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Magnetic field penetration through a circular aperture in a perfectly conducting plate excited by a coaxial loop

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Abstract

The transmission of an electromagnetic field produced by a current loop of finite radius through a coaxial circular aperture in a perfectly conducting plate is evaluated through a rapidly convergent formulation in an exact form. By applying the equivalence principle, the problem is first formulated in the Hankel transform domain, obtaining a set of dual integral equations in which the equivalent surface magnetic current density defined on the aperture is not known. The set of dual integral equations is regularised in a second-kind Fredholm integral equation by applying the Abel integral-transform technique. The solution is achieved by expanding the unknown in a set of orthogonal basis functions that correctly reproduce the behaviour of the equivalent magnetic current at the edge of the aperture. Finally, under particular assumptions, a low-frequency solution is extracted in a closed form. Numerical results are reported to validate the accuracy and efficiency of the proposed formulations.

1 | INTRODUCTION

The transmission of an electromagnetic (EM) field through an aperture in a planar conducting screen is a canonical problem that has attracted great attention in the EM community [1, 2]. In general, an analytical solution is available only for an incident plane wave impinging on an infinitesimally thin circular aperture (in this case, the solution is expressed as an expansion in terms of oblate spheroidal vector wave functions [3]).

In the low-frequency region, Lord Rayleigh was the first to propose a solution to the problem [4]: procedure of the solution was based on a series expansion in ascending powers of the wavenumber of certain quantities (the so-called Rayleigh series), and it has been shown that it leads to a sequence of simple integral equations with a kernel of the electrostatic type [2]. In a famous paper, Bethe studied low-frequency EM scattering by a small circular hole cut in an infinite perfectly conducting (PEC) plane [5]. Using a scalar potential approach, he derived the leading terms of the Rayleigh series. Bouwkamp studied the same problem in a more rigorous way using a complicated system of integrodifferential equations, and he found some errors and incorrect results in Bethe's solution (in particular, the first-order approximation in Bethe's solution does not satisfy the condition for which the normal component of the electric field has to be zero in the aperture) [1, 6]. Later, several numerical studies based on the Bethe–Bouwkamp model were presented in Eviatan [7, 8]. An alternative use of Rayleigh series expansion has been discussed [9–11]. An interestingly elegant variational formulation of EM diffraction problems for planar apertures, which allowed for approximate but accurate numerical evaluations of the scattered fields in a wide range of frequencies, was provided by Levine and Schwinger [12]. Finally, in connection with near-field scanning optical microscopy, Michalski presented rigorous spectral-domain formulations (based on the Bethe–Bouwkamp model) to determine the plane-wave field transmitted through small circular apertures to include the effect of a nearby (possibly anisotropic) material sample [13, 14] and deriving useful closed-form expressions, especially in the near-field region [15].

In this article, we address the problem of the transmission of the magnetic field radiated by a finite source through a circular aperture in a planar PEC plate. In particular, the finite source consists of a current loop of finite radius coaxial to the circular aperture (none of the previous works dealt with such an excitation). The adopted approach belongs to the family of analytical regularisation procedures aimed at formulating the problem in terms of Fredholm integral equations of the second kind, and is based on the Abel integral-transform technique.

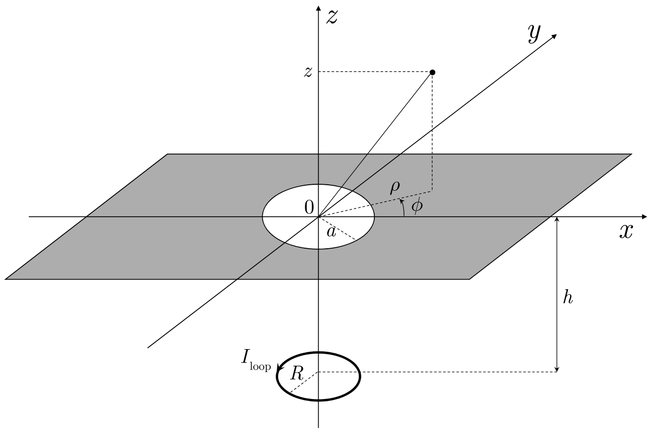

FIGURE 1 Configuration under analysis: a current loop of radius R excites a circular aperture of radius a in a PEC plate of negligible thickness. The loop is coaxial to the aperture and placed at distance h from it

The related configuration composed of a PEC disk excited by a vertical magnetic dipole is considered in Ovat et al. [16], where the problem is solved through a standard spectral-domain approach with basis functions that reproduce the correct singular behaviour of the current density at the disk edge. In both cases, diagonalisation of the static kernel of the relevant integral operator is achieved.

The formulation of the problem is first presented in Section 2. After applying the equivalence principle, operating in the Hankel transform domain, a set of dual integral equations is derived whose unknown is the equivalent magnetic surface current density on the aperture. An exact numerical solution is then obtained in Section 3, where the Abel integral transformation technique is applied to convert the set of dual integral equations into a second-kind Fredholm integral equation. The convergence properties of the obtained numerical solution are further improved by a suitable series representation of the elements of the impedance matrix. In Section 4, as in Lovat et al. [16], under certain assumptions and owing to some integral identities, a low-frequency solution is extracted in a closed form. In Section 5, numerical results that assess the accuracy and the validity of the proposed formulations are provided. Finally, in Section 6, conclusions are drawn.

2 | FORMULATION OF THE PROBLEM

The configuration under analysis is reported in Figure 1 and consists of a current loop of radius R coaxial with a circular aperture of radius a etched in an infinitesimally thin PEC plate placed along the $z = 0$ plane of a cylindrical coordinate system (ρ, ϕ, z). The origin of the coordinate system is located at the centre of the aperture and the loop is placed in the half-space $z < 0$ at distance h from the conducting plate. The EM problem is axially symmetric so that all field quantities depend only on ρ and z. A time-harmonic dependence $e^{j\omega t}$ is assumed and suppressed throughout.

To obtain an integral equation that solves the problem, the equivalence principle is first applied and equivalent magnetic current densities $\boldsymbol{M}_S$ and $-\boldsymbol{M}_S$ are introduced on the aperture area for $z = 0^-$ and $z = 0^+$, respectively. The problem is thus split into two half-space problems (for $z < 0$ and $z > 0$, respectively). The magnetic field on both sides of the screen can be expressed as a function of $\boldsymbol{M}_S$ and the continuity of the tangential component of the electric field through the aperture is automatically fulfilled. Therefore, the only condition that must be imposed is continuity of the magnetic field through the aperture: the sought-for integral equation follows directly from such a constraint. In particular, based on the equivalence principle, circular aperture A is short-circuited (i.e. completely replaced by a PEC plate), and it constitutes the domain for equivalent surface magnetic current density $\boldsymbol{M}_S$, which accounts for a nonzero value of the electric field at $z = 0^-$ (i.e. $\boldsymbol{M}_S = \boldsymbol{u}_z \times \boldsymbol{E}$). By means of image theory, the PEC plate then can be removed and the magnetic current density is doubled. For $z > 0$, the magnetic field is therefore the scattered magnetic field $\boldsymbol{H}^{\text{scat}}$ owing to magnetic current density $-2\boldsymbol{M}_S$ radiating in free space, whereas for $z < 0$, it is scattered magnetic field $\boldsymbol{H}^{\text{scat}}$ owing to magnetic current density $2\boldsymbol{M}_S$ radiating in free space plus the magnetic field radiated by the current loop in the presence of an infinite PEC plane at $z = 0$ (so-called short-circuited magnetic field $\boldsymbol{H}^{\text{SC}}$). As mentioned, the key condition that has to be imposed is continuity of the tangential magnetic field through the aperture. This is accomplished by enforcing

$$\lim_{z\to 0^+}[\boldsymbol{H}^+(\boldsymbol{r}) \times \boldsymbol{u}_z] = \lim_{z\to 0^-}[\boldsymbol{H}^-(\boldsymbol{r}) \times \boldsymbol{u}_z] \qquad \rho \le a \qquad (1)$$

which can be expressed as

$$\boldsymbol{H}^{\text{FS}}\{-2\boldsymbol{M}_S\} \times \boldsymbol{u}_z = \boldsymbol{H}^{\text{FS}}\{2\boldsymbol{M}_S\} + \boldsymbol{H}^{\text{SC}} \times \boldsymbol{u}_z \qquad \rho \le a. \quad (2)$$

where $\boldsymbol{H}^{\text{FS}}\{\boldsymbol{M}\}$ is the magnetic field radiated in free space by magnetic current $\boldsymbol{M}$. Because, by virtue of image theory, $\boldsymbol{H}^{\text{SC}} \times \boldsymbol{u}_z = 2\boldsymbol{H}^{\text{inc}} \times \boldsymbol{u}_z$ (where $\boldsymbol{H}^{\text{inc}}$ is the magnetic field radiated by the current loop in free space), from Equation (2), we obtain

$$-2\boldsymbol{H}^{\text{FS}}(\boldsymbol{M}_S) \times \boldsymbol{u}_z = \boldsymbol{H}^{\text{inc}} \times \boldsymbol{u}_z \qquad \rho \le a. \qquad (3)$$

Equation (3) represents the sought-for integral equation whose solution evaluates the field everywhere in space.

2.1 | Incident and scattered magnetic fields

We first consider an electric current loop of current I_0 and radius R placed over plane $z = -h$. Therefore, the impressed current source has the following expression:

$$\boldsymbol{J}_{\text{i}}(\rho, z) = J_\phi(\rho, z)\boldsymbol{u}_\phi = I_0 R\,\frac{\delta(\rho - R)}{\rho}\,\delta(z + h)\boldsymbol{u}_\phi \qquad (4)$$

where $\delta(\cdot)$ indicates the Dirac distribution.

By symmetry, vector potential $\boldsymbol{A}^{\mathrm{inc}}$ generated by the current loop has only azimuthal component A_ϕ^{inc}. As shown in Lovat et al. [16], passing through the Hankel-transform domain, an integral representation for A_ϕ^{inc} can be obtained as

$$A_\phi^{\mathrm{inc}}(\rho,z)=\frac{\mu_0 I_0 R}{2\mathrm{j}}\int_0^\infty \frac{\mathrm{e}^{-\mathrm{j}k_z|z+h|}}{k_z}\mathrm{J}_1(\lambda R)\mathrm{J}_1(\lambda\rho)\lambda\,\mathrm{d}\lambda \quad (5)$$

where μ_0 is the free-space magnetic permeability, $k_z=-\mathrm{j}\sqrt{\lambda^2-k_0^2}$ (with $\mathrm{Re}\left[\sqrt{\lambda^2-k_0^2}\right]>0$ and k_0 indicates the free-space wavenumber), whereas $\mathrm{J}_n(\cdot)$ is the first-kind Bessel function of order n.

Because $\boldsymbol{H}^{\mathrm{inc}}=\nabla\times\boldsymbol{A}^{\mathrm{inc}}/\mu_0$, for the incident magnetic field, it results in

$$H_\rho^{\mathrm{inc}}(\rho,z)=\frac{I_0 R}{2}\int_0^\infty \mathrm{e}^{-\mathrm{j}k_z|z+h|}\mathrm{J}_1(\lambda R)\mathrm{J}_1(\lambda\rho)\lambda\,\mathrm{d}\lambda \quad (6)$$

and

$$H_z^{\mathrm{inc}}(\rho,z)=\frac{I_0 R}{2\mathrm{j}}\int_0^\infty \frac{\mathrm{e}^{-\mathrm{j}k_z|z+h|}}{k_z}\mathrm{J}_1(\lambda R)\mathrm{J}_0(\lambda\rho)\lambda^2\,\mathrm{d}\lambda. \quad (7)$$

On the other hand, the aperture acts like a disk of radius a in $z=0$ with a surface magnetic current density $\boldsymbol{M}_S(\rho,\phi)$. Because of the azimuthal symmetry of the problem and the TE_z nature of the radiated field, it results in $\boldsymbol{E}(\rho,\phi,z)=E_\phi(\rho,z)\boldsymbol{u}_\phi$, so that $\boldsymbol{M}_S(\rho,\phi)=M_{S\rho}(\rho)\boldsymbol{u}_\rho$. The relevant vector potential $\boldsymbol{F}^{\mathrm{scat}}(\rho,z)=F_\rho^{\mathrm{scat}}(\rho,z)\boldsymbol{u}_\rho$ can easily be obtained through an analysis in the Hankel spectral domain [21], that is,

$$F_\rho^{\mathrm{scat}}(\rho,z)=\frac{\varepsilon 0}{2\mathrm{j}}\int_0^\infty \frac{\mathrm{e}^{-\mathrm{j}k_z|z|}}{k_z}\widetilde{M}_{S\rho}(\lambda)\mathrm{J}_1(\lambda\rho)\lambda\,\mathrm{d}\lambda \quad (8)$$

where ε_0 is the free-space permittivity and $\widetilde{M}_{S\rho}(\lambda)$ is the Hankel transform of the magnetic current density $M_{S\rho}(\rho)$, defined as

$$\widetilde{M}_{S\rho}(\lambda)=\mathscr{H}_1\{M_{S\rho}(\rho)\}=\int_0^\infty M_{S\rho}(\rho)\mathrm{J}_1(\lambda\rho)\rho\,\mathrm{d}\rho. \quad (9)$$

Because $\boldsymbol{H}^{\mathrm{scat}}=-\mathrm{j}\omega\boldsymbol{F}^{\mathrm{scat}}+\nabla\nabla\cdot\boldsymbol{F}^{\mathrm{scat}}/(\mathrm{j}\omega\mu_0\varepsilon_0)$, we have

$$H_\rho^{\mathrm{scat}}(\rho,z)=-\frac{1}{2k_0\zeta_0}\int_0^\infty k_z\,\mathrm{e}^{-\mathrm{j}k_z|z|}\widetilde{M}_{S\rho}(\lambda)\mathrm{J}_1(\lambda\rho)\lambda\,\mathrm{d}\lambda \quad (10)$$

and

$$H_z^{\mathrm{scat}}(\rho,z)=\pm\frac{\mathrm{j}}{2k_0\zeta_0}\int_0^\infty \mathrm{e}^{-\mathrm{j}k_z|z|}\widetilde{M}_{S\rho}(\lambda)\mathrm{J}_0(\lambda\rho)\lambda^2\,\mathrm{d}\lambda, \quad (11)$$

where the plus or minus sign holds for $z>0$ and $z<0$, respectively, and ζ_0 is the free-space characteristic impedance.

2.2 | Boundary condition

From Equation (3) and the expressions of the tangential fields to apertures (6) and (10), we thus obtain for $\rho<a$

$$\frac{1}{k_0\zeta_0}\int_0^\infty k_z\widetilde{M}_{S\rho}(\lambda)\mathrm{J}_1(\lambda\rho)\lambda\,\mathrm{d}\lambda - \frac{I_0 R}{2}\int_0^\infty \mathrm{e}^{-\mathrm{j}k_z h}\mathrm{J}_1(\lambda R)\mathrm{J}_1(\lambda\rho)\lambda\,\mathrm{d}\lambda=0. \quad (12)$$

This equation and the condition for which the magnetic current density vanishes for $\rho>a$ constitute a system of dual integral equations. In particular, by rearranging, we obtain

$$\int_0^\infty\left[k_z\widetilde{M}_{S\rho}(\lambda)-\frac{I_0 R k_0\zeta_0}{2}\mathrm{e}^{-\mathrm{j}k_z h}\mathrm{J}_1(\lambda R)\right]\cdot\mathrm{J}_1(\lambda\rho)\lambda\,\mathrm{d}\lambda=0\,,\qquad \rho<a \quad (13)$$

$$\int_0^\infty\widetilde{M}_{S\rho}(\lambda)\mathrm{J}_1(\lambda\rho)\lambda\,\mathrm{d}\lambda=0\,,\qquad \rho>a. \quad (14)$$

3 | ABEL TRANSFORM APPROACH

To solve the system of dual integral Equations (13) and (14), the method of the Abel transformation technique is applied [17]. In particular, by defining $V(\lambda)=\lambda\widetilde{M}_{S\rho}(\lambda)$, the system (13) and (14) can be first recast in the following form:

$$\int_0^\infty\sqrt{\lambda^2-k_0^2}\,V(\lambda)\mathrm{J}_1(\lambda\rho)\,\mathrm{d}\lambda = \mathrm{j}\frac{I_0 R k_0\zeta_0}{2}\int_0^\infty \mathrm{J}_1(\lambda R)\mathrm{J}_1(\lambda\rho)\,\mathrm{e}^{-\sqrt{\lambda^2-k_0^2}\,h}\lambda\,\mathrm{d}\lambda\,,\ \rho<a, \quad (15)$$

$$\int_0^\infty V(\lambda)\mathrm{J}_1(\lambda\rho)\,\mathrm{d}\lambda=0\,,\qquad \rho>a. \quad (16)$$

By means of the integral identity (derived from the first integral representation of Abel type for Bessel functions [18]):

$$\frac{1}{\alpha}\mathrm{J}_{\nu+1}(\alpha x)=\frac{1}{x^{\nu+1}}\int_0^x \mathrm{J}_\nu(\alpha y)y^{\nu+1}\,\mathrm{d}y, \quad (17)$$

Equation (15) can also be written as

$$\int_0^\infty\{V(\lambda)[1+h(\lambda,k_0)]-I(\lambda)\}\mathrm{J}_2(\lambda\rho)\lambda\,\mathrm{d}\lambda=0\,,\ \rho<a, \quad (18)$$

where

$$I(\lambda)=\mathrm{j}I_0 R k_0\zeta_0\,\frac{\mathrm{e}^{-\sqrt{\lambda^2-k_0^2}\,h}}{2\lambda}\mathrm{J}_1(\lambda R) \quad (19)$$

and

$$h(\lambda, k_0) = \frac{\sqrt{\lambda^2 - k_0^2}}{\lambda} - 1 \tag{20}$$

is a small parameter in the limit $\lambda \to +\infty$. By using the first and the second Abel integral representations of the Bessel functions [18],

$$\mathrm{J}_2(\lambda\rho) = \frac{2^{1/2}\lambda^{1/2}}{\rho^2\Gamma(1/2)} \int_0^\rho \frac{x^{5/2}\mathrm{J}_{3/2}(\lambda x)}{(\rho^2 - x^2)^{1/2}}\,\mathrm{d}x, \tag{21}$$

$$\mathrm{J}_1(\lambda\rho) = \frac{2^{1/2}\lambda^{1/2}\rho}{\Gamma(1/2)} \int_\rho^\infty \frac{\mathrm{J}_{3/2}(\lambda x)}{x^{1/2}(\rho^2 - x^2)^{1/2}}\,\mathrm{d}x, \tag{22}$$

Equations (18) and (16) are transformed into the Abel integral equations:

$$\int_0^\rho \frac{x^{5/2}}{(\rho^2 - x^2)^{1/2}} \left\{\int_0^\infty \lambda^{1/2}[V(\lambda)(1+h) - I(\lambda)] \cdot \mathrm{J}_{3/2}(\lambda x)\,\mathrm{d}\lambda\right\} \mathrm{d}x = 0 \quad \rho < a, \tag{23}$$

$$\int_\rho^\infty \frac{1}{x^{1/2}(\rho^2 - x^2)^{1/2}} \cdot \left\{\int_0^\infty \lambda^{1/2} V(\lambda)\mathrm{J}_{3/2}(\lambda x)\,\mathrm{d}\lambda\right\} \mathrm{d}x = 0 \quad \rho > a, \tag{24}$$

each of which possesses a unique zero solution, so that

$$\int_0^\infty \lambda^{1/2} V(\lambda)\mathrm{J}_{3/2}(\lambda\rho)\,\mathrm{d}\lambda = \int_0^\infty \lambda^{1/2} I(\lambda)\mathrm{J}_{3/2}(\lambda\rho)\,\mathrm{d}\lambda - \int_0^\infty \lambda^{1/2} V(\lambda) h(\lambda, k_0)\mathrm{J}_{3/2}(\lambda\rho)\,\mathrm{d}\lambda \quad \rho < a, \tag{25}$$

$$\int_0^\infty \lambda^{1/2} V(\lambda)\mathrm{J}_{3/2}(\lambda\rho)\,\mathrm{d}\lambda = 0 \quad \rho > a. \tag{26}$$

Multiplying Equations (25) and (26) by $\rho \mathrm{J}_{3/2}(\nu\rho)$ and integrating over $(0, \infty)$, a second-kind Fredholm integral equation is obtained as

$$V(\nu) + \nu^{1/2}\int_0^\infty \lambda^{1/2} V(\lambda) h(\lambda, k_0) G(\lambda, \nu)\,\mathrm{d}\lambda = \nu^{1/2}\int_0^\infty \lambda^{1/2} I(\lambda) G(\lambda, \nu)\,\mathrm{d}\lambda, \tag{27}$$

where

$$G(\lambda, \nu) = \int_0^a \rho \mathrm{J}_{3/2}(\lambda\rho)\mathrm{J}_{3/2}(\nu\rho)\,\mathrm{d}\rho. \tag{28}$$

A (different) regularised second-kind Fredholm integral equation could also have been obtained through another method of analytical regularisation [19–21], based on extracting the static part of the original integral operator, following the same procedure as in Lovat et al. [16].

In any case, the system (25–26) can also be converted into a second-kind matrix system by expanding unknown function $V(\nu)$ in the Neumann series. This can be accomplished considering that the unknown current density $M_{S\rho}$ can be expanded through a set of basis functions $b_n(\rho)$, that is,

$$M_{S\rho}(\rho) = \sum_{n=1}^{+\infty} v_n b_n(\rho). \tag{29}$$

Functions $b_n(\rho)$ should correctly reproduce the behaviour of the equivalent magnetic current in $\rho = a$. In particular, $M_{S\rho}$ is proportional to the component of the tangential electric field parallel to an infinitesimally thin PEC edge that behaves as $(a^2 - \rho^2)^{1/2}$ as $\rho \to a$ [22]. Moreover, the magnetic current has to be finite or, better, identically zero at the origin. We can thus adopt the radial parts of the generalized Zernike functions or generalized spherical harmonics [23] as a set of basis functions, that is,

$$b_n(\rho) = \frac{(n-1)!}{\sqrt{2}\,\Gamma(n+1/2)a^2}\,\rho\sqrt{a^2 - \rho^2} \cdot P_{n-1}^{(1,1/2)}\left(1 - \frac{2\rho^2}{a^2}\right) u_{-1}(a - \rho), \quad n = 1, 2, \ldots, \tag{30}$$

where $u_{-1}(\cdot)$ is the Heaviside unit-step function, $P_n^{(\alpha,\beta)}(\cdot)$ are the Jacobi polynomials of order n, and $\Gamma(\cdot)$ is the Gamma function. The adopted basis functions $b_n(\rho)$ are normalized in such a way that the relevant Hankel transforms are

$$\tilde{b}_n(\lambda) = \sqrt{a}\,\frac{\mathrm{J}_{2n+1/2}(\lambda a)}{\lambda^{3/2}}, \quad n = 1, 2, \ldots, \tag{31}$$

so that

$$V(\lambda) = \frac{a^{1/2}}{\lambda^{1/2}} \sum_{n=1}^{+\infty} v_n \mathrm{J}_{2n+1/2}(\lambda a). \tag{32}$$

By substituting Equation (32) into Equations (25) and (26) and using the Weber–Schafheitlin integrals [24, Section 13.4]:

$$\int_0^\infty \mathrm{J}_{3/2}(\lambda\rho)\mathrm{J}_{2n+1/2}(\lambda a)\,\mathrm{d}\lambda = \frac{\rho^{3/2}}{a^{5/2}} P_{n-1}^{(3/2,0)}\left(1 - \frac{2\rho^2}{a^2}\right) \quad \rho < a, \tag{33}$$

$$\int_0^\infty \mathrm{J}_{3/2}(\lambda\rho)\mathrm{J}_{2n+1/2}(\lambda a)\,\mathrm{d}\lambda = 0 \quad \rho > a, \tag{34}$$

we obtain

$$\sum_{n=1}^{+\infty} v_n \frac{\rho^{3/2}}{a^2} P_{n-1}^{(3/2,0)}\left(1-\frac{2\rho^2}{a^2}\right) = u_{-1}(a-\rho)\left\{\int_0^\infty \lambda^{1/2} I(\lambda) \mathrm{J}_{3/2}(\lambda\rho)\,\mathrm{d}\lambda - \sum_{n=1}^{+\infty} v_n\sqrt{a}\int_0^\infty \mathrm{J}_{2n+1/2}(\lambda a) h(\lambda,k_0)\mathrm{J}_{3/2}(\lambda\rho)\,\mathrm{d}\lambda\right\}. \tag{35}$$

Now, by multiplying both sides of Equation (35) by $\rho^{5/2}P_{m-1}^{(3/2,0)}(1-2\rho^2/a^2)$ and integrating over $(0,a)$, we have

$$v_n \frac{a^3}{4n+1} = \int_0^\infty I(\lambda)\frac{a^{5/2}}{\lambda^{1/2}}\mathrm{J}_{2n+1/2}(\lambda a)\,\mathrm{d}\lambda - \sum_{n=1}^{+\infty} v_n a^3 \int_0^\infty \frac{h(\lambda,k_0)}{\lambda}\mathrm{J}_{2m+1/2}(\lambda a)\mathrm{J}_{2n+1/2}(\lambda a)\,\mathrm{d}\lambda, \tag{36}$$

where the orthogonality property of the radial part of Zernike polynomials [25]:

$$\int_0^a \rho^4 P_{m-1}^{(3/2,0)}\left(1-\frac{2\rho^2}{a^2}\right)P_{n-1}^{(3/2,0)}\left(1-\frac{2\rho^2}{a^2}\right)\mathrm{d}\rho = \frac{\delta_{mn}a^5}{4n+1} \tag{37}$$

has been used together with the identity [25].

$$\int_0^a \rho^{5/2} P_{m-1}^{(3/2,0)}\left(1-\frac{2\rho^2}{a^2}\right)\mathrm{J}_{3/2}(\lambda\rho)\,\mathrm{d}\rho = \frac{a^{5/2}}{\lambda}\mathrm{J}_{2m+1/2}(\lambda a). \tag{38}$$

From Equation (36), we thus obtain the matrix system:

$$v_n + \sum_{m=1}^{\infty} Y_{nm} v_m = I_n, \qquad n = 1,2,\ldots \tag{39}$$

where

$$Y_{nm} = (4n+1) \cdot \int_0^\infty \frac{\sqrt{\lambda^2-k_0^2}}{\lambda^2}\mathrm{J}_{2m+1/2}(\lambda a)\mathrm{J}_{2n+1/2}(\lambda a)\,\mathrm{d}\lambda \tag{40}$$

and

$$I_n = \mathrm{j}\frac{I_0 R k_0 \zeta_0}{2\sqrt{a}}(4n+1) \cdot \int_0^\infty \mathrm{J}_{2n+1/2}(\lambda a)\mathrm{J}_1(\lambda R)\frac{\mathrm{e}^{-\sqrt{\lambda^2-k_0^2}\,h}}{\lambda^{1/2}}\,\mathrm{d}\lambda. \tag{41}$$

The solution of the algebraic system in Equation (32) furnishes coefficients v_n and magnetic current density $M_{S\rho}$ is recovered through Equation (29).

In general, the improper integrals in Equation (40) are highly oscillating and slowly decaying and may be difficult to compute. However, it can be shown that they can be evaluated through a rapidly converging series as

$$Y_{nm} = \frac{(4n+1)}{4}\sum_{p=0}^{\infty}(-1)^{p/2}\frac{\left(-\frac{p}{2}+\frac{3}{2}\right)_{m+n-1}}{\left(\frac{p}{2}+\frac{1}{2}\right)_{m+n+1}} \cdot \frac{1}{\Gamma\left(\frac{p}{2}+n-m+1\right)\Gamma\left(\frac{p}{2}+m-n+1\right)}(k_0 a)^p \tag{42}$$

where the Pochhammer symbol $(x)_y$ is defined as [26]

$$(x)_y = \frac{\Gamma(x+y)}{\Gamma(x)}, \tag{43}$$

In general, few terms are needed to reach high accuracy. The result in Equation (42) seems to be new and original; therefore, its proof is reported in Appendix.

Once the v_n coefficients are known, spectral magnetic current $M_{S\rho}(\lambda)$ is

$$\widetilde{M}_{S\rho}(\lambda) = \sum_{n=1}^{+\infty} v_n \widetilde{b}_n(\lambda), \tag{44}$$

and therefore, the radiated tangential magnetic field beyond the aperture (i.e. in $z > 0$) is given by

$$H_\rho^{\text{scat}}(\rho,z) = \mathrm{j}\frac{\sqrt{a}}{2k_0\zeta_0}\sum_{n=1}^{+\infty} v_n \cdot \int_0^\infty \sqrt{\lambda^2-k_0^2}\,\frac{\mathrm{J}_{2n+1/2}(\lambda a)\mathrm{J}_1(\lambda\rho)}{\sqrt{\lambda}}\,\mathrm{e}^{-\sqrt{\lambda^2-k_0^2}\,z}\,\mathrm{d}\lambda \tag{45}$$

Regarding the z-components of the magnetic fields, for $z > 0$, we instead have

$$H_z^{\text{scat}}(\rho,z) = \frac{\mathrm{j}\sqrt{a}}{2k_0\zeta_0}\sum_{n=1}^{+\infty} v_n \cdot \int_0^\infty \mathrm{e}^{-\sqrt{\lambda^2-k_0^2}\,z}\mathrm{J}_{2n+1/2}(\lambda a)\mathrm{J}_0(\lambda\rho)\sqrt{\lambda}\,\mathrm{d}\lambda \tag{46}$$

4 | LOW-FREQUENCY SOLUTION

In the low-frequency limit (i.e. $k_0 \to 0$), elements Y_{mn} in Equation (40) become:

$$Y_{mn} = (4n+1)\int_0^\infty \frac{J_{2m+1/2}(\lambda a)J_{2n+1/2}(\lambda a)}{\lambda}\,\mathrm{d}\lambda \tag{47}$$

Such an integral, again of the Weber–Schafheitlin type [24], can be evaluated in a closed form using identity [26, Section 6.574] (in any case, the integral in Equation (47) is the product of two orthogonal functions) and it results in $Y_{mn} = 0$ for $m \neq n$, whereas

$$Y_{nn} = 1 \tag{48}$$

Regarding I_n, from Equation (41), in the low-frequency limit, we have

$$I_n = \mathrm{j}\,\frac{I_0 R k_0 \zeta_0 (4n+1)}{2\sqrt{a}} \cdot \int_0^\infty \mathrm{e}^{-\lambda h}\,\frac{\mathrm{J}_{2m+1/2}(\lambda a)\mathrm{J}_1(\lambda R)}{\sqrt{\lambda}}\,\mathrm{d}\lambda \tag{49}$$

Also, the latter integral can be expressed in a closed form by using the following identity [26, Section 6.626]:

$$\int_0^\infty x^{\lambda-1}\,e^{-\alpha x}J_\mu(\beta x)J_\nu(\gamma x)\,\mathrm{d}x = \frac{\beta^\mu\gamma^\nu}{\Gamma(\nu+1)}\,2^{-\nu-\mu}\,\alpha^{-\lambda-\mu-\nu}\sum_{m=0}^{\infty}\frac{\Gamma(\lambda+\mu+\nu+2m)}{m!\,\Gamma(\mu+m+1)}\,F\left(-m,-\mu-m;\nu+1;\frac{\gamma^2}{\beta^2}\right)\left(-\frac{\beta^2}{4\alpha^2}\right)^m \tag{50}$$

where $F(\cdot,\cdot;\cdot;\cdot)$ is the Gauss hypergeometric function [26]. By letting $\lambda = 1/2$, $\alpha = h$, $\mu = 2n + 1/2$, $\beta = a$, $\nu = 1$, and $\gamma = R$, we thus have

$$I_n = \mathrm{j}\,\frac{I_0 k_0 \zeta_0 (4n+1) a^{2n} R^2}{2^{2n+5/2} h^{2n+2}}\sum_{q=0}^{\infty}\frac{(2n+2q+1)!}{q!\,\Gamma\left(2n+\frac{3}{2}+q\right)}\,F\left(-q,-2n-\frac{1}{2}-q;2;\frac{R^2}{a^2}\right)\left(-\frac{a^2}{4\,h^2}\right)^q \tag{51}$$

Other ways to express these kinds of integrals analytically can be found in Fabrikant [27].

Because the system is diagonal, we immediately obtain:

$$v_n = \frac{I_n}{1+Y_{nn}} = \frac{I_n}{2} \tag{52}$$

From Equations (6) and (7), and (45) and (46), in the low-frequency limit, we also have

$$H_\rho^{\mathrm{inc}}(\rho,z) = \frac{I_0 R}{2}\int_0^\infty \mathrm{e}^{-\lambda|z+h|}\,\mathrm{J}_1(\lambda R)\mathrm{J}_1(\lambda\rho)\lambda\,\mathrm{d}\lambda \tag{53}$$

$$H_z^{\mathrm{inc}}(\rho,z) = \frac{I_0 R}{2}\int_0^\infty \mathrm{e}^{-\lambda|z+h|}\,\mathrm{J}_1(\lambda R)\mathrm{J}_0(\lambda\rho)\lambda\,\mathrm{d}\lambda \tag{54}$$

and

$$H_\rho^{\mathrm{scat}}(\rho,z) = \mathrm{j}\,\frac{\sqrt{a}}{2k_0\zeta_0}\sum_{n=1}^{+\infty} v_n \cdot \int_0^\infty \mathrm{e}^{-\lambda z}\,\mathrm{J}_{2n+1/2}(\lambda a)\mathrm{J}_1(\lambda\rho)\sqrt{\lambda}\,\mathrm{d}\lambda \tag{55}$$

$$H_z^{\mathrm{scat}}(\rho,z) = \frac{\mathrm{j}\sqrt{a}}{2k_0\zeta_0}\sum_{n=1}^{+\infty} v_n \cdot \int_0^\infty \mathrm{e}^{-\lambda|z|}\,\mathrm{J}_{2n+1/2}(\lambda a)\mathrm{J}_0(\lambda\rho)\sqrt{\lambda}\,\mathrm{d}\lambda \tag{56}$$

All of these integrals are of Lipschitz–Hankel type and can be evaluated in a closed form. In particular, for the integrals in Equations (53) and (54), using the expressions for $I(1,1,1)$ and $I(1,0,1)$ in Ason et al. [28], we have

$$H_\rho^{\mathrm{inc}}(\rho,z) = \frac{I_0 R|z+h|k}{8\pi R^{3/2}\rho^{3/2}}\left[-\boldsymbol{K}(k) + \left(1-\frac{k^2}{2}\right)\frac{1}{\left(1-k^2\right)}\boldsymbol{E}(k)\right] \tag{57}$$

and

$$H_z^{\mathrm{inc}}(\rho,z) = \frac{I_0 R}{2}\left[\frac{k}{2\pi R^{3/2}\rho^{1/2}}\boldsymbol{K}(k) + \frac{k^3\left(R^2-\rho^2-|z+h|^2\right)}{8\pi\left(1-k^2\right)R^{5/2}\rho^{3/2}}\boldsymbol{E}(k)\right], \tag{58}$$

where $\boldsymbol{K}(\cdot)$ and $\boldsymbol{E}(\cdot)$ are the complete elliptic functions of the first and second kind [26], respectively, and

$$k^2 = \frac{4R\rho}{(R+\rho)^2+|z+h|^2}\,. \tag{59}$$

Using Equation (59), we finally obtain

$$H_\rho^{\mathrm{inc}}(\rho,z) = \frac{I_0}{2\pi}\,\frac{|z+h|}{\rho\sqrt{(R+\rho)^2+|z+h|^2}} \cdot \left[\frac{R^2+\rho^2+|z+h|^2}{(R-\rho)^2+|z+h|^2}\boldsymbol{E}(k) - \boldsymbol{K}(k)\right] \tag{60}$$

and

$$H_z^{\mathrm{inc}}(\rho,z)=\frac{I_0}{2\pi}\,\frac{1}{\sqrt{(R+\rho)^2+|z+h|^2}}\cdot\left[\frac{R^2-\rho^2-|z+h|^2}{(R-\rho)^2+|z+h|^2}\boldsymbol{E}(k)+\boldsymbol{K}(k)\right]. \tag{61}$$

The expressions for the incident field in Equations (60) and (61) are in exact agreement with those given in Smythe [29], Section 7.10, Eqs. (6) and (7), obtained through the vector magnetic potential using the free-space Green's function in the space domain.

For observation points along the z axis (i.e. for $\rho=0$ and thus $k=0$), the incident field simply becomes

$$H_\rho^{\mathrm{inc}}(0,z)=0 \tag{62}$$

and

$$H_z^{\mathrm{inc}}(0,z)=\frac{I_0R^2}{2\left(R^2+|z+h|^2\right)^{3/2}}, \tag{63}$$

For the scattered field, we instead have

$$H_\rho^{\mathrm{scat}}(0,z)=0 \tag{64}$$

and

$$H_z^{\mathrm{scat}}(0,z)=\frac{\mathrm{j}\sqrt{a}}{2k_0\zeta_0}\sum_{n=1}^{+\infty}v_n\int_0^\infty \mathrm{e}^{-\lambda|z|}\mathrm{J}_{2n+1/2}(\lambda a)\sqrt{\lambda}\,\mathrm{d}\lambda \tag{65}$$

The integral in Equation (65) can be solved in a closed form by using identity [26, Section 6.621] with $x=\lambda$, $\alpha=|z|$, $\nu=2n+1/2$, $\beta=a$, and $\mu=3/2$, thus obtaining

$$H_z^{\mathrm{scat}}(0,z)=\frac{\mathrm{j}\sqrt{a}}{2k_0\zeta_0}\sum_{n=1}^{\infty}v_n\cdot\frac{(2n+1)!}{\left(|z|^2+a^2\right)^{3/4}}P_{1/2}^{-(2n+1/2)}\left(\frac{|z|}{\sqrt{|z|^2+a^2}}\right) \tag{66}$$

where $P_\mu^\nu(\cdot)$ are the associated Legendre functions of the first kind.

For sources sufficiently far from the disk, only one basis function is sufficient to reach an excellent convergence, so in these cases, a closed form result can be obtained. In fact, from [26, Section 6.621.10]

$$\begin{aligned}I_1&=\mathrm{j}\frac{5I_0Rk_0\zeta_0}{2\sqrt{a}}\int_0^\infty \mathrm{e}^{-\lambda h}\frac{\mathrm{J}_{5/2}(\lambda a)\mathrm{J}_1(\lambda R)}{\sqrt{\lambda}}\,\mathrm{d}\lambda\\&=\mathrm{j}\frac{5I_0k_0\zeta_0}{2\sqrt{2\pi}}\frac{h}{a^3}\cdot\left[L\sqrt{R^2-L^2}+\frac{2R^2L}{\sqrt{R^2-L^2}}-3R^2\arcsin\left(\frac{L}{R}\right)\right]\end{aligned} \tag{67}$$

where

$$L=\frac{1}{2}\left[\sqrt{(a+R)^2+h^2}-\sqrt{(a-R)^2+h^2}\right] \tag{68}$$

and from Equation (66) (using Equations [52] and [67]), we thus obtain

$$H_z^{\mathrm{scat}}(0,z)\simeq-\frac{15I_0}{2\sqrt{2\pi}}\frac{h}{a^{5/2}}\cdot\left[L\sqrt{R^2-L^2}+\frac{2R^2L}{\sqrt{R^2-L^2}}-3R^2\arcsin\left(\frac{L}{R}\right)\right]\cdot\frac{1}{\left(|z|^2+a^2\right)^{3/4}}P_{1/2}^{-5/2}\left(\frac{|z|}{\sqrt{|z|^2+a^2}}\right) \tag{69}$$

For the considered configuration and in an electromagnetic compatibility (EMC) context, an important parameter is so-called magnetic shielding effectiveness SE_H [30], which along the z axis is defined as

$$\mathrm{SE}_H=20\log\frac{|H_z^{\mathrm{inc}}(0,z)|}{|H_z^{\mathrm{tot}}(0,z)|} \tag{70}$$

The total magnetic field is equal to the scattered magnetic field for $z>0$ (i.e. $\mathrm{H}^{\mathrm{tot}}=\mathrm{H}^{\mathrm{scat}}$) and the sum of the incident and scattered magnetic field for $z<0$ (i.e. $\mathrm{H}^{\mathrm{tot}}=\mathrm{H}^{\mathrm{inc}}+\mathrm{H}^{\mathrm{scat}}$). In particular, for $z>0$, low-frequency approximations (63) and (69) yield

$$\mathrm{SE}_H\simeq20\log\frac{\sqrt{2\pi a^5}R^2}{15\,h}\frac{\left(|z|^2+a^2\right)^{3/4}}{\left(R^2+|z+h|^2\right)^{3/2}}\cdot\left[L\sqrt{R^2-L^2}+\frac{2R^2L}{\sqrt{R^2-L^2}}-3R^2\arcsin\left(\frac{L}{R}\right)\right]^{-1}\cdot\left[P_{1/2}^{-5/2}\left(\frac{|z|}{\sqrt{|z|^2+a^2}}\right)\right]^{-1} \tag{71}$$

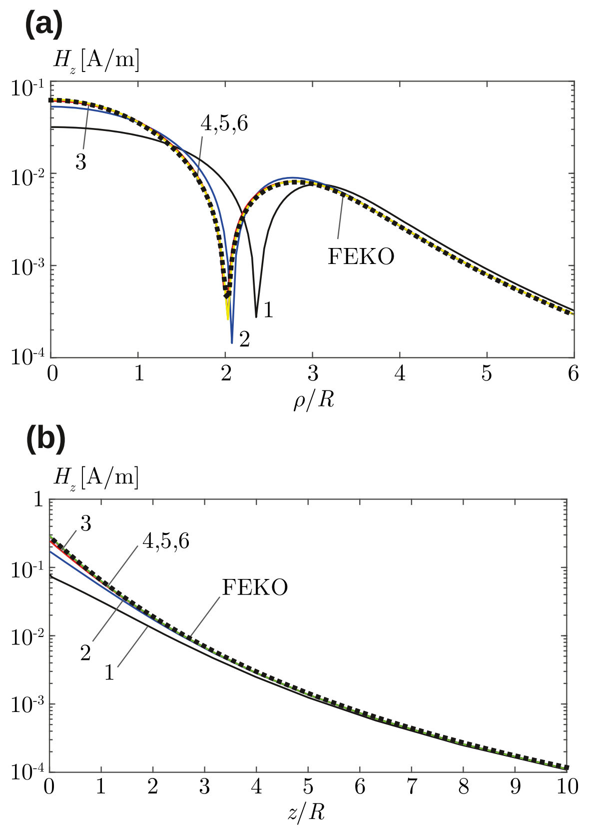

FIGURE 2 Amplitude of the z-component of magnetic field H_z obtained with the proposed formulation and the commercial software FEKO as a function of ρ/R at $z = R$ (a) and as a function of z/R at $\rho = 0$ (b). Other parameters: $f = 10$ MHz, $2R = 30.4$ cm, $h = R$, and $a = 3R$

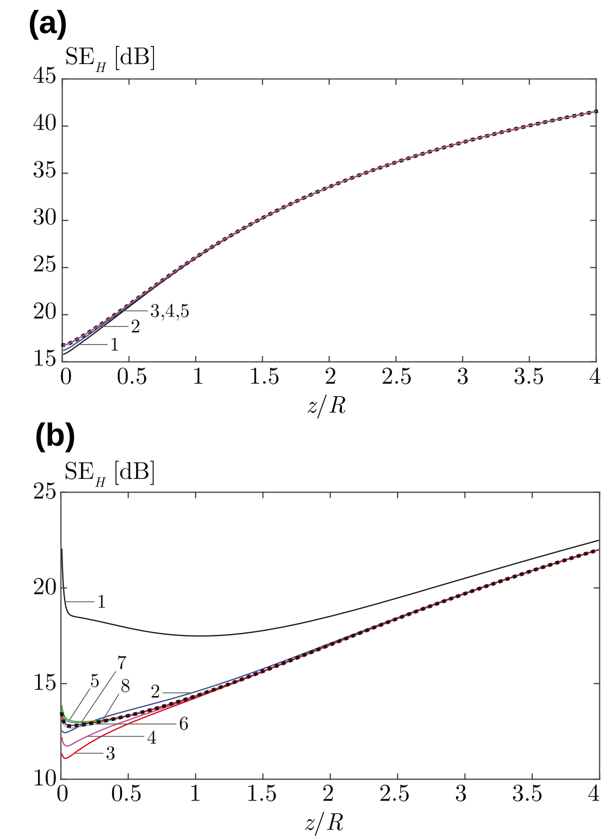

FIGURE 3 Convergence trend of SE_H on the z-axis with respect to the number of basis functions ($2R = 30.4$ cm, $h = R/2, f = 1$ kHz): (a) $a = R$; (b) $a = 2R$

5 | NUMERICAL RESULTS

In this section, numerical results are provided to illustrate the main features of the shielding configuration under consideration.

In Figure 2, results obtained through the proposed formulation and the full-wave results obtained with the commercial software FEKO [31] are compared. In particular, the z-component of transmitted magnetic field H_z is reported as a function of normalized radial distance ρ/R (Figure 2a) and of normalized vertical distance z/R (Figure 2b) at operating frequency $f = 10$ MHz for a configuration with $2R = 30.4$ cm, $h = R$ and $a = 3R$. A rapid convergence is obtained with few basis functions and for observation points close to the aperture (for sufficiently large distances, only one basis function is sufficient to obtain accurate results).

In Figure 3, SE_H is reported as a function of z for points on the z-axis, calculated with the proposed formulation by employing different numbers N of basis functions indicated in the figure. The current loop has diameter $2R = 30.4$ cm (as in typical low frequency [LF] EMC configurations), its current oscillates at $f = 1$ kHz, and it is placed at distance $h = R/2$ from the aperture centre; in particular, two aperture radii are considered: $a = R$ (in Figure 3a) and $a = 2R$ (in Figure 3b). In the former case, a single ($N = 1$) basis function is sufficient to obtain excellent results for observation points with z as small as a few centimeters (i.e. $z/R > 0.3$). In the latter case, conversely, at least $N = 2$ basis functions are required, but this guarantees accurate results for any $z/R > 0.8$; for observation points arbitrarily close to the aperture, convergence is achieved by using $N = 6$ basis functions.

To assess the accuracy of the proposed approximate low-frequency solution (71), in Figure 4 SE_H is reported as a function of frequency f (or normalized frequency k_0a) and for different distances from the aperture for a structure with $2R = 30.4$ cm and $h = a = R$ (in Figure 4a) and $2R = 13.3$ cm and $h = a = R$ (in Figure 4b). In both cases, the approximate formulation (LF approx.) is superimposed to the exact one (Exact) for $k_0a < 10^{-2}$, whereas it provides acceptable results up to $k_0a < 10^{-1}$ (above tens of megahertz for practical dimensions); in the highest frequency range, the approximate formulation slightly underestimates the exact SE_H.

In Figure 5 both the exact and approximate formulations are used to calculate SE_H for $2R = 30.4$ cm, $f = 1$ MHz, and different values of z/R, as a function of the dimensional ratios h/R (in Figure 5a) and a/R (in Figure 5b). In the former case, the two formulations are in perfect agreement for all considered values of h/R, whereas in the latter case, they agree only for a/R smaller than a threshold that increases with z/R.

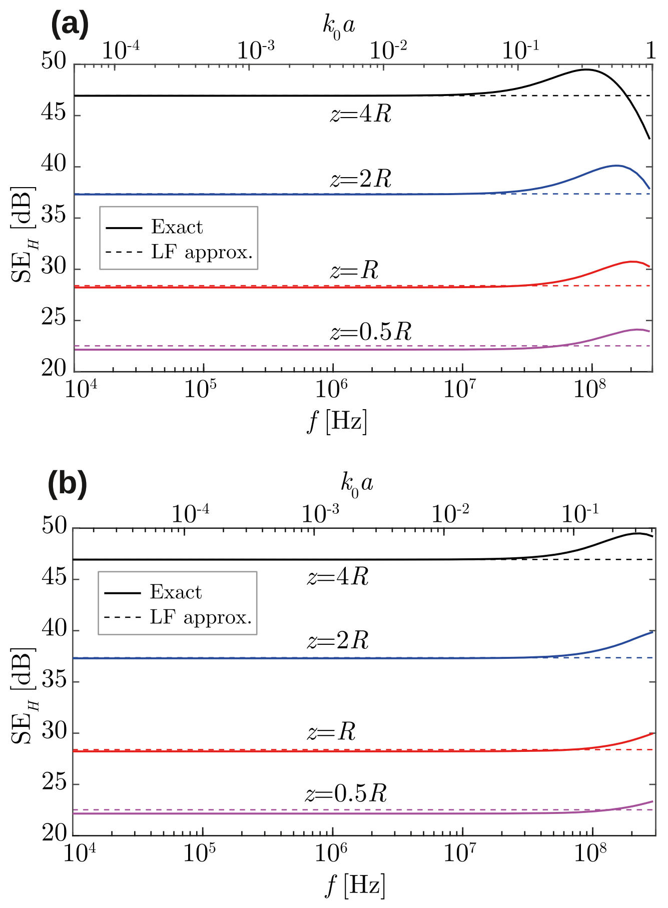

FIGURE 4 SE_H as a function of the normalized radius k_0a for $h = a = R$: (a) $2R = 30.4$ cm; (b) $2R = 13.3$ cm

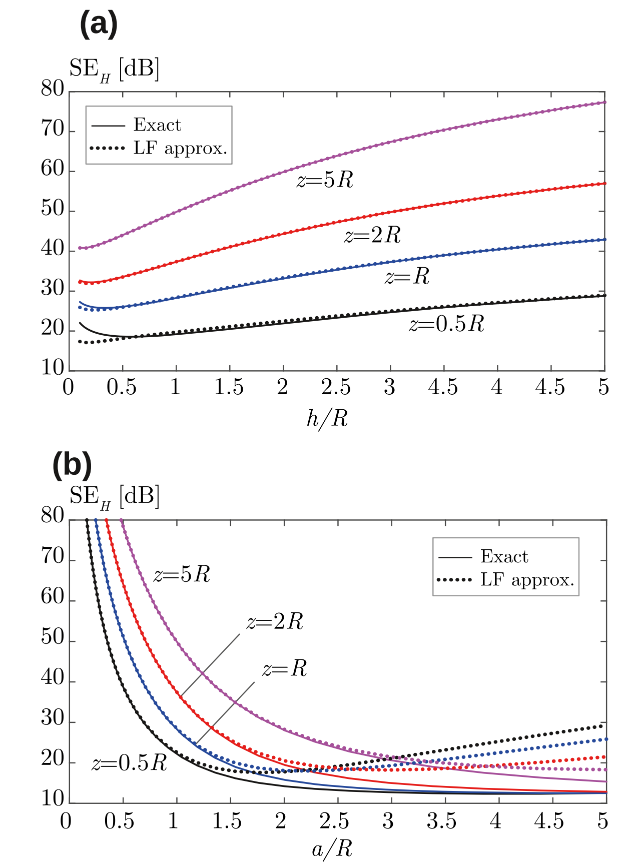

FIGURE 5 SE_H at $f = 1$ MHz as a function of h/R (a) ($2R = 30.4$ cm, $a = R$) and a/R (b) for $2R = 30.4$ cm, $h = R$

Finally, the proposed general formulation in Section 3 achieves a computational speed of about one order of magnitude with respect to the purely numerical FEKO simulations. For instance, to calculate 301 points for the curves in Figure 2 with a desktop PC Intel i9-9900-K, FEKO in the parallel module takes 216 s compared with 25 s for the proposed formulation. On the other hand, the low-frequency formulation of Section 4 provides instantaneous results with negligible computational time, thus completely removing any computation burden.

6 | CONCLUSION

An effective formulation for evaluating the magnetic shielding effectiveness of an infinite PEC planar screen with a circular aperture has been presented, applicable to the case of a current loop source with a finite radius coaxial with the aperture. By applying the Abel transform technique, the original set of dual integral equations for the equivalent magnetic currents defined on the aperture has been transformed into a single Fredholm integral equation of the second kind. Appropriate basis functions have been introduced that consider the edge conditions. The resulting formulation is rapidly convergent and accurate. In particular, for low frequencies, a closed form expression is extracted and numerical results are presented to assess its limits of validity. Work is in progress to develop a formulation for circular apertures in planar screens with finite conductivity.

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APPENDIX

In this appendix, Integral (40) is evaluated as the rapidly convergent series (42). The proof reported here was inspired by evaluating infinite integrals containing Bessel functions in Atson [24] and in Arts [32].

First, the integral in Equation (40) can be written as

$$I_{mn} = I^{\mathrm{R}}_{mn} + \mathrm{j} I^{\mathrm{J}}_{mn} \tag{A1}$$

where

$$I^{\mathrm{R}}_{mn} = \int_{k_0}^{\infty} \frac{\sqrt{\lambda^2 - k_0^2}}{\lambda^2} \mathrm{J}_{2m+1/2}(\lambda a) \mathrm{J}_{2n+1/2}(\lambda a)\, \mathrm{d}\lambda \tag{A2}$$

and

$$I^{\mathrm{J}}_{mn} = \int_{0}^{k_0} \frac{\sqrt{k_0^2 - \lambda^2}}{\lambda^2} \mathrm{J}_{2m+1/2}(\lambda a) \mathrm{J}_{2n+1/2}(\lambda a)\, \mathrm{d}\lambda \tag{A3}$$

At the basis of the proof, there is the integral identity [24]:

$$\mathrm{J}_{\mu}(\lambda a)\mathrm{J}_{\nu}(\lambda a) = \frac{1}{2\pi \mathrm{j}} \cdot \int_{c-\mathrm{j}\infty}^{c+\mathrm{j}\infty} \frac{\Gamma(-s)\Gamma(\mu+\nu+2s+1)\left(\frac{1}{2}\lambda a\right)^{\mu+\nu+2s}}{\Gamma(\mu+s+1)\Gamma(\nu+s+1)\Gamma(\mu+\nu+s+1)}\, \mathrm{d}s \tag{A4}$$

where the singularities of the integrand in Equation (A4) are the poles of $\Gamma(-s)$ at $s_p = p$ $(p = 0, 1, 2\ldots)$ and the poles of $\Gamma(\mu+\nu+2s+1)$ at $s_q = -(q+\mu+\nu+1)/2$ $(q = 0, 1, 2\ldots)$ [26]. Parameter c is therefore a real number such that $-(\mu+\nu+1)/2 < c < 0$.

Let us start with the integral in Equation (A3), which can thus be written as

$$I^{\mathrm{J}}_{mn} = \int_0^{k_0} \frac{\sqrt{k_0^2-\lambda^2}}{\lambda^2} \frac{1}{2\pi \mathrm{j}} \int_{c-\mathrm{j}\infty}^{c+\mathrm{j}\infty} \frac{\Gamma(-s)}{\Gamma\left(2m+s+\frac{3}{2}\right)} \cdot \frac{\Gamma(2m+2n+2s+2)\left(\frac{1}{2}\lambda a\right)^{2m+2n+2s+1}}{\Gamma\left(2n+s+\frac{3}{2}\right)\Gamma(2m+2n+s+2)}\, \mathrm{d}s\, \mathrm{d}\lambda \tag{A5}$$

By interchanging the two integrals in Equation (A5), we have

$$I^{\mathrm{J}}_{mn}=\frac{1}{2\pi \mathrm{j}}\int_{c-\mathrm{j}\infty}^{c+\mathrm{j}\infty}\frac{\Gamma(-s)}{\Gamma\left(2m+s+\frac{3}{2}\right)\Gamma\left(2n+s+\frac{3}{2}\right)}\cdot\frac{\Gamma(2m+2n+2s+2)}{\Gamma(2m+2n+s+2)}\left(\frac{1}{2}a\right)^{2m+2n+2s+1}\cdot\int_0^{k_0}\sqrt{k_0^2-\lambda^2}\,\lambda^{2m+2n+2s-1}\,\mathrm{d}\lambda\,\mathrm{d}s \tag{A6}$$

From the integral [26, Section 3.251],

$$\int_0^{k_0}\sqrt{x^2-y^2}\,y^\alpha\,\mathrm{d}y=\frac{x^{2+\alpha}\sqrt{\pi}\,\Gamma\left(\frac{\alpha+1}{2}\right)}{4\Gamma\left(\frac{\alpha}{2}+2\right)}\qquad \alpha>-1 \tag{A7}$$

it follows that

$$I^{\mathrm{J}}_{mn}=\frac{1}{2\pi \mathrm{j}}\int_{c-\mathrm{j}\infty}^{c+\mathrm{j}\infty}\frac{\Gamma(-s)}{\Gamma\left(2m+s+\frac{3}{2}\right)}\cdot\frac{\Gamma(2m+2n+2s+2)}{\Gamma\left(2n+s+\frac{3}{2}\right)\Gamma(2m+2n+s+2)}\left(\frac{1}{2}a\right)^{2m+2n+2s+1}\cdot\frac{k_0^{2m+2n+2s+1}\sqrt{\pi}\,\Gamma(m+n+s)}{4\Gamma\left(m+n+s+\frac{3}{2}\right)}\,\mathrm{d}s\qquad \mathrm{Re}[s]>-(m+n). \tag{A8}$$

In addition to poles $s_p=p$ ($p=0, 1, 2\ldots$) and $s_q=-m-n-q/2$ ($q=0, 1, 2\ldots$), the integrand function in Equation (A8) also has pole singularities in $s_r=-(r+m+n)$ ($r=0, 1, 2, \ldots$). Parameter c is thus chosen in the interval $-(m+n)<c<0$ so that all poles s_p are on the right of the integration path in Equation (A8) and all poles s_q and s_r are on the left. The integration path can then be closed to its right enclosing all poles s_p that have residues $R_p=(-1)^p/p!$ [26]. By letting $\sigma=m+n$ and $\delta=m-n$, from the Cauchy theorem, it thus follows:

$$I^{\mathrm{J}}_{mn}=\frac{\sqrt{\pi}}{4}\left(\frac{k_0a}{2}\right)^{2\sigma+1}\sum_{p=0}^{\infty}\frac{\Gamma(\sigma+p)}{\Gamma(2\sigma+p+2)\Gamma\left(\sigma+p+\frac{3}{2}\right)}\cdot\frac{\Gamma(2\sigma+2p+2)}{\Gamma\left(\sigma-\delta+p+\frac{3}{2}\right)\Gamma\left(\sigma+\delta+p+\frac{3}{2}\right)}\frac{(-1)^p}{\Gamma(p+1)}\left(\frac{k_0a}{2}\right)^{2p} \tag{A9}$$

and, by letting $l=p+\sigma$, we have

$$I^{\mathrm{J}}_{mn}=\frac{\sqrt{\pi}}{4}(-1)^{-\sigma}\sum_{l=\sigma}^{\infty}\frac{\Gamma(l)}{\Gamma(l+\sigma+2)\Gamma\left(l+\frac{3}{2}\right)}\cdot\frac{\Gamma(2l+2)}{\Gamma\left(l-\delta+\frac{3}{2}\right)\Gamma\left(l+\delta+\frac{3}{2}\right)}\frac{(-1)^l}{\Gamma(l-\sigma+1)}\left(\frac{k_0a}{2}\right)^{2l+1} \tag{A10}$$

By using the Legendre duplication formula for the Gamma function [26, Section 8.335], that is,

$$\Gamma(2x)=\frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right), \tag{A11}$$

Equation (A10) becomes

$$I^{\mathrm{J}}_{mn}=\frac{1}{4}(-1)^{-\sigma}\sum_{l=\sigma}^{\infty}\frac{(-1)^l2^{2l+1}\Gamma(l+1)}{\Gamma(l+\sigma+2)}\cdot\frac{\Gamma(l)}{\Gamma(l-\sigma+1)}\frac{1}{\Gamma\left(l-\delta+\frac{3}{2}\right)\Gamma\left(l+\delta+\frac{3}{2}\right)}\left(\frac{k_0a}{2}\right)^{2l+1} \tag{A12}$$

By using the Pochhammer symbol and the property:

$$(x-y)_y=\frac{\Gamma(x)}{\Gamma(x-y)}=(-1)^y(1-x)_y, \tag{A13}$$

we can express

$$I^{\mathrm{J}}_{mn}=\frac{1}{4}\sum_{l=\sigma}^{\infty}(-1)^{l-1}\frac{(-l+1)_{\sigma-1}}{(l+1)_{\sigma+1}}\cdot\frac{1}{\Gamma\left(l-\delta+\frac{3}{2}\right)\Gamma\left(l+\delta+\frac{3}{2}\right)}(k_0a)^{2l+1} \tag{A14}$$

Finally, considering $\mathrm{j}=(-1)^{1/2}$ and $(-l+1)_{\sigma-1}=0$ for $l=0, 1,\ldots, \sigma-1$, the summation in Equation (A14) can start at $j=0$, that is,

$$I^{\mathrm{J}}_{mn}=-\mathrm{j}\,\frac{1}{4}\sum_{l=0}^{\infty}(-1)^{l+1/2}\frac{(-l+1)_{\sigma-1}}{(l+1)_{\sigma+1}}\cdot\frac{1}{\Gamma\left(l-\delta+\frac{3}{2}\right)\Gamma\left(l+\delta+\frac{3}{2}\right)}(k_0a)^{2l+1} \tag{A15}$$

Using similar arguments for the integral in Equation (A2) and using the identity [26, Section 3.251]:

$$\int_{k_0}^{\infty}\sqrt{x^2-y^2}\,y^{\alpha}\,dy=\frac{x^{2+\alpha}\sqrt{\pi}\,\Gamma\left(-1-\frac{\alpha}{2}\right)}{4\Gamma\left(\frac{1-\alpha}{2}\right)}\qquad \alpha<-2 \tag{A16}$$

we have

$$I_{mn}^{\mathrm{R}}=\frac{\sqrt{\pi}}{4}\left(\frac{k_0a}{2}\right)^{2m+2n+1}\frac{1}{2\pi\mathrm{j}}\int_{c-\mathrm{j}\infty}^{c+\mathrm{j}\infty}\frac{\Gamma(-s)}{\Gamma(1-m-n-s)}$$
$$\cdot\frac{\Gamma\left(-m-n-\frac{1}{2}-s\right)\Gamma(2m+2n+2s+2)}{\Gamma\left(2n+s+\frac{3}{2}\right)\Gamma(2m+2n+s+2)\Gamma\left(2m+s+\frac{3}{2}\right)}$$
$$\cdot\left(\frac{k_0a}{2}\right)^{2s}\mathrm{d}s,\qquad \mathrm{Re}[s]<-\left(m+n+\frac{1}{2}\right). \tag{A17}$$

However, contrarily to what happens in Equation (A8), the poles of $\Gamma(-s)$ are cancelled by the zeros of $1/\Gamma(1-m-s)$; thus, it can be shown that the integrand function in Equation (A17) has instead pole singularities at $s_q=-m-n-1-q$ ($q=0,1,\ldots$) and in $s_p=p-m-n-1/2$ ($p=0,1,\ldots$). Parameter c is thus chosen in interval $-(m+n+1)<c<-(m+n+1/2)$ so that all poles s_p are on the right of the integration path in Equation (A17) and all poles s_q are on the left. The integration path can then be closed to its right enclosing all poles s_p that have residues $R_p=(-1)^p/p!$ [26]. From the Cauchy theorem, it thus follows:

$$I_{mn}^{\mathrm{R}}=\frac{\sqrt{\pi}}{4}\sum_{p=0}^{\infty}\frac{\Gamma(\sigma+1/2-p)}{\Gamma(-p+3/2)}\frac{\Gamma(2p+1)}{\Gamma(\sigma+p+3/2)}$$
$$\cdot\frac{(-1)^p}{\Gamma(p+1)}\frac{1}{\Gamma(p-\delta+1)\Gamma(p+\delta+1)}\left(\frac{k_0a}{2}\right)^{2p} \tag{A18}$$

By using Legendre duplication formula (A11) together with the Pochhammer symbols (43) and their properties, Equations (A13) and (A18) can finally be expressed as

$$I_{mn}^{\mathrm{R}}=\frac{1}{4}\sum_{p=0}^{\infty}(-1)^p\frac{\left(-p+\frac{3}{2}\right)_{\sigma-1}}{\left(p+\frac{1}{2}\right)_{\sigma+1}}$$
$$\cdot\frac{1}{\Gamma(p-\delta+1)\Gamma(p+\delta+1)}(k_0a)^{2p} \tag{A19}$$

Finally, combining Equations (A19) and (A15) in Equation (A1), a single power series in k_0a as in (42) is obtained, that is,

$$I_{mn}=\frac{1}{4}\sum_{i=0}^{\infty}(-1)^{i/2}\frac{\left(-\frac{i}{2}+\frac{3}{2}\right)_{m+n-1}}{\left(\frac{i}{2}+\frac{1}{2}\right)_{m+n+1}}$$
$$\cdot\frac{1}{\Gamma\left(\frac{i}{2}+n-m+1\right)\Gamma\left(\frac{i}{2}+m-n+1\right)}(k_0a)^{i} \tag{A20}$$

where the odd terms ($i=2L+1$) come from Equation (A15) and the even terms ($i=2p$) from Equation (A19).