,U}(\boldsymbol{v}^x) \Big)^{\frac{1}{|\mathbb{T}_L|}}, \tag{3.4}$$

where $\boldsymbol{v}^x = (v^x_z)_{z \in \mathbb{T}_L}$ is a vector that is obtained from $\boldsymbol{v} := (v_z)_{z \in \mathbb{T}_L}$ by copying the value v_x at each original vertex. Since for each $x \in \mathbb{T}_L$, the term of order $O(\varphi^2)$ is 0 when we look at the vectors $\boldsymbol{v}^x$, i.e.,

$$\mathcal{Z}_{L,N,\lambda,H}(\varphi \boldsymbol{v}^x) = \lambda^{|\mathbb{T}_L|} \mathbb{Z}^{\ell}_{L,N,\frac{1}{\lambda}} + o(\varphi^2), \tag{3.5}$$

we deduce from (3.3), (3.4), and (3.5) and from a Taylor expansion of the root in (3.4) that *the term of order* $O(\varphi^2)$ *in* (3.3) *is nonpositive.* This is the desired concavity property.

Remark 3.2. The random path model, which depends on an arbitrary weight function U, is related by the expansion (3.3) to lattice permutations when a specific choice for U is made. Our method can be adapted to any weight function U satisfying the general assumptions in Definitions 4.1 and 4.2 below. For example, there exists a special choice of the weight function U such that the random path model is a *representation* of the spin $O(N)$ model [3], and our method can be used to derive the famous result of Fröhlich, Simon, and Spencer [30] on the spin $O(N)$ model directly from its representation as a random loop model. Since our method applies to the random path models with weight function U for which no spin representation exists or is known, for example, lattice permutations and the dimer model, it can be viewed as a generalisation of [30].

3.2 Description of Part II: Derivation of a version of the infrared bound

We now give a brief overview to the second part of the proof, which is presented in Section 5 and uses Fourier transforms. To begin, we define the *dual torus*,

$$\mathbb{T}^*_L := \Big\{ \frac{2\pi}{L}(n_1, \ldots, n_d) \in \mathbb{R}^d : n_i \in \big(-\tfrac{L}{2}, \tfrac{L}{2}\big] \cap \mathbb{Z} \Big\}.$$

We denote the elements of $\mathbb{T}_L^*$ by $k = (k_1, \dots, k_d)$, and we keep using the notation o for $(0, \dots, 0) \in \mathbb{T}_L$ or $(0, \dots, 0) \in \mathbb{T}_L^*$. Given a function $f \in \ell^2(\mathbb{T}_L)$, we define its Fourier transform

$$\forall k \in \mathbb{T}_L^* \quad \hat{f}(k) := \sum_{x \in \mathbb{T}_L} e^{-ik\cdot x} f(x). \tag{3.6}$$

It follows from this definition that

$$\forall x \in \mathbb{T}_L \quad f(x) = \frac{1}{|\mathbb{T}_L|} \sum_{k \in \mathbb{T}_L^*} e^{ik\cdot x} \hat{f}(k). \tag{3.7}$$

The next lemma, which will be proved in the appendix and which is an immediate consequence of (3.6) and (3.7), allows us to explain the strategy of the proof.

LEMMA 3.3. *Define the Fourier mode $p := (\pi, \pi, \dots, \pi) \in \mathbb{T}_L^*$. We have that, for any $L \in \mathbb{N}_{>0}$, $\rho, N \in [0, \infty)$,*

$$\frac{2}{|\mathbb{T}_L|} \sum_{x \in \mathbb{T}_L^o} \mathbb{G}_{L,N,\rho}(x) = \mathbb{G}_{L,N,\rho}(\boldsymbol{e}_1) - \frac{1}{|\mathbb{T}_L|} \sum_{k \in \mathbb{T}_L^* \setminus \{o,p\}} e^{ik\cdot \boldsymbol{e}_1} \hat{\mathbb{G}}_{L,N,\rho}(k). \tag{3.8}$$

The goal is to bound away from 0 uniformly in L the quantity on the left-hand side of (3.8), obtaining (2.13). This quantity corresponds to the difference between the $(0, \dots, 0)$ and the $(\pi, \dots, \pi)$ Fourier mode of the two-point function (note that the sum involves only odd vertices). When $\rho = 0$, the first term on the right-hand side of (3.8) satisfies

$$\mathbb{G}_{L,N,0}(\boldsymbol{e}_1) = \frac{1}{dN} \tag{3.9}$$

for any even L, as we prove in Section 7 (and it is easy to show). Section 5 is devoted to showing that uniformly in L,

$$\limsup_{L\to\infty} \frac{1}{|\mathbb{T}_L|} \sum_{k \in \mathbb{T}_L^* \setminus \{o,p\}} e^{ik\cdot \boldsymbol{e}_1} \hat{\mathbb{G}}_{L,N,0}(k) \le \frac{r_d}{4d}. \tag{3.10}$$

This is the point where we use the key inequality under specific choices of the vector $\boldsymbol{v}$, and the symmetry properties of the even and odd Fourier two-point functions (which will be defined below), and we make use of the assumption $\rho = 0$ in a crucial way. By replacing (3.9) and (3.10) in (3.8), we obtain the desired uniform lower bound for the Cesàro sum, (2.13). Fortunately for us the value of r_d, which was computed exactly and rigorously by Watson [53] when $d = 3$, is small enough to imply, by monotonicity, nontrivial results for any $d \geq 3$. Indeed, contrary to the spin systems case, where a factor $\frac{1}{\beta}$ on the right-hand side of 'the analogue of' (3.10) makes the bound better and better as one takes the inverse temperature parameter β larger, in our case the bound does not improve arbitrarily by taking ρ arbitrarily close to 0 (and there is no reason to expect this should be the case). Hence there is no way to ensure a priori that the method will lead to nontrivial results until one derives the optimal constant $\frac{r_d}{4d}$ and proves that it is strictly less than (3.9) for a nonempty range of strictly positive integers N in any dimension $d \geq 3$. We refer to Remark 5.2 for further general comments on this part of the proof and for a comparison with the classical case of spin systems with continuous symmetry.

3.3 From lattice permutations to dimers: proof of Theorem 2.1 given Theorem 2.3

We now prove (2.12) formally. This will be the last time the dimer model appears in this paper, since our main result on the dimer model follows from its representation as a 'fully packed' lattice permutation model in the special case $N = 2$, and the next sections are devoted to the study of lattice permutations. In this special case, lattice permutations can be viewed as a different formulation of the *double dimer model* [17,41]. Here, by 'fully packed' π we mean that π is such that $\mathcal{M}(\pi) = 0$.

PROOF OF (2.12). We claim that there exist two bijections,

$$\Pi^1 : \mathcal{D}(\varnothing) \times \mathcal{D}(\{o, z\}) \mapsto \{\pi \in \Omega_{o,z} : \mathcal{M}(\pi) = 0\},$$
$$\Pi^2 : \mathcal{D}(\varnothing) \times \mathcal{D}(\varnothing) \mapsto \{\pi \in \Omega^{\ell} : \mathcal{M}(\pi) = 0\}.$$

Indeed, note the following: If we superimpose two dimer covers, $\eta^1 \in \mathcal{D}(\varnothing)$ and $\eta^2 \in \mathcal{D}(\{o, z\})$, which we call blue and red, respectively, we obtain a system of mutually disjoint, self-avoiding loops, double dimers, and a self-avoiding walk from o to z, as in Figure 2.1, where the double dimer corresponds to the superposition of a blue and a red dimer on the same edge, while the loops and walk consist of alternating blue and red dimers. Note also that any loop might appear with two different colourings. Indeed, given a pair (η_1, η_2) and some arbitrary loops of such a pair, one might obtain a new pair (η_1', η_2') that is identical to (η_1, η_2) except for the fact that the selected loops appear with the opposite colours. Thus, we can associate to (η_1, η_2) an element $\pi \in \Omega_{o,z}$ that is such that π has a double edge at $\{x, y\}$ if both η_1 and η_2 have a dimer at $\{x, y\}$, every loop of π corresponds to a loop of (η_1, η_2) (each loop of π has two possible orientations, similarly every loop of (η_1, η_2) has two possible alternations of red and blue dimers; hence one can define a convention to associate loops of π with a given orientation to loops of (η_1, η_2) with a given alternation of blue and red dimers). This defines the bijection Π^1. The bijection Π^2 is defined analogously (the only difference is that we have no walk starting at o and ending at z). Since we have two bijections, we deduce that

$$\forall z \in \mathbb{T}_L \quad \mathbb{G}_{L,2,0}(o, z) = \frac{|\{\pi \in \Omega_{o,z} : \mathcal{M}(\pi) = 0\}|}{|\{\pi \in \Omega^{\ell} : \mathcal{M}(\pi) = 0\}|} = \frac{|\mathcal{D}(\{o, z\})||\mathcal{D}(\varnothing)|}{|\mathcal{D}(\varnothing)|^2} = \Xi_L(z). \tag{3.11}$$

This leads to our claim. □

PROOF OF THEOREM 2.1 GIVEN THEOREM 2.3. By Theorem 2.3 and (2.12), we deduce Theorem 2.1. □

Notation

$\boldsymbol{e}_i$	Cartesian vector, with $i \in \{1, \dots d\}$ or $i \in \{1, \dots d\}$
$\mathcal{G} = (\mathcal{V}, \mathcal{E})$	an undirected, simple, finite graph
$e \in \mathcal{E}$ or $\{x, y\} \in \mathcal{E}$	undirected edges
$(x, y) \in \mathcal{E}$	edge directed from x to y
$(\mathbb{T}_L, \mathbb{E}_L)$	graph corresponding to the torus $\mathbb{Z}^d / L\mathbb{Z}^d$
$(\mathcal{T}_L, \mathcal{E}_L)$	extended torus, with original and virtual vertices
$\mathbb{T}_L^{(2)} \subset \mathcal{T}_L$	set of virtual vertices
$\mathbb{T}_L^*$	Fourier dual torus
$o \in \mathbb{T}_L$, $o \in \mathcal{T}_L$, or $o \in \mathbb{T}_L^*$	origin
$x \sim y$	pair of neighbouring vertices in $(\mathbb{T}_L, \mathbb{E}_L)$
$N \in \mathbb{N}_{>0}$	number of colours
$\lambda, \rho \in \mathbb{R}_{\geq 0}$	edge-parameter and monomer activity, respectively
$U = (U_x)_{x \in \mathcal{V}}$	weight function
$m = (m_e)_{e \in \mathcal{E}}$	link cardinalities
m_e	number of links on the edge e
$c = (c_e)_{e \in \mathcal{E}}$	link colourings, with $c_e : \{1, \dots, m_e\} \mapsto \{1, \dots, N\}$
$\gamma = (\gamma_x)_{x \in \mathcal{V}}$	pairings, with γ_x pairing the links touching the vertex x
$\mathcal{W}_{\mathcal{G}}$	the set of configurations in $\mathcal{G}$, with $w = (m, c, \gamma) \in \mathcal{W}_G$
n_x	number of pairings at x
u_x	number of links touching x that are unpaired at x
$\mathbb{Z}^{\ell}_{L,N,\rho}$	loop partition function
$\mathbb{Y}^{\ell}_{L,N,\lambda}$	loop partition function times a constant
$\mathbb{Z}_{L,N,\rho}(x, y)$	directed partition function
$\mathbb{Y}_{L,N,\lambda}(x, y)$	directed partition function times a constant
$\mathbb{G}_{L,N,\rho}(x, y)$	two-point function
$\mathbb{G}_{L,N,\rho}(x)$	equivalent to $\mathbb{G}_{L,N,\rho}(o, x)$
$\widehat{\mathbb{G}}_{L,N,\rho}(k)$	Fourier transform of $\mathbb{G}_{L,N,\rho}(x)$
$\boldsymbol{v} = (v_x)_{x \in \mathbb{T}_L}$	vector with coordinates corresponding to $\mathbb{T}_L$
$\boldsymbol{h} = (h_x)_{x \in \mathcal{T}_L}$	vector with coordinates corresponding to $\mathcal{T}_L$
$\mathcal{Z}_{L,N,\lambda,U}(\boldsymbol{h})$	the central quantity
$\mathcal{Z}^{(2)}_{L,N,\lambda,U}(\boldsymbol{h})$	second term of the polynomial expansion

4 Derivation of the Key Inequality

This section is devoted to the proof of Theorem 3.1. Before starting, it will be convenient to introduce a different parametrisation of the partition functions. More precisely, let $x, y \in \mathbb{T}_L$ be arbitrary vertices for any $\pi \in \Omega^{\ell}$ or $\pi \in \Omega_{x,y}$, and define $\mathcal{H}(\pi) := |E_\pi|$, the number of directed edges in the graph $\pi = (\mathbb{T}_L, E_\pi)$. Define the *edge-parameter* $\lambda \geq 0$ and define the partition functions parametrised by λ,

$$
\begin{aligned}
\mathbb{Y}^{\ell}_{L,N,\lambda} &:= \sum_{\pi \in \Omega^\ell} \lambda^{\mathcal{H}(\pi)} \left(\frac{N}{2}\right)^{\mathcal{L}(\pi)}, \\
\mathbb{Y}_{L,N,\lambda}(x,y) &:= \sum_{\pi \in \Omega_{x,y}} \lambda^{\mathcal{H}(\pi)} \left(\frac{N}{2}\right)^{\mathcal{L}(\pi)},
\end{aligned}
\tag{4.1}
$$

which for any $\lambda \in (0, \infty)$ and $L \in 2\mathbb{N}$ are related to the partition functions (2.9) and (2.10) by

$$\mathbb{Y}^{\ell}_{L,N,\lambda} = \lambda^{|\mathbb{T}_L|} \mathbb{Z}^{\ell}_{L,N,\frac{1}{\lambda}}, \quad \mathbb{Y}_{L,N,\lambda}(x,y) = \lambda^{|\mathbb{T}_L|-1} \mathbb{Z}_{L,N,\frac{1}{\lambda}}(x,y),$$

(for this, we use that $\mathcal{H}(\pi) + \mathcal{M}(\pi) = |\mathbb{T}_L|$ if $\pi \in \Omega^\ell$ and that $\mathcal{H}(\pi) + \mathcal{M}(\pi) = |\mathbb{T}_L| - 1$ if $\pi \in \Omega$) and thus satisfy for any $\lambda \in (0, \infty)$

$$\mathbb{G}_{L,N,\frac{1}{\lambda}}(x,y) = \frac{\lambda \mathbb{Y}_{L,N,\lambda}(x,y)}{\mathbb{Y}^{\ell}_{L,N,\lambda}}. \tag{4.2}$$

The edge-parameter λ will play a similar role to the inverse temperature in spin systems.

4.1 The random path model

In this section we introduce the random path model on an arbitrary graph (this section is similar to section 2.1 in [43]). Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected, simple, finite graph, and assume that $N \in \mathbb{N}_{>0}$. We refer to N as the *number of colours*. A realisation of the random path model can be viewed as a collection of undirected paths (which might be closed or open).

Links, colourings, pairings. To define a realisation we need to introduce links, colourings, and pairings. We represent a *link configuration* by $m \in \mathcal{M}_{\mathcal{G}} := \mathbb{N}^{\mathcal{E}}$. More specifically,

$$m = (m_e)_{e \in \mathcal{E}},$$

where $m_e \in \mathbb{N}$ represents the number of links *on the edge* e. Intuitively, a link represents a 'visit' at the edge from a path. The links are ordered and receive a label between 1 and m_e. See also Figure 4.1. No constraint concerning the parity of m_e is introduced. If a link is on the edge $e = \{x, y\}$, then we say that *it touches* x and y.

Given a link configuration $m \in \mathcal{M}_{\mathcal{G}}$, a *colouring* $c \in \mathcal{C}_{\mathcal{G}}(m) := \{1, \ldots, N\}^m$ is a realisation that assigns an integer in $\{1, \ldots, N\}$ to each link, which will be called its *colour*. More precisely,

$$c = (c_e)_{e \in \mathcal{E}}$$

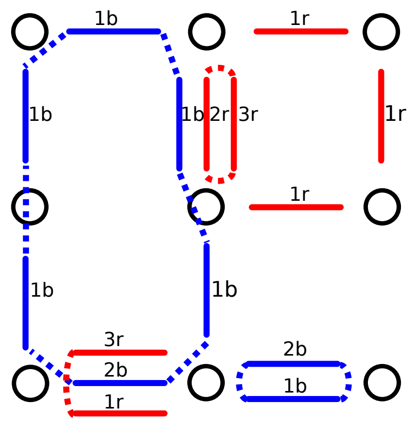

FIGURE 4.1. A configuration $w = (m, c, \gamma) \in \mathcal{W}_{\mathcal{G}}$, where $\mathcal{G}$ corresponds to the graph $\{1, 2, 3\} \times \{1, 2, 3\}$ with edges connecting nearest neighbours and the lowest leftmost vertex corresponds to $(1, 1)$. On every edge e, the links are ordered and receive a label from 1 to m_e. In the figure, the numbers $1, 2, \ldots$ are used for the identification of the links, and the letters b and r are used for the colours that are assigned to the links by c (we assume that $N = 2$ and that each link might be either blue or red). Paired links are connected by a dotted line. For example, the first link on the edge connecting the vertices $(1, 1)$ and $(2, 1)$ is coloured by red; it is paired at $(1, 1)$ with the third link on the same edge and it is unpaired at $(2, 1)$. Moreover, both links touching the vertex $(3, 3)$ are red and are unpaired at $(3, 3)$. Finally, no link is on the edge that connects the vertices $(1, 2)$ and $(2, 2)$.

is such that $c_e \in \{1, \ldots, N\}^{m_e}$, where $c_e(p) \in \{1, \ldots, N\}$ is the colour of the p^{th} link which is on the edge $e \in \mathcal{E}$, with $p \in \{1, \ldots, m_e\}$. See Figure 4.1 for an example, where $N = 2$ and the colors are represented by a label in $\{r, b\}$.

Given a link configuration, $m \in \mathcal{M}_{\mathcal{G}}$, and a colouring $c \in \mathcal{C}_{\mathcal{G}}(m)$, a pairing $\gamma = (\gamma_x)_{x \in \mathcal{V}}$ for m and c pairs links touching x in such a way that, if two links are paired, then they have the same colour. A link touching x can be paired to at most another link touching x, and it is not necessarily the case that all links touching x are paired to another link at x. If a link touching x is paired at x to no other link touching x, then we say that the link is *unpaired at* x. Given two links, if there exists a vertex x such that such links are *paired at* x, then we say that such links are *paired*. It follows from these definitions that a link can be paired to at most two other links. We remark that, by definition, a link cannot be paired to itself. We denote by $\mathcal{P}_{\mathcal{G}}(m, c)$ the set of all such pairings for $m \in \mathcal{M}_{\mathcal{G}}$, $c \in \mathcal{C}_{\mathcal{G}}(m)$.

A *configuration* of the random path model is an element $w = (m, c, \gamma)$ such that $m \in \mathcal{M}_{\mathcal{G}}$, $c \in \mathcal{C}_{\mathcal{G}}(m)$, $\gamma \in \mathcal{P}_{\mathcal{G}}(m, c)$. We let $\mathcal{W}_{\mathcal{G}}$ be the set of such configurations.

It follows from these definitions that any $w \in \mathcal{W}_{\mathcal{G}}$ can be viewed as a collection of closed and open paths. These will be defined in Section 4.4 formally, and will be divided into four classes: ℓ-loops, double links, ℓ-walks, and segments.

For any $w = (m, c, \gamma) \in \mathcal{W}_{\mathcal{G}}$, we denote (with a slight abuse of notation) by $m_e(w)$ the random variable corresponding to the number of links on the edge e, i.e., the element of the vector $m = (m_{\tilde{e}})_{\tilde{e} \in \mathcal{E}}$ such that $\tilde{e} = e$. For any $x \in \mathcal{V}$, let $u_x : \mathcal{W}_{\mathcal{G}} \mapsto \mathbb{N}$ be the number of links touching x that are unpaired at x. Moreover, let $n_x : \mathcal{W}_{\mathcal{G}} \mapsto \mathbb{N}$ be the number of pairings at x, namely

$$n_x(w) := \frac{1}{2} \sum_{(x,z) \in \mathcal{E}} m_{\{x,z\}}(w) - \frac{u_x(w)}{2}, \tag{4.3}$$

which corresponds to the number of pairings at x (i.e., the number of links touching x and paired at x to another link divided by 2).

Domains, restrictions, measure. We now introduce the notions of *domain* and *restriction* and, after that, we introduce *reflections*. Intuitively, a function with domain $D \subset \mathcal{V}$ is a function that depends only on how $w \in \mathcal{W}_{\mathcal{G}}$ looks in D. More precisely, the function might only depend on how many links emanate from the vertices of D, on the direction in which they emanate, on which colour they have, and on the pairings on vertices in D. A function $f : \mathcal{W}_{\mathcal{G}} \mapsto \mathbb{R}$ has *domain* $D \subset \mathcal{V}$ if, for any pair of configurations $w = (m, c, \gamma)$, $w' = (m', c', \gamma') \in \mathcal{W}_{\mathcal{G}}$ such that

$$\forall e \in \mathcal{E} : e \cap D \neq \emptyset, \quad \forall z \in D, \quad m_e = m'_e, \quad c_e = c'_e, \quad \gamma_z = \gamma'_z,$$

one has that $f(w) = f(w')$. Moreover, for any $w = (m, c, \gamma) \in \mathcal{W}_{\mathcal{G}}$, define the *restriction* of w to $D \subset \mathcal{V}$, $w_D = (m_D, c_D, \gamma_D)$ with $c_D \in \mathcal{C}_{\mathcal{G}}(m_D)$, $\gamma_D \in \mathcal{P}_{\mathcal{G}}(m_D, c_D)$, by

(i) $(m_D)^i_e = m^i_e$ for any edge $e \in \mathcal{E}$ that has at least one endpoint in D and $(m_D)^i_e = 0$ otherwise;
(ii) $(c_D)_e = c_e$ for any edge e that has at least one endpoint in D and $(c_D)_e = \emptyset$ otherwise;
(iii) $(\gamma_D)_x = \gamma_x$ for any $x \in D$, and for $x \in \mathcal{V} \setminus D$ we set $(\gamma_D)_x$ as the pairing that leaves all links touching x unpaired (if any).

We now introduce a measure on $\mathcal{W}_{\mathcal{G}}$.

DEFINITION 4.1. Let $N \in \mathbb{N}_{>0}$, let $U = \big(U_x\big)_{x \in \mathcal{V}}$ be a vector of real-valued functions such that, for any $x \in \mathbb{T}_L$, U_x has domain $\{x\}$. We refer to U as the *weight function.* We introduce the (unnormalized, possibly signed) measure of the random path model on $\mathcal{W}_{\mathcal{G}}$, which depends on the parameter $\lambda \in [0, \infty)$ and on the weight function U,

$$\forall w = (m, c, \gamma) \in \mathcal{W}_{\mathcal{G}} \quad \mu_{\mathcal{G},N,\lambda,U}(w) := \Big(\prod_{e \in \mathcal{E}} \frac{\lambda^{m_e}}{m_e!} \Big) \Big(\prod_{x \in \mathcal{V}} U_x(w) \Big) \tag{4.4}$$

Given a function $f : \mathcal{W}_{\mathcal{G}} \to \mathbb{R}$, we represent its unnormalized average by

$$\mu_{\mathcal{G},N,\lambda,U}(f) = \sum_{w \in \mathcal{W}_{\mathcal{G}}} \mu_{\mathcal{G},N,\lambda,U}(w) f(w).$$

We always assume that the choice of the weight function U is such that the measure $\mu_{N,\lambda,U}$ has finite mass. The role played by the normalisation factor $\frac{1}{m_e!}$ in (4.4) will be explained at the beginning of Section 8.

4.2 Reflection positivity and virtual vertices

In this section we introduce the extended torus, a graph that contains the torus $(\mathbb{T}_L, \mathbb{E}_L)$, and the important notion of reflection positivity. From now on we consider the random path model on such a graph.

Extended torus, virtual and original vertices. Recall that $(\mathbb{T}_L, \mathbb{E}_L)$ was defined as the graph corresponding to a d-dimensional torus with edges connecting nearest-neighbour vertices. We will now view $(\mathbb{T}_L, \mathbb{E}_L)$ as the subgraph of a larger graph that will be denoted by $(\mathcal{T}_L, \mathcal{E}_L)$ and will be referred to as *extended torus*. The extended torus is obtained from the d-dimensional torus by duplicating the vertex set and by adding an edge between every vertex in $\mathbb{T}_L$ and its copy.

More precisely, we define the vertex set of the extended torus as

$$\mathcal{T}_L := \mathbb{T}_L \cup \mathbb{T}_L^{(2)},$$

where $\mathbb{T}_L^{(2)}$ is disjoint from $\mathbb{T}_L$ and is such that there exists a *bijection* $g : \mathbb{T}_L \mapsto \mathbb{T}_L^{(2)}$. The vertex $g(x) \in \mathbb{T}_L^{(2)}$ will be referred to as *the ghost of* x. Recall that $\mathbb{E}_L$ is defined as the set of edges connecting pairs of nearest-neighbour vertices and boundary vertices in $\mathbb{T}_L$ so that the $(\mathbb{T}_L, \mathbb{E}_L)$ can be identified with the d-dimensional torus and define the edge set,

$$\mathcal{E}_L := \mathbb{E}_L \cup \big\{\{x, y\} \in \mathbb{T}_L \cup \mathbb{T}_L^{(2)} : x \in \mathbb{T}_L, y = g(x)\big\}.$$

This defines the extended torus $(\mathcal{T}_L, \mathcal{E}_L)$. We will refer to the vertices in $\mathbb{T}_L \subset \mathcal{T}_L$ as *original* and to the vertices in $\mathbb{T}_L^{(2)} \subset \mathcal{T}_L$ as *virtual*. From now on, we take $\mathcal{G} = (\mathcal{T}_L, \mathcal{E}_L)$ for $L \in \mathbb{N}_{>0}$, and we omit the subscript $\mathcal{G}$ in all the quantities that were defined above or replace it by L when appropriate. In this setting we will keep referring to o, corresponding to the vertex $(0, \dots, 0) \in \mathbb{T}_L$, as the origin. From now on the current section is an adaptation of [43][sec. 3] to the extended torus.

Reflection through edges. We say that the plane R is through the edges of $(\mathcal{T}_L, \mathcal{E}_L)$ if it is orthogonal to one of the Cartesian vectors $\boldsymbol{e_i}$ for $i \in \{1, \dots, d\}$, and it intersects the midpoint of L^{d-1} edges of the graph $(\mathcal{T}_L, \mathcal{E}_L)$, i.e., $R = \{z \in \mathbb{R}^d : z \cdot \boldsymbol{e_i} = u\}$ for some u such that $u - \frac{1}{2} \in \mathbb{Z} \cap (-\frac{L}{2}, \frac{L}{2}]$ and $i \in \{1, \dots, d\}$. See Figure 3.1 for an example.

Given such a plane R, we denote by $\Theta : \mathcal{T}_L \to \mathcal{T}_L$ the reflection operator that reflects the vertices of $\mathcal{T}_L$ with respect to R; i.e., for any $x = (x_1, x_2, \dots, x_d) \in \mathcal{T}_L$,

$$\Theta(x)_k := \begin{cases} x_k & \text{if } k \neq i, \\ 2u - x_k \bmod L & \text{if } k = i. \end{cases} \tag{4.5}$$

Let $\mathcal{T}_L^+, \mathcal{T}_L^- \subset \mathcal{T}_L$ be the corresponding partition of the extended torus into two disjoint halves such that $\Theta(\mathcal{T}_L^\pm) = \mathcal{T}_L^\mp$, as in Figure 3.1. Let $\mathcal{E}_L^+, \mathcal{E}_L^- \subset \mathcal{E}_L$ be the set of edges $\{x, y\}$ with at least one of x, y in $\mathcal{T}_L^+$, respectively $\mathcal{T}_L^-$. Moreover, let $\mathcal{E}_L^R := \mathcal{E}_L^+ \cap \mathcal{E}_L^-$. Note that this set contains $2L^{d-1}$ edges, half of them intersecting the plane R, and all of them belonging to $\mathbb{E}_L$. Further, let $\Theta : \mathcal{W} \to \mathcal{W}$ denote the reflection operator reflecting the configuration $w = (m, c, \gamma)$ with respect to R (we commit an abuse of notation by using the same letter). More precisely, we define $\Theta w = (\Theta m, \Theta c, \Theta \gamma)$ where $(\Theta m)_{\{x,y\}} = m_{\{\Theta x, \Theta y\}}$, $(\Theta c)_{\{x,y\}} = c_{\{\Theta x, \Theta y\}}$, $(\Theta \gamma)_x = \gamma_{\Theta x}$. Given a function $f : \mathcal{W} \to \mathbb{R}$, we also use the letter Θ to denote the reflection operator Θ that acts on f as $\Theta f(w) := f(\Theta w)$. We denote by $\mathcal{A}^\pm$ the set of functions with domain $\mathcal{T}_L^\pm$ and denote by $\mathcal{W}^\pm$ the set of configurations $w \in \mathcal{W}$ that are obtained as a restriction of some $w' \in \mathcal{W}$ to $\mathcal{T}_L^\pm$.

DEFINITION 4.2. The weight function $U = (U_x)_{x \in \mathcal{T}_L}$, which was defined in Definition 4.1, is *invariant under reflections* if for any reflection plane R through edges (which is orthogonal to one of the Cartesian vectors $\boldsymbol{e}_i$ for $i \in \{1, \dots d\}$), it holds that

$$\forall x \in \mathcal{T}_L \quad \Theta(U_x) = U_{\Theta(x)},$$

where Θ is the reflection operator associated to the reflection plane R.

The next proposition introduces an important tool. The proposition states that the random path model with weight function U satisfying the assumptions in Definition 4.1 and which is invariant under reflections, as defined in Definition 4.2, is reflection positive.

THEOREM 4.3 (Reflection positivity). *Consider the torus $(\mathcal{T}_L, \mathcal{E}_L)$ for $L \in 2\mathbb{N}$. Let R be a reflection plane through edges, which is orthogonal to one of the Cartesian vectors $\boldsymbol{e}_i$, $i \in \{1, \dots, d\}$, and let Θ be the corresponding reflection operator. Consider the random path model with $N \in \mathbb{N}_{>0}$, $\lambda \in \mathbb{R}_{>0}$, and weight function U invariant under reflections. For any pair of functions $f, g \in \mathcal{A}^+$, we have that*

(1) $\mu_{L,N,\lambda,U}(f \Theta g) = \mu_{L,N,\lambda,U}(g \Theta f)$,
(2) $\mu_{L,N,\lambda,U}(f \Theta f) \geq 0$.

From this we obtain that

$$\mu_{L,N,\lambda,U}(f\ \Theta g) \leq \mu_{L,N,\lambda,U}(f\ \Theta f)^{\frac{1}{2}} \mu_{L,N,\lambda,U}(g\ \Theta g)^{\frac{1}{2}}. \tag{4.6}$$

PROOF OF THEOREM 4.3. This proof is similar to the proof of proposition 3.2 in [43]; the difference is that here we deal with an extended torus in place of the graph $(\mathbb{T}_L, \mathbb{E}_L)$. The main step of the proof is as follows: we condition on the link cardinalities on edges that are crossed by the reflection plane. Under this conditioning, the rest factorizes over the two halves by the spatial Markov property and both contributions are identical by reflection symmetry of the model. This leads to a 'square' and cannot be negative.

To begin, we introduce the notion of *projection*. We denote by $\mathcal{W}^R$ the set of configurations $w = (m, c, \gamma)$ such that $m_e = 0$ whenever $e \notin \mathcal{E}_L^R$ and, for all $x \in \mathcal{T}_L$, γ_x leaves all links touching x unpaired. We also denote by $P_R : \mathcal{W} \to \mathcal{W}^R$ the projection such that, for any $w = (m, c, \gamma) \in \mathcal{W}$, $P_R(w) = (m^R, c^R, \gamma^R)$ is defined as the configuration such that $m_e^R = \mathbb{1}_{\{e \in \mathcal{E}_L^R\}} m_e$ and $c_e^R = c_e$ if $e \in \mathcal{E}_L^R$ and $c_e^R = \varnothing$ otherwise, and all links are unpaired at every vertex. The following remark will be useful.

Remark 4.4. Recall the definition of *restriction* from Section 4.1. Given a triplet of configurations $w' \in \mathcal{W}^R$, $w_1 \in \mathcal{W}^+$, $w_2 \in \mathcal{W}^-$ such that

$$P_R(w_1) = P_R(w_2) = w',$$

there exists a unique configuration $w \in \mathcal{W}$ such that

$$w_{\mathcal{T}_L^+} = w_1, \quad w_{\mathcal{T}_L^-} = w_2, \quad P_R(w) = w'.$$

This configuration is formed by concatenating w_1 and w_2 (concatenation includes the pairing structures of each w_j).

Throughout the proof we write $\mu = \mu_{L,N,\lambda,U}$. To begin, we note that (4.6) follows in the standard way as properties (1) and (2) show that we have a positive semidefinite, symmetric bilinear form. To prove (1) we note that, by Definition 4.1 and due to the symmetries of the torus and the fact that U is invariant under reflections, $\mu(w) = \mu(\Theta w)$ for any $w \in \mathcal{W}$. Hence

$$\begin{aligned} \mu(f\Theta g) &= \sum_{w \in \mathcal{W}} f(w)\Theta g(w)\mu(w) = \sum_{\Theta w \in \mathcal{W}} f(\Theta w)\Theta g(\Theta w)\mu(w) \\ &= \sum_{\Theta w \in \mathcal{W}} g(w)\Theta f(w)\mu(w) = \sum_{w \in \mathcal{W}} g(w)\Theta f(w)\mu(w) = \mu(g\Theta f). \end{aligned} \tag{4.7}$$

For (2) we condition on the number of links in w crossing the reflection plane and on their colours. We write

$$\mu(f\Theta f) = \sum_{w \in \mathcal{W}^R} \mu(f\,;\,w), \tag{4.8}$$

where, for any $w' \in \mathcal{W}^R$,

$$\mu(f;w') := \sum_{\substack{w\in\mathcal{W}\\ P_R(w)=w'}} f(w)\Theta f(w)\mu(w)$$

$$= \prod_{e\in\mathcal{E}_L^R} \frac{m_e(w')!}{\lambda^{m_e(w')}} \sum_{\substack{w\in\mathcal{W}\\ P_R(w)=w'}} f(w) \prod_{e\in\mathcal{E}_L^+} \frac{\lambda^{m_e(w)}}{m_e(w)!} \prod_{x\in\mathcal{T}_L^+} U_x(w) \tag{4.9}$$

$$\Theta f(w) \prod_{e\in\mathcal{E}_L^-} \frac{\lambda^{m_e(w)}}{m_e(w)!} \prod_{x\in\mathcal{T}_L^-} U_x(w).$$

Now, any $w \in \mathcal{W}$ such that $P_R(w) = w'$ uniquely defines $w_{\mathcal{T}_L^\pm}$, the restriction of w to $\mathcal{T}_L^\pm$. Thus, from Remark 4.4 we deduce that we can split the sum over $w \in \mathcal{W}$ with $P_R(w) = w'$ as the product of two independent sums and continue:

$$\begin{aligned}
\mu(f;w') &= \Big(\prod_{e\in\mathcal{E}_L^R} \frac{m_e(w')!}{\lambda^{m_e(w')}} \Big) \\
&\quad \times \Big(\sum_{\substack{w_1\in\mathcal{W}^+\\ P_R(w_1)=w'}} f(w_1) \prod_{e\in\mathcal{E}_L^+} \frac{\lambda^{m_e(w_1)}}{m_e(w_1)!} \prod_{x\in\mathcal{T}_L^+} U_x(w_1) \Big) \\
&\quad \times \Big(\sum_{\substack{w_2\in\mathcal{W}^-\\ P_R(w_2)=w'}} \Theta f(w_2) \prod_{e\in\mathcal{E}_L^-} \frac{\lambda^{m_e(w_2)}}{m_e(w_2)!} \prod_{x\in\mathcal{T}_L^-} U_x(w_2) \Big) \\
&= \prod_{e\in\mathcal{E}_L^R} \frac{m_e(w')!}{\lambda^{m_e(w')}} \\
&\quad \times \Big(\sum_{\substack{w_1\in\mathcal{W}^+\\ P_R(w_1)=w'}} f(w_1) \prod_{e\in\mathcal{E}_L^+} \frac{\lambda^{m_e(w_1)}}{m_e(w_1)!} \prod_{x\in\mathcal{T}_L^+} U_x(w_1) \Big)^2.
\end{aligned} \tag{4.10}$$

The last equality holds true by the symmetry of the extended torus. Since the last expression is nonnegative, from (4.8) we conclude the proof of (2) and, thus, the proof of the proposition. $\square$

4.3 Chessboard estimate

We now introduce the notion of support. Contrary to the notion of domain, which was introduced in Section 4.2, the notion of support is defined only for subsets of the original torus. We say that the function $f : \mathcal{W} \mapsto \mathbb{R}$ has *support* in $D \subset \mathbb{T}_L$ if it has domain in $D \cup D^{(2)}$, where $D^{(2)}$ is defined as the set of sites that

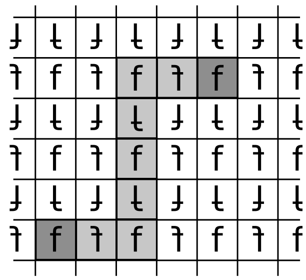

FIGURE 4.2. The function $f^{[t]} := \Theta_k \circ \Theta_{k-1} \circ \cdots \circ \Theta_1 \circ \cdots \circ (f)$ does not depend on the chosen path.

are 'on the top' of those in D,

$$D^{(2)} := \left\{ z \in \mathbb{T}_L^{(2)} : z = g(x) \text{ for some } x \in D \right\}.$$

Fix an arbitrary site $t \in \mathbb{T}_L$ and let $t_0 = o$, $t_1, \ldots, t_k = t$ be a self-avoiding nearest-neighbour path from o to t, and for any $i \in \{1, \ldots, k\}$, let Θ_i be the reflection with respect to the plane going through the edge $\{t_{i-1}, t_i\}$. Let f be a function having support in $\{o\}$ and define

$$f^{[t]} := \Theta_k \circ \Theta_{k-1} \circ \cdots \circ \Theta_1 (f).$$

Observe that the function $f^{[t]}$ does not depend on the chosen path (a glance at Figure 4.2 might be useful).

PROPOSITION 4.5 (Chessboard estimate). *Let $f = (f_t)_{t \in \mathbb{T}_L}$ be real-valued functions with support $\{o\}$ and that are either all bounded or all nonnegative. Under the same assumptions as in Theorem* 4.3*, we have that*

$$\mu_{L,N,\lambda,U}\Big(\prod_{t \in \mathbb{T}_L} f_t^{[t]} \Big) \leq \Big(\prod_{t \in \mathbb{T}_L} \mu_{L,N,\lambda,U}\Big(\prod_{s \in \mathbb{T}_L} f_t^{[s]} \Big) \Big)^{\frac{1}{|\mathbb{T}_L|}}.$$

The proof of Proposition 4.5 for a measure μ satisfying (4.6) is classical and was first presented in [29]. Since we only use reflections with respect to the reflection planes that are orthogonal to the Cartesian vectors $\boldsymbol{e}_i$, $i \in \{1, \ldots, d\}$, virtual vertices play no role in the proof and thus the same proof for [29] applies to our case directly. For the proof of Proposition 4.5 we refer to the original paper [29] or to the overviews [10, theorem 5.8] or [26, theorem 10.11].

We now introduce the *central quantity.* Recall that, for any vertex $x \in \mathcal{T}_L$ and any configuration $w \in \mathcal{W}$, $u_x(w)$ denotes the number of links touching $x \in \mathcal{T}_L$ that are unpaired at x.

DEFINITION 4.6 (Central quantity). *For any $L \in \mathbb{N}$, $\lambda \in \mathbb{R}_{\geq 0}$, $N \in \mathbb{N}_{>0}$, any U as in Definition* 4.1*, and any vector of real numbers $\boldsymbol{h} = (h_x)_{x \in \mathcal{T}_L}$, we define*

$$\mathcal{Z}_{L,N,\lambda,U}(\boldsymbol{h}) := \mu_{L,N,\lambda,U}\Big(\prod_{x \in \mathcal{T}_L} h_x^{u_x}\Big). \tag{4.11}$$

In other words, the function $h_x^{u_x}$ in Definition 4.6 assigns a multiplicative factor h_x to each link touching x that is unpaired at x. We assume that the weight function U is such that the quantity (4.11) is finite for any vector $\boldsymbol{h}$ as in Definition 4.6 and for any $L \in 2\mathbb{N}$. The next proposition is an immediate consequence of Proposition 4.5.

PROPOSITION 4.7. *Fix arbitrary $L \in 2\mathbb{N}$, $\lambda \geq 0$, $N \in \mathbb{N}_{>0}$. Suppose that the weight function U is invariant under reflections. Let $\boldsymbol{h} = (h_z)_{z \in \mathcal{T}_L}$ be a real-valued vector such that $|h_z| \leq 1$ for every $z \in \mathcal{T}_L$. For any $x \in \mathbb{T}_L$ define the new real-valued vector $\boldsymbol{h}^x = (h_z^x)_{z \in \mathcal{T}_L}$ that is obtained from $\boldsymbol{h}$ by copying the value h_x at each original vertex and the value $h_{g(x)}$ at each virtual vertex, namely,*

$$\forall z \in \mathcal{T}_L \quad h_z^x := \begin{cases} h_x & \text{if } z \in \mathbb{T}_L, \\ h_{g(x)} & \text{if } z \in \mathbb{T}_L^{(2)}. \end{cases}$$

We have that

$$\mathcal{Z}_{L,N,\lambda,U}(\boldsymbol{h}) \leq \Big(\prod_{x \in \mathbb{T}_L} \mathcal{Z}_{L,N,\lambda,U}(\boldsymbol{h}^x)\Big)^{\frac{1}{|\mathbb{T}_L|}}.$$

PROOF. The proof follows from an immediate application of Proposition 4.5. Define

$$\forall x \in \mathbb{T}_L \quad f_{\boldsymbol{h},x} := (h_x)^{u_o}(h_{g(x)})^{u_{g(o)}};$$

note that this function has support $\{o\}$. Moreover, note that for any $x \in \mathbb{T}_L$,

$$f_{\boldsymbol{h},x}^{[x]} = (h_x)^{u_x}(h_{g(x)})^{u_{g(x)}}, \tag{4.12}$$

which has support $\{x\}$. From this we deduce that

$$\mathcal{Z}_{L,N,\lambda,U}(\boldsymbol{h}) = \mu_{L,N,\lambda,U}\Big(\prod_{x \in \mathcal{T}_L} f_{\boldsymbol{h},x}^{[x]}\Big)$$

and that, for any $x \in \mathbb{T}_L$,

$$\mathcal{Z}_{L,N,\lambda,U}(\boldsymbol{h}^x) = \mu_{L,N,\lambda,U}\Big(\prod_{z \in \mathcal{T}_L} f_{\boldsymbol{h},x}^{[z]}\Big).$$

The claim now follows from a direct application of Proposition 4.5. $\square$

4.4 Polynomial expansion

This subsection presents an important step in the proof of the key inequality, namely Proposition 4.9 below, which states a relation between the values of any vector $\boldsymbol{h}$, the partition function $\mathcal{Z}(\varphi \boldsymbol{h})$ in the limit $\varphi \to 0$, where $\varphi \in \mathbb{R}$, and the partition functions that were defined in (4.1). To make this connection we choose an appropriate weight function, which is denoted by H and is introduced in the next definition, and expand $\mathcal{Z}_{L,N,\lambda,H}(\varphi \boldsymbol{h})$ as a polynomial in φ. Recall that n_x denotes the number of pairings at x (i.e., one half the number of links touching x that are paired at x to another link touching x).

DEFINITION 4.8. We define the weight functions $H = (H_x)_{x \in \mathcal{T}_L}$ as follows:

$$\forall x \in \mathbb{T}_L \quad H_x := \begin{cases} 1 & \text{if } n_x \leq 1,\, u_x \leq 2, \text{ and no link on } \{x, g(x)\} \\ & \text{is unpaired at } x, \\ \frac{1}{2} & \text{if } n_x \leq 1,\, u_x \leq 2, \text{ and precisely one link on} \\ & \{x, g(x)\} \text{ is unpaired at } x, \\ 0 & \text{otherwise.} \end{cases} \tag{4.13}$$

$$\forall x \in \mathbb{T}_L^{(2)} \quad H_x := \mathbb{1}_{\{n_x = 0\}} \tag{4.14}$$

Moreover, we define $\mathcal{W}^1$ to be the set of configurations $w \in \mathcal{W}$ such that

$$\prod_{x \in \mathcal{T}_L} H_x(w) > 0.$$

Each configuration $w \notin \mathcal{W}^1$ has weight 0 under $\mu_{L,N,\lambda,H}$, and thus ignoring it costs nothing. See Figure 4.3 for an example of two realisations w that are not in $\mathcal{W}^1$. The upper bound $u_x \leq 2$ in Definition 4.8 is only necessary to guarantee that $|\mathcal{W}^1| < \infty$ and 2 might be replaced by any other integer greater than 2; this replacement would only affect the terms of smaller order than $O(\varphi^2)$ in the polynomial expansion. From the boundedness of $|\mathcal{W}^1|$ we deduce that $\mathcal{Z}_{L,N,\lambda,H}(\boldsymbol{h}) < \infty$ for any $L \in \mathbb{N}$, $N, \lambda \in [0, \infty)$, and $\boldsymbol{h} \in \mathbb{R}^{\mathcal{T}_L}$. Note also that H_x has domain $\{x\}$ and that $H = (H_x)_{x \in \mathcal{T}_L}$ is invariant under reflections; thus all the results stated in Sections 4.2 and 4.3 apply to $\mu_{L,N,\lambda,U}$ under the choice of $U = H$. As we will explain in Section 8, the choice of H is such that any closed path in w lies entirely in the original torus and is vertex-self-avoiding; moreover, closed paths are mutually vertex disjoint (paths will be defined later, but the reader might already have an intuition of what they are). Contrary to closed paths, open paths are not entirely vertex self-avoiding, since they are allowed to touch themselves or other paths at their endpoints. The open paths might start (or end) at virtual vertices or at original vertices, and they are allowed to touch the virtual vertices only at their endpoints. These details and further technical aspects are fundamental for the validity of the next proposition and will be discussed in Section 8. For the statement of the next proposition recall the definition of the partition functions (4.1).

FIGURE 4.3. Two examples of realisations $w \in \mathcal{W} \backslash \mathcal{W}^1$ on the extended torus $(\mathcal{T}_L, \mathcal{E}_L)$ in dimension $d = 1$, where $L = 6$, with the upper row representing the virtual vertices. The realisation on the left is not in $\mathcal{W}^1$, since there exists a vertex x with $n_x = 2$; the realisation on the right is not in $\mathcal{W}^1$, since there exists a virtual vertex y with $n_y = 1$.

PROPOSITION 4.9 (Polynomial expansion). *For any fixed $L \in 2\mathbb{N}$, $N \in \mathbb{N}_{>0}$, $\lambda \in \mathbb{R}_{>0}$, any vector of real numbers $\boldsymbol{h} = (h_x)_{x \in \mathcal{T}_L}$, and $\varphi \in \mathbb{R}$, we have that*

$$\mathcal{Z}_{L,N,\lambda,H}(\varphi \boldsymbol{h}) = \mathbb{Y}^{\ell}_{L,N,\lambda} + \varphi^2 \mathcal{Z}^{(2)}_{L,N,\lambda,H}(\boldsymbol{h}) + o(\varphi^2), \tag{4.15}$$

in the limit as $\varphi \to 0$, where

$$\mathcal{Z}^{(2)}_{L,N,\lambda,H}(\boldsymbol{h}) := N\lambda \mathbb{Y}^{\ell}_{L,N,\lambda}\Big(\sum_{\{x,y\} \in \mathbb{E}_L} h_x h_y + \frac{1}{2} \sum_{x \in \mathbb{T}_L} h_x h_{g(x)} \Big) + N \frac{\lambda^2}{2} \sum_{x,y \in \mathbb{T}_L} \mathbb{Y}_{L,N,\lambda}(x,y) \Big(\sum_{\substack{q \in \mathcal{T}_L \\ \{x,q\} \in \mathcal{E}_L}} h_q \Big) \Big(\sum_{\substack{r \in \mathcal{T}_L \\ \{y,r\} \in \mathcal{E}_L}} h_r \Big)$$

The key inequality will follow from a concavity property of the central quantity at $\boldsymbol{h} = 0$; namely, the term of order $O(\varphi^2)$ in the polynomial expansion is nonpositive for a large class of choices of $\boldsymbol{h}$. Such a concavity property will follow from reflection positivity. Note that the terms in the expansion are slightly different than in (3.3), since here we use the partition functions parametrised by λ, which were defined in (4.1), and the entries of the vector $\boldsymbol{h}$ are associated to the vertices of the extended torus (later we will relate the vector $\boldsymbol{h}$ to a vector $\boldsymbol{v}$, whose entries are associated to the vertices of the original torus, obtaining an expression that is similar to (3.3)).

The remainder of the current subsection is devoted to the proof of Proposition 4.9. Before presenting the proof, we will provide some definitions and state a preparatory lemma. All the definitions below are functional to the proof of Proposition 4.9. Section 4.5, which contains the proof of Theorem 3.1, can be read independently from what follows in the rest of the current subsection.

Paths. Given $w \in \mathcal{W}$, we use $(\{x,y\}, p)$ to denote the p^{th} link of w, which is on the edge $\{x,y\}$, with $p \in \{1, \dots, m_{\{x,y\}}(w)\}$. We say that a set of links S in w,

$$S = \big\{ (\{x_1, y_1\}, p_1), (\{x_2, y_2\}, p_2), \dots, (\{x_\ell, y_\ell\}, p_\ell) \big\},$$

is *pairing-connected in* w if, for any pair of links, $(\{x,y\},p), (\{x',y'\},p') \in S$, there exists an ordered sequence of links in S,

$$\big((\{x'_1,y'_1\},p'_1),(\{x'_2,y'_2\},p'_2),\dots(\{x'_k,y'_k\},p'_k)\big) \subset S,$$

such that the following two conditions hold at the same time:

(i) $(\{x,y\},p) = (\{x'_1,y'_1\},p'_1)$ and $(\{x',y'\},p') = (\{x'_k,y'_k\},p'_k)$,

(ii) for any $i \in \{1,\dots,k-1\}$, $y'_i = x'_{i+1}$, and $(\{x'_i,y'_i\},p'_i)$ is paired to $(\{x'_{i+1},y'_{i+1}\},p'_{i+1})$ at $y'_i = x'_{i+1}$.

Paths are maximal pairing-connected sets. More precisely, a set of links S of w is a *path* in w if it is pairing-connected and there exists no pairing-connected set of links in w, S', which is such that $S' \supset S$ and $S' \neq S$. It is necessarily the case that all links belonging to the same path have the same colour. For example, the configuration represented in Figure 4.1 contains seven paths, two of them are coloured by blue and five by red.

ℓ-loops, double links, ℓ-walks, segments, extremal links. We will now distinguish between different types of paths. A path S of w is called a *loop of links*, or just *ℓ-loop*, if any link $(\{x,y\},p) \in S$ is paired to another link at both its endpoints and $|S| > 2$. A path S of w is called a *double link* if any link $(\{x,y\},p) \in S$ is paired at both its endpoints and $|S| = 2$. It is necessarily the case that both links belonging to the double link are on the same edge. A path S of w is called a *walk of links*, or just *ℓ-walk*, if $|S| > 1$ and there exist precisely two distinct links in S such that each of them is unpaired at one endpoint and paired at the other endpoint. Such two links will be called *extremal links* for the ℓ-walk or extremal links for w. A path S of w is called a *segment* if $|S| = 1$. If S is a segment, then the unique link that belongs to S is unpaired at both of its endpoints. From these definitions it follows that any path is either an ℓ-loop, a double link, an ℓ-walk, or a segment. There are no other possibilities. For example, the configuration w in Figure 4.1 is composed of one ℓ-loop, two double links, three segments, and one ℓ-walk that is composed of two links. The two links belonging to such an ℓ-walk are the only two extremal links of the configuration in Figure 4.1.

Subsets of $\mathcal{W}^1$. We now define several subsets of $\mathcal{W}^1 \subset \mathcal{W}$, where the set $\mathcal{W}^1$ was defined in Definition 4.8.

- Let $\mathcal{A}^\ell$ be the set of realisations $w \in \mathcal{W}^1$ such that no path of w is an ℓ-walk or a segment. In other words, each link of w is paired at both of its endpoints. This also means that each path of $w \in \mathcal{A}^\ell$ is either an ℓ-loop or a double link and, by definition of H, that no link of $w \in \mathcal{A}^\ell$ touches a virtual vertex.
- For any $\{x,y\} \in \mathcal{E}_L$, let $\mathcal{A}^s(\{x,y\})$ be the set of realisations $w \in \mathcal{W}^1$ such that exactly one path of w is a segment, and this segment is composed of a link that is on the edge $\{x,y\}$, and no connected component of w is an ℓ-walk. In other words, each link of w except for the one that belongs to

the segment is paired at both its endpoints. A realisation $w \in \mathcal{A}^s(\{x, y\})$ is represented in Figure 8.1-left.

- For any pair of (directed, not necessarily distinct) edges (x, q), $(y, r) \in \mathcal{E}_L$, let $\mathcal{A}^w((x, q), (y, r))$ be the set of realisations $w \in \mathcal{W}^1$ such that the following three conditions hold true at the same time: (1) there exists a unique ℓ-walk in w; (2) the two extremal links of this walk are on the edges $\{x, q\}$, $\{y, r\}$, respectively, one of them unpaired at q and the other one unpaired at r; (3) no path of w is a segment. These three conditions and the definition of H imply that the following properties hold for any $w \in \mathcal{A}^w((x, q), (y, r))$:
  - (i) The unique ℓ-walk in w has endpoints q and r, where q and r might coincide (see some examples in Figure 8.1-right, Figure 8.2, and Figure 8.3, where x is taken to be the origin and $d = 1$),
  - (ii) There are precisely two extremal links, which are on the edges $\{x, q\}$ and $\{y, r\}$, respectively (it is possible that $\{x, q\} = \{y, r\}$), and all the remaining links are paired at both their endpoints.
  - (iii) Any link of w that is not extremal is on an edge in $\mathbb{E}_L$.
  - (iv) Both x and y belong to the original torus; q and r may be original or virtual.

In the statement of the next lemma, recall that (x, y) represents an edge directed from x to y, while $\{x, y\}$ represents a undirected edge.

LEMMA 4.10. *Under the same assumptions as in Proposition* 4.9*, for any* (x, q)*,* (y, r)*,* $\{u, b\} \in \mathcal{E}_L$*, we have that*

$$\mu_{L,N,\lambda,H}(\mathcal{A}^\ell) = \mathbb{Y}^\ell_{L,N,\lambda}, \tag{4.16}$$

$$\mu_{L,N,\lambda,H}(\mathcal{A}^s(\{u, b\})) = \begin{cases} \lambda N \mathbb{Y}^\ell_{L,N,\lambda} & \text{if } \{u, b\} \in \mathbb{E}_L, \\ \frac{\lambda}{2} N \mathbb{Y}^\ell_{L,N,\lambda} & \text{if } \{u, b\} \in \mathcal{E}_L \setminus \mathbb{E}_L, \end{cases} \tag{4.17}$$

$$\begin{aligned} &\mu_{L,N,\lambda,H}\big(\mathcal{A}^w\big((x, q), (y, r)\big)\big) = \\ &\begin{cases} \lambda^2 N \mathbb{Y}_{L,N,\lambda}(x, y) & \text{if } x, y \in \mathbb{T}_L \text{ and } (x, q) \neq (y, r), \\ \frac{\lambda^2}{2} N \mathbb{Y}_{L,N,\lambda}(x, x) & \text{if } x, y \in \mathbb{T}_L \text{ and } (x, q) = (y, r), \\ 0 & \text{if } \{x, y\} \cap \mathbb{T}^{(2)}_L \neq \varnothing. \end{cases} \end{aligned} \tag{4.18}$$

The proof of the lemma is postponed to Section 8 and is crucial. We will now present the proof of Proposition 4.9 given Lemma 4.10 .

PROOF OF PROPOSITION 4.9 GIVEN LEMMA 4.10. Fix $L \in 2\mathbb{N}$, $N \in \mathbb{N}_{>0}$, $\lambda > 0$, and a vector of real numbers $\boldsymbol{h} = (h_x)_{x \in \mathcal{T}_L}$. We have that

$$\mathcal{Z}_{L,N,\lambda,H}(\varphi \boldsymbol{h}) = \sum_{i=0}^{\infty} \varphi^i \mathcal{C}^{(i)}_{L,N,\lambda,H}(\boldsymbol{h}), \tag{4.19}$$

where

$$\mathcal{C}^{(i)}_{L,N,\lambda,H}(\boldsymbol{h}) := \mu_{L,N,\lambda,H}\Big(\mathbb{1}_{\{M=i\}}\Big(\prod_{z\in\mathcal{T}_L} h_z^{u_z}\Big)\Big)$$

and $M := \sum_{z\in\mathcal{T}_L} u_z$ is the number of endpoints of links that are unpaired in the whole graph.

First of all, note that

$$\forall i \in 2\mathbb{N}+1 \quad \mathcal{C}^{(i)}_{L,N,\lambda,H}(\boldsymbol{h}) = 0. \tag{4.20}$$

since any path has either no link with unpaired endpoints or two links with precisely one unpaired endpoint each, or one link with two unpaired endpoints. Thus, $M(w)$ is even for any $w \in \mathcal{W}^1$. Moreover, note that

$$\mathcal{C}^{(0)}_{L,N,\lambda,H}(\boldsymbol{h}) = \mu_{L,N,\lambda,H}(\mathcal{A}^{\ell}) = \mathbb{Y}^{\ell}_{L,N,\lambda}, \tag{4.21}$$

where the first identity holds true since $w \in \{M = 0\}$ if and only if each path of w is an ℓ-loop or a double link and the second identity follows from Lemma 4.10. Furthermore, note that $w \in \{M = 2\} \cap \mathcal{W}^1$ if and only if precisely one path of w is a segment or an ℓ-walk and all the remaining paths of w are ℓ-loops or double links. In the next expression, the first term in the right-hand side corresponds to a sum over all possible edges on which the segment might be located, and the second term in the right-hand side corresponds to a sum over all (directed) edges on which the extremal links might be located (recall the definitions provided before the statement of Lemma 4.10),

$$\begin{aligned}
&\mathcal{C}^{(2)}_{L,N,\lambda,H}(\boldsymbol{h}) \\
&\quad = \sum_{\{x,y\}\in\mathcal{E}_L} \mu_{L,N,\lambda,H}\big(\mathcal{A}^{s}(\{x,y\})\big) h_x h_y \\
&\qquad + \sum_{\{(x,q),(y,r)\}\subset\mathcal{E}_L} \mu_{L,N,\lambda,H}\big(\mathcal{A}^{w}\big((x,q),(y,r)\big)\big) h_q h_r.
\end{aligned} \tag{4.22}$$

Note that the second sum in the right-hand side is over all *unordered* pairs of (not necessarily distinct) *directed* edges. Now we apply Lemma 4.10, and we rewrite the second term in the right-hand side of the previous expression as follows:

$$
\begin{aligned}
&\frac{1}{2}\sum_{\substack{x,y\in\mathcal{T}_L:\\ x\neq y}}\ \sum_{\substack{q,r\in\mathcal{T}_L:\\ \{x,q\},\{y,r\}\in\mathcal{E}_L}} h_q h_r \mu_{L,N,\lambda,H}\big(\mathcal{A}^w\big((x,q),(y,r)\big)\big)\\
&\quad+\frac{1}{2}\sum_{x\in\mathbb{T}_L}\ \sum_{\substack{q,r\in\mathcal{T}_L:\\ \{x,q\},\{x,r\}\in\mathcal{E}_L, q\neq r}} h_q h_r \mu_{L,N,\lambda,H}\big(\mathcal{A}^w\big((x,q),(x,r)\big)\big)\\
&\quad+\sum_{x\in\mathbb{T}_L}\ \sum_{\substack{q\in\mathbb{T}_L:\\ \{x,q\}\in\mathcal{E}_L}} h_q^2 \mu_{L,N,\lambda,H}\big(\mathcal{A}^w\big((x,q),(x,q)\big)\big)\\
&\qquad=\frac{1}{2}N\lambda^2\sum_{x,y\in\mathbb{T}_L}\mathbb{Y}_{L,N,\lambda}(x,y)\Big(\sum_{\substack{q\in\mathcal{T}_L:\\ \{x,q\}\in\mathcal{E}_L}} h_q\Big)\Big(\sum_{\substack{r\in\mathcal{T}_L:\\ \{y,r\}\in\mathcal{E}_L}} h_r\Big)
\end{aligned}
\tag{4.23}
$$

By replacing (4.23) with the second term in the right-hand side of (4.22), applying Lemma 4.10 for the first term in the right-hand side of (4.22), and using (4.20) and (4.21), we conclude the proof. □

4.5 Proof of Theorem 3.1 given Lemma 4.10

All the ingredients have been introduced and we can now combine them to present the proof of Theorem 3.1 given Lemma 4.10, whose proof is postponed to the end of the paper.

PROOF OF THEOREM 3.1. Fix arbitrary finite integers $L \in 2\mathbb{N}_{>0}$ and $N \in \mathbb{N}_{>0}$, and fix an edge-parameter $\lambda \in (0,\infty)$. Recall that $x \sim y$ denotes that x and y are nearest neighbours in $(\mathbb{T}_L, \mathbb{E}_L)$, and recall that $\sum_{(x,y)\in\mathcal{E}_L}$ is the sum over directed edges while $\sum_{\{x,y\}\in\mathcal{E}_L}$ is the sum over undirected edges. Recall also that $(\mathbb{T}_L, \mathbb{E}_L)$ corresponds to the torus $\mathbb{Z}^d/L\mathbb{Z}^d$, while $(\mathcal{T}_L, \mathcal{E}_L)$ is the extended torus. For any real-valued vector $\boldsymbol{v} = (v_x)_{x\in\mathbb{T}_L}$, let $\boldsymbol{h}^{\boldsymbol{v}} = (h_x^{\boldsymbol{v}})_{x\in\mathcal{T}_L}$ be obtained from $\boldsymbol{v}$ as follows:

$$
\forall x \in \mathcal{T}_L \quad h_x^{\boldsymbol{v}} := \begin{cases} v_x & \text{if } x \in \mathbb{T}_L,\\ -2d v_{g^{-1}(x)} & \text{if } x \in \mathbb{T}_L^{(2)}. \end{cases}
\tag{4.24}
$$

Using the fact that for any real-valued vector $\boldsymbol{v} = (v_x)_{x\in\mathbb{T}_L}$,

$$
2d\sum_{x\in\mathbb{T}_L} v_x^2 = \sum_{\{x,y\}\in\mathbb{E}_L}\big(v_x^2+v_y^2\big),
\tag{4.25}
$$

we deduce that

$$\begin{aligned}\sum_{\{x,y\}\in\mathbb{E}_L} h^{\boldsymbol{v}}_x h^{\boldsymbol{v}}_y + \frac{1}{2}\sum_{x\in\mathbb{T}_L} h^{\boldsymbol{v}}_x h^{\boldsymbol{v}}_{g(x)} &= \sum_{\{x,y\}\in\mathbb{E}_L} v_x v_y - d\sum_{x\in\mathbb{T}_L} v_x^2 \\ &= \frac{1}{2}\sum_{\{x,y\}\in\mathbb{E}_L}\big(2v_x v_y - v_x^2 - v_y^2\big) \\ &= -\frac{1}{2}\sum_{\{x,y\}\in\mathbb{E}_L}(v_x - v_y)^2.\end{aligned} \tag{4.26}$$

Moreover,

$$\sum_{q\in\mathcal{T}_L:(x,q)\in\mathcal{E}_L} h^{\boldsymbol{v}}_q = (\triangle\boldsymbol{v})_x. \tag{4.27}$$

From (4.26), (4.27), and the definition in Proposition 4.9, we deduce that for any $\boldsymbol{v} = (v_x)_{x\in\mathbb{T}_L}$,

$$\begin{aligned}\mathcal{Z}^{(2)}_{L,N,\lambda,H}(\boldsymbol{h}^{\boldsymbol{v}}) = &-\frac{\lambda N}{2}\mathbb{Y}^{\ell}_{L,N,\lambda}\sum_{\{x,y\}\in\mathbb{E}_L}(v_y - v_x)^2 \\ &+ \frac{\lambda^2 N}{2}\sum_{x,y\in\mathbb{T}_L}\mathbb{Y}_{L,N,\lambda}(x,y)(\triangle\boldsymbol{v})_x\,(\triangle\boldsymbol{v})_y.\end{aligned} \tag{4.28}$$

Moreover, recall that, as defined in Section 4.3, for any original vertex $x \in \mathbb{T}_L$, $(\boldsymbol{h}^{\boldsymbol{v}})^x$ is defined as the vector that is obtained from $\boldsymbol{h}^{\boldsymbol{v}}$ by copying the value $h^{\boldsymbol{v}}_x = v_x$ at any original vertex and the value $h^{\boldsymbol{v}}_{g(x)} = -2dv_x$ at any virtual vertex and deduce from this and from (4.28) that

$$\forall\boldsymbol{v} = (v_z)_{z\in\mathbb{T}_L},\ \forall x\in\mathbb{T}_L,\quad \mathcal{Z}^{(2)}_{L,N,\lambda,H}\big((\boldsymbol{h}^{\boldsymbol{v}})^x\big) = 0. \tag{4.29}$$

We have that, in the limit as $\varphi \to 0$,

$$\begin{aligned}\mathcal{Z}_{L,N,\lambda,H}(\varphi\boldsymbol{h}^{\boldsymbol{v}}) &= \mathbb{Y}^{\ell}_{L,N,\lambda} + \varphi^2\mathcal{Z}^{(2)}_{L,N,\lambda,H}(\boldsymbol{h}^{\boldsymbol{v}}) + o(\varphi^2) \\ &\le \Big(\prod_{x\in\mathbb{T}_L}\mathcal{Z}_{L,N,\lambda,H}\big((\varphi\boldsymbol{h}^{\boldsymbol{v}})^x\big)\Big)^{\frac{1}{|\mathbb{T}_L|}} \\ &= \Big(\prod_{x\in\mathbb{T}_L}\big(\mathbb{Y}^{\ell}_{L,N,\lambda} + o(\varphi^2)\big)\Big)^{\frac{1}{|\mathbb{T}_L|}} \\ &= \mathbb{Y}^{\ell}_{L,N,\lambda} + o(\varphi^2),\end{aligned}$$

For the first step above we used Proposition 4.9, for the second step we used Proposition 4.7, for the third step Proposition 4.9 and (4.29), and for the last step we performed the Taylor expansion around $x = 0$ of the function $(1+x)^{1/|\mathbb{T}_L|} =$

$1 + x/|\mathbb{T}_L| + O(x^2)$, where in our case $x = o(\varphi^2)$. Thus we proved that, for any $\boldsymbol{v} \in \mathbb{R}^{\mathbb{T}_L}$, in the limit as $\varphi \to 0$,

$$\mathbb{Y}^{\ell}_{L,N,\lambda} + \varphi^2 \mathcal{Z}^{(2)}_{L,N,\lambda,H}(\boldsymbol{h}^{\boldsymbol{v}}) + o(\varphi^2) \leq \mathbb{Y}^{\ell}_{L,N,\lambda} + o(\varphi^2),$$

where $\boldsymbol{h}^{\boldsymbol{v}}$ was defined in (4.24) as a function of $\boldsymbol{v}$, and this can only hold true if

$$\mathcal{Z}^{(2)}_{L,N,\lambda,H}(\boldsymbol{h}^{\boldsymbol{v}}) \leq 0. \tag{4.30}$$

By replacing (4.28) on the left-hand side of (4.30), dividing the whole expression by $\frac{\lambda N}{2}\mathbb{Y}^{\ell}_{L,N,\lambda}$, and plugging in (4.2), we deduce that, for any finite strictly positive λ,

$$\sum_{x,y \in \mathbb{T}_L} \mathbb{G}_{L,N,\frac{1}{\lambda}}(x,y)(\triangle v)_x(\triangle v)_y \leq \sum_{\{x,y\} \in \mathbb{E}_L} (v_y - v_x)^2.$$

Since the previous relation holds for any strictly positive λ and since for any finite L, $\lim_{\lambda \to \infty} \mathbb{G}_{L,N,1/\lambda}(x,y) = \mathbb{G}_{L,N,0}(x,y)$, we deduce that the same inequality holds true also with $\frac{1}{\lambda}$ replaced by 0, and thus the proof is concluded. $\square$

5 A Version of the Infrared Bound

The main goal of this section is to state and prove Theorem 5.1 below, which provides a uniform lower bound for the Cesàro sum of the two-point function. Recall the definition of the *odd* and *even* sublattices and (2.2), and define the *odd* and *even two-point functions*,

$$\mathbb{G}^{o}_{L,N,\rho}(x,y) := \mathbb{G}_{L,N,\rho}(x,y)\ \mathbb{1}_{\{y \in \mathbb{T}^{o}_{L}\}}, \tag{5.1}$$

$$\mathbb{G}^{e}_{L,N,\rho}(x,y) := \mathbb{G}_{L,N,\rho}(x,y)\ \mathbb{1}_{\{y \in \mathbb{T}^{e}_{L}\}}. \tag{5.2}$$

We will use the notation

$$\begin{aligned}
\mathbb{G}_{L,N,\rho}(x) &:= \mathbb{G}_{L,N,\rho}(o,x),\\
\mathbb{G}^{o}_{L,N,\rho}(x) &:= \mathbb{G}^{o}_{L,N,\rho}(o,x),\\
\mathbb{G}^{e}_{L,N,\rho}(x) &:= \mathbb{G}^{e}_{L,N,\rho}(o,x),
\end{aligned}$$

for any $x \in \mathbb{T}_L$, and we will omit the subscripts when possible. Recall that r_d is the expected number of returns of a simple random walk in $\mathbb{Z}^d$.

THEOREM 5.1 (Infrared-ultraviolet bound). *For any $d, N \in \mathbb{N}_{>0}$, $L \in 2\mathbb{N}_{>0}$, and $\rho \in [0,\infty)$, we have that*

$$\begin{aligned}
\sum_{x \in \mathbb{T}^{o}_{L}} \frac{\mathbb{G}^{o}_{L,N,\rho}(x)}{|\mathbb{T}^{o}_{L}|} \geq \mathbb{G}_{L,N,\rho}(\boldsymbol{e}_1) - \mathcal{I}_L(d) - \sum_{x \in \mathbb{T}_L} \frac{\mathbb{G}^{e}_{L,N,\rho}(x)}{|\mathbb{T}^{e}_{L}|}\\
+ \sum_{\substack{x \in \mathbb{T}_L:\\ x_2 = \cdots = x_d = 0}} \Upsilon_L(x)\mathbb{G}^{e}_{L,N,\rho}(x)
\end{aligned} \tag{5.3}$$

where $\big(\mathcal{I}_L(d)\big)_{L\in\mathbb{N}}$ *is a sequence of real numbers, which is defined in* (5.9) *below, whose limit* $L \to \infty$ *exists and satisfies*

$$\lim_{L\to\infty} \mathcal{I}_L(d) = \frac{r_d}{4d}, \tag{5.4}$$

and $(\Upsilon_L)_{L\in\mathbb{N}}$ *is a sequence of real-valued functions, which are defined in* (5.12) *below and converges pointwise with* L *to a finite function* Υ.

This theorem will be applied under the assumption that $\rho = 0$, in which case the last two terms on the right-hand side of (5.3) equal 0, as we will prove in Lemma 5.4 below. Although we will apply the theorem under the assumption $\rho = 0$, in this section we will allow ρ to take positive values for the sake of generality.

Remark 5.2. A similar lower bound for the Cesàro sum of two-point functions was obtained in the framework of spin systems with continuous symmetry [27, 28, 30]. Our analysis differs from the spin systems case in some important aspects. In the spin systems case one obtains the Key Inequality with $\mathbb{G}_{L,N,\rho}(x, y)$ replaced by the correlation between the spins, which is typically denoted by $\langle S_o \cdot S_x\rangle_{L,N,\beta}$, where N there represents the number of components of the spins and β is the inverse temperature. There, the Key Inequality leads to a uniformly positive lower bound for the Cesàro sum of two-point functions, similarly to our case. This bound is usually referred to as an *infrared bound*, since the quantity that one bounds from below corresponds to the zero (i.e., low-frequency) Fourier mode of the two-point function. The same approach as in the classical case of spin systems with continuous symmetry would work in our case if the term $\mathbb{G}_{L,N,\rho}(o)$ was strictly positive (and large enough) uniformly in L and in the limit of small ρ. Unfortunately, this is not the case, since it is shown in Lemma 5.4 below that $\mathbb{G}_{L,N,0}(o) = 0$ (more precisely, when $\rho = 0$, the two-point function equals 0 at any even site). For this reason, we proceed differently than in [27,28,30]: The term $\mathbb{G}_{L,N,\rho}(o)$ is replaced by the term $\mathbb{G}_{L,N,\rho}(\boldsymbol{e}_1)$, and we use the symmetry properties of the Fourier odd two-point function to deal with the presence of the factor $e^{ik\cdot \boldsymbol{e}_1}$ on the right-hand side of (3.8), which is not present in [27,28,30]. We refer to the resulting bound as the *infrared-ultraviolet bound*, since the quantity that we bound from below, which is on the left-hand side of (3.8), involves not only the lowest-, but also the highest-frequency Fourier mode (more precisely, it equals the difference of the two).

We now start to introduce the arguments that lead to the proof of Theorem 5.1. To begin, we define the central quantity,

$$\forall k \in \mathbb{T}_L^* \quad \varepsilon(k) := 2\sum_{j=1}^{d} \big(1 - \cos(k_j)\big). \tag{5.5}$$

Recall also the definitions of the Fourier transform and inverse Fourier transform, which were provided in Section 3.2.

PROPOSITION 5.3 (High-frequency upper bound). *Under the same assumptions as in Theorem* 5.1, *for any* $L \in 2\mathbb{N}_{>0}$,

$$\forall k \in \mathbb{T}_L^* \setminus \{o\} \quad \widehat{\mathbb{G}}_{L,N,\rho}(k) = \widehat{\mathbb{G}}^o_{L,N,\rho}(k) + \widehat{\mathbb{G}}^e_{L,N,\rho}(k) \leq \frac{1}{\varepsilon(k)}. \tag{5.6}$$

PROOF. To begin, we fix an arbitrary $k \in \mathbb{T}_L^* \setminus \{o\}$ and choose the vector $\boldsymbol{v} = (v_x)_{x \in \mathbb{T}_L}$ such that, for any $x \in \mathbb{T}_L$, $v_x := \cos(k \cdot x)$. We note that under this choice the following facts hold true:

(i) For any $x \in \mathbb{T}_L$, $(\triangle v)_x = -\varepsilon(k) v_x$.
(ii) $\sum_{\{x,y\} \in \mathbb{E}_L} (v_y - v_x)^2 = \varepsilon(k) \sum_{x \in \mathbb{T}_L} v_x^2$.
(iii) $\sum_{x,y \in \mathbb{T}_L} v_x v_y \mathbb{G}(x,y) = \widehat{\mathbb{G}}(k) \sum_{x \in \mathbb{T}_L} v_x^2$.

These computations are classical, and we present their proof in the appendix. The proof of Proposition 5.3 follows from Theorem 3.1 and from these computations. We first apply (i) to the left-hand side of (3.2), and then we apply (ii) to the right-hand side of (3.2), thus obtaining that

$$\varepsilon^2(k) \sum_{x,y \in \mathbb{T}_L} v_x v_y \mathbb{G}(x,y) \leq \varepsilon(k) \sum_{x \in \mathbb{T}_L} v_x^2.$$

Now we apply (iii) to the left-hand side and divide everything by $\varepsilon^2(k) \sum_{x \in \mathbb{T}_L} v_x^2$. This concludes the proof. $\square$

The next lemma states some properties of the two-point functions and of their Fourier transforms.

LEMMA 5.4. *We have the following*:

(i) *For any* $k \in \mathbb{T}_L^*$, $\widehat{\mathbb{G}}_{L,N,\rho}(k), \widehat{\mathbb{G}}^e_{L,N,\rho}(k), \widehat{\mathbb{G}}^o_{L,N,\rho}(k) \in \mathbb{R}$.
(ii) *For any* $\boldsymbol{u} \in \{-1,1\}^d$, *if* $k, k + \pi\boldsymbol{u} \in \mathbb{T}_L^*$, *then*

$$\widehat{\mathbb{G}}^o_{L,N,\rho}(k + \pi\boldsymbol{u}) = -\widehat{\mathbb{G}}^o_{L,N,\rho}(k),$$

(iii) *For any* $\boldsymbol{u} \in \{-1,1\}^d$, *if* $k, k + \pi\boldsymbol{u} \in \mathbb{T}_L^*$, *then*

$$\widehat{\mathbb{G}}^e_{L,N,\rho}(k + \pi\boldsymbol{u}) = \widehat{\mathbb{G}}^e_{L,N,\rho}(k).$$

(iv) *For any* $L \in 2\mathbb{N}$ *and* $x \in \mathbb{T}_L$, *we have that* $\mathbb{G}^e_{L,N,0}(o,x) = 0$.

PROOF. The first property follows from the definition of Fourier transform and the symmetries of $\mathbb{Z}^d / L\mathbb{Z}^d$. Properties (ii) and (iii) follow from the definition of Fourier transform and the fact that, if $x \in \mathbb{T}_L^o$, then $\sum_{i=1}^d x_i \in 2\mathbb{Z}+1$; if $x \in \mathbb{T}_L^e$, then $\sum_{i=1}^d x_i \in 2\mathbb{Z}$. The fourth property holds true since, if the walk in $\pi \in \Omega$ ends at an even site, then it contains an odd number of sites, and since the total number of sites in $\mathbb{T}_L$ is even and since each loop or double edge contains an even number of sites, this implies that at least one monomer is present in π and thus that the weight of π is zero since $\rho = 0$. $\square$

We now have all the ingredients we need for proving Theorem 5.1.

PROOF OF THEOREM 5.1. Note that, since $\widehat{\mathbb{G}}_{L,N,0}(k)$ is real, it follows from (3.8) that the term on the left-hand side of the next expression is real; hence we deduce that

$$\begin{aligned}\sum_{k\in\mathbb{T}_L^*\setminus\{o,p\}} e^{ik\cdot\boldsymbol{e}_1}\widehat{\mathbb{G}}_{L,N,\rho}(k) &= \sum_{k\in\mathbb{T}_L^*\setminus\{o,p\}} Re\big(e^{ik\cdot\boldsymbol{e}_1}\widehat{\mathbb{G}}_{L,N,\rho}(k)\big)\\ &= \sum_{k\in\mathbb{T}_L^*\setminus\{o,p\}} \cos(k\cdot\boldsymbol{e}_1)\widehat{\mathbb{G}}_{L,N,\rho}(k).\end{aligned}\tag{5.7}$$

Our goal is to provide an upper bound for this expression, which by Lemma 5.4 gives a lower bound to the Cesáro sum of the odd two-point function. For this we use the symmetry properties of the odd and even Fourier two-point functions to transform the previous sum into a sum over sites where the cosine in (5.7) takes nonnegative values. This makes possible the application of Proposition 5.3 to bound $\widehat{\mathbb{G}}_{L,N,\rho}(k)$ from above. More precisely, we define the subset of $\mathbb{T}_L^*$,

$$\mathbb{H} := \big\{k\in\mathbb{T}_L^* : k_1\in(-\tfrac{\pi}{2},\tfrac{\pi}{2}]\big\},$$

and we note that there exists a bijection $\Psi : \mathbb{H}\setminus\{o\}\mapsto\mathbb{T}_L^*\setminus(\mathbb{H}\cup\{p\})$ which is such that, for any $k\in\mathbb{H}$, the following properties hold true:

$$\begin{aligned}\cos(k\cdot\boldsymbol{e}_1) &= -\cos(\Psi(k)\cdot\boldsymbol{e}_1),\\ \widehat{\mathbb{G}}^{o}_{L,N,\rho}(k) &= -\widehat{\mathbb{G}}^{o}_{L,N,\rho}\big(\Psi(k)\big),\\ \widehat{\mathbb{G}}^{e}_{L,N,\rho}(k) &= \widehat{\mathbb{G}}^{e}_{L,N,\rho}\big(\Psi(k)\big).\end{aligned}\tag{5.8}$$

The bijection Ψ consists in translating a vertex $x\in\mathbb{H}$ by an appropriate vector $\pi\boldsymbol{u}$, where $\boldsymbol{u}$ is an element of $\{-1,1\}^d$ that depends on x. See also Figure 5.1 for a representation of Ψ in the (simpler) case of $d=2$. Thus, (5.8) follows from Lemma 3.3.

More precisely, the bijection Ψ is defined as follows. To begin, we split $\mathbb{T}_L^*$ into 2^{d+1} disjoint subregions, by first defining the set of indices

$$\mathbb{B} := \{-1,-\tfrac{1}{2},\tfrac{1}{2},1\}\times\{0,1\}\times\cdots\times\{0,1\}\subset\frac{1}{2}\mathbb{Z}^d$$

and then, for any $b=(b_1,\dots,b_d)\in\mathbb{B}$, we define

$$\begin{aligned}\mathbb{H}^b := \big\{k\in\mathbb{T}_L^* : k_1&\in\big(\pi\big(b_1-\tfrac{1}{2}\big),\pi b_1\big],\\ k_i&\in\big(\pi(b_i-1),\pi b_i\big]\text{ for } i=2,\dots,d\big\}.\end{aligned}$$

Note that $\mathbb{H}^b\subset\mathbb{H}$ only if $b_1\in\{-\frac{1}{2},\frac{1}{2}\}$. For any $x\in\mathbb{H}\setminus\{o\}$, let b be the unique element of $\mathbb{B}$ such that $x\in\mathbb{H}^b$. Then,

$$\Psi(x) := x+\pi\boldsymbol{u},$$

where $\boldsymbol{u}\in\mathbb{U}$ depends on b and is defined as follows: If $b_1=\pm\frac{1}{2}$, then $u_1:=\mp 1$. This guarantees that $\Psi(x)\in\mathbb{T}_L\setminus\mathbb{H}$. Moreover, for any $i\in\{2,\dots,d\}$, if $b_i=0$,

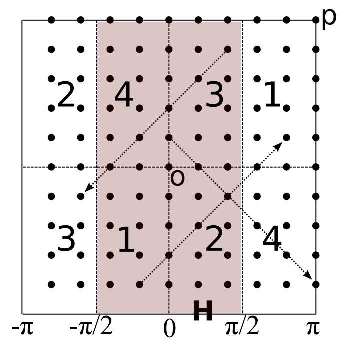

FIGURE 5.1. A representation of the dual torus $\mathbb{T}_L^*$ and the 2^{d+1} regions $\mathbb{H}^b$, $b \in \mathbb{B}$, which are delimited by the torus boundary or by the dotted lines, where $d = 2$. The bijection Ψ maps the sites where the dotted arrows start at the sites where the dotted arrows end and, for each $i \in \{1, \ldots, 4\}$, it maps the darker region with label i to the lighter region with the same label.

then $u_i := 1$, while if $b_i = 1$, then $u_i := -1$. This defines the bijection Ψ. Note that it follows from this definition that $p \notin \Psi(\mathbb{H} \setminus \{o\})$ as required. We continue using the properties (5.8) and apply Proposition 5.3, using the fact that $\cos(k \cdot \boldsymbol{e}_1)$ is nonnegative for $k \in \mathbb{H}$, obtaining

$$
\begin{aligned}
&\sum_{k \in \mathbb{T}_L^* \setminus \{o,p\}} \cos(k \cdot \boldsymbol{e}_1) \widehat{\mathbb{G}}_{L,N,\rho}(k) \\
&= \sum_{k \in \mathbb{H} \setminus \{o\}} \big(\cos(k \cdot \boldsymbol{e}_1) \widehat{\mathbb{G}}_{L,N,\rho}(k) + \cos(\Psi(k) \cdot \boldsymbol{e}_1) \widehat{\mathbb{G}}_{L,N,\rho}\big(\Psi(k)\big) \big) \\
&= 2 \sum_{k \in \mathbb{H} \setminus \{o\}} \cos(k \cdot \boldsymbol{e}_1) \widehat{\mathbb{G}}^o_{L,N,\rho}(k) \\
&\le \frac{1}{2d} \sum_{k \in \mathbb{H} \setminus \{o\}} \frac{2 \cos(k \cdot \boldsymbol{e}_1)}{1 - \frac{1}{d} \sum_{i=1}^d \cos(k \cdot \boldsymbol{e}_1)} - 2 \sum_{k \in \mathbb{H} \setminus \{o\}} \cos(k \cdot \boldsymbol{e}_1) \widehat{\mathbb{G}}^e_{L,N,\rho}(k).
\end{aligned}
$$

Since the previous quantity corresponds to the right-hand side of (3.8), Theorem 5.1 now follows from (3.8) and from the fact that

$$
\mathcal{I}_L(d) := \frac{1}{2d} \frac{1}{|\mathbb{T}_L|} \sum_{k \in \mathbb{H} \setminus \{o\}} \frac{2 \cos(k \cdot \boldsymbol{e}_1)}{1 - \frac{1}{d} \sum_{i=1}^d \cos(k \cdot \boldsymbol{e}_i)} \tag{5.9}
$$

satisfies

$$
\lim_{L \to \infty} \mathcal{I}_L(d) = \frac{r_d}{4d} \tag{5.10}
$$

and that

$$
\begin{aligned}
&-\frac{2}{|\mathbb{T}_L|}\sum_{k\in\mathbb{H}\setminus\{o\}}\cos(k\cdot\boldsymbol{e}_1)\widehat{\mathbb{G}}^{e}_{L,N,\rho}(k)\\
&\quad=\frac{2}{|\mathbb{T}_L|}\sum_{x\in\mathbb{T}_L}\mathbb{G}^{e}_{L,N,\rho}(x)-\sum_{x\in\mathbb{T}_L}\Upsilon_L(x)\mathbb{G}^{e}_{L,N,\rho}(x),
\end{aligned}
\tag{5.11}
$$

where

$$
\forall x\in\mathbb{Z}^d\quad \Upsilon_L(x):=\frac{2}{|\mathbb{T}_L|}\sum_{k\in\mathbb{H}}e^{-ik\cdot(x-\boldsymbol{e}_1)}.
\tag{5.12}
$$

Thus, to conclude the proof of Theorem 5.1, it remains to prove (5.10) and (5.11).

PROOF OF (5.10). To begin, we define the set of vectors $\mathcal{N}:=\{\pm\frac{\boldsymbol{e}_1}{2},\pm\boldsymbol{e}_2,\dots,\pm\boldsymbol{e}_d\}$, and the function

$$
J(k):=\frac{1}{d}\left(\cos\left(\frac{k_1}{2}\right)+\sum_{i=2}^{d}\cos(k_i)\right)=\frac{1}{2d}\sum_{\boldsymbol{e}\in\mathcal{N}}e^{i\boldsymbol{e}\cdot k}.
$$

Below, we first use the fact that the sum is Riemann, and after that we perform the change of variable $k_1'=2k_1$ (and call again k_1 the new variable):

$$
\begin{aligned}
&\lim_{L\to\infty}\frac{1}{|\mathbb{T}_L^*|}\sum_{k\in\mathbb{H}\setminus\{o\}}\frac{2\cos(k_1)}{1-\frac{1}{d}\sum_{i=1}^{d}\cos(k_i)}\\
&\quad=\frac{1}{2}\frac{1}{(2\pi)^d}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}dk_1\int_{-\pi}^{\pi}dk_2\cdots\int_{-\pi}^{\pi}dk_d\,\frac{2\cos(k_1)}{1-\frac{1}{d}\sum_{i=1}^{d}\cos(k_i)}\\
&\quad=\frac{1}{4}\frac{1}{(2\pi)^d}\int_{-\pi}^{\pi}dk_1'\int_{-\pi}^{\pi}dk_2\cdots\int_{-\pi}^{\pi}dk_d\,\frac{2\cos(\frac{k_1}{2})}{1-\frac{1}{d}\cos(\frac{k_1}{2})-\frac{1}{d}\sum_{i=2}^{d}\cos(k_i)}\\
&\quad=\frac{1}{2}\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}dk\,\frac{\cos(\frac{k_1}{2})}{1-J(k)}.
\end{aligned}
$$

We will now relate the previous quantity to the Green's function of the simple random walk. For this, let $\widetilde{S}_n$ be a random walk with i.i.d. increments on $\frac{1}{2}\mathbb{Z}^d$ with jump distribution $\widetilde{P}$ satisfying

$$
\forall x\in\frac{1}{2}\mathbb{Z}^d\quad \widetilde{P}(\widetilde{S}_1=x)=\frac{1}{2d}\mathbb{1}_{\{x\in\mathcal{N}\}},
$$

and denote by $\widetilde{E}$ its expectation. In other words, the simple random walk $\widetilde{S}_n$ performs half-unit jumps in the $\pm\boldsymbol{e}_1$ directions and unit jumps in all the other directions. By independence of the simple random walk increments we deduce that

$$
\widetilde{E}(e^{ik\cdot\widetilde{S}_n})=\widetilde{E}(e^{ik\cdot\widetilde{S}_1})^n=J(k)^n.
\tag{5.13}
$$

Using the fact that

$$\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} dk e^{ik\cdot x} = \mathbb{1}_{\{x=o\}},$$

and using (5.13), we deduce that

$$\begin{aligned}\widetilde{P}\left(\widetilde{S}_n = -\frac{\boldsymbol{e}_1}{2}\right) &= \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} dk \widetilde{E}\left[e^{ik\cdot(\widetilde{S}_n+\frac{\boldsymbol{e}_1}{2})}\right] \\ &= \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} dk e^{ik\cdot\frac{\boldsymbol{e}_1}{2}} J(k)^n.\end{aligned}$$

Recalling that P is the distribution of a simple random walk S_n on $\mathbb{Z}^d$, we deduce by an obvious coupling of the random walks S_n and $\widetilde{S}_n$ that

$$\forall n \in \mathbb{N} \quad P(S_n = \boldsymbol{e}_1) = \widetilde{P}\left(\widetilde{S}_n = \pm\frac{\boldsymbol{e}_1}{2}\right).$$

From the previous two expressions we deduce that, for any arbitrary finite $m \in \mathbb{N}$,

$$\sum_{n=0}^{m} P(S_n = \boldsymbol{e}_1) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} dk \frac{\cos(\frac{k_1}{2})(1 - J(k)^{m+1})}{1 - J(k)}. \tag{5.14}$$

Define for any $x \in \mathbb{Z}^d$, $N_x := \sum_{n=0}^{\infty} \mathbb{1}\{S_n = x\}$, and recall that $N_+ = \sum_{n>0} \mathbb{1}\{S_n = o\}$. We have that the following limit exists and satisfies

$$\lim_{m\to\infty}\sum_{n=0}^{m} P\big(S_n = \boldsymbol{e}_1\big) = E[N_{\boldsymbol{e}_1}] = E[N_+]. \tag{5.15}$$

For the second identity we used the fact that, every time the simple random walk jumps from a nearest neighbour of the origin, it has a chance $\frac{1}{2d}$ to hit the origin at the next step. Thus we deduce that $\frac{1}{2d}E[\sum_{y\sim o} N_y] = E[N_+]$ and the claim thus follows from rotational symmetry. To conclude the proof, we need to show that we can exchange the limit $m \to \infty$ with the integral on the right-hand side of (5.14). For this, note first that for any $0 < \delta < \pi/2$, we have that the integrand is positive for any $m \in \mathbb{N}$ and any $k \in [-\delta,\delta]^d$, and thus by the monotone convergence theorem the limit can be taken inside the integral. To deal with the integral in $[-\pi,\pi]^d \setminus [-\delta,\delta]^d$, note that the integrand is uniformly bounded and converges pointwise as $m \to \infty$ in $[-\pi,\pi]^d \setminus [-\delta,\delta]^d$, thus by the dominated convergence theorem the limit can be taken inside the integral. This concludes the proof. □

6 Proof of Equation 5.11

For the first identity we use the facts that the term on the left-hand side is real and the function $\widehat{\mathbb{G}}^e(k)$ is real, and the definition of Fourier transform, (3.7),

$$\begin{aligned} &-2 \sum_{k\in\mathbb{H}\setminus\{o\}} \cos(k\cdot \boldsymbol{e}_1)\widehat{\mathbb{G}}^e_{L,N,\rho}(k) \\ &= -2\,\mathrm{Re}\Big[\sum_{x\in\mathbb{T}_L} \mathbb{G}^e_{L,N,\rho}(x) \sum_{k\in\mathbb{H}\setminus\{o\}} e^{-ik\cdot(x-\boldsymbol{e}_1)}\Big] \\ &= -2\,\mathrm{Re}\Big[\sum_{x\in\mathbb{T}_L} \mathbb{G}^e_{L,N,\rho}(x)\big(-1+\sum_{k\in\mathbb{H}} e^{-ik\cdot(x-\boldsymbol{e}_1)}\big)\Big] \\ &= 2\sum_{x\in\mathbb{T}_L} \mathbb{G}^e_{L,N,\rho}(x) - |\mathbb{T}_L| \sum_{x\in\mathbb{T}_L} \mathbb{G}^e_{L,N,\rho}(x)\Upsilon_L(x). \end{aligned}$$

An exact and standard computation shows that the function $\Upsilon_L(x)$, which was defined in (5.12), takes nonzero (negative or positive) values only at even sites along the $\boldsymbol{e}_1$-axis, and that it converges pointwise to a function $\Upsilon(x)$, which decays like $|\Upsilon(x)| \sim \frac{1}{|x_1|}$. This concludes the proof of Theorem 5.1. $\square$

7 Proof of Theorems 2.2 and 2.3

In this section we present the proofs of Theorems 2.2 and 2.3.

PROOF OF (2.13) IN THEOREM 2.3. To begin, we claim that, for any $L \in 2\mathbb{N}$,

$$\mathbb{G}_{L,N,0}(o,\boldsymbol{e}_1) = \frac{1}{dN}. \tag{7.1}$$

To see why this is true, define the map $\Pi : \Omega_{o,\boldsymbol{e}_1} \mapsto \{\pi \in \Omega^\ell : (o,\boldsymbol{e}_1) \in E_\pi\}$, which associates to any $\pi \in \Omega_{o,\boldsymbol{e}_1}$ an element $\Pi(\pi)$ that is obtained from π by adding to π an edge directed from $\boldsymbol{e}_1$ to o. Note that, by definition of $\Omega_{o,\boldsymbol{e}_1}$, such a directed edge cannot be already present in $\pi \in \Omega_{o,\boldsymbol{e}_1}$ (but an edge directed from o to $\boldsymbol{e}_1$ might be present!), and that this map is one-to-one. Thus, we deduce that

$$\begin{aligned} \mathbb{Z}_{L,N,\rho}(o,\boldsymbol{e}_1) &= \sum_{\pi\in\Omega_{o,\boldsymbol{e}_1}} \rho^{\mathcal{M}(\pi)}\Big(\frac{N}{2}\Big)^{\mathcal{L}(\pi)} = \frac{2}{N} \sum_{\substack{\pi\in\Omega^\ell:\\ (o,\boldsymbol{e}_1)\in E_\pi}} \rho^{\mathcal{M}(\pi)}\Big(\frac{N}{2}\Big)^{\mathcal{L}(\pi)} \\ &= \frac{2}{N}\frac{1}{2d} \sum_{\substack{\pi\in\Omega^\ell:\\ \exists i\in[1,d]:(o,\boldsymbol{e}_i)\in E_\pi}} \rho^{\mathcal{M}(\pi)}\Big(\frac{N}{2}\Big)^{\mathcal{L}(\pi)}, \end{aligned}$$

where $\mathcal{L}(\Pi(\pi)) = \mathcal{L}(\pi)+1$, and the last step follows from reflection and rotational symmetry. From this and (4.2) we deduce that

$$\mathbb{G}_{L,N,\rho}\big(o, \boldsymbol{e}_1\big) = \frac{1}{dN} \frac{\sum\limits_{\substack{\pi\in\Omega^\ell: \\ \exists i\in[1,d]:(o,\boldsymbol{e}_i)\in E_\pi}} \rho^{\mathcal{M}(\pi)}\big(\frac{N}{2}\big)^{\mathcal{L}(\pi)}}{\sum\limits_{\pi\in\Omega^\ell} \rho^{\mathcal{M}(\pi)}\big(\frac{N}{2}\big)^{\mathcal{L}(\pi)}}.$$

Since for any finite $L \in 2\mathbb{N}$, the second factor equals 1 when $\rho = 0$ (the origin is not a monomer almost surely), the proof of (7.1) is concluded. From a direct application of our infrared-ultraviolet bound, Theorem 5.1 above, from point (iv) of Lemma 5.4 and from (7.1), we deduce that

$$\frac{1}{|\mathbb{T}_L^o|} \sum_{x\in\mathbb{T}_L^o} \mathbb{G}_{L,N,0}(x) \geq \mathbb{G}_{L,N,0}(\boldsymbol{e}_1) - \mathcal{I}_L(d) = \frac{1}{dN} - \mathcal{I}_L(d).$$

Since by Theorem 5.1 we have that $\lim_{L\to\infty} \mathcal{I}_L(d) = \frac{1}{2d}\frac{r_d}{2}$, we obtain (2.13) and conclude the proof. □

PROOF OF (2.14) IN THEOREM 2.3. To begin, note that the monotonicity properties in [43, theorem 2.4] imply that, for any $L \in 2\mathbb{N}$, any $N \in \mathbb{N}_{>0}$, any Cartesian vector $\boldsymbol{e}_i$, and any $z \in \mathbb{T}_L$ such that $\boldsymbol{e}_i \cdot z \in (2\mathbb{N}+1) \cap (0, \frac{L}{2})$, for any odd integer $n \in (3, z \cdot \boldsymbol{e}_i)$

$$\begin{aligned} \mathbb{G}^o_{L,N,0}(o, z) \leq \mathbb{G}^o_{L,N,0}(o, n\boldsymbol{e}_i) &\leq \mathbb{G}^o_{L,N,0}(o, (n-2)\boldsymbol{e}_i) \\ &\leq \mathbb{G}^o_{L,N,0}(o, \boldsymbol{e}_i) = \frac{1}{dN}, \end{aligned} \tag{7.2}$$

where the identity follows from (7.1). By the torus symmetry and the fact that for any $z \in \mathbb{T}_L^o$ there exists $\boldsymbol{e}_i$ such that $z \cdot \boldsymbol{e}_i \in 2\mathbb{Z}+1$; this implies that

$$\forall z \in \mathbb{T}_L \quad \mathbb{G}^o_{L,N,0}(o, z) \leq \frac{1}{dN}. \tag{7.3}$$

We now deduce the pointwise lower bound (2.14) from (2.13) and (7.3). To begin, for any $k \in \mathbb{N}$, we define the set

$$\mathbb{S}_{k,L} := \big\{z \in \mathbb{T}_L^o : \exists i \in \{1, \dots, d\} \text{ s.t. } |z \cdot \boldsymbol{e}_i| < k\big\}.$$

Note that, for any $L \in 2\mathbb{N}$ and $k \in (0, L/2) \cap \mathbb{N}$,

$$\big|\mathbb{T}_L^o \setminus \mathbb{S}^o_{k,L}\big| = \frac{1}{2}(L-2k)^d.$$

We now choose an arbitrary $\varphi \in (0, \frac{1}{2d}(\frac{2}{N} - \frac{r_d}{2}))$. We claim that

$$\begin{aligned} &\exists c = c(d, \varphi, N) \in (0, \tfrac{1}{2}) : \forall L \in 2\mathbb{N} \text{ large enough} \\ &\qquad \exists\, z_L \in \mathbb{T}_L^o \setminus \mathbb{S}_{cL,L} \text{ s.t. } \mathbb{G}_{L,N,0}(z_L) \geq \varphi. \end{aligned} \tag{7.4}$$

We first conclude the proof using (7.4) and then prove (7.4). Choose c as in (7.4) and deduce that, for any large enough $L \in 2\mathbb{N}$, since $z_L \in \mathbb{T}_L^o$, there exists a Cartesian vector $\boldsymbol{e}_i$ such that $m_L := z_L \cdot \boldsymbol{e}_i \in 2\mathbb{Z}+1$. Moreover, since $z_L \in \mathbb{T}_L^o \setminus \mathbb{S}_{cL,L}$, we deduce that $|m_L| \geq c\,L$. Thus, from the monotonicity properties (7.3) and symmetry, we deduce that, for any odd integer $n \in (-|m_L|, |m_L|)$ and any Cartesian vector $\boldsymbol{e}_i$,

$$\mathbb{G}_{L,N,\mathbf{0}}(o, \boldsymbol{e}_i n) \geq \mathbb{G}_{L,N,\mathbf{0}}(o, \boldsymbol{e}_i m_L) > \varphi.$$

This concludes the proof of (2.4) given (7.4).

Now we prove (7.4) by contradiction. Assume that (7.4) is false, namely, that for any $c \in (0, \frac{1}{2})$ there exists a infinite sequence of even integers $(L_n)_{n\in\mathbb{N}}$ such that $\mathbb{G}_{L_n,N,\mathbf{0}}(z) < \varphi$ for any $z \in \mathbb{T}_{L_n}^o \setminus \mathbb{S}_{cL_n,L_n}$. From this, (2.3), and (7.3), we deduce that, for any $c \in (0, \frac{1}{2})$ (define $q := (1-2c)^d$), there exists an infinite sequence $(L_n)_{n\in\mathbb{N}}$ such that

$$\begin{aligned}\sum_{z\in\mathbb{T}_L^o} \mathbb{G}^o_{L_n,N,\mathbf{0}}(z) &< \varphi\big|\mathbb{T}_L^o \setminus \mathbb{S}_{cL_n,L_n}\big| + \frac{1}{dN}\big(\big|\mathbb{T}^o_{L_n}\big| - \big|\mathbb{T}_L^o \setminus \mathbb{S}_{cL_n,L_n}\big|\big)\\ &= \frac{1}{2}L_n^d\bigg[\frac{1}{dN} - (1-2c)^d\bigg(\frac{1}{dN} - \varphi\bigg)\bigg]\\ &= \frac{1}{2}L_n^d\bigg[\frac{1}{dN}(1-q) + q\varphi\bigg] = |\mathbb{T}^o_{L_n}|\bigg[\frac{1}{dN}(1-q) + q\varphi\bigg].\end{aligned}$$

Since we chose $\varphi \in (0, \frac{1}{2d}(\frac{2}{N} - \frac{r_d}{2}))$, we see that the previous inequality cannot hold for any constant c and for an infinite sequence $(L_n)_{n\in\mathbb{N}}$ unless violating (2.13) (by choosing c small enough, namely q close enough to 1, we bound the quantity inside the square bracket away from $\frac{1}{2d}(\frac{2}{N} - \frac{r_d}{2})$, uniformly in L_n), which was proved to hold true. Thus, we obtained the desired contradiction and conclude the proof. □

PROOF OF THEOREM 2.2. Theorem 2.2 is an immediate consequence of Theorem 2.3. For any $A \subset \mathbb{T}_L$, we have that

$$\begin{aligned}\mathbb{P}_{L,N,\mathbf{0}}(X \in A) = \sum_{x\in A}\mathbb{P}_{L,N,\mathbf{0}}(X = x) &= \frac{\sum_{x\in A}\mathbb{Z}_{L,N,\mathbf{0}}(o,x)}{\sum_{x\in\mathbb{T}_L}\mathbb{Z}_{L,N,\mathbf{0}}(o,x)}\\ &= \frac{\sum_{x\in A}\mathbb{G}_{L,N,\mathbf{0}}(o,x)}{\sum_{x\in\mathbb{T}_L}\mathbb{G}_{L,N,\mathbf{0}}(o,x)},\end{aligned}$$

where the last identity follows after dividing the numerator and the denominator by $\mathbb{Z}^\ell_{L,N,\mathbf{0}}$. Now the claim follows from (2.13), which provides a lower bound for the denominator on the right-most term, and from (7.1), which provides an upper bound for the numerator on the rightmost term. Using both bounds we obtain (2.13). □

8 Proof of Lemma 4.10

In this section we prove Lemma 4.10, which is a fundamental step in the proof of the polynomial expansion. The proof of (4.16) is the easiest. Indeed, our choice of the weight function H imposes that any configuration in the set $\mathcal{A}^\ell$ consists of mutually vertex-disjoint ℓ-loops and double links that lie entirely in the original torus; these can be identified with loops and double edges of the configurations in Ω^ℓ taking the same positions. The proofs of (4.17) and (4.18) are more elaborate. The proof requires defining a map that maps sets of configurations in $\mathcal{A}^s(\{x, y\})$ to sets of configurations in Ω^ℓ and sets of configurations in $\mathcal{A}^w(\{(x, q), (y, r)\})$ to sets of configurations in $\Omega_{x,y}$ and consists of a comparison of the weights taken by such sets.

Informally the map works as follows: For the proof of (4.17), we take any configuration in $\mathcal{A}^s(\{x, y\})$ and 'remove' the link that is unpaired at both its endpoints. Such a removal has a cost λ (whose corresponding factor appears on the right-hand side of (4.17)) and leads to a configuration in $\mathcal{A}^\ell$. After that, we compare the sets of configurations $\mathcal{A}^\ell$ obtained after such a removal with sets of configurations in Ω^ℓ similarly to the previous case.

For the proof of (4.18) we remove from any configuration in $\mathcal{A}^w(\{(x, q), (y, r)\})$ the two extremal links (which, by definition, are on $\{x, q\}$ and on $\{y, r\}$, respectively, and are unpaired at q and r, respectively), paying a cost λ^2 (which appears on the right-hand side of (4.18)) and obtain a configuration with an ℓ-walk having endpoints x and y and possibly double links and ℓ-loops, with all these objects being vertex self-avoiding, mutually vertex self-avoiding, and lying entirely in the original torus by our choice $U = H$. Such objects (ℓ-walk, double links, and ℓ-loops) can be identified with the corresponding objects of the configurations in $\Omega_{x,y}$ (walk, double edges, and loops, respectively) taking the same positions. Such an identification allows the comparison of the weights of the set $\mathcal{A}^w(\{(x, q), (y, r)\})$ under μ and the weights taken by the configurations in $\Omega_{x,y}$ in the partition function $\mathbb{Y}_{L,N,\lambda}(x, y)$.

It is important for the comparison to ensure that the removal of the links does not leave a 'hole': For this reason the definition of the weight function H implies that the ℓ-walk is not entirely vertex self-avoiding, namely at the vertices where its two extremal links are unpaired, q and r, it might touch itself or other paths. Here by 'no hole' we mean that, when the two extremal links are removed, one obtains configurations whose paths are 'free' to use the vertices that are touched by the links which get removed.

Other technical aspects in the proofs of (4.17) and (4.18) are that such a removal is a many-to-one map, since the links that get removed might occupy different positions on the same edge, and the removal maps several input configurations with different positions of such links to the same output. For this reason we need to compute the factor corresponding to the number of such possible positions, which also depends on the pairing of the other links on that edge. Fortunately for us, the factor $\frac{1}{m_e!}$ in the definition of the measure $\mu_{L,N,\lambda,U}$ assigns a higher weight to the configuration obtained after the removal and such an energy gain matches the corresponding entropy loss *perfectly*, giving a total factor that equals precisely 1.

PROOF OF LEMMA 4.10. For the formal proof it will be convenient to deal with undirected subgraphs of the torus. For this reason we introduce the set Σ, which can be viewed as an 'intermediate object' between the sets $\mathcal{W}^1$ and $\Omega \cup \Omega^{\ell}$, whose respective subsets must be compared.

Definition of the set Σ. Let Σ be the set of spanning subgraphs of $(\mathbb{T}_L, \mathbb{E}_L)$ such that every vertex has degree 0, 1 or 2. Any connected component of $\sigma \in \Sigma$ is called a *monomer* if it consists of a single vertex, an *isolated edge* if it consists of two vertices connected by one edge, a *loop* if the set of its edges is isomorphic to a simple closed curve in $\mathbb{R}^d$, and a *walk* if the set of its edges is isomorphic to an open simple curve in $\mathbb{R}^d$. Thus, an isolated edge is also a walk.

For $x \neq y$, let $\Sigma_{x,y}$ be defined as the set of graphs $\sigma \in \Sigma$ such that there exists a walk with endpoints x and y and any other connected component is a monomer, an isolated edge, or a loop. Let Σ^{ℓ} be defined as the set of graphs $\sigma \in \Sigma$ such that any connected component is a monomer, a isolated edge, or a loop. Let $\Sigma_{x,x}$ be the set of graphs $\sigma \in \Sigma^{\ell}$ such that x is monomer. For any $\sigma \in \Sigma$, let $\mathcal{L}(\sigma)$ be the number of connected components in σ that are not monomers (by a slight abuse of notation, since we already defined the related quantity $\mathcal{L}(\pi)$ in the introduction), let $\mathcal{D}(\sigma)$ be the number of isolated edges in σ, let $\mathcal{D}'(\sigma)$ be the number of isolated edges in σ that do not contain the origin, and let $|\sigma|$ be the number of edges in σ.

Recall the definitions of the partition functions (4.1) parametrised by λ. We have that, for any $y \in \mathbb{T}_L \setminus \{o\}$,

$$\mathbb{Y}^{\ell}_{L,N,\lambda} = \sum_{\sigma \in \Sigma^{\ell}} \lambda^{|\sigma|} N^{\mathcal{L}(\sigma)} \left(\frac{\lambda}{2}\right)^{\mathcal{D}(\sigma)}, \tag{8.1}$$

$$N\mathbb{Y}_{L,N,\lambda}(o,y) = \sum_{\sigma \in \Sigma_{o,y}} \lambda^{|\sigma|} N^{\mathcal{L}(\sigma)} \left(\frac{\lambda}{2}\right)^{\mathcal{D}'(\sigma)}, \tag{8.2}$$

$$\mathbb{Y}_{L,N,\lambda}(o,o) = \sum_{\sigma \in \Sigma_{o,y}} \lambda^{|\sigma|} N^{\mathcal{L}(\sigma)} \left(\frac{\lambda}{2}\right)^{\mathcal{D}'(\sigma)}. \tag{8.3}$$

To see why the previous relations hold true, note that there is an obvious correspondence between the elements $\pi \in \Omega^{\ell}$ and the elements $\sigma \in \Sigma^{\ell}$ and between the elements $\pi \in \Omega_{o,x}$ and the elements $\sigma \in \Sigma_{o,x}$. Indeed, for each π, we obtain a unique element σ that is associated to π by replacing any double edge, directed loop, or directed walk by an isolated edge, undirected loop, or undirected walk, respectively, which is composed of the same edges and sites. We deduce (8.1) and (8.2) from the definitions (4.1), considering that directed loops have two possible orientations and that double edges in π consist of two (directed) edges while the isolated edges in σ just of one edge. Note that the factor N on the left-hand side of (8.2) is due to the fact that $\mathcal{L}(\pi)$, which was defined in Section 2, does not count the walk, while $\mathcal{L}(\sigma)$ counts the number of connected components that are not monomers and thus also the walk.

Finally, note that in (8.2) and (8.3) we have $\mathcal{D}'$ in place of $\mathcal{D}$ since, if the walk consists of just one edge, we don't want to assign to it a factor $\frac{\lambda}{2}$. Now that the partition functions have been defined in terms of sums over elements of Σ, we can proceed with the comparison between the elements of $\mathcal{W}^1$ and the elements of Σ. This comparison will require introducing a map between such sets and studying its multiplicity properties.

Below we will keep adopting the following terminology: *double links*, *ℓ-loops*, *ℓ-walks*, and *segments* for the paths of the realisations $w \in \mathcal{W}^1$; and *isolated edges*, *loops*, *walks*, and *monomers* for the connected components of the realisation $\sigma \in \Sigma$. Moreover, we write that $\{x, y\} \in \sigma$ if $\{x, y\}$ belongs to the edge set of $\sigma \in \Sigma$.

Definition and properties of the map $Q : \mathcal{W}^1 \mapsto \Sigma$. For any $w \in \mathcal{W}^1$, let $\mathcal{Q}(w)$ be the set of edges $\{x, y\} \in \mathbb{E}_L$ such that there exists a link on $\{x, y\}$ in w that is paired both at x and y. We define a map Q that associates to each realisation $w \in \mathcal{W}^1$ the realisation $Q(w) := (\mathbb{T}_L, \mathcal{Q}(w))$.

To begin note that

$$\forall w \in \mathcal{W}^1 \quad Q(w) \in \Sigma. \tag{8.4}$$

This holds true since, by definition of $\mathcal{W}^1$, for each realisation $w \in \mathcal{W}^1$, each vertex of $Q(w)$ has degree 0, 1, or 2. For any $\sigma \in \Sigma$, define the set $Q^{-1}(\sigma) := \{w \in \mathcal{W}^1 : Q(w) = \sigma\}$. From the definition of the map Q we deduce that, for any pair of graphs $\sigma_1, \sigma_2 \in \Sigma$,

$$\sigma_1 \neq \sigma_2 \implies Q^{-1}(\sigma_1) \cap Q^{-1}(\sigma_2) = \varnothing. \tag{8.5}$$

Note that for any $w \in \mathcal{W}^1$, a loop is present in $Q(w)$ if and only if an ℓ-loop with precisely one link located on each edge of the loop is present in w. Moreover, note that an isolated edge is present in $Q(w)$ if and only if a double link whose two links are on that edge is present in w. Moreover, suppose that $x \neq y$. Note that for any $w \in \mathcal{W}^1$, a walk with endpoints x and y is present in $Q(w)$ if and only if an ℓ-walk with extremal links (x, q) and (y, r) for some $q, r \in \mathcal{T}_L$ and with precisely a nonextremal link on each edge of that walk is present in w.

Finally, suppose that $x = y$. Note that, by definition of H, for any $w \in \mathcal{W}^1$, a ℓ-walk with extremal links (x, q) and (x, r) can only consist of two links that are paired to each other at x and are both extremal in w. Thus, $Q(w)$ has a monomer at $x = y$ if and only if either an ℓ-walk composed of just two links paired at x and on the edges $\{x, q\}$ and $\{x, r\}$ for some $q, r \in \mathcal{T}_L$ (with possibly $q = r$) is present in w or if no link of w is paired at $x = y$. See also Figures 8.1, 8.2, and 8.3 for examples.

From all these considerations we deduce that

$$\forall w \in \mathcal{A}^{\ell} \quad Q(w) \in \Sigma^{\ell}, \tag{8.6}$$

$$\forall \{x,y\} \in \mathcal{E}_L,\ \forall w \in \mathcal{A}^s(\{x,y\}) \quad Q(w) \in \Sigma^{\ell}, \tag{8.7}$$

$$\begin{gathered} \forall (x,q),(y,r) \in \mathcal{E}_L : x,y \in \mathbb{T}_L, \\ \forall w \in \mathcal{A}^w((x,q),(y,r)) \quad Q(w) \in \Sigma_{x,y}. \end{gathered} \tag{8.8}$$

Moreover, by definition of $\mathcal{W}^1$ we also have that

$$\forall (x,q),(y,r) \in \mathcal{E}_L : \{x,y\} \cap \mathbb{T}_L^{(2)} \neq \emptyset, \quad \mathcal{A}^w\big((x,q),(y,r)\big) = \emptyset. \tag{8.9}$$

We will now prove all the claims in the statement of Lemma 4.10 one by one using such properties.

Proof of (4.16). From (8.6) and from the considerations made in the paragraph before (8.6), we deduce that

$$\forall \sigma \in \Sigma^{\ell} \quad \mu_{L,N,\lambda,H}(\mathcal{A}^{\ell} \cap \{Q(w) = \sigma\}) = \left(\frac{1}{2}\right)^{\mathcal{D}(\sigma)} \lambda^{|\sigma| + \mathcal{D}(\sigma)} N^{\mathcal{L}(\sigma)}. \tag{8.10}$$

The factor $N^{\mathcal{L}(\sigma)}$ above takes into account the fact that if w' is obtained from w by changing the colour of all the links belonging to the same path, then $Q(w) = Q(w')$, the term $|\sigma| + \mathcal{D}(\sigma)$ corresponds to the number of links in each configuration $w \in \mathcal{A}^{\ell}$ such that $Q(w) = \sigma$, and the factor $(\frac{1}{2})^{\mathcal{D}(\sigma)}$ comes from the term $\frac{1}{m_e!}$ in the definition (4.4). Now note that

$$\begin{aligned} \mu_{L,N,\lambda,H}(\mathcal{A}^{\ell}) &= \sum_{\sigma \in \Sigma^{\ell}} \mu_{L,N,\lambda,H}(\mathcal{A}^{\ell} \cap \{Q(w) = \sigma\}) \\ &= \sum_{\sigma \in \Sigma^{\ell}} \left(\frac{\lambda}{2}\right)^{\mathcal{D}(\sigma)} \lambda^{|\sigma|} N^{\mathcal{L}(\sigma)} = \mathbb{Y}^{\ell}_{L,N,\lambda}. \end{aligned}$$

For the first identity we used (8.5) and (8.6); for the second identity we used (8.1). This concludes the proof of (4.16). □

Proof of (4.17) Recall that, if $\{x,y\}$ belongs to the edge set of $\sigma \in \Sigma$, we write $\{x,y\} \in \sigma$. In the whole proof we fix an arbitrary undirected edge $\{x,y\} \in \mathcal{E}_L$. To begin, we claim that for any $\sigma \in \Sigma^{\ell}$,

$$\begin{aligned} &\big|\{w \in \mathcal{A}^s(\{x,y\}) : Q(w) = \sigma\}\big| \\ &\quad = \begin{cases} 3N^{\mathcal{L}(\sigma)+1} & \text{if } \sigma \text{ has a isolated edge at } \{x,y\} \text{ and } \{x,y\} \in \mathbb{E}_L, \\ 2N^{\mathcal{L}(\sigma)+1} & \text{if } \{x,y\} \text{ belongs to a loop of } \sigma \text{ and } \{x,y\} \in \mathbb{E}_L, \\ 1N^{\mathcal{L}(\sigma)+1} & \text{if } \{x,y\} \notin \sigma, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{8.11}$$

We now prove (8.11), starting from the *fourth case* of (8.11) ('otherwise'), which is when $\{x,y\} \in \mathcal{E}_L \setminus \mathbb{E}_L$ and $\{x,y\}$ belongs to a loop or an isolated edge of σ. In

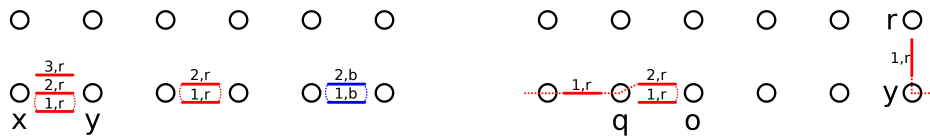

FIGURE 8.1. Two copies of the vertex set of the graph $(\mathcal{T}_L, \mathcal{E}_L)$ when $d = 1$, with $\mathcal{T}_L = \{-2, \ldots, 3\} \times \{1, 2\}$. On each copy a realisation $w \in \mathcal{W}^1$ is represented, each link has two possible colours, red or blue, and a dotted line connects endpoints of paired links. *Left*: A realisation in $w \in \mathcal{A}^s\big(\{x, y\}\big)$ such that $Q(w) \in \Sigma$ consists of three isolated edges and six monomers. *Right*: A realisation $w \in \mathcal{A}^w\big((o, q), (y, r)\big)$ such that $Q(w) \in \Sigma$ consists of one walk composed of two edges and eight monomers.

this case $\mathcal{A}^w(\{x, y\}) \cap \{Q(w) = \sigma\} = \varnothing$, since for any $w \in \mathcal{W}^1$, no double link or ℓ-loop is allowed to *touch* a virtual vertex. This explains why we get 0 in the fourth case of (8.11).

We now consider the first three cases. To begin, note that the factor $N^{\mathcal{L}(\sigma)+1}$ in the first three cases takes into account the fact that if w' is obtained from w by changing the colour of all the links belonging to the same path, then $Q(w) = Q(w')$.

The factors 3, 2, or 1 in the first three cases above take into account the number of possible labels of the link belonging to the segment and which is on $\{x, y\}$. We explain this starting from the first case. In the *first case*, when σ has an isolated edge at $\{x, y\}$, each configuration $w \in Q^{-1}(\sigma) \cap \mathcal{A}^s(\{x, y\})$ has three links on $\{x, y\}$, where two of these three links are paired to each other and compose a double link, while the third link is unpaired at both its endpoints. Such an unpaired link might be the first, the second, or the third link on $\{x, y\}$. This situation is represented, for example, on the left of Figure 8.1. Thus, the factor 3 takes into account the fact that the unpaired link might have three distinct possible labels (in other words, it might occupy three distinct possible positions on $\{x, y\}$), with each label corresponding to a distinct configuration w such that $Q(w) = \sigma$.

In the *second case*, when $\{x, y\}$ belongs to a loop of σ, each $w \in Q^{-1}(\sigma) \cap \mathcal{A}^s(\{x, y\})$ has two links on $\{x, y\}$, with one link belonging to the segment and thus being unpaired at both its endpoints and the other link being paired both at x and y. Thus, the factor 2 takes into account the fact that there are two choices for which link on $\{x, y\}$ belongs to the segment and which link on $\{x, y\}$ is paired at both its endpoints.

Finally, in the *third case* we have no entropy factor. From these considerations and from the definition of μ, which is given in Definition 4.1 and the definition of H, which is given in Definition 4.8, we also deduce that, for any $\sigma \in \Sigma^\ell$, for any $w \in \mathcal{A}^s(\{x, y\})$ such that $Q(w) = \sigma$,

$$\mu_{L,N,\lambda,H}(w) = \begin{cases} \frac{1}{3!}\frac{1}{2^{\mathcal{D}(\sigma)-1}}\lambda^{|\sigma|+\mathcal{D}(\sigma)+1} & \text{if } \sigma \text{ has a isolated edge at } \{x,y\} \text{ and } \{x,y\} \in \mathbb{E}_L, \\ \frac{1}{2}\frac{1}{2^{\mathcal{D}(\sigma)}}\lambda^{|\sigma|+\mathcal{D}(\sigma)+1} & \text{if } \{x,y\} \text{ belongs to a loop of } \sigma \text{ and } \{x,y\} \in \mathbb{E}_L, \\ \frac{1}{2}\frac{1}{2^{\mathcal{D}(\sigma)}}\lambda^{|\sigma|+\mathcal{D}(\sigma)+1} & \text{if } \{x,y\} \notin \sigma \text{ and } \{x,y\} \in \mathcal{E}_L \setminus \mathbb{E}_L, \\ \frac{1}{2^{\mathcal{D}(\sigma)}}\lambda^{|\sigma|+\mathcal{D}(\sigma)+1} & \text{if } \{x,y\} \notin \sigma \text{ and } \{x,y\} \in \mathbb{E}_L. \end{cases} \tag{8.12}$$

In all the cases above, the last factor corresponds to the weight of the links, whose number is $|\sigma| + \mathcal{D}(\sigma) + 1$. The first two factors in the first two cases, the second factor in the third case, and the first factor in the last case follows from the term $\frac{1}{m_e!}$ in the definition of μ, the first factor $\frac{1}{2}$ in the third case comes from the fact that the weight function H_x, $x \in \mathbb{T}_L$, assigns a factor $\frac{1}{2}$ whenever there is a link on $\{x, g(x)\}$ that is unpaired at x, and this can only happen when such a link is unpaired at x and at $\{x, g(x)\}$, and thus is a segment.

From (8.11) and (8.12) we deduce that, for any $w \in \mathcal{A}^s(\{x, y\})$, for any $\sigma \in \Sigma^\ell$,

$$\mu_{L,N,\lambda,H}\big(\mathcal{A}^s(\{x,y\}) \cap \{Q(w) = \sigma\}\big) = \begin{cases} \lambda N \lambda^{|\sigma|+\mathcal{D}(\sigma)}(\frac{1}{2})^{\mathcal{D}(\sigma)} N^{\mathcal{L}(\sigma)} & \text{if } \{x,y\} \in \mathbb{E}_L, \\ \frac{\lambda}{2} N \lambda^{|\sigma|+\mathcal{D}(\sigma)}(\frac{1}{2})^{\mathcal{D}(\sigma)} N^{\mathcal{L}(\sigma)} & \text{if } \{x,y\} \in \mathcal{E}_L \setminus \mathbb{E}_L. \end{cases} \tag{8.13}$$

From (8.5), (8.7), (8.12), and (8.13) we deduce that, when $\{x, y\} \in \mathbb{E}_L$,

$$\begin{aligned} \mu_{L,N,\lambda,H}\big(\mathcal{A}^s(\{x,y\})\big) &= \sum_{\sigma\in\Sigma^\ell} \mu_{L,N,\lambda,H}\big(\mathcal{A}^s(\{x,y\}) \cap \{Q(w) = \sigma\}\big) \\ &= \lambda N \sum_{\sigma\in\Sigma^\ell} \left(\frac{\lambda}{2}\right)^{\mathcal{D}(\sigma)} \lambda^{|\sigma|} N^{\mathcal{L}(\sigma)} = \lambda N \mathbb{Y}^\ell_{L,N,\lambda}, \end{aligned}$$

and that the same holds true with a factor of $\frac{1}{2}$ in front of the two last terms when $\{x, y\} \in \mathcal{E}_L \setminus \mathbb{E}_L$. □

Proof of (4.18) *when* $\{x, y\} \cap \mathbb{T}^{(2)}_L \neq \varnothing$. In this case, the proof follows immediately from (8.9). □

Proof of (4.18) *when* $\{x, y\} \subset \mathbb{T}_L$. Suppose that $\{x, y\} \subset \mathbb{T}_L$ (possibly $x = y$). Without loss of generality (by translation invariance) fix $x = o$. From (8.8) and from the properties of the map Q we claim that, under these assumptions, for any $y \in \mathbb{T}_L$ and $\sigma \in \Sigma_{o,y}$, we have that

$$\big|\{w \in \mathcal{A}^w((o,q),(y,r)) : Q(w) = \sigma\}\big| = \begin{cases} 2^{\mathbb{1}\{\{o,q\}\in\sigma\}} 2^{\mathbb{1}\{\{y,r\}\in\sigma\}} N^{\mathcal{L}(\sigma)} & \text{if } y \neq o \text{ and } (y,r) \neq (q,o), \\ 6N^{\mathcal{L}(\sigma)} & \text{if } y \neq o,\ (y,r) = (q,o) \text{ and } \{o,y\} \in \sigma, \\ 2N^{\mathcal{L}(\sigma)} & \text{if } y \neq o,\ (y,r) = (q,o) \text{ and } \{o,y\} \notin \sigma, \\ N^{\mathcal{L}(\sigma)+1} & \text{if } y = o. \end{cases} \tag{8.14}$$

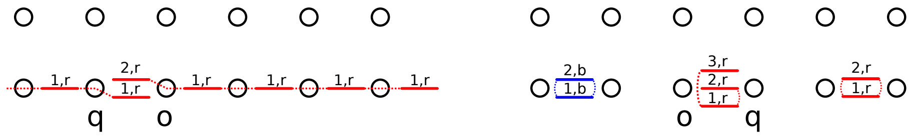

FIGURE 8.2. Same setting as in Figure 8.1. *Right*: A realisation $w \in \mathcal{A}^w((o,q),(q,o))$ such that $\{o,q\} \notin Q(w)$ and $Q(w)$ consists of one walk composed of five edges and six monomers. *Left*: A realisation $w \in \mathcal{A}^w((o,q),(q,o))$ such that $\{o,q\} \in Q(w)$ and such that $Q(w) \in \Sigma_{o,q}$ consists of three isolated edges.

We now explain (8.14). The factors $N^{\mathcal{L}(\sigma)}$ and $N^{\mathcal{L}(\sigma)+1}$ in all the cases above take into account the fact that if w' is obtained from w by changing the colour of all the links belonging to the same path, then $Q(w) = Q(w')$. We now explain the remaining factors considering case by case.

- Let us explain the first case: $y \neq o$, and $(y,r) \neq (q,o)$. Note that, from the properties of the map Q, it follows that for any $w \in \mathcal{A}^w((o,q),(y,r))$ such that $Q(w) = \sigma$, $\{o,q\} \in \sigma$ if and only if two links of the unique ℓ-walk in w are on $\{o,q\}$, one of which is extremal. Note also that the same claim holds true if we replace $\{o,q\}$ by $\{y,r\}$. Thus, the factors $2^{\mathbb{1}\{\{o,q\}\in\sigma\}}$ and $2^{\mathbb{1}\{\{y,r\}\in\sigma\}}$ account for the fact that there are two possibilities for choosing which of the two links is the extremal one (the other link belongs to the ℓ-walk, but it is not extremal). For example, if w^1 is the configuration on the right of Figure 8.1, σ is such that $Q(w^1) = \sigma$, and w^2 is the configuration that is obtained from w^1 by exchanging the pairing at the vertex q in such a way that the link $(\{q,o\},1)$ is paired at q to the link $(\{q-\boldsymbol{e}_1,q\},1)$, and $(\{q,o\},2)$ is unpaired at q, then also $Q(w^2) = \sigma$. From these considerations we also deduce that, if $y \neq o$ and $(y,r) \neq (q,o)$ for any $\sigma \in \Sigma_{o,y}$ and $w \in \mathcal{A}((o,q),(y,r))$ such that $Q(w) = \sigma$,

$$\mu_{L,N,\lambda,H}(w) = \frac{1}{2^{\mathbb{1}\{\{o,q\}\in\sigma\}+\mathbb{1}\{\{y,r\}\in\sigma\}}} \frac{1}{2^{\mathcal{D}'(\sigma)}} \lambda^2 \lambda^{|\sigma|+\mathcal{D}'(\sigma)} \tag{8.15}$$

  where the first and the second factor follows from the term $\frac{1}{m_e!}$ in the definition of μ, the factor λ^2 corresponds to the weight of the two extremal links, and the last factor corresponds to the weight of all the remaining links.
- Let us explain the second case: $y \neq o$, $(y,r) = (q,o)$, and $\{o,q\} \in \sigma$. In this case, any $w \in \mathcal{A}^w((o,q),(q,o))$ is such that the ℓ-walk consists of three links that are on $\{o,q\}$, and there are precisely three links on $\{o,q\}$. Thus, one link of the ℓ-walk must be paired at both its endpoints to the two

other links of the ℓ-walk, while the two remaining links are paired at one endpoint and unpaired at the other endpoint. An example of such a configuration is represented on the right of Figure 8.2. The factor 6 on the right-hand side of (8.14) accounts for the fact that there are three distinct possibilities for choosing which of these three links is paired at both endpoints. Once this has been chosen, there are two possibilities for choosing which of the two remaining links is paired at o and unpaired at q. From these considerations we also deduce that for any $\sigma \in \Sigma_{o,q}$ such that $\{o, q\} \in \mathbb{E}_L$ and $\{o, q\} \in \sigma$ for any $w \in \mathcal{A}^w((o, q), (q, o))$ such that $Q(w) = \sigma$,

$$\mu_{L,N,\lambda,H}(w) = \frac{1}{3!} \frac{1}{2^{\mathcal{D}'(\sigma)}} \lambda^2 \lambda^{|\sigma|+\mathcal{D}'(\sigma)}, \tag{8.16}$$

where the first and the second factor follow from the term $\frac{1}{m_e!}$ in the definition of μ; the factor λ^2 corresponds to the weight of the two extremal links of the ℓ-walk, and the last factor corresponds to the weight of all the remaining links.

- Let us explain the third case: $y \neq o$, $(y, r) = (q, o)$, and $\{o, q\} \notin \sigma$. In this case, any $w \in \mathcal{A}^w((o, q), (q, o))$ is such that two links are on $\{o, q\}$, where one of them is unpaired at o and is paired to another link of the walk at q, while the second one is unpaired at q and is paired to another link of the walk at o. An example of such a configuration is represented in Figure 8.2-left. The factor 2 on the right-hand side of (8.14) accounts for the fact that there are two possibilities for choosing which of the two links is paired at o and which at q. From these considerations we also deduce that, for any $\sigma \in \Sigma_{o,q}$ such that $\{o, q\} \in \mathbb{E}_L$ and $\{o, q\} \in \sigma$, for any $w \in \mathcal{A}^w((o, q), (q, o))$ such that $Q(w) = \sigma$,

$$\mu_{L,N,\lambda,H}(w) = \frac{1}{2!} \frac{1}{2^{\mathcal{D}'(\sigma)}} \lambda^2 \lambda^{|\sigma|+\mathcal{D}'(\sigma)}, \tag{8.17}$$

where the first and the second factor follows from the term $\frac{1}{m_e!}$ in the definition of μ, the factor λ^2 corresponds to the weight of the two extremal links of the ℓ-walk, and the last factor corresponds to the weight of all the remaining links.

- Let us explain the last case: $y = o$. An example of a configuration $w \in \mathcal{A}^w((o, q), (o, r))$ is represented on the left of Figure 8.3 when $q \neq r$ and on the right of Figure 8.3 when $q = r$. In this case, for any $w \in \mathcal{A}^w((o, q), (o, r))$ the unique ℓ-walk in w consists of just two links that are paired to each other at o. When $q = r$, these links are the only two links on $\{o, q\} = \{o, r\}$, while when $q \neq r$, each link of the two is the unique link on $\{o, q\}$ and $\{o, r\}$. Since all the other paths are double links or ℓ-loops, we deduce (8.14). From these considerations we also deduce that, for any

FIGURE 8.3. Same setting as in Figure 8.1. *Left:* A realisation $w \in \mathcal{A}^w(\{(o,q),(o,r)\})$, $r \neq q$, such that $Q(w)$ consists of two isolated edges and eight monomers. *Right:* A realisation $w \in \mathcal{A}^w(\{(o,q),(o,q)\})$ such that $Q(w) \in \Sigma_{o,o}$ consists of two isolated edges and eight monomers.

$\sigma \in \Sigma_{o,o}$, for any $w \in \mathcal{A}((o,q),(o,r))$, we have that

$$\mu_{L,N,\lambda,H}(w) = \begin{cases} \frac{1}{2^{\mathcal{D}'(\sigma)}} \lambda^2 \lambda^{|\sigma| + \mathcal{D}'(\sigma)} & \text{if } q \neq r, \\ \frac{1}{2!} \frac{1}{2^{\mathcal{D}'(\sigma)}} \lambda^2 \lambda^{|\sigma| + \mathcal{D}'(\sigma)} & \text{if } q = r. \end{cases} \tag{8.18}$$

where the first factor in the first case and the first two factors in the second case follows from the term $\frac{1}{m_e!}$ in the definition of μ, the factor λ^2 corresponds to the weight of the two unique links the ℓ-walk is composed of, and the last factor corresponds to the weight of all the remaining links.

Now that the multiplicity properties of the map and that the weights assigned by μ to the configurations w in each of the four cases above have been considered, we can put all the cases together to conclude the proof of (4.18). Below, we use the general properties of the map Q, (8.5), and (8.9) for the first identity, (8.14), (8.15), (8.16), (8.17), and (8.18) for the three cases of the second identity, and (8.2) and (8.3) for the three cases of the third and last identity, obtaining that, for any pair of directed edges $(o,q),(y,r) \in \mathcal{E}_L$,

$$\begin{aligned} &\mu_{L,N,\lambda,H}\big(\mathcal{A}^w\big((o,q),(y,r)\big)\big) \\ &\quad = \sum_{\sigma \in \Sigma_{o,y}} \mu_{L,N,\lambda,H}\big(\mathcal{A}^w\big((o,q),(y,r)\big) \cap \{Q(w) = \sigma\}\big) \\ &\quad = \begin{cases} \lambda^2 \sum\limits_{\sigma \in \Sigma_{o,y}} \big(\frac{\lambda}{2}\big)^{\mathcal{D}'(\sigma)} \lambda^{|\sigma|} N^{\mathcal{L}(\sigma)} = \lambda^2 N \, \mathbb{Y}_{L,N,\lambda}(o,y) & \text{if } y \neq o, \\ N\lambda^2 \sum\limits_{\sigma \in \Sigma_{o,y}} \big(\frac{\lambda}{2}\big)^{\mathcal{D}'(\sigma)} \lambda^{|\sigma|} N^{\mathcal{L}(\sigma)} = \lambda^2 N \, \mathbb{Y}_{L,N,\lambda}(o,y) & \text{if } y = o, \, (o,q) \neq (y,r), \\ \frac{N}{2}\lambda^2 \sum\limits_{\sigma \in \Sigma_{o,y}} \big(\frac{\lambda}{2}\big)^{\mathcal{D}'(\sigma)} \lambda^{|\sigma|} N^{\mathcal{L}(\sigma)} = \frac{\lambda^2}{2} N \, \mathbb{Y}_{L,N,\lambda}(o,y) & \text{if } y = o, \, (o,q) = (y,r). \end{cases} \end{aligned}$$

This concludes the proof of Lemma 4.10. $\square$

Appendix

PROOFS OF LEMMA 3.3. We omit the subscripts for convenience. To begin, note that it follows from (3.7) that

$$\mathbb{G}(\boldsymbol{e}_1) = \frac{1}{|\mathbb{T}_L|}\widehat{\mathbb{G}}(o) - \frac{1}{|\mathbb{T}_L|}\widehat{\mathbb{G}}(p) + \frac{1}{|\mathbb{T}_L|}\sum_{k\in\mathbb{T}_L^*\setminus\{o,p\}} e^{ik\cdot\boldsymbol{e}_1}\widehat{\mathbb{G}}(k), \tag{A.1}$$

and it follows from (3.6) that $\frac{1}{|\mathbb{T}_L|}\widehat{\mathbb{G}}(o) = \frac{1}{|\mathbb{T}_L|}\sum_{x\in\mathbb{T}_L}\mathbb{G}(x)$ and that

$$\frac{1}{|\mathbb{T}_L|}\widehat{\mathbb{G}}(p) = -\frac{1}{|\mathbb{T}_L|}\sum_{x\in\mathbb{T}_L}\mathbb{G}^o(x) + \frac{1}{|\mathbb{T}_L|}\sum_{x\in\mathbb{T}_L}\mathbb{G}^e(x).$$

Combining the equations above, we conclude the proof. $\square$

PROOF OF (i), (ii), AND (iii) IN THE PROOF OF PROPOSITION 5.3. These computations are classical and can be extracted, for example, from the computations in [50]. We present them for the reader's convenience. The proof of (i) consists of the following computation:

$$\begin{aligned}
(\triangle v)_x &= \sum_{y\sim o}\big(\cos\big((x+y)\cdot k\big) - v_x\big)\\
&= \sum_{y\sim o}\big(\cos(x\cdot k)\cos(y\cdot k) - \sin(x\cdot k)\sin(y\cdot k) - v_x\big)\\
&= \sum_{y\sim o}\big(v_x\cos(y\cdot k) - v_x\big) = -\varepsilon(k)v_x.
\end{aligned}$$

The proof of (ii) follows from the first Green identity, which states that, for any pair of real-valued vectors, $\boldsymbol{a} = (a_x)_{x\in\mathbb{T}_L}$, $\boldsymbol{b} = (b_x)_{x\in\mathbb{T}_L}$, when $(\mathbb{T}_L, \mathbb{E}_L)$ is the torus,

$$\sum_{\{x,y\}\in\mathbb{E}_L}(b_y - b_x)(a_y - a_x) = -\sum_{x\in\mathbb{T}_L} a_x(\triangle b)_x.$$

The proof of such an identity can be found in [26][lemma 8.7]. Applying such an identity with $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{v}$ and using (i), we obtain (ii).

It remains to prove (iii). For this, we use the fact that, by lattice symmetries, $\widehat{\mathbb{G}}(k)$ is real, and we obtain:

$$\begin{aligned}
&\sum_{x,y\in\mathbb{T}_L}\cos(k\cdot x)\cos(k\cdot y)\mathbb{G}(x,y)\\
&\quad= \sum_{x\in\mathbb{T}_L}\Big(\cos(k\cdot x)\,\mathrm{Re}\Big[\sum_{y\in\mathbb{T}_L}\cos(k\cdot y)\mathbb{G}(y-x)\Big]\Big)\\
&\quad= \sum_{x\in\mathbb{T}_L}\Big(\cos(k\cdot x)\,\mathrm{Re}\Big[e^{ik\cdot x}\sum_{y\in\mathbb{T}_L}e^{ik\cdot(y-x)}\mathbb{G}(y-x)\Big]\Big)\\
&\quad= \sum_{x\in\mathbb{T}_L}\big(\cos(k\cdot x)\,\mathrm{Re}[e^{ik\cdot x}\widehat{\mathbb{G}}_k]\big) = \sum_{x\in\mathbb{T}_L}\cos^2(k\cdot x)\widehat{\mathbb{G}}(k).
\end{aligned}$$

This concludes the proof. $\square$

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