# Existence of EOQ and its Evaluation: Some Cases of Stock Blow Down Dynamics Depending on its Level

**Alessandro Gambini, Giovanni Mingari Scarpello and Daniele Ritelli**

**Abstract** The EOQ mathematical models usually deal with the problem of a wholesaler who has to manage a goods restocking policy, settling his best amount of goods to be procured. *Best* means capable of minimizing all the costs concerning the trade of the stored goods. The relevant seminal contributions are due to Harris, and Wilson, who analyzed an easy scenario with a certain demand uniform all over the time so that its instantaneous change rate is fixed, with stocking charges not dependent on time. In such a field, our own contribution consists of establishing sufficient conditions on the well posedness to the minimum cost problem and relationships providing either closed form solutions or, alternatively, quadrature formulae—without *ex ante* approximations. All this allows a numerical solution to the transcendental (or algebraical of high degree) equation solving to the most economical batch. In short, such our paper is concerning the special family of EOQ mathematical models with different deterministic time-dependent demands.

## 1 Introduction

The EOQ mathematical models usually deal with the problem of a wholesaler who has to manage a goods restocking policy, settling his best amount of goods to be procured. *Best* means capable of minimizing all the costs concerning the trade of

A. Gambini · D. Ritelli (✉)
Dipartimento di Scienze Statistiche, via Belle Arti 41 40126 Bologna, Italy
e-mail: daniele.ritelli@unibo.it

A. Gambini
e-mail: alessandro.gambini4@unibo.it

G. M. Scarpello
Via Negroli 6 20136 Milan, Italy
e-mail: giovannimingari@yahoo.it

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the stored goods. The relevant seminal contributions are due to Harris (1913, 1915) and Wilson (1934), where an easy scenario is analyzed with a certain demand uniform all over the time so that its instantaneous change rate is fixed, with stocking charges not dependent on time. In subsequent years, the subject attracted the authors' continuous efforts to improve the assumptions of the ingenuous early models about the stored goods' demand, the charges due to goods stocking and to their perishability, if any.

The model main features considered by us concern the store blow-down which will depend on the products demand and on perishability, as for food or medicines or vaporizing liquids. The theoretical models presented hereinafter provide single mathematical representations of the blow-down and of charges. For example Benkherouf (1997) and Sarma (1987) considering the case of a perishable good stored in two different warehouses, get blow-down dynamics ruled by two different time laws. In Balki and Benkherouf (1996) and Raafat and Wolfe (1991) time changes of production/demand rates not due to perishability are taken into account, while Bhunia and Maiti (1998) analyzes a frame where the store level decrement is a function of its own level. Anyway the effort of providing a full overview on the main contributions is out of our purpose for being giant the relevant literature. Very often some Journals publish review articles on the subject like Goyal and Giri (2001), Nahamias (1982), Pentico and Drake (2011) and Raafat (1991); alternatively monographs are available as Zipkin (2000). The theoretical treatments reviewed throughout this article are concerning a stocks blowdown dynamics depending on their level itself. For an extended overview see Urban (2005).

In such a field, our own contribution consists of establishing sufficient conditions ensuring the well posedness to the problem of minimum cost and relationships providing either closed form solutions or, alternatively, quadrature formulae—without *ex ante* approximations—allowing a numerical solution to the transcendental (or algebraical of high degree) equation providing the most economical batch. Let us introduce the unified notation used throughout all the paper:

$q(t)$ store level at time $t$
$f(t, q)$ demand level ruled by time $t$ and store level $q$
$\hat{h}(t) > 0$ holding cost, assumed as a positive function of $t$
$\hat{k}(q)$ a factor affecting the holding cost as a increasing function of $q$ such that $\hat{k}(q) \to \infty$ as $q \to \infty$
$A > 0$ costs for delivery

Let the stored goods blow-down according to:

$$\begin{cases} \dot{q}(t) = -f(t, q(t)) \\ q(0) = Q > 0 \end{cases} \tag{1}$$

where the function $f : [0, \infty[ \times [0, \infty[ \to \mathbb{R}$ is assumed positive, so that the solution to (1) fulfills $q(t) \leq Q$ for each $t \geq 0$. We call *reordering time* generated by the batch $Q$ the real positive value $T(Q)$ solution of $q(t) = 0$ where $q(t)$ solves (1). If

$A > 0$ is the delivery cost, $\hat{h}(t) > 0$ models the holding cost at time $t$ as a *continuous* function, so that $\hat{h}(0) > 0$, if $\hat{k}(q)$ denotes a continuous and positive function of $q$ so that $\hat{k}(q) \to \infty$ for $q \to \infty$ and that $\hat{k}(0) = 0$, then the total cost for reordering an amount $Q > 0$ of goods is:

$$C(Q) = \frac{A}{T(Q)} + \frac{1}{T(Q)} \int_0^{T(Q)} \hat{h}(t)\,\hat{k}(q(t))\,dt. \tag{2}$$

The Wilson originary treatment, Wilson (1934) follows putting $f(t,q) = \delta > 0$, $\hat{h}(t) = h$, $\hat{k}(q) = q$. Several literature models: Ferguson et al. (2007), Giri and Chaudhuri (1998), Goh (1994) and Weiss (1982) are all particular cases of what above, being there $f(t,q) = aq + bq^{\beta}$, $\hat{h}(t) = ht^{\alpha}$. For some models, the function $f(t,q)$ is piecewise defined, e.g., Balki and Benkherouf (1996), Chang et al. (2006), Dye and Ouyang (2005), Hou (2006) and Roy (2007). In Bernardi et al. (2009) is treated the case $f(t,q) = \delta(t)$ where $\delta(t)$ is a given positive and continuous function of time, $\hat{h}(t) = h$, $\hat{k}(q) = q$.

The statement of the problem is quite clear: find $Q^*$ such that

$$C(Q^*) = \inf_{Q>0} C(Q) \tag{3}$$

The general problem (3) can be solved explicitly when:

(a) one succeeds in solving the differential equation (1) finding $q(t)$
(b) one succeeds in solving explicitly the equation $q(T) = 0$
(c) one succeeds in solving explicitly the critical point equation

$$C'(Q) = \frac{T'(Q)}{T^2(Q)} \left\{ T(Q)\hat{h}(T(Q))\hat{k}(q(T(Q))) - A - \int_0^{T(Q)} \hat{h}(t)\hat{k}(q(t))\,dt \right\} \tag{4}$$

In the Wilson model, the blow-down law will be: $q(t) = Q - \delta\, t$ and $T(Q) = Q/\delta$ and the cost

$$C(Q) = \frac{A}{T(Q)} + \frac{h}{T(Q)} \int_0^{T(Q)} q(t)dt = \frac{\delta A}{Q} + \frac{h}{2}\, Q.$$

In more elaborate models one shall solve, either exactly or numerically, the Eq. (4), but a previous knowledge is needed whether problem (3) is well posed—existence of solution—or not; so that a numerical treatment for solving Eq. (4) has a meaning. When possible, some uniqueness conditions for the solution will be provided. Let us notice that in Bernardi et al. (2009), an example is provided of a not-unique solution taking

$$f(t,q) = t^2 - \frac{9}{2}t + \frac{13}{2}, \quad \hat{h} = 1, A = 1, \hat{k}(q) = q.$$

We will provide existence-uniqueness conditions following different demand good dynamics. We will follow Bernardi et al. (2009), Gambini et al. (2013), Mingari Scarpello and Ritelli (2008 and 2010), as far as it concerns the store costs given by functions $\hat{h}(t)$ e $\hat{k}(q)$. For each theoretical case, we will provide applications leading—even if not always—either to closed form solutions by means of Special Functions (e.g., Gauss hypergeometric function, Lambert $W$ function) or to quadrature formulae allowing a direct settlement of the best batch $Q^*$. Furthermore, the problem of backordering will be embodied: it has been recently tackled by several authors, but always under a constant rate of store level change, (Cárdenas-Barrón et al. 2010; Cárdenas-Barrón 2010a, b; Teng 2009). They try to detect the optimal batch backordering levels without calculus, but founding upon classic inequalities such that they are between the arithmetic and geometric means powered by the methods in Garver (1935) and Niven (1981). Anyway in our very general frame where the stock inventory level is ruled by a nonlinear dynamics, the classic approach through the infinitesimal calculus is compulsory.

## 2 Demand Depending on the Stock Level Only

Let us start with (1) when $f(t,q) = f(q)$ and let the stored goods blow-down behaves according to law:

$$\begin{cases} \dot{q}(t) = -f(q(t)) \\ q(0) = Q > 0 \end{cases} \tag{5}$$

The autonomous structure of (5) allows a closed form solution: defining

$$F(q) := \int_q^Q \frac{1}{f(u)}\, du = t \tag{6}$$

then, inverting $F(q)$ we find that $q(t) = F^{-1}(t)$ solves (5).

The *reordering time* generated by the batch $Q$ is the positive value $T(Q)$ solution of $q(t) = 0$:

$$T(Q) = F(0) = \int_0^Q \frac{1}{f(u)}\, du.$$

The total cost for reordering an amount $Q > 0$ of goods is here:

$$C(Q) = \frac{A}{T(Q)} + \frac{1}{T(Q)} \int_0^{T(Q)} \hat{h}(t)\, \hat{k}(q(t))\, dt. \tag{7}$$

The Wilson early treatment, Wilson (1934) follows putting $f(u) = \delta$, $\hat{h}(t) = h$, *and* $\hat{k}(q) = q$. Notice that the several literature models: Giri and Chaudhuri (1998), Goh (1994) and Weiss (1982) are nothing else but particular cases of what above, being there $f(u) = au + bh^{\beta}$, $\hat{k}(q) = q$, *and* $\hat{h}(t) = ht^{\alpha}$. In Giri et al. (1996) $f(q)$ is defined as $f(q) = -\theta q - \alpha q^{\beta}$ for $0 \le t \le t_1$ and $f(q) = -\theta q - D$ for $\le t_1 \le t \le T$. In Mingari Scarpello and Ritelli (2008) is treated as the case for arbitrary $f(u)$.

**Theorem 2.1** *Suppose that function* $f$ *in* (1) *is such that*

$$\lim_{v\to\infty} \int_0^v \frac{du}{f(u)} = \infty \tag{8}$$

*Moreover, we assume that if* $f(0) = 0$, *the integrability in* $u = 0$ *of both functions*:

$$\frac{1}{f(u)}, \quad \frac{\hat{k}(u)}{f(u)}. \tag{9}$$

*Then the cost function of* (7) *attains its absolute minimum at* $Q^* > 0$, *which is unique*.

*Proof* In the integral at the right hand side of (7), we do the change $t = F(u)$. Minding that $t = 0 \Rightarrow u = Q$, $t = T(Q) \Rightarrow u = 0$, and that $dt = -(1/f(u))du$, and $q(t) = F^{-1}(t)$, we get:

$$\begin{aligned} C(Q) &= \frac{A}{T(Q)} + \frac{1}{T(Q)} \int_0^Q \hat{h}(F(u))\, \hat{k}(F^{-1}(F(u)))\, \frac{du}{f(u)} \\ &= \frac{A}{T(Q)} + \frac{1}{T(Q)} \int_0^Q \hat{h}(F(u)) \frac{\hat{k}(u)}{f(u)} du \end{aligned} \tag{10}$$

The good position of (10) follows from (9). The structure of (10) implies that $Q \mapsto C(Q)$, $Q > 0$ has exactly one minimizer. First we observe that:

$$\lim_{Q\to 0^+} C(Q) = \infty.$$

Then from (8) we see that the cost function (10) diverges when $Q \to \infty$, as immediately checked through De l'Hospital rule:

$$\lim_{Q\to\infty} C(Q) = \lim_{Q\to\infty} \frac{\dfrac{\hat{h}(F(Q))\, \hat{k}(Q)}{f(Q)}}{\dfrac{1}{f(Q)}} = \lim_{Q\to\infty} \hat{h}(0)\, \hat{k}(Q) = \infty$$

Thus $C(Q)$ is bounded from below: so it has at least one stationary value. The extremum will be attained at only one value since the first derivative of $C(Q)$ vanishes if and only if the batch $Q$ solves the equation:

$$\hat{h}(0)\,\hat{k}(Q)\int_0^Q \frac{du}{f(u)} - \left\{A + \int_0^Q \hat{h}(F(u))\frac{\hat{k}(u)}{f(u)}\,du\right\} = 0. \tag{11}$$

But the function

$$\mathcal{N}(Q) := \hat{h}(0)\,\hat{k}(Q)\int_0^Q \frac{du}{f(u)} - \left\{A + \int_0^Q \hat{h}(F(u))\frac{\hat{k}(u)}{f(u)}\,du\right\}$$

is the difference of two increasing functions; thus this minimizing batch is unique.

Through a similar way it is possible to prove that thesis of Theorem 2.1 holds with slightly different assumptions on $f$.

**Corollary 2.2** *The same conclusion of Theorem 2.1 holds if*:

$$\int_0^\infty \frac{du}{f(u)} \in \mathbb{R}, \qquad \int_0^\infty \hat{h}(F(u))\frac{\hat{k}(u)}{f(u)}\,du = \infty$$

*and*

$$\int_0^\infty \frac{du}{f(u)} \in \mathbb{R}, \qquad \int_0^\infty \hat{h}(F(u))\frac{\hat{k}(u)}{f(u)}\,du \in \mathbb{R}$$

## 2.1 Applications to Known Models

For the model (Goh 1994), where $f(q) = \delta q^\beta$, $\hat{h}(t) = h,$ *and* $\hat{k}(q) = q$ the optimum condition (11) gives:

$$hQ^{2-\beta} - A\left(\beta^2 - 3\beta + 2\right)\delta = 0.$$

Finally, in the model (Giri and Chaudhuri 1998), being there $f(q) = \theta q + \delta q^\beta, \hat{h}(t) = h,$ *and* $\hat{k}(q) = q$, the detection of the optimum batch, first order condition (4) leads at the $Q$-equation:

$$\frac{hQ}{(1-\beta)\vartheta}\ln\left(1 + \frac{\vartheta}{\delta}Q^{1-\beta}\right) - A - \frac{hQ}{\vartheta}\left(1 - {}_2F_1\left(1, \frac{1}{1-\beta}; \frac{2-\beta}{1-\beta}; -\frac{\vartheta}{\delta}Q^{1-\beta}\right)\right) = 0.$$

Anyway the above formula involving the Gauss hypergeometric function ${}_2F_1$ is not present in the article (Giri and Chaudhuri 1998); it is found upon the integral identities:

$$\int_0^Q \frac{du}{\vartheta\, u + \delta\, u^\beta} = \frac{1}{\vartheta(1-\beta)} \ln\left(1 + \frac{\vartheta}{\beta} Q^{1-\beta}\right),$$

$$\int_0^Q \frac{u\, du}{\vartheta\, u + \delta\, u^\beta} = \frac{1}{\vartheta}\left[Q - Q\, {}_2F_1\left(1, \frac{1}{1-\beta}; \frac{2-\beta}{1-\beta}; -\frac{\vartheta}{\delta} Q^{1-\beta}\right)\right].$$

We limit here to recall that ${}_2F_1$ is the Gauss hypergeometric function defined as a $|x| < 1$ power series:

$${}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!},$$

where $(a)_k$ is a Pochhammer symbol: $(a)_k = a(a+1)\cdots(a+k-1)$. ${}_2F_1$ is analytically continued by the integral representation theorem:

$${}_2F_1(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} \int_0^1 \frac{t^{a-1}(1-t)^{c-a-1}}{(1-xt)^b}\, dt,$$

whose validity ranges are: $\mathrm{Re}\, c > \mathrm{Re}\, a > 0$, $|x| < 1$. It provides the way for extending the region where the (complex) hypergeometric function is defined, namely for its analytical continuation to the (almost) whole complex plane excluding the half-line $]1, \infty[$.

Let us now introduce some $f(q)$ not considered up to this time. Notice that $f(q)$ could be known as experimental data set to be fitted in some reliable analytical expression: this explains the theoretical laws we are going to study.

## 2.2 More Applications

### 2.2.1 Affine Demand

The demand function which provides the most immediate generalization to the old one (Wilson and Harris), consists of modeling the inventory blow-down through an affine function of the stock level $q$, namely $f(q) = \delta + \varepsilon\, q$ with $\delta$, $\varepsilon > 0$. The optimum condition (11) in such a case will lead to the transcendental $Q$-equation

$$\frac{h}{\varepsilon^2}\left[(\delta + \varepsilon Q) \ln\left(\frac{\delta + \varepsilon Q}{\delta}\right) - \varepsilon Q\right] - A = 0. \tag{12}$$

Equation (12) was treated by Warburton, see Warburton (2009).

### 2.2.2 Rational Demand

By rational demand functions, we find algebraic first order conditions, in fact, if the inventory blow-down is rational,

$$f(q) = \frac{a}{b+q},$$

$a, b > 0$, then the optimum condition leads (11) to a cubic $Q$-equation:

$$hQ^3 + 3bhQ^2 - 6aA = 0.$$

If:

$$f(q) = \frac{a}{b^2+q^2}$$

from (11) we get:

$$hQ^4 + 6b^2hQ^2 - 12aA = 0.$$

### 2.2.3 Quadratic Demand

Let the instantaneous inventory stock level be ruled by (1) with $0 < f(u) = (u-a)(u-b)$, $a, b < 0$. In such a way the optimum condition (11) specializes in:

$$\frac{hQ}{a-b}\ln\left[\frac{b(Q-a)}{a(Q-b)}\right] - \frac{h}{a-b}\left[a\ln\left(\frac{a-Q}{a}\right) - b\ln\left(\frac{b-Q}{b}\right)\right] - A = 0, \qquad (13)$$

being (13) to be solved to $Q$, the only possible approach is numerical. For example, the left hand side of (13) as a function of $Q$ is plotted below, showing the unique optimal solution $Q \simeq 11.6987$.

### 2.2.4 Exponential

The inventory manager is faced with aperiodic demand which either is always increasing or decreasing: for instance $f(q) = a\,e^{q}$, or, $f(q) = a\,e^{-q}$. Even if the integrals in (11) are all elementary for the exponential situation, the relevant $Q$-equations:

$$hQe^{Q} - (aA+h)e^{Q} + h = 0, \qquad (14)$$

$$-hQe^{-Q} - (aA+h)e^{-Q} + h = 0, \qquad (15)$$

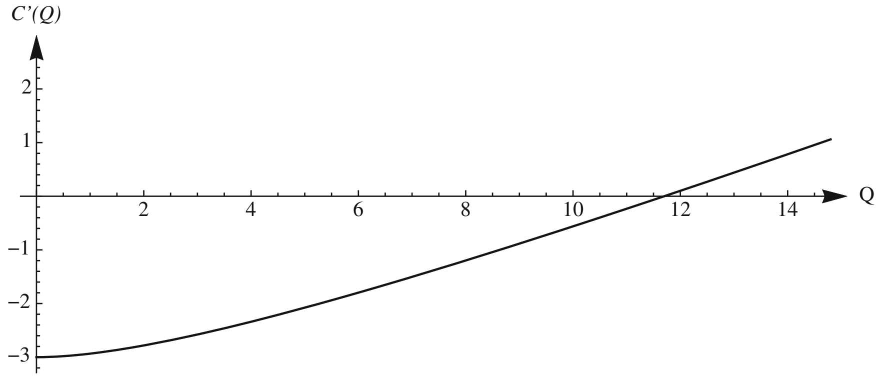


**Fig. 1** The solution to quadratic demand with $a = -3$, $b = -2$, $h = 1$, $A = 3$

are transcendental yet. Nevertheless they can be solved through a special function, not being compulsory a numerical solution any more.

The Lambert function(s) $W(y)$ can be achieved starting from $\ell(x) = xe^x$ and after taking its inverse. Of course $\ell(x) = xe^x$ is a not monotonic function of $x \in \mathbb{R}$, and then its inverse is multivalued. So that, we do not have *one* Lambert $W$-function but *two* Lambert functions on the real line, both coming from the relationship $W(y)e^{W(y)} = y$ where the discriminating point, in order to decide the branch, is $x = -e^{-1}$, $W = -1$. Some special values are $W(0) = 0$; $W(-1) = -e^{-1}$; $W(e) = 1$; and $W(1) = \Omega = 0.67143\ldots$. In such a way, looking at the Fig. 2, four behaviors are possible:

- if $y \geq 0$, we move on the *principal* branch, say $W_0(y)$, or simply $W(y)$, when no ambiguity can occur;
- if $-e^{-1} \leq y \leq 0$, we move on the principal branch again if $W(u) \geq -1$;
- if $-e^{-1} \leq y \leq 0$, but $W(y) < -1$, we move on the *secondary* branch, say $W_{-1}(y)$;
- if $y < -e^{-1}$, we do not have at all real values of $W$ any more.

Anyway, there is no possibility of expressing $W(y)$ in terms of elementary functions. A method for computing $W(y)$ for each $y$ could be: to develop $\ell(x) = xe^x$ in a power series, what we know has a sum equal to $y$; and to revert such a series by the Lagrange inversion theorem. In such a way, one obtains $W$ expanded in ascending powers of $y$:

$$W(y) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} y^n$$

whose convergence radius is $e^{-1}$. As far as we are concerned, the first appearance of $W$ function in an economics context was in Mingari Scarpello and Ritelli

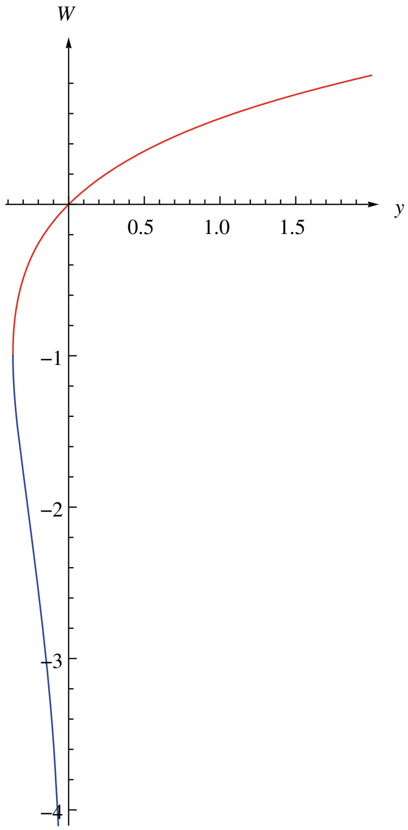


**Fig. 2** The couple of real branches of the Lambert $W$-function: $W_0(y)$ and $W_{-1}(y)$

(2007), a paper where we generalized a Goodwin microeconomic model, while the first use of this function in the EOQ contest is due to Warburton (2009). An almost exhaustive survey on Lambert functions can be read at Corless et al. (1996).

After this short synopsis, let us go back to our Eqs. (14) and (15) and solve them by means of $W$. Let us begin writing (14) as: $e^Q(Q-b) = -1$ with the obvious meaning of $b > 1$. We change variable putting $Q - b = R$ obtaining:

$$R\,e^R = -e^{-b}. \tag{16}$$

It is worth noting that (16) is well posed, i.e., has two real roots because $b > 1$: they are $W_{-1}(-e^{-b})$ and $W_0(-e^{-b})$. Only $W_0(-e^{-b})$ has economic meaning, in fact recalling that, $-1 < W_0(-e^{-b}) < 0$ and going back to the original $Q$ we find:

$$Q_+^* = b + W_0(-e^{-b}),$$

where the index $+$ reminds we started from a positive exponential. In order to solve (15), we observe that it can be written as:

$$e^{-Q-b}(-Q-b) = -e^{-b}, \tag{17}$$

where the meaning of $b$ does not change. The solution, of economic interest, i.e., positive, is then:

$$Q_-^* = -b - W_{-1}(-e^{-b}),$$

since $-\infty < W_{-1}(-e^{-b}) < -1$.

The $W$ Lambert function is available by several computer algebra packages like Mathematica®, for automatic computing. In addition, the $f(q)$ exponential nature is not an analytical oddness, but has a deep market meaning.

## 3 Backordering

In order to take the backordering into account, we present our recent contribution (Gambini et al. 2013). Assume $f(t,q) = f(q)$, $\hat{h} = const.$ and $\hat{k}(q) = q$. The quantity $Q$ ordered at each cycle undergoes two different uses: a first share $Q - R$ covers the demand of the previous cycle, and then does not enter the inventory; while $R$ is the residual share which enters the store so that the outstanding amount is again $Q - R$, and so on. As a consequence, the reordering time becomes:

$$T(Q) = F(R-Q) = \int_{R-Q}^{R} \frac{1}{f(u)}\, du,$$

where the function $f : [R-Q, \infty[ \to \mathbb{R}$ is assumed positive, and the total cost is:

$$C(R,Q) = \frac{A}{T(Q)} + \frac{h}{T(Q)} \int_0^{T(R)} q(t)\, dt - \frac{b}{T(Q)} \int_{T(R)}^{T(Q)} q(t)\, dt. \tag{18}$$

It is possible to get easier (18), by the following Lemma.

**Lemma 3.1** *Let $f(q)$ be the law describing the q-blowdown dynamics: then the total cost is given by*:

$$C(R,Q) = \frac{A + h\displaystyle\int_0^R \frac{u}{f(u)}du - b\int_{R-Q}^0 \frac{u}{f(u)}du}{\displaystyle\int_{R-Q}^R \frac{du}{f(u)}}, \tag{19}$$

*where, if $f(0) = 0$ we assume the integrability of both functions:*

$$\frac{1}{f(u)}, \quad \frac{u}{f(u)}.$$

*Proof* Putting in (18) $t = F(u)$, notice that $t = 0 \Rightarrow u = R$, $t = T(R) \Rightarrow u = 0$, $t = T(Q) \Rightarrow u = R - Q$, and $dt = -(1/f(u))du$, so that, minding that $q(t) = F^{-1}(t)$ one finds:

$$C(R,Q) = \frac{A}{T(Q)} + \frac{h}{T(Q)} \int_{F(R)}^{F(0)} F^{-1}(t)\,dt - \frac{b}{T(Q)} \int_{F(0)}^{F(R-Q)} F^{-1}(t)\,dt$$

$$= \frac{A}{T(Q)} + \frac{h}{T(Q)} \int_{0}^{R} F(q)\,dt - \frac{b}{T(Q)} \left( (R-Q)F(R-Q) + \int_{R-Q}^{0} F(q)\,dt \right)$$

writing $F$ in terms of $f$ we find out:

$$C(R,Q) = \frac{1}{T(Q)} \left( A + h \int_{0}^{R} \left( \int_{q}^{R} \frac{du}{f(u)} \right) dq \right.$$
$$\left. - b \left( (R-Q) \int_{R-Q}^{R} \frac{du}{f(u)} + \int_{R-Q}^{0} \left( \int_{q}^{R} \frac{du}{f(u)} \right) dq \right) \right)$$

exchanging the integrations order and computing the inner one

$$C(R,Q) = \frac{1}{T(Q)} \left[ A + h \int_{0}^{R} \frac{u}{f(u)} du \right.$$
$$\left. - b \left( (R-Q) \int_{R-Q}^{R} \frac{du}{f(u)} + \int_{R-Q}^{0} \frac{u-(R-Q)}{f(u)} du + \int_{0}^{R} \frac{Q-R}{f(u)} du \right) \right]$$
$$= \frac{A + h \int_0^R \frac{u}{f(u)} du + b \left( (Q-R) \int_{R-Q}^R \frac{du}{f(u)} - \int_{R-Q}^0 \frac{u-(R-Q)}{f(u)} du - \int_0^R \frac{Q-R}{f(u)} du \right)}{\int_{R-Q}^R \frac{du}{f(u)}}$$

A numerator straightforward reduction completes the proof.

**Theorem 3.1** The cost function introduced in (19) attains its absolute minimum at proper positive values $(Q^*, R^*)$. Such a minimizing batch is unique.

*Proof* Recall that

$$\lim_{(R,Q) \to (0,0)} C(R,Q) = \infty$$

and that if $R \to Q$, we go back to the originary model, furthermore, by De l'Hospital rule one finds that:

$$\lim_{Q\to+\infty} C(R,Q) = +\infty$$

Let us change variables passing from $C(R,Q)$ to $C(R,Q-R)$: accordingly, the total cost $C(R,Q-R)$ is:

$$\frac{A + h\int_0^R \frac{u}{f(u)}du - b\int_{R-Q}^0 \frac{u}{f(u)}du}{\int_{R-Q}^R \frac{du}{f(u)}} \tag{20}$$

Partial derivatives of the total cost with respect to $R$ and $Q$ provide:

$$\frac{\partial C}{\partial Q} = \frac{-A + b\,(Q-R)\,\int_{R-Q}^R \frac{1}{f(u)}\,du - h\,\int_0^R \frac{u}{f(u)}\,du + b\,\int_{R-Q}^0 \frac{u}{f(u)}\,du}{f(R-Q)\left(\int_{R-Q}^R \frac{1}{f(u)}\,du\right)^2}$$

$$\frac{\partial C}{\partial R} = \frac{-A + h R\,\int_{R-Q}^R \frac{1}{f(u)}\,du - h\,\int_0^R \frac{u}{f(u)}\,du + b\,\int_{R-Q}^0 \frac{u}{f(u)}\,du}{f(R)\left(\int_{R-Q}^R \frac{1}{f(u)}\,du\right)^2} - \frac{\partial C}{\partial Q}$$

Imposing partial derivatives to vanish:

$$\frac{\partial C}{\partial Q} = \frac{-A + b\,(Q-R)\,\int_{R-Q}^R \frac{1}{f(u)}\,du - h\,\int_0^R \frac{u}{f(u)}\,du + b\,\int_{R-Q}^0 \frac{u}{f(u)}\,du}{f(R-Q)\left(\int_{R-Q}^R \frac{1}{f(u)}\,du\right)^2} = 0$$

$$\frac{\partial C}{\partial R} = \frac{-A + h R\,\int_{R-Q}^R \frac{1}{f(u)}\,du - h\,\int_0^R \frac{u}{f(u)}\,du + b\,\int_{R-Q}^0 \frac{u}{f(u)}\,du}{f(R)\left(\int_{R-Q}^R \frac{1}{f(u)}\,du\right)^2} = 0$$

We assume $f > 0$ for each $u$, then both the denominators are strictly positive; setting the numerators to be zero, first order conditions will provide the critical point system:

$$g(R,Q-R) = -A + h\,R \int_{R-Q}^{R} \frac{1}{f(u)}\,du - h\int_{0}^{R} \frac{u}{f(u)}\,du + b\int_{R-Q}^{0} \frac{u}{f(u)}\,du = 0$$

$$m(R,Q-R) = -A + b\,(Q-R) \int_{R-Q}^{R} \frac{1}{f(u)}\,du - h\int_{0}^{R} \frac{u}{f(u)}\,du + b\int_{R-Q}^{0} \frac{u}{f(u)}\,du = 0. \tag{21}$$

To solve (21), subtracting side by side, one finds:

$$Q = \frac{hR}{b} + R$$

$$m\left(R, \frac{hR}{b}\right) = -A + hR \int_{-\frac{hR}{b}}^{R} \frac{1}{f(u)}\,du - h\int_{0}^{R} \frac{u}{f(u)}\,du + b\int_{-\frac{hR}{b}}^{0} \frac{u}{f(u)}\,du = 0$$

$m(R, hR/b)$ is an increasing function being:

$$\frac{d}{dR} m(R, hR/b) = h \int_{-\frac{hR}{b}}^{R} \frac{1}{f(u)}\,du > 0,$$

so observing that $m(0,0) < 0$, then $m(R, hR/b)$ has a unique real root, and we have one and only one critical point for the cost function (19). Let us show it is a minimum. The Hessian determinant at the critical point is:

$$H = \frac{bh}{f(R)f\left(-\frac{hR}{b}\right)\left(\int_{-\frac{hR}{b}}^{R} \frac{1}{f(u)}\,du\right)^2} \tag{22}$$

In fact being:

$$H = \frac{\partial^2}{\partial R^2} C(R,Q) \frac{\partial^2}{\partial Q^2} C(R,Q) - \left(\frac{\partial^2}{\partial R \partial Q} C(R,Q)\right)^2$$

minding that $g(R, Q-R) = m(R, Q-R) = 0$, we have

$$\frac{\partial^2 C}{\partial Q^2} = \frac{\frac{\partial m}{\partial Q}}{Z^2 f(R-Q)}, \quad \frac{\partial^2 C}{\partial R^2} = \frac{\frac{\partial g}{\partial R}}{Z^2 f(R)} - \frac{\frac{\partial m}{\partial R}}{Z^2 f(R-Q)}, \quad \frac{\partial^2 C}{\partial Q \partial R} = \frac{\frac{\partial m}{\partial R}}{Z^2 f(R-Q)}$$

where we put:

$$Z = \int_{R-Q}^{R} \frac{1}{f(u)}\,du.$$

Eventually, recalling that

$$Q = \frac{hR}{b} + R, \quad \frac{\partial m}{\partial R} = -bZ, \quad \frac{\partial m}{\partial Q} = bZ, \quad \frac{\partial g}{\partial R} = hZ$$

we find (22) proving the stationary point to be a minimum.

## 4 Sample Problems

We provide now some applications of above to known models of the literature extended to backorders, getting in any case a transcendental (or algebraic) $R$-resolvent equation. The following conditions are assumed to be true in any case:

$$h > 0,\ Q > 0,\ A > 0\ R > 0,\ 0 < p < 1,\ b > 0,\ \delta > 0$$

### *Wilson model*

$$f(u) = \delta \Rightarrow C(R,Q) = \frac{\delta}{Q}\left(A + \frac{b\,(Q-R)^2}{2\,\delta} + \frac{h\,R^2}{2\,\delta}\right)$$

Such a case has a theoretical interest due to its final (not transcendental and) exactly solvable resolvent: the minimizing batch is found to be:

$$Q^* = \sqrt{\frac{2A\delta(b+h)}{bh}}; \qquad R^* = \sqrt{\frac{2Ab\delta}{h(b+h)}}$$

In such a way, the minimized cost will be:

$$C^* = \sqrt{\frac{2Ab\delta h}{b+h}}$$

### *Goh's model,* $p = 1/2$

$$f(u) = \sqrt{|u|} \Rightarrow C(R,Q) = \frac{3\,A + 2\,h\,R^{\frac{3}{2}} + 2\,b\,(Q-R)^{\frac{3}{2}}}{6\left(\sqrt{R} + \sqrt{Q-R}\right)}$$

By $Q - R = \frac{hR}{b}$ and $g(R, Q - R) = 0$ we get:

$$-3\,A + 4\,h\left(1 + \sqrt{\frac{h}{b}}\right)R^{\frac{3}{2}} = 0$$

### *Goh's model,* p *general*

If $f(u) = |u|^p$ then

$$C(R,Q) = \frac{(1-p)\left(b\,R^p\,(Q-R)^2 + h\,R^2\,(Q-R)^p + A\,(2-p)\,(R\,(Q-R))^p\right)}{(2-p)\,(R^p(Q-R) + R(Q-R)^p)}$$

By $Q - R = \frac{hR}{b}$ and $g(R, Q - R) = 0$ we get:

$$h^{1-p}R^{2-p}(hb^p + bh^p) - Ab(p-2)(p-1) = 0$$

solving to $R^{2-p}$ we get:

$$R^{2-p} = A\,\frac{(p-1)(p-2)}{h\left(1 + (bh)^{p-1}\right)}$$

#### Exponentials

$$f(u) = e^{-u} \Rightarrow C(R,Q) = \frac{e^{Q-R}\,\{A + h\,[1 + e^R\,(R-1)]\} + b\,(e^{Q-R} - Q + R - 1)}{e^Q - 1}$$

If $Q - R = \frac{hR}{b}$ e $g(R, Q - R) = 0$, we get the transcendental $R$-equation:

$$b - e^{\frac{hR}{b}}\left(A + b + h - e^R\,h\right) = 0$$

It is then provided a simulation for $A = 1,\ b = 1/3,\ and\, h = 1/4$. Figure 3 shows the iso-cost curves, highlighting the minimizer $(Q^*, R^*)$ numerically detected.

Figure 4 shows where is the intersection of lines obtained putting to zero the single first partial derivatives.

And finally in Fig. 5, a 3-D plot of the global cost function. We have a similar behavior for $f(u) = e^u$.

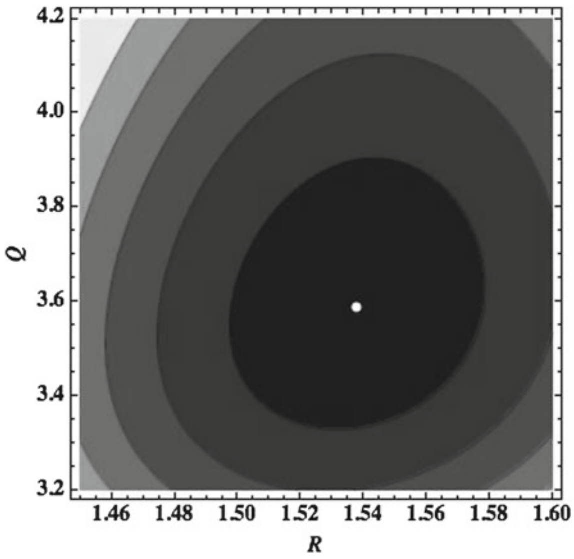


**Fig. 3** Level curves relevant to the cost function

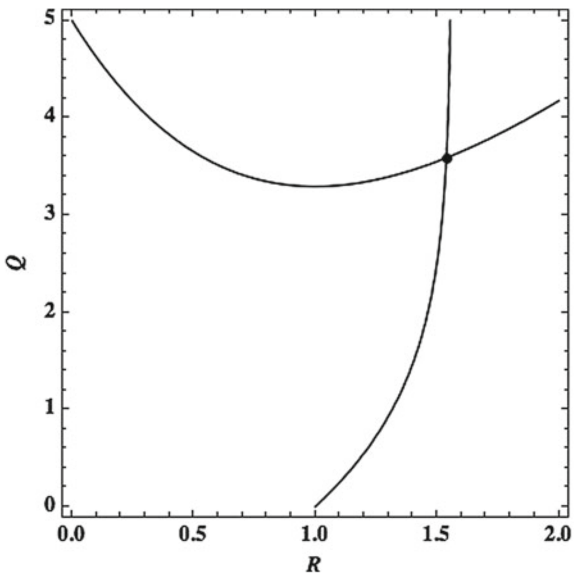


**Fig. 4** Crossing of the loci of roots of the single partial derivatives

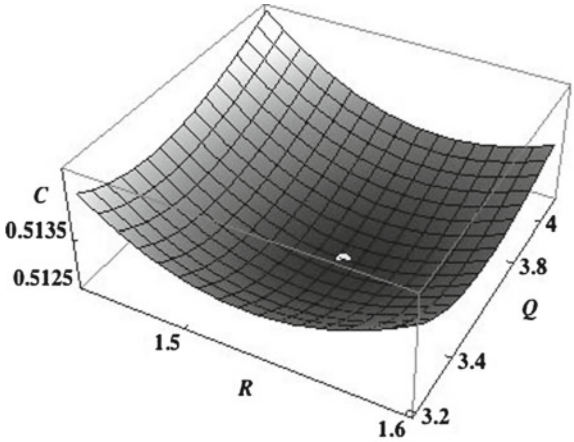


**Fig. 5** a 3-D plot of $C(Q, R)$ for $f(u) = e^{-u}$

## *Rational (First)*

$$f(u) = \frac{k}{n+u} \Rightarrow C(R, Q) = \frac{6\,A\,k + h\,R^2\,(3\,n + 2\,R) + b\,(3\,n - 2\,Q + 2R)\,(Q - R)^2}{3Q\,(2\,n + 2R - Q)}$$

By $Q - R = \frac{hR}{b}$ and $g(R, Q - R) = 0$ one obtains:

$$-6\,A\,b^2\,k + h\,(b + h)\,R^2\,[b\,(3\,n + R)h\,R] = 0$$

which is providing promptly the batch.

### Rational (Second)

$$f(u) = \frac{k}{n+u^2} \Rightarrow C(R, Q) = \frac{3\left(4\,A\,k + h\,(2\,n\,R^2 + R^4) + b\,(Q - R)^2\left(2\,n + (Q - R)^2\right)\right)}{4\left(R^3 + (Q - R)^3 + 3\,n\,Q\right)}$$

By $Q - R = \frac{hR}{b}$ and $g(R, Q - R) = 0$ one finds:

$$-12\,A\,k + \frac{6\,h\,(b + h)\,n\,R^2}{b} + \frac{h\,(b^3 + h^3)\,R^4}{b^3} = 0$$

biquadratic equation.

## 5 Conclusions

We proved two existence-uniqueness Theorems 2.1 and 3.1 about a minimum cost batch for a class of EOQ models with perishable inventory and nonlinear cost and with sole backordering, leading to a set of sufficient conditions which require to check the convergence of some improper integrals, and form the article's main theoretical effort. As application, several cases have been treated of demand $f(q)$ as a continuous function of the stock level $q$. Being one of the sufficient conditions met in any case, the economic order quantity is unique, and the relevant computations lead to transcendental equations. In some cases, the plot of the global cost function is provided, and, even if the optimality condition can be written in closed (but transcendental) form, its solution shall mostly be faced numerically. Mind that the reordering time, the global cost function and the minimum cost (optimum) condition are here detected without any previous approximation, being a numerical treatment required—if any—only at the end, in order to solve the (often) transcendental equation for the economic batch.

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## References

Axsäter, S.: Inventory Control. Kluwer, Boston (2000).

Balki, Z. T., & Benkherouf, L.: A production lot size inventory for deteriorating items and arbitrary production and demand rates. Eur. J. Oper. Res. **92**, 302–309 (1996)

Benkherouf, L.: A deterministic model for deteriorating items with two storage facilities. International journal of production economics. **48**, 167–175 (1997)

Bernardi, E., Mingari Scarpello, G., Ritelli, D.: EOQ under exogenous and periodic demand. Advanced Modeling and. Optimization. **11**, 279–287 (2009)

Bhunia, A. K., & Maiti, M.: Determinsitic inventory model for deteriorating items with finite rate of replenishment dependent on inventory level. Computers Ops Res. **25**, 997–1006 (1998)

Cáárdenas-Barrón, L. E.: An easy method to derive EOQ and EPQ inventory models with backorders. Comp. Math. Appl. **59**, 948–952 (2010a)

Cárdenas-Barrón, L. E.: A simple method to compute economic order quantities: Some observations. Appl. Math. Model. **34**, 1684–1688 (2010b)

Cárdenas-Barrón, L. E., Smith, R. N., & Goyal, S. K.: Optimal order size to take advantage of a one-time discount offer with allowed backorders. Appl. Math. Model. **34**, 1642–1652 (2010)

Chang, C. Y., Goyal, S. K., & Teng, J. T.: On "An EOQ model for perishable items under stock-dependent selling rate and time-dependent partial backlogging" by Dye and Ouyang. Eur. J. Oper. Res. **174**, 923–929 (2006)

Corless, R. M., Goonnet, G. H., Jeffrey, D. J., & Knutt, D. E.: On the Lambert W Function. Advances in Computational Mathematics. **4**, 329–359 (1996)

Dye, C. Y., & Ouyang, L. Y.: An EOQ model for perishable items under stock-dependent selling rate and time-dependent partial backlogging. Eur. J. Oper. Res. **163**, 776–783 (2005)

Ferguson, M., Jayaraman, V., & Souza, G. C.: Note: An application of the EOQ model with nonlinear holding cost to inventory managment of perishables. European Journal of Operational Resesarch. **180**, 485–490 (2007)

Gambini, A., Mingari Scarpello, G., Ritelli, D.: Mathematical properties of EOQ models with special cost structure. App. Math. Model. **37**, 656–666 (2013)

Garver, R.: The solution of problems in maxima and minima by algebra. Amer. Math. Monthly. **42**, 435–437 (1935)

Giri, B. C., & Chaudhuri, K. S.: Deterministic models of perishable inventory with stock-dependent demand rate and nonlinear holding cost. Eur. J. Oper. Res. **105**, 467–474 (1998)

Giri, B. C., Pal, S., Goswami, A., & Chaudhuri, K. S.: An inventory model for deteriorating items with stock-dependent demand rate. Eur. J. Oper. Res. **95**, 604–610 (1996)

Goh, M.: EOQ models with general demand and holding cost function. Eur. J. Oper. Res. **73** 50–54 (1994)

Goyal, S. K., & Giri, B. C.: Recent trends in modeling of deteriorating inventory. Eur. J. Oper. Res. **134**, 1–16 (2001)

Harris, F. W.: How Many Parts To Make At Once Factory. The Magazine of Management. **10**, 135–136 (1913)

Harris, F. W.: Operations and Cost Factory Management Series. A.W. Shaw Co, Chicago (1915)

Hou, K. L.: An inventory model for deteriorating items with stock-dependent consumption rate and shortages under ination and time discounting. Eur. J. Oper. Res. **168**, 463–474 (2006)

Mingari Scarpello, G., Ritelli, D: EOQ problem well-posedness: an alternative approach establishing sufficient conditions. Appl. Math. Scie. **4**, 1213–1219 (2010)

Mingari Scarpello, G., Ritelli, D: EOQ when holding costs grow with the stock level: well-posedness and, solutions. Advanced Modeling and. Optimization. **10**, 233–239 (2008)

Mingari Scarpello, G., Ritelli, D.: The Goodwin cycle improved with generalized wages: phase portrait, periodic behaviour. Journal of Interdisciplinary Mathematics. **10**, 71–87 (2007).

Nahamias, S.: Perishable inventory theory: A review. Oper. Res. **30**, 680–708 (1982)

Niven, I.: Maxima and minima without calculus. The Dolciani Mathematical Expositions No. 6. The Mathematical Association of, America (1981).

Pentico, D. W., & Drake, M. J.: A survey of deterministic models for the EOQ and EPQ with partial backordering. Eur. J. Oper. Res. **214**, 179–198 (2011)

Raafat, P. M., & Wolfe, H. K. Eddin, An inventory model for deteriorating items, Computers and Engineering **20**, 89–94 (1991)

Raafat, F.: Survey of literature on continuosly deteriorating inventory models. Journal of the Operational Research Society. **42**, 27–37 (1991)

Roy, T., & Chaudhury, K. S.: A finite time horizon deterministic EOQ model with stock dependent demand, effects of inflation amd time value of money with shortages in all cycles. Yugoslav Journal of Operations Research. **17**, 195–207 (2007)

Sarma, K. V. S.: Deterministic order level inventory model for deteriorating items with two storage facilities. Eur. J. Oper. Res. **29**, 70–73 (1987)

Teng, J. T.: A simple method to compute economic order quantities. Eur. J. Oper. Res. **198**, 351–353 (2009)

Urban, T. L.: Inventory models with inventory-level-dependent demand: A comprehensive review and unifying theory. Eur. J. Oper. Res. **162**, 792–804 (2005)

Warburton, R. D. H.: EOQ extensions exploiting the Lambert W function. European J. Industrial Engineering **3**, 45–69 (2009)

Weiss, H. J.: Economic Order Quantity Models with Nonlinear Holding Costs. Eur. J. Oper. Res. **9**, 56–60 (1982)

Wilson, R. H.: A Scientific Routine for Stock Control. Harvard Business Review **13**, 116–128 (1934)

Zipkin, P.: Foundations of Inventory Managment. Mac Graw Hill, New York (2000)