

Critical Probabilities and Convergence Time of Percolation Probabilistic Cellular Automata

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Abstract This paper considers a class of probabilistic cellular automata undergoing a phase transition with an absorbing state. Denoting by $\mathcal{U}(x)$ the neighbourhood of site x , the transition probability is $T(\eta_x = 1 | \eta_{\mathcal{U}(x)}) = 0$ if $\eta_{\mathcal{U}(x)} = \mathbf{0}$ or p otherwise, $\forall x \in \mathbb{Z}$. For any \mathcal{U} there exists a non-trivial critical probability $p_c(\mathcal{U})$ that separates a phase with an absorbing state from a fluctuating phase. This paper studies how the neighbourhood affects the value of $p_c(\mathcal{U})$ and provides lower bounds for $p_c(\mathcal{U})$. Furthermore, by using dynamic renormalization techniques, we prove that the expected convergence time of the processes on a finite space with periodic boundaries grows exponentially (resp. logarithmically) with the system size if $p > p_c$ (resp. $p < p_c$). This provides a partial answer to an open problem in Toom et al. (Stochastic Cellular Systems: Ergodicity, Memory, Morphogenesis, pp. 1–182. Manchester University Press, Manchester, 1990; Topics in Contemporary Probability and its Applications, pp. 117–157. CRC Press, Boca Raton, 1995).

Keywords Probabilistic cellular automata · Percolation systems · Convergence time · Stavskaya's process · Absorbing-state phase transition

1 Introduction

Probabilistic cellular automata (PCA) are discrete-time Markov processes modelling the time evolution of a multicomponent system. Their main feature is the synchronous update of the states of the components, which take values in a finite set and interact with their neighbours according to a given probabilistic interaction rule.

Probabilistic cellular automata are favourable models to study non-equilibrium phenomena. Indeed, on the one hand, their definition is simple, as the space of realizations is discrete

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and interactions are local. On the other hand, despite this simplicity, they show a variety of complex behaviours.

One of the interesting phenomena involving probabilistic cellular automata is that they show a transition from an *ergodic* to a *non-ergodic* regime. Namely, after setting a free parameter above or below a certain critical threshold, at infinite time the process preserves part of the information on its initial state (non-ergodic behaviour). This means that the probability measure at infinite time depends on the initial state of the dynamics. On the contrary, if the process is ergodic, it admits a unique, attracting invariant measure. In [12] it has been shown that the non-ergodic regime of a PCA is connected to the existence of a phase transition for the PCA, considered as a statistical mechanics system.

Over the last 50 years, PCA have undergone intense analytical and numerical investigations (e.g. [4, 12, 24, 25]). However, as far as we know, many questions involving the rate of convergence to equilibrium or the characterisation of the invariant measures still remain open, even for the simplest models (see e.g. [23, 25] for a survey).

In this paper we consider a class of PCA that has a correspondence with percolation. These models are referred to as *Percolation Systems* in [23] and as *Percolation Operators* in [25]. From now on we will refer to them as *Percolation PCA*. This class includes the well studied *Stavskaya's process* (see e.g. [5, 6, 19–21, 23–25]), in which the neighbourhood of every site corresponds to the site itself and its right nearest neighbour on the one dimensional lattice. On the contrary in Percolation PCA the neighbourhood of every site could be any finite (translation invariant) set.

The reason why we decided to consider Percolation PCA is that their simplicity, combined with the presence of a phase transition, make them an interesting test case for attempts to characterise transient behaviour and stationary measures for spatially extended stochastic dynamics. Namely, Percolation PCA are a prominent model for studying *absorbing state phase transitions* [10], i.e. there exists a phase characterised by almost sure convergence into an “absorbing state” (a realisation where the process remains for ever whenever reached) and a fluctuating phase, where the process remains active at all times.

In this paper we discuss two distinct aspects of Percolation PCA. In Sect. 3 we consider Percolation PCA on a finite one dimensional lattice with periodic boundaries and we study the time the process needs to reach the absorbing state (*absorption time*). Our main result is stated in Theorem 2.1. We show that at the critical probability p_c there exists a transition from a fast to a slow convergence regime. Namely, we prove that the expected absorption time of the model grows exponentially (resp. logarithmically) as the size of the system grows if $p > p_c(\mathcal{U})$ (resp. $p < p_c(\mathcal{U})$). This provides a partial answer to the *Unsolved Problem 5.3.3* in [23] and to an open problem mentioned in [25, pp. 80–83]. If compared with [25], where the fast (resp. slow) convergence behaviour is proved for p small enough (resp. close enough to 1), our result provides a sharp estimation. The slow convergence regime can be interpreted as a metastable behaviour of the model, as the process spends an exceptionally long time into a non-stable state before falling into the absorbing state. Similar studies on the metastable behaviour of PCA models were recently presented also in [2, 3, 13], although the methods used there do not apply in our case, as Percolation PCA are not reversible and do not have a naturally associated potential. Numerical estimations of $p_c(\mathcal{U})$ (e.g. [11, 14, 18]) are obtained assuming that the metastable regime (the actual regime observed in numerical simulations, as there is no way to really simulate “infinite space” in computers) is observed only for all values of p at which the infinite process is in the fluctuating phase. Although this fact might appear obvious in terms of physical intuition, Theorem 2.1 provides a justification for this assumption from a rigorous mathematical point of view.

The proof of our result relies almost entirely on the correspondence between Percolation PCA and oriented percolation in two dimensions. This connection has been described for the first time in [22]. The proof of the statement of the theorem involving the case of $p < p_c$ is an application of some percolation estimations presented in [8]. We generalize these estimates to the percolation model considered here, which differs from [8] as here the neighbourhood is an arbitrary finite set and because sites (instead of bonds) can be open or closed. The proof of the statement involving the case of $p > p_c$ is more technical and is based on (1) the generalization of the dynamic-block argument presented in [8] to the case of non symmetric neighbourhood with more than two elements and (2) the estimation of the probability of a certain event involving a dual lattice construction provided by [22].

In Sect. 4 we study how the neighbourhood affects the critical probability. We provide a lower bound for critical probabilities $p_c(\mathcal{U})$ and our result is stated in Theorem 2.2. With our estimations we improve the previous lower bound [17] showing that $p_c(\mathcal{U}) > 1/2$ strictly if the neighbourhood $\mathcal{U} = \{-1, 0, 1\}$. Furthermore, we provide new bounds in case of neighbourhoods not considered before (as far as we know). The comparison with numerical estimations, provided in the last section of this article, shows that our bounds are sharp. In order to derive the lower bound we studied the temporal evolution of “absorbed sets” (sets of adjacent sites all in state “zero”). If these sets on average are expanding, the realisation at infinite time is “all zeros” almost surely. This idea comes from [25, Chapter 6]. Our estimations take into account a certain aspect of the dynamics, i.e. absorbed sets can dynamically merge one with the other. This leads to an improvement of the bound.

We shall end this introductory section by presenting the structure of the paper. In Sect. 2 we define the model and we present our main results, Theorems 2.1 and 2.2. In Sect. 3, divided into three subsections, we prove Theorem 2.1. In Sect. 3.1 we describe the correspondence between Percolation PCA and oriented percolation in two dimension, following [22,25]. In Sect. 3.2 we present several percolation estimations from [8] used to prove of the theorem. Finally in Sect. 3.3 we prove Theorem 2.1. In Sect. 4 we prove Theorem 2.2.

2 Definition and Results

Probabilistic Cellular Automata (PCA) are discrete-time Markov chains on a product space, $\Sigma = X^S$. In this paper we consider both the case of infinite space, $S = \mathbb{Z}$, and of finite space, $S = \mathbb{S}_n, \mathbb{S}_n := \{-n, -n + 1, \dots, n - 2, n - 1\}$.

We consider the case of boolean variables, $X = \{0, 1\}$. Realisations of the process are denoted by $\eta \in \Sigma$. For any $x \in S$ and any $K \subset S$, use η_x to denote the x -th component of the vector η and η_K to designate the set of components corresponding to the sites of K .

We introduce a *neighbourhood function* on S . We first fix a finite set $\mathcal{U} = \{s_1, s_2, \dots, s_u\} \subset \mathbb{Z}$, assuming that $s_1 < s_2 < \dots < s_u$. If $S = \mathbb{Z}, \forall x \in S$ we define the neighbourhood of x as $\mathcal{U}(x) = \{s_1, s_2, \dots, s_u\} + x$. If $S = \mathbb{S}_n$ we consider periodic boundaries. Namely, for any $\forall x \in \mathbb{S}_n$ we define the neighbourhood of x as

$$\mathcal{U}(x) = \{|x + s_1 + n|_{2n} - n, |x + s_2 + n|_{2n} - n, \dots, |x + s_u + n|_{2n} - n\}, \tag{1}$$

where $|x|_{2n}$ denotes $x \pmod{2n}$. For example, if $\mathcal{U} = \{0, 1\}$, the neighbourhood of the site $n - 1$ is $\mathcal{U}(n - 1) = \{n - 1, -n\}$. For any set $K \subset S$, we define the neighbourhood of K as $\mathcal{U}(K) = \bigcup_{x \in K} \mathcal{U}(x)$.

In Percolation PCA the states of the process are synchronously updated at every site according to the following *transition probability*,

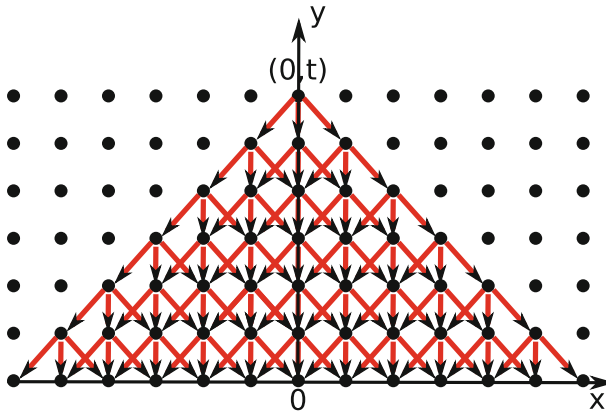


Fig. 1 Representation of the graph $\mathcal{G}_{\mathcal{U}}$ with neighbourhood $\mathcal{U} = \{-1, 0, 1\}$. In this figure only edges between vertices belonging to the evolution cone of $(0, t)$ have been drawn

$$T_x(\eta'_x = 1 | \eta_{\mathcal{U}(x)}) = \begin{cases} 0 & \text{if } \eta_{\mathcal{U}(x)} = \mathbf{0} \\ p & \text{otherwise} \end{cases}, \tag{2}$$

where $p \in [0, 1]$ is a free parameter.¹

The temporal evolution of the process can be represented by introducing a linear operator \mathcal{P} , which acts on the space of probability measures $\mathcal{M}(\Sigma)$. For any $\mu \in \mathcal{M}(\Sigma)$, we use $\mu\mathcal{P}$ to denote the measure obtained applying \mathcal{P} to μ . By using $C_{\eta'_K}$ to denote the cylinder set $C_{\eta'_K} = \{\eta \in \Sigma : \eta_K = \eta'_K\}$, with $K \subset S$, the measure $\mu\mathcal{P}$ is defined as

$$\mu\mathcal{P}(C_{\eta'_K}) = \sum_{\eta_{\mathcal{U}(K)} \in \{0,1\}^{\mathcal{U}(K)}} \mu(C_{\eta_{\mathcal{U}(K)}}) \prod_{x \in K} T_x(\eta'_x | \eta_{\mathcal{U}(x)}). \tag{3}$$

In order to characterise the time evolution of PCA, it is useful to introduce the set of *space-time realisations*, $\tilde{\Sigma} = \{0, 1\}^V$, where $V = S \times \mathbb{N}$ is the *space-time set*. The elements of $\tilde{\Sigma}$ are the realisations of the process at all times, $\tilde{\eta} = (\eta^t)_{t=0}^\infty \in \tilde{\Sigma}$. We then introduce an oriented graph $\mathcal{G}_{\mathcal{U}} = (V, \mathbf{E}_{\mathcal{U}})$, whose edges connect any vertex $(x, t) \in V$ to the vertices $(k, t - 1) \in V$, where $k \in \mathcal{U}(x)$. The vertices that can be reached from $(x, t) \in V$ through a path on $\mathcal{G}_{\mathcal{U}}$ constitute the *evolution cone* of $(0, t)$ (Fig. 1).

We now introduce some definitions that will be used along the whole article.

Definition 1 (*Evolution Measure*) Consider the Percolation PCA (3) with $S = \mathbb{Z}$ (respectively $S = S_n$ and periodic boundaries). For every $\mu \in \mathcal{M}(\Omega)$, we define the *evolution measure* \mathcal{E}_μ (respectively \mathcal{E}_μ^n) as the joint probability distribution of measures $\mu, \mu\mathcal{P}^1, \mu\mathcal{P}^2, \dots$

For example, we use $\mathcal{E}_{\delta_1^n}$ to denote the evolution measure of the Percolation PCA on finite space, starting from the realisation “all ones”.

Definition 2 (*Expectation on the evolution space*) Consider the Percolation PCA (3) with $S = \mathbb{Z}$ (respectively $S = S_n$ and periodic boundaries). We use $\mathbb{E}_\mu[\cdot]$ (respectively $\mathbb{E}_\mu^{(n)}[\cdot]$) to denote the expectation in relation to the evolution measure \mathcal{E}_μ (respectively \mathcal{E}_μ^n).

¹ We use a different notation from [23–25]: here p corresponds to $1 - \epsilon$ and zeroes and ones are inverted.

Monotonicity It is immediate from the definition of transition probability that the Dirac measure $\delta_{\mathbf{0}}$, where $\mathbf{0} = (0, 0, 0, \dots)$, is stationary, i.e. $\delta_{\mathbf{0}} = \delta_{\mathbf{0}}\mathcal{P}$. Furthermore, the operator \mathcal{P} of this stochastic process is *monotone*. Monotonicity of \mathcal{P} means that it preserves partial order among elements of $\mathcal{M}(\Sigma)$. We first introduce partial order “ $<$ ” in Σ by defining for any two realizations $\eta, \eta' \in \Sigma, \eta < \eta' \Leftrightarrow \forall x \in S \eta_x \leq \eta'_x$. We then introduce the functions $\varphi : \Sigma \mapsto \mathbb{R}$, which only depend on a finite number of sites. We call φ *monotone iff* for any $\eta, \eta' \in \Sigma, \eta < \eta' \Rightarrow \varphi(\eta) \leq \varphi(\eta')$. We then introduce partial order in $\mathcal{M}(\Sigma)$ by defining $\mu < \mu' \Leftrightarrow$ for any *monotone* function $\varphi, \int \varphi d\mu \leq \int \varphi d\mu'$. Finally, we introduce an order relation between operators and we introduce the notion of monotone operator.

Definition 3 (Monotone operator) An operator $P : \mathcal{M}(\Sigma) \mapsto \mathcal{M}(\Sigma)$ is called monotone if for any pair of measures $\mu, \mu' \in \mathcal{M}(\Sigma), \mu < \mu' \Rightarrow \mu P < \mu' P$.

The operator (3) of the Percolation PCA is monotone. This property follows from the fact that the transition probability (2) preserves order locally, i.e. for any $x \in S,$

$$\eta_{\mathcal{U}(x)}^1 < \eta_{\mathcal{U}(x)}^2 \Rightarrow T_p(\eta_x = 1 | \eta_{\mathcal{U}(x)}^1) \leq T_p(\eta_x = 1 | \eta_{\mathcal{U}(x)}^2),$$

(see for example [25, p. 28] for a proof of this). Monotonicity of \mathcal{P} implies that the probability measure,

$$v_p := \lim_{t \rightarrow \infty} \delta_{\mathbf{1}} \mathcal{P}^t, \tag{4}$$

exists and it is invariant.

Definition 4 (Critical Probability) Consider the Percolation PCA on \mathbb{Z} with finite neighbourhood $\mathcal{U} \subset \mathbb{Z}$. We define the *critical probability* as,

$$p_c(\mathcal{U}) = \sup_{p \in [0,1]} \{v_p = \delta_{\mathbf{0}}\}. \tag{5}$$

Definition 5 (Ergodic Operator) An operator $\mathcal{P} : \mathcal{M}(\Sigma) \rightarrow \mathcal{M}(\Sigma)$ is ergodic if the two following conditions hold: **(a)** there exists a unique $\varphi \in \mathcal{M}(\Sigma)$ such that $\varphi P = \varphi$ and **(b)** $\forall \mu \in \mathcal{M}(\Sigma), \lim_{t \rightarrow \infty} \mu P^t = \varphi$.

For any $p > p_c$ the evolution operator of the Percolation PCA is not ergodic. Indeed, in this case $\delta_{\mathbf{0}}$ and $v_p \neq \delta_{\mathbf{0}}$ (defined in 4) are two distinct invariant measures. For any $p < p_c$, the Percolation PCA (3) is ergodic.

In [21,22] it has been proved that

$$p_c(\mathcal{U}) \in (0, 1)$$

for the Stavskaya’s process ($\mathcal{U} = \{0, 1\}$) and a more general proof in case of general neighbourhood can be found in [25]. The proofs are based on two methods widely used in statistical mechanics, namely, the counting path method and the Peierls argument.

Our main result is stated in Theorem 2.1 and it involves the convergence time into the absorbing state of the Percolation PCA with finite space and periodic boundaries, as defined at the beginning of this section.

When S is finite, the process is always ergodic (Definition 5). Indeed, for any realisation of the process $\eta^t \in \Sigma$ at time t , the probability that η^{t+1} = “all zeroes” is bounded from below by the constant $(1 - p)^{|S|}$. This implies that there exists almost surely a finite time $\tau \in \mathbb{N}$ such that η^t = “all zeroes” for all $t \geq \tau$. Hence, for any $\mu \in \mathcal{M}(\Sigma), \lim_{t \rightarrow \infty} \mu P^t = \delta_{\mathbf{0}}$.

We define the *absorption time* $\tau_k \in \mathbb{N}_0$, representing the first time all sites in the segment $\{-k, -k + 1, \dots, k - 1\}$ are in state zero for η^τ .

Definition 6 For all $k \in \mathbb{N}$, we call the absorption time of the interval $\{-k, -k+1, \dots, k-1\}$ the random variable $\tau_k : \tilde{\Sigma} \rightarrow \mathbb{N}$,

$$\tau_k(\tilde{\eta}) = \min\{t \in \mathbb{N}_0 \text{ s.t. } \tilde{\eta}_x^t = 0 \quad \forall x \in [[-k, k-1]]\}, \tag{6}$$

where $[[-k, k - 1]]$ denotes the set of integers in $[-k, k - 1]$.

In case $S = \mathbb{S}_n$, this random variable is well defined only if $k \leq n$.

We recall Definitions 1 and 2 and we state our main result.

Theorem 2.1 Consider the Percolation PCA with space \mathbb{S}_n , periodic boundaries and finite neighbourhood $\mathcal{U} = \{s_1, s_2, \dots, s_u\}$, where s_1, s_2, \dots, s_u are some distinct elements of \mathbb{Z} . For every $p \in [0, 1]$ there exist $n_0 \in \mathbb{N}$ and some positive constants $K_1, K_2, K_3, K_4, c_1, c_2, c_3, c_4$ (dependent on p) such that for all $n > n_0$,

- (a) if $p < p_c$, $K_1 \log(c_1 n) \leq \mathbb{E}_{\delta_1}^{(n)}[\tau_n] \leq K_2 \log(c_2 n)$,
- (b) if $p > p_c$, $K_3 \exp(c_3 n) \leq \mathbb{E}_{\delta_1}^{(n)}[\tau_n] \leq K_4 \exp(c_4 n)$.

The proof of the theorem is presented in Sect. 3.

Our second result is stated in the following theorem and it involves the estimation of p_c as a function of \mathcal{U} .

Theorem 2.2 Consider the Percolation PCA on \mathbb{Z} with finite neighbourhood $\mathcal{U} = \{s_1, s_2, s_3, \dots, s_u\}$, where $s_1, s_2, \dots, s_u \in \mathbb{Z}$ are such that $s_1 < s_2 < \dots < s_u$. Define $p_1 := \frac{2}{2+s_u-s_1}$ and p_2 as the unique solution in the interval $(0, 1]$ of the following equation,

$$p = p_1 \cdot \frac{1}{1 - \frac{\varphi(p)}{s_u-s_1+2}}, \tag{7}$$

where $\varphi(p) = \frac{(1-p)^6+(1-p)^{2(s_u-s_1)}}{p(2-p)}$. Then $p_c(\mathcal{U}) \geq p_2$.

The proof of the theorem is presented in Sect. 4. From (7) it follows that $p_2 > p_1$, as $\varphi(p)$ is positive in $(0, 1)$. Our analytical lower bounds can be compared with numerical estimations in Fig. 2, where Percolation PCA with neighbourhood $\mathcal{U} = \{0, 1, 2, \dots, k\}$, for some positive integers k have been considered. Results of numerical simulations can be found also in the appendix of this article. The numerical estimation of p_c in case $\mathcal{U} = \{-1, 0\}$ can be found in [15] and the numerical estimation in case $\mathcal{U} = \{-1, 0, 1\}$ can be found both in the appendix and in [17].

3 Absorption Time

In this section we prove Theorem 2.1. In Sect. 3.1 we describe the connection between Percolation PCA and oriented percolation. We mainly follow [22,25], although propositions and statement have been reformulated emphasising the differences between Percolation PCA on a finite and infinite space. In Sect. 3.2 we list some percolation estimates. Some of these percolation estimates have been proved in [7–9] in the case of oriented bond percolation with symmetric neighbourhood. In this article we consider a similar model, namely, oriented site percolation with arbitrary neighbourhood. The proofs of these estimates in our case are substantially the same of those provided in [7,8]. We sketch them illustrating the small differences. In Sect. 3.3 we prove the theorem. The proof of the right inequality of statement (a) of the theorem is an application of the estimates presented in Sect. 3.2. The proof of the

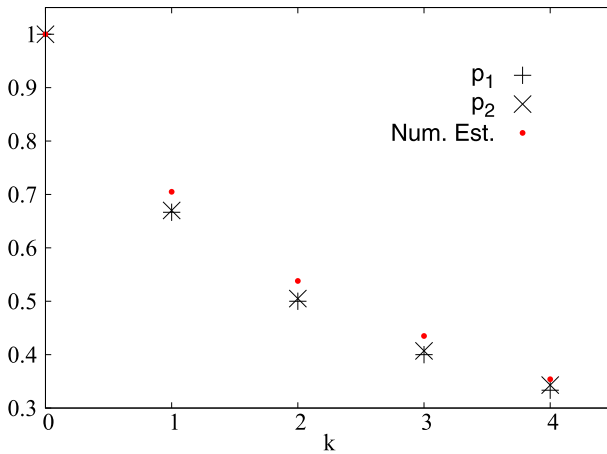


Fig. 2 Numerical and analytical estimations of critical probabilities of Percolation PCA with neighborhood $\mathcal{U} = \{0, 1, \dots, k\}$

left inequality can be found in [21]. The proof of the right inequality of statement (b) is trivial. The proof of the left inequality uses some of the percolation estimates and the estimation provided Proposition 8, which is stated in the same section. The original contribution of the author consists in the proof and the application of Proposition 8 to the proof of the statement (b), in the estimations based on path constructions used in the proof of statement (b) and in the generalization of the percolation estimates to the proof of the statement (a).

3.1 Relations with Oriented Percolation

In this section we describe a connection between the Percolation PCA and a certain percolation model. This connection has been pointed out for the first time in [22], as far as we know. We consider a Percolation PCA with space $S \in \{\mathbb{S}_n, \mathbb{Z}\}$, as defined in Sect. 2. We define an auxiliary space $\Omega = \{0, 1\}^V$, we denote by $\omega \in \Omega$ its elements and we introduce in this space the Bernoulli product measure \mathbb{P}_p . Namely, the state of every component is 1 with probability p and 0 with probability $1 - p$ independently. We declare a vertex $(x, y) \in V$ “open” if $\omega_{x,y} = 1$ and “closed” otherwise. Percolation PCA are related to percolation as the probability that the state of the site $x \in S$ is 1 at time $t \in \mathbb{N}_0$ for equals the probability that the site $(x, t) \in V$ is connected by a path of open vertices in $\mathcal{G}_{\mathcal{U}}$ to the line $y = 0$. This is precisely the meaning of the statement of Proposition 2, which is stated below.

In order to describe this connection rigorously, we represent the Percolation PCA starting from an initial realisation $\eta^i \in \Sigma$ by introducing a deterministic mapping

$$D : \Omega \times \Sigma \longrightarrow \tilde{\Sigma}.$$

For every $(x, t) \in V$, the component $D_x^t : \Omega \times \Sigma \rightarrow \{0, 1\}$ of D is defined as

$$D_x^t := \begin{cases} \min \{ \omega_{x,t-1}, \max_{k \in \mathcal{U}(x)} \{ D_k^{t-1} \} \}, & \text{if } t \in \mathbb{Z}_+ \\ \eta_x^i, & \text{if } t = 0, \end{cases} \tag{8}$$

where $(\omega_{x,t})_{x \in S, y \in \mathbb{N}}$ are elements of Ω . This mapping defines any $D_z^T, z \in S, T \in \mathbb{Z}_+$, as a function of the variables $\omega_{x,y}$ associated to vertices belonging to the evolution cone of

$(z, T) \in V$, and of the initial realisation η^i . One should observe that, recalling (2) and using independence, for any $x \in S, t \in \mathbb{Z}_+, a \in \{0, 1\}, \eta_{\mathcal{U}(x)} \in \{0, 1\}^{\mathcal{U}(x)}, \eta^i \in \Sigma$,

$$T_x(a|\eta_{\mathcal{U}(x)}) = \mathbb{P}_p(\omega \in \Omega \text{ s.t. } D_x^t(\omega, \eta^i) = a | \omega \in \Omega \text{ s.t. } D_{\mathcal{U}(x)}^{t-1}(\omega, \eta^i) = \eta_{\mathcal{U}(x)}) \\ := \mathbb{P}_p^{\eta^i}(D_x^t = a | \eta_{\mathcal{U}(x)}), \tag{9}$$

where in the last expression we rewrote the second quantity in a more compact form. This notation will be used also in the proof of the next proposition.

Proposition 1 *Consider Percolation PCA with space $S \in \{\mathbb{S}_n, \mathbb{Z}\}$, represented by the operator $\mathcal{P} : \mathcal{M}(\Sigma) \rightarrow \mathcal{M}(\Sigma)$. Then, for any $\eta^i \in \Sigma, a \in \{0, 1\}$,*

$$\delta_{\eta^i} \mathcal{P}^t(\eta_x = a) = \mathbb{P}_p(w \in \Omega \text{ s.t. } D_x^t(\omega, \eta^i) = a). \tag{10}$$

Proof For any $x \in S, t \in \mathbb{Z}^+$, we define

$$\mathcal{U}^t(x) = \overbrace{\mathcal{U} \circ \mathcal{U} \circ \dots \circ \mathcal{U}}^t(x).$$

By using Eq. (9), we observe that the following equalities hold,

$$\mathbb{P}_p(D_x^t(\omega, \eta^i) = a) = \sum_{\eta_{\mathcal{U}(x)}^{t-1} \in \{0, 1\}^{\mathcal{U}(x)}} \mathbb{P}_p^{\eta^i}(D_x^t = a | \eta_{\mathcal{U}(x)}^{t-1}) \mathbb{P}_p(D_{\mathcal{U}(x)}^{t-1}(\omega, \eta^i) = \eta_{\mathcal{U}(x)}^{t-1}) \\ = \sum_{\eta_{\mathcal{U}(x)}^{t-1} \in \{0, 1\}^{\mathcal{U}(x)}} T_x(a | \eta_{\mathcal{U}(x)}^{t-1}) \mathbb{P}_p(D_{\mathcal{U}(x)}^{t-1}(\omega, \eta^i) = \eta_{\mathcal{U}(x)}^{t-1}) \\ = \sum_{\eta_{\mathcal{U}(x)}^{t-1} \in \{0, 1\}^{\mathcal{U}(x)}} \sum_{\eta_{\mathcal{U}^2(x)}^{t-2} \in \{0, 1\}^{\mathcal{U}^2(x)}} T_x(a | \eta_{\mathcal{U}(x)}^{t-1}) \\ \times \mathbb{P}_p^{\eta^i}(\eta_{\mathcal{U}(x)}^{t-1} | \eta_{\mathcal{U}^2(x)}^{t-2}) \mathbb{P}_p(D_{\mathcal{U}^2(x)}^{t-2}(\omega, \eta^i) = \eta_{\mathcal{U}^2(x)}^{t-2}) \\ = \sum_{\eta_{\mathcal{U}(x)}^{t-1} \in \{0, 1\}^{\mathcal{U}(x)}} \sum_{\eta_{\mathcal{U}^2(x)}^{t-2} \in \{0, 1\}^{\mathcal{U}^2(x)}} T_x(a | \eta_{\mathcal{U}(x)}^{t-1}) \\ \times \prod_{y \in \mathcal{U}(x)} T(\eta_y^{t-1} | \eta_{\mathcal{U}(y)}^{t-2}) \mathbb{P}_p^{\eta^i}(D_{\mathcal{U}^2(x)}^{t-2}(\omega, \eta^i) = \eta_{\mathcal{U}^2(x)}^{t-2}). \tag{11}$$

By proceeding with the expansion, we obtain the next formula,

$$\mathbb{P}_p(D_x^t(\omega, \eta^i) = a) = \sum_{\eta_{\mathcal{U}(x)}^{t-1} \in \{0, 1\}^{\mathcal{U}(x)}} \sum_{\eta_{\mathcal{U}^2(x)}^{t-2} \in \{0, 1\}^{\mathcal{U}^2(x)}} \dots \sum_{\eta_{\mathcal{U}^t(x)}^0 \in \{0, 1\}^{\mathcal{U}^t(x)}} T_x(\eta_x^t = a | \eta_{\mathcal{U}(y)}^{t-1}) \\ \prod_{k=1}^{t-1} \left[\prod_{y \in \mathcal{U}^k(x)} T_y(\eta_y^{t-k} | \eta_{\mathcal{U}(y)}^{t-k-1}) \right] \mathbb{P}_p(D_{\mathcal{U}^t(x)}^0(\omega, \eta^i) = \eta_{\mathcal{U}^t(x)}^0). \tag{12}$$

We observe that by definition of $D, \mathbb{P}_p(D^0(\omega, \eta^i) = \eta) = 1$ if $\eta = \eta^i$ and 0 otherwise. By using the same expansion for the measure $\delta_{\eta^i} \mathcal{P}^t$ and cylinder set $\{\eta \in \Omega : \eta_x = a\}$ in (3), one derives again formula (12) and concludes that Eq. (10) holds. \square

The next proposition has been proved in [22].

Proposition 2 *The function $D_x^t : \Omega \times \Sigma \mapsto \{0, 1\}$ is such that $D_x^t = 1$ iff there exists a sequence $\{x_0, x_1, x_2, \dots, x_t\} \subset \mathbb{Z}$ satisfying the three following properties,*

1. $x_t = x$ and $x_{i-1} \in \mathcal{U}(x_i)$ for any $i \in \{1, 2, \dots, t\}$,
2. $\omega_{i-1, x_i} = 1$ for any $i \in \{1, 2, \dots, t\}$,
3. $\eta_{x_0}^t = 1$.

Proof We sketch the proof of the proposition. Assume $D_x^t = 1$ and assume that properties 1, 2, 3 hold for a sequence of sites $x_{t-k}, x_{t-k+1}, \dots, x_t$. From (8) it follows that $D_{x_{t-k}}^{t-k} = 1 \Leftrightarrow \omega_{x_{t-k-1}, t-k} = 1$ and $\exists x_{t-k-1} \in \mathcal{U}(x_{t-k})$ s.t. $D_{x_{t-k-1}}^{t-k-1} = 1$. This implies that there exists an element $x_{t-k-1} \in S$ such that properties 1, 2, 3 hold for the sequence $x_{t-k-1}, x_{t-k}, \dots, x_t$. The proof of the proposition follows by induction. □

If we consider the case of infinite space, from the previous proposition it follows that ergodicity for the probabilistic cellular automaton is associated with the existence of an infinite path of open vertices in the auxiliary space. Indeed, recall Definitions 4 and 5 and observe that,

$$p > p_c \implies \lim_{t \rightarrow \infty} \delta_1 \mathcal{P}^t(\eta_x = 1) > 0 \tag{13}$$

$$p < p_c \implies \lim_{t \rightarrow \infty} \delta_1 \mathcal{P}^t(\eta_x = 1) = 0, \tag{14}$$

Thus the probabilistic cellular automaton is non-ergodic if and only if the limit $t \rightarrow \infty$ of the probability that a vertex $(0, t)$ is connected to the line $y = 0$ by an open path is positive.

If we consider the case of finite space with periodic boundaries, the previous proposition shows that there is a connection between the absorption time of the probabilistic cellular automaton and the existence of an open path in the auxiliary space. This connection is clarified in the next proposition. Before its statement we introduce some more definitions.

From now on we use $\mathbb{P}_p^n(\cdot)$ to denote the Bernoulli product measure in the finite space and $\mathbb{P}_p(\cdot)$ to denote the Bernoulli product measure in the infinite space.

Definition 7 Consider $S \in \{\mathbb{S}_n, \mathbb{Z}\}$ and consider the event,

$$\left\{ \omega \in \Omega \text{ s.t. there exists a path of open vertices in } \mathcal{G}_{\mathcal{U}} \text{ that connects } (x, t) \text{ to one of the vertices belonging to the line } y = 0 \right\}. \tag{15}$$

If $S = \mathbb{Z}$ we denote this event by $\{(s, t) \xrightarrow{\mathcal{G}_{\mathcal{U}}} S^0\}$ and if $S = \mathbb{S}_n$ we denote this event by $\{(s, t) \xrightarrow{\mathcal{G}_{\mathcal{U}}^{(n)}} S^0\}$.

Recall the definition of evolution measure (Definition 1) and of absorption time (Eq. 6). Recall that τ_k can be considered as a function $\tau_k : \Omega \times \Sigma \rightarrow \mathbb{N}$, as, from (8), $(D_x^t)_{x \in S, t \in \mathbb{N}}$ it is a mapping from $\Omega \times \Sigma$ to $\tilde{\Sigma}$.

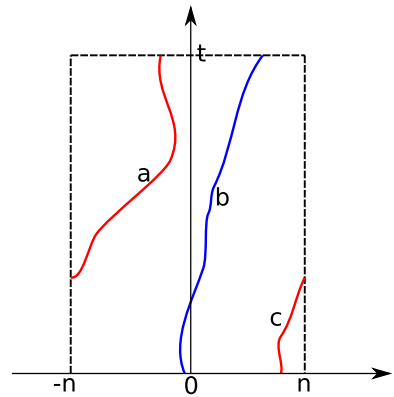
Proposition 3 *Consider the Percolation PCA on a finite space with periodic boundaries. For every $t \in \mathbb{N}_0$,*

$$\mathcal{E}_{\delta_1}^n(\tau_n > t) = \mathbb{P}_p^n \left(\exists x \in [[-n, n - 1]] \text{ s.t. } (x, t) \xrightarrow{\mathcal{G}_{\mathcal{U}}^{(n)}} S^0 \right), \tag{16}$$

where S^0 denotes the set of vertices of V belonging to the line $y = 0$.

Proof By the definition of τ_n (see Definition 6), $\tau_n > t$ if and only if $\exists x \in [[-n, n - 1]]$ such that $D_x^t = 1$. From Proposition 2, it follows that $D_x^t = 1$ if and only if $(x, t) \xrightarrow{\mathcal{G}_{\mathcal{U}}^{(n)}} S^0$. □

Fig. 3 The event $\{\tau_n > t\}$ (recall Definition 6) occurs if at least one open path joins one of the sites (x, t) such that $x \in [[-n, n - 1]]$ to one of the sites $(y, 0)$, $y \in S$. If the neighbourhood is periodic, then the path can leave from one the two vertical lines $x = -n$ or $x = n - 1$ and re-appear at the same high on the other line (e.g. see the path $a \circ c$)



Remark Recall the definition of the neighbourhood in the case of finite space with periodic boundaries, provided in Eq. (1). As boundaries are periodic, the site (x, t) is connected to the line $y = 0$ also if the path of open vertices leaves one of the vertical boundaries ($x = -n$ or $x = n - 1$) from one side and it re-appears at the same high on the other side (see for example the path $a \circ c$ in Fig. 3).

3.2 Percolation Estimates

In this section we list some properties involving the cluster of vertices belonging to an open path in \mathcal{G}_U starting from $(0, t)$. These properties have been proved in [7–9] in case of a bond percolation model with symmetric neighbourhood of two elements. In this article we consider a slightly different percolation model, as sites instead of bonds can be open or closed and the neighbourhood is an arbitrary (translation invariant) finite set. The proofs of these propositions in the case considered in this article are similar to those provided in [7–9]. We sketch their proof describing the small differences.

We start with some definitions. From now on we will consider $S = \mathbb{Z}$. For every $t, m \in \mathbb{N}$ we define the sets,

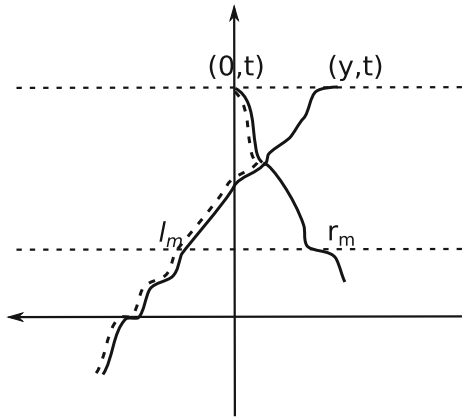
$$\begin{aligned} \xi_m^t &= \left\{ x \in \mathbb{Z} : (0, t) \xrightarrow{\mathcal{G}_U} (x, t - m) \right\}, \\ \bar{\xi}_m^t &= \left\{ x \in \mathbb{Z} : \exists z \leq 0 \text{ s.t. } (z, t) \xrightarrow{\mathcal{G}_U} (x, t - m) \right\}, \\ \bar{\chi}_m^t &= \left\{ x \in \mathbb{Z} : \exists z \geq 0 \text{ s.t. } (z, t) \xrightarrow{\mathcal{G}_U} (x, t - m) \right\}, \end{aligned} \tag{17}$$

Note that $\xi_m^t \subset \{s_1 m, s_1 m + 1, s_1 m + 2, \dots, s_u m\}$. We define then the variables,

$$\begin{aligned} r_m^t &= \sup \{ \xi_m^t \}, \\ \ell_m^t &= \inf \{ \bar{\xi}_m^t \}, \\ \bar{r}_m^t &= \sup \{ \bar{\xi}_m^t \}, \\ \bar{\ell}_m^t &= \inf \{ \bar{\chi}_m^t \}, \end{aligned} \tag{18}$$

and we set $r_m^t = -\infty, \ell_m^t = \infty$ if $\xi_m^t = \emptyset$. As the distributions of $r_m^t, \ell_m^t, \bar{r}_m^t, \bar{\ell}_m^t, \xi_m^t, \bar{\xi}_m^t$ and $\bar{\chi}_m^t$ depend only on the difference $t - m$, from now on we will omit the dependence on t , that will be some positive integer. Furthermore, we consider the space \mathcal{G}_U as before, but with

Fig. 4 Representation of open paths



vertices $\mathbb{Z} \times \mathbb{Z}$ instead of $\mathbb{Z} \times \mathbb{N}$. In the former case, if we consider only paths starting from $(0, t)$, we allow $(0, t)$ to belong to an infinite open path. Thus we recover the notation of [8] $(r_m, \ell_m, \bar{r}_m, \xi_m)$, with the difference that in this article paths are oriented from up to down.

We observe that for every t, m , the probability that $\bar{\xi}^m = \emptyset$ is zero, as every vertex in $\{(x, y) : \text{s.t. } y = t, x \geq 0\}$ has a non-zero probability of being connected to S^0 by an open path in \mathcal{G}_U . The same holds for the event $\bar{\chi}^{m,t} = \emptyset$. By definition,

$$\begin{aligned} r_m &\leq \bar{r}_m, \\ \ell_m &\geq \bar{\ell}_m. \end{aligned} \tag{19}$$

The following relations hold,

$$\xi_m = \bar{\xi}_m \cap [\ell_m, +\infty) = \bar{\chi}_m^t \cap (-\infty, r_m], \tag{20}$$

$$\text{on } \{\xi_m \neq \emptyset\}, \quad r_m = \bar{r}_m, \tag{21}$$

$$\text{on } \{\bar{\xi}_m \neq \emptyset\}, \quad \ell_m = \bar{\ell}_m. \tag{22}$$

Proof Equation (20) is a corollary of Eqs. (21) and (22). We sketch an argument for (22), that can be also found in [8, Section 3]. By reflection the same argument holds also for (21). It is trivial from the definition that $\xi_m \subset \bar{\xi}_m$ and that $\xi_m \subset (-\infty, r_m]$. We have to show that $\bar{\xi}_m \cap (-\infty, r_m] \subset \xi_m$. In this case it is clear from Fig. 4 that if there is a path from some site $(y, t), y > 0$ to $(x, t - m), x \leq r_m$, then there is also a path from $(0, t)$ to $(x, t - m)$. Then $x \in \xi_m$. \square

We introduce the following quantities, for all integers $n \geq m \geq 0$,

$$\begin{aligned} \bar{r}_{m,n} &= \sup \{x - \bar{r}_m : x \in \mathbb{Z} \text{ and } \exists z \in \mathbb{Z} \text{ s.t.} \\ &\quad z \leq \bar{r}_m \text{ and } (z, t - m) \xrightarrow{\mathcal{G}_U} (x, t - n)\}. \end{aligned} \tag{23}$$

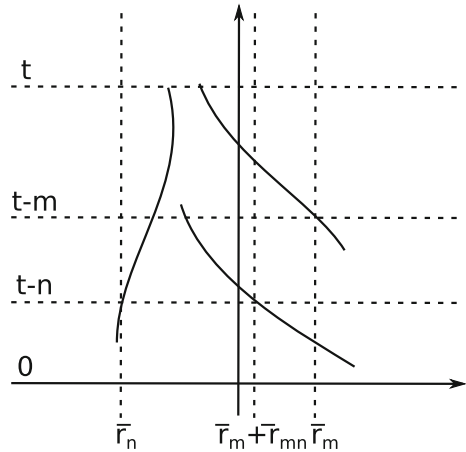
$$\begin{aligned} \bar{\ell}_{m,n} &= \inf \{x - \bar{\ell}_m : x \in \mathbb{Z} \text{ and } \exists z \in \mathbb{Z} \text{ s.t.} \\ &\quad z \geq \bar{\ell}_m \text{ and } (z, t - m) \xrightarrow{\mathcal{G}_U} (x, t - n)\}. \end{aligned} \tag{24}$$

The following relations holds.

$$\bar{r}_m + \bar{r}_{m,n} \geq \bar{r}_n. \tag{25}$$

$$\bar{\ell}_m + \bar{\ell}_{m,n} \leq \bar{\ell}_n. \tag{26}$$

Fig. 5 Curves represent open paths



Proof We prove (25) and a similar argument holds for (26). One should observe that $\bar{r}_m + \bar{r}_{m,n}$ is the rightmost point on the line $y = t - n$ which can be reached from any of the points $(x, t - m)$ with $x \leq \bar{r}_m$. Instead \bar{r}_n is the rightmost point on the line $y = t - n$ which can be reached from any of the points $(x, t - m)$ with $x \leq \bar{r}_m$ and with the additional restriction that there exists an open path in \mathcal{G}_U from (z, t) to $(x, t - m)$ for some $z \leq 0$. See also Fig. 5. \square

The next proposition involves the random variables defined above and it corresponds to [7, Theorem 2.1]. It holds for a class of model called *growth processes* that is more general than the class of models considered here. We refer to [7] for its proof, which is based on the subadditivity property of (25) and some arguments similar to those used in the proof of the Kingman’s Subadditive Ergodic Theorem.

Proposition 4 *Let \bar{r}_m and $\bar{\ell}_m$ be the quantities defined above. Then there exist two constants $\alpha \in [-\infty, s_u]$ and $\beta \in [s_1, +\infty]$ such that,*

$$\bar{r}_m/m \rightarrow \alpha \text{ almost surely,} \tag{27}$$

$$\bar{\ell}_m/m \rightarrow \beta \text{ almost surely.} \tag{28}$$

Let $E^t \subset \Omega$ be the following event,

$$E^t := \{ \text{“there exists an infinite open path starting from } (0,t)\text{”} \}.$$

Then, if $p > p_c$, conditioning on E^t , from Proposition 4 and from Eqs. (20–22) the following properties hold,

$$\lim_{m \rightarrow \infty} r_m/m = \alpha \text{ almost surely,} \tag{29}$$

$$\lim_{m \rightarrow \infty} \ell_m/m = \beta \text{ almost surely,} \tag{30}$$

$$\beta \leq \alpha. \tag{31}$$

Proof If $p > p_c$ then the event E^t occurs with positive probability. Conditioning on E^t , for all $m \geq 0$ $r_m \geq \ell_m$. Furthermore, from Eqs. (21) and (22) it follows that $r_m = \bar{r}_m$ and $\ell_m = \bar{\ell}_m$. \square

We define now the variable,

$$\gamma := \alpha - \beta, \tag{32}$$

which plays the role played by α in [8]. The proof of the next proposition can be found in [8, Section 3], in case of bond percolation with symmetric neighbourhood. As the statement is needed for the proof of Theorem 2.1, we sketch its proof, adapting it to the model considered in this article.

Proposition 5 *Let γ be the variable defined in Eq. (32). Then,*

$$p_c = \inf \{p : \gamma(p) > 0\}. \tag{33}$$

Proof Observe that Eq. (31) implies that,

$$\alpha < \beta \implies p \leq p_c. \tag{34}$$

Then, to prove Eq. (33), first it is necessary to show that,

$$\gamma > 0 \implies p > p_c. \tag{35}$$

Indeed, Eqs. (34) and (35) imply that

$$\sup \{p : \gamma(p) < 0\} \leq p_c \leq \inf \{p : \gamma(p) > 0\}. \tag{36}$$

Hence, it remains to exclude the possibility that the interval $\{p : \gamma(p) = 0\}$ has positive length. This fact is a consequence of the following property,

$$p > p' \text{ and } \alpha(p') > -\infty \implies \alpha(p) - \alpha(p') \geq (p - p'). \tag{37}$$

and of the fact that $\beta(p)$ is non-decreasing with p . For the proof of (37) we refer to [8, Section 3], as the symmetry of the neighbourhood does not play any role in the proof. The proof is based on the construction of two systems with parameter p and p' on the same space by assigning an independent random variable $U_{x,y}$ to each vertex $(x, y) \in V$ which is uniformly distributed on $(0, 1)$. The vertex is open if $U_{x,y} < p$ and closed otherwise. The only difference from [8] is that these random variables are assigned to bonds and that the set of vertices of the graph is different, i.e. $\{(x, y) \text{ s.t. } x + y \text{ is even}\}$.

In the remaining part of the proof we prove Eq. (35). Observe that if $\gamma > 0$, then $\bar{r}_m - \alpha m + \frac{\gamma}{2}m = \bar{r}_m - \frac{\alpha + \beta}{2}m \rightarrow \infty$ and $\bar{\ell}_m - \beta m - \frac{\gamma}{2}m = \bar{\ell}_m - \frac{\alpha + \beta}{2}m \rightarrow -\infty$ almost surely. Then there exists an integer $M < \infty$ such that,

$$\mathbb{P}_p \left(\forall m, \bar{r}_m > \frac{\alpha + \beta}{2}m - M \right) \geq 0.51, \tag{38}$$

$$\mathbb{P}_p \left(\forall m, \bar{\ell}_m < \frac{\alpha + \beta}{2}m + M \right) \geq 0.51. \tag{39}$$

Secondly we introduce the following notation. If $A \subset (-\infty, +\infty)$, then we let

$$\xi_m^A := \{x : \exists y \in A \text{ s.t. } (y, t) \rightarrow (x, t - m)\}, \tag{40}$$

$$r_m^A := \sup \xi_m^A, \tag{41}$$

$$\ell_m^A := \inf \xi_m^A, \tag{42}$$

$$\tau^A := \inf \{m : \xi_m^A = \emptyset\}. \tag{43}$$

Repeating the proof of (20–22) (see also [8, Section 3, Equation 10]), it follows that

$$\begin{aligned} \tau^{[-M, M]} &= \inf \left\{ m : r_m^{[-M, M]} < \ell_m^{[-M, M]} \right\} \\ &= \inf \left\{ m : r_m^{[-\infty, M]} < \ell_m^{[-M, \infty]} \right\}. \end{aligned} \tag{44}$$

The previous equality implies that,

$$\left\{ \tau^{[-M, M]} = \infty \right\} \supset \left\{ \ell_m^{[-M, \infty)} \leq \frac{\alpha + \beta}{2} m \leq r_m^{(-\infty, M]}, \quad \forall m \right\}.$$

As

$$\mathbb{P}_p \left(r_m^{(-\infty, M]} > \frac{\alpha + \beta}{2} m, \quad \forall m \right) = \mathbb{P}_p \left(r_m^{(-\infty, 0]} > \frac{\alpha + \beta}{2} m - M, \quad \forall m \right),$$

and

$$\mathbb{P}_p \left(\ell_m^{[-M, +\infty)} < \frac{\alpha + \beta}{2} m, \quad \forall m \right) = \mathbb{P}_p \left(\ell_m^{[0, +\infty)} > \frac{\alpha + \beta}{2} m + M, \quad \forall m \right),$$

it follows that

$$\mathbb{P}_p \left(\xi_m^{[-M, M]} \neq \emptyset, \quad \forall m \right) \geq 0.02.$$

Since, $\forall M > 0$,

$$\mathbb{P}_p \left(\xi_M^0 \supset \mathbb{Z} \cap [-M, M] \right) > 0,$$

it follows that $\mathbb{P}_p(E^t) > 0$. Then $p > p_c$. □

The next estimates have been proved in [9]. The proof can be found also in [8, Section 7, Equations (1) and (2)]. In particular Eq. (45) holds for a wide class of percolation models in the subcritical regime (see [1] for a proof in a very general setting).

Proposition 6 *Recall Definition 7. For every p , let $a(p) > \alpha(p)$ and $b(p) < \beta(p)$. If $p < p_c$ there exist some positive constants h, h_2, h_3, C_2, C_3 (dependent on p) such that,*

$$\mathbb{P}_p \left((0, m) \xrightarrow{\mathcal{G}\mathcal{U}} S^0 \right) \leq \exp(-h m), \tag{45}$$

$$\mathbb{P}_p \left(\bar{r}_m > a m \right) \leq C_2 \exp(-h_2 m), \tag{46}$$

$$\mathbb{P}_p \left(\bar{\ell}_m < b m \right) \leq C_3 \exp(-h_3 m). \tag{47}$$

Proof We sketch the proof of (45), which is similar to the proof of (46) and (47). If $p < p_c$, then from Eq. (33) $\alpha < \frac{\alpha + \beta}{2} < \beta$. Thus there exists an N large enough such that $E[\bar{r}_{0, N}] < \frac{\alpha + \beta}{2} N$, $E[\bar{\ell}_{0, N}] > \frac{\alpha + \beta}{2} N$. By using the subadditivity property of $\bar{r}_{m, n}$ and $\bar{\ell}_{m, n}$ one can see that,

$$\begin{aligned} \bar{r}_{mN} - \frac{\alpha + \beta}{2} mN &\leq S_m := \bar{r}_{0, N} - \frac{\alpha + \beta}{2} N + \bar{r}_{N, 2N} - \frac{\alpha + \beta}{2} N \\ &\quad + \dots + \bar{r}_{(m-1)N, mN} - \frac{\alpha + \beta}{2} N, \\ \frac{\alpha + \beta}{2} mN - \bar{\ell}_{mN} &\leq S'_m := \frac{\alpha + \beta}{2} N - \bar{\ell}_{0, N} + \frac{\alpha + \beta}{2} N - \bar{\ell}_{N, 2N} \\ &\quad \dots + \frac{\alpha + \beta}{2} N - \bar{\ell}_{(m-1)N, mN}. \end{aligned} \tag{48}$$

The right hand side of the two previous inequalities is a random walk with expectation respectively $E[S_1] < 0$, $E[S'_1] < 0$. As $S_1 \leq s_u N$, $S'_1 \leq s_u N$, then $\varphi(\theta) := E[\exp(\theta S_1)] < \infty$

and $\varphi'(\theta) := E[\exp(\theta S'_1)] < \infty$ for all $\theta > 0$. From the considerations in [8] it follows that we can pick $\theta_0 > 0$ with $\varphi(\theta_0) < 1$ and $\varphi'(\theta_0) < 1$ such that,

$$\begin{aligned} \mathbb{P}_p(S_m \geq 0) &\leq E[\exp(\theta_0 S_m)] = \varphi(\theta_0)^m, \\ \mathbb{P}_p(S'_m \geq 0) &\leq E[\exp(\theta_0 S_m)] = \varphi'(\theta_0)^m. \end{aligned}$$

This implies that $\mathbb{P}_p(\bar{r}_{mN} \geq \frac{\alpha+\beta}{2}mN) \rightarrow 0$ and $\mathbb{P}_p(\bar{\ell}_{mN} \leq \frac{\alpha+\beta}{2}mN) \rightarrow 0$ exponentially fast. Observe also that as $\mathbb{P}_p(\xi_m = \emptyset) \geq \mathbb{P}_p(\bar{r}_m < \frac{\alpha+\beta}{2}m < \bar{\ell}_m)$, then

$$\mathbb{P}_p(\xi_m \neq \emptyset) \leq \mathbb{P}_p\left(\bar{r}_m \geq \frac{\alpha + \beta}{2}m\right) + \mathbb{P}_p\left(\bar{\ell}_m \leq \frac{\alpha + \beta}{2}m\right).$$

This implies (45). □

We end this section recalling a property proved in [16]. As the reference is in Russian, we sketch its proof below.

Proposition 7 *Recall Definition 7. For every $t, n \in \mathbb{N}$,*

$$\mathbb{P}_p^n\left((0, t) \xrightarrow{\mathcal{G}\mathcal{U}^{(n)}} S^0\right) \leq \mathbb{P}_p\left((0, t) \xrightarrow{\mathcal{G}\mathcal{U}} S^0\right). \tag{49}$$

Proof Observe that in $\mathbb{Z} \times \mathbb{Z}$ all paths of length t starting from $(0, t)$ lie within $\Delta = [s_1t, s_ut] \times [0, t - 1]$. At each point we have a random variable $\omega_{x,y}$ that is equal to 1 with probability p and to 0 with probability $1 - p$ and these random variables are mutually independent. We consider the same set Δ but with a different set of random variables $z_{x,y}$. Each $z_{x,y}$ is equal to 1 with probability p and 0 with probability $1 - p$, but these random variables are not independent. Namely, for all (x, y) , the random variables $z_{x+2kn,y}$ for all integers k such that $(x + 2kn, y) \in \Delta$ have the same outcome (i.e. they are “synchronized”). This model is equivalent to the model on the cylinder $\Delta_n \times [0, t - 1]$ (i.e. with periodic boundaries), where $\omega_{x,y}$ are independent, because in these two models their probabilistic spaces and sets of open paths starting at $(0, t)$ are isomorphic.

Let then \mathcal{W} be the set of all possible paths of length t from $(0, 0)$. We will show that “synchronization” does not increase the probability of the existence of an open path of length t on Δ .

Let then $\theta_{x,y}$ be some random variables with values 0 or 1 associated with $(x, y) \in \Delta$. Consider the function Z , with arguments $\theta_{x,y}$,

$$Z = \sum \prod_{h \in \mathcal{W}(x,y) \in h} \theta_{x,y}.$$

Then $Z \geq 0$ and $Z > 0$ if and only if there exists an open path. Suppose that at the beginning $\theta_{x,y} = \xi_{x,y}$, for all $(x, y) \in \Delta$ and at each step we “synchronize” the variables $\theta_{a+2kn,b}$ for a certain (a, b) until we get $\theta_{x,y} = z_{x,y}$ for all $(x, y) \in \Delta$. We will show that each synchronization step does not increase Z . To do this, we write

$$Z = \sum_{k:(a+2kn,b) \in \Delta} \theta_{a+2kn,b} f_k(\tilde{\theta}) + g(\tilde{\theta}),$$

where $\tilde{\theta}$ is the set of all $(x, y) \neq (a + 2kn, b)$, i.e. they are independent from the group $\theta_{a+2kn,b}$. The f_k and g are some functions with non-negative integer values. Here we use the

fact that a path can contain only one point of the form $(a + 2kn, b)$, so different $\theta_{a+2kn,b}$ don't multiply. Before the "synchronization" step,

$$Z = Z_1 = \sum_{k:(a+2kn,b) \in \Delta} \omega_{a+2kn,b} f_k(\tilde{\theta}) + g(\tilde{\theta}),$$

and after it,

$$Z = Z_2 = z_{a,b} \cdot \sum_{k:(a+2kn,b) \in \Delta} f_k(\tilde{\theta}) + g(\tilde{\theta}).$$

It is easy to show that, fixing any value of the set $\tilde{\theta}$, $P(Z_1 > 0) \geq P(Z_2 > 0)$. Hence, the same is true when $\tilde{\theta}$ is not fixed. □

3.3 Proof of Theorem 2.1

Recall the definitions provided just before the statement of the theorem. Along the whole proof we denote by $\mathbb{P}^n(\cdot)$ the Bernoulli product measure in Σ , where the space is finite, and by $\mathbb{P}(\cdot)$ the Bernoulli product measure in Σ , where the space is infinite. The proof is based on the estimation of $\mathcal{E}_{\delta_1}^n(\tau_n > t)$, which gives the expectation,

$$\mathbb{E}_{\delta_1}^{(n)}[\tau_n] = \sum_{t=0}^{\infty} \mathcal{E}_{\delta_1}^n(\tau_n > t), \tag{50}$$

We prepare the reader to the proof of the left inequality of the statement (b). The proofs of the other inequalities do not need an introduction, as they are simpler. The proof is based on the estimation of the probability of the event $\{\tau_n > t\}$. In order to provide this estimation, first we define the event $\mathcal{D}_{n,t,a}$, whose probability is less than the probability of $\{\tau_n > t\}$. The event occurs if a path connects $[-n, n - 1] \times \{t\}$ to the line $y = 0$ without crossing the diagonal sides of a parallelogram (a rigorous definition is given later). This allows to reduce the estimation of $\{\tau_n > t\}$ to the estimation of the probability of an event that is simpler to study, as periodic boundaries play no role.

As the neighbourhood of the model is in general non symmetric, the cluster of vertices belonging to an open path starting from $(0, t)$ (which is infinite with positive probability, as $p > p_c$) will typically have a drift. Indeed, recall Proposition 4 and the fact that $\bar{r}_t \sim t(\frac{\alpha+\beta}{2} + \frac{\gamma}{2})$ and $\bar{\ell}_t \sim t(\frac{\alpha+\beta}{2} - \frac{\gamma}{2})$, $\beta \leq \alpha$. Thus, as p is slightly larger than p_c , then typically the cluster of vertices will be centred around $\sim \frac{\alpha+\beta}{2}t$. Hence, the diagonal sides of the parallelogram is chosen in such a way that in the limit $t \rightarrow \infty$ the cluster has typically a non-empty intersection with the parallelogram. With this choice, the probability of the event $\mathcal{D}_{n,t,a}$ does not go to zero too fast as t grows.

Later we introduce a change of coordinates T_b^t that allows to simplify the notation, by transforming the graph in a new graph, where the cluster of vertices connected by an open path starting from $(0, t)$ (namely, in the new graph $\frac{\alpha+\beta}{2} = 0$). We provide a lower bound for the probability of $\mathcal{D}_{n,t,a}$ by introducing a new event \mathcal{H}_n and by using the FKG inequality to bound the probability of $\mathcal{D}_{n,t,a}$ with a product of probabilities of events \mathcal{H}_n .

In the last part of the proof we estimate the probability of the event \mathcal{H}_n , showing that it goes to 1 fast enough with n for any $p > p_c$. This estimation is stated in Proposition 8.

Proof of part (a) The proof of the left inequality of the statement (a) can be found in [21, Section 2] together with an estimation of the constants, so we do not provide it here. Indeed,

the left inequality holds for any $p \in [0, 1]$. The proof of the right inequality of statement (a) is an application of the estimates presented in Sect. 3.2. Starting from (50),

$$\begin{aligned}
 \mathbb{E}_{\delta_1}^{(n)}[\tau_n] &= \sum_{t=1}^{\infty} \mathbb{P}_p^n \left(\bigcup_{s=-n}^{n-1} \left\{ (s, t) \xrightarrow{\mathcal{G}_U^{(n)}} S^0 \right\} \right) \\
 &\leq \sum_{t=1}^{\infty} \min \left\{ 1, 2n \mathbb{P}^n \left((0, t) \xrightarrow{\mathcal{G}_U^{(n)}} S^0 \right) \right\} \\
 &\leq \sum_{t=1}^{\infty} \min \{ 1, 2n \exp(-ht) \} \\
 &\leq \frac{\log(2n)}{h} + K,
 \end{aligned} \tag{51}$$

where K is some positive constant. In the first equality we used Proposition 3, in the second inequality we used the union bound and translation invariance, in the second-last inequality we used (45) and (49). The algebraic tricks of (51) have been used also in the proof of [25, Proposition 8.6].

Proof of part (b) The proof of the right inequality of the statement (b) is trivial. We define a new process $(q_x^t)_{x \in \mathbb{S}_n}$ where every q_x^t is 1 with probability p and 0 with probability $1 - p$ independently. Observe that for all $x \in \mathbb{S}_n, t \in \mathbb{Z}_+, q_x^t \geq \eta_x^t$, as long as the two processes are driven by the same random process. Hence, the expected convergence time for the former is larger than the convergence time for the latter. By a simple computation, the expected convergence time for the system q^t is $\frac{1}{1-p} 2n$. This implies the inequality.

We start with the proof of the left inequality. For every $a \in \mathbb{R}$ we define the event,

$$\begin{aligned}
 \mathcal{D}_{n,t,a} &:= \{ \exists x \in [[-n, n - 1]], \text{ such that } (x, t) \text{ is connected to } S^0 \\
 &\quad \text{by an open path in } \mathcal{G}_U \text{ that never crosses the lines} \\
 &\quad y = n - 1 - a(x - t), y = -n - a(x - t) \ x \in \mathbb{R} \},
 \end{aligned} \tag{52}$$

which is a subset of Ω , recalling that S^0 denotes the set of vertices belonging to the line $y = 0$. See also Fig. 6—up for a representation.

Recall Definition 7 and observe that,

$$\mathbb{P}_p(\mathcal{D}_{n,t,a}) \leq \mathbb{P}_p^n \left(\exists x \in [[-n, n - 1]] \text{ s.t. } (x, t) \xrightarrow{\mathcal{G}_U^{(n)}} S^0 \right) = \mathcal{E}_{\delta_1}^n(\tau_n > t). \tag{53}$$

Observe that the quantity on the left is defined in the infinite system and the quantities in the middle and on the right are defined on the finite system with periodic boundaries. We provide a proof of the statement below.

Proof of (53) Consider two graphs, \mathcal{G}_U^i and \mathcal{G}_U^f . The former is defined on the infinite space $\mathbb{Z} \times \mathbb{N}_0$ and the latter on the finite space $\mathbb{S}_n \times \mathbb{N}_0$ with periodic boundaries, as defined in Sect. 2. Let $Q_{a,t} \subset \mathbb{Z} \times \mathbb{N}_0$ be the region inside the parallelogram identified by the points $(-n, t), (n - 1, t), (-n + at, 0), (n - 1 + at, 0)$ (see Fig. 7). The event $\mathcal{D}_{n,a,t} \subset \{0, 1\}^{\mathbb{Z} \times \mathbb{N}_0}$ occurs if an open path connects $[-n, n - 1] \times \{t\}$ to $[-n + at, n - 1 + at] \times \{0\}$ without ever crossing the diagonal sides of the parallelograms. We couple the model on the finite space and the model on the infinite space in the following way. Namely, call $\omega_{x,y}$, for all $(x, y) \in \mathbb{Z} \times \mathbb{N}_0$, and $z_{x,y}$, for all $(x, y) \in \mathbb{S}_n \times \mathbb{N}_0$, the random variables taking values 0 or 1 independently. The coupling is such that for all $(x, y) \in Q_{a,t}, \omega_{x,y} = z_{x',y'}$, where $x' = |x + n|_{2n} - n, y' = y$, where $|x|_n$ denotes $x \bmod n$. The random variables $\omega_{x,y}$ associated to sites (x, y) not contained

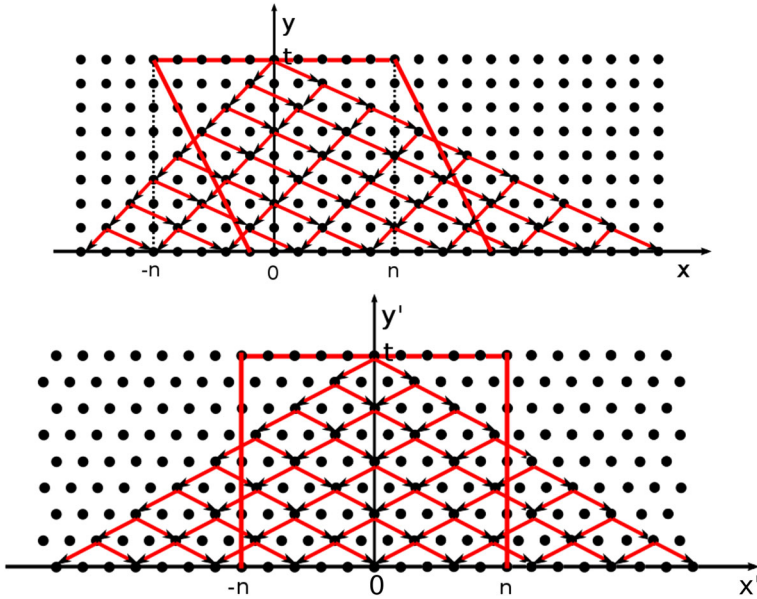
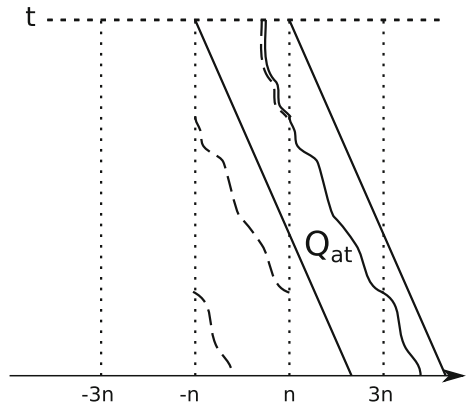


Fig. 6 Up Representation of $\mathcal{G}_{\mathcal{U}}$ in case of neighbourhood $\mathcal{U} = \{-1, 2\}$. For graphical reasons only edges belonging to the evolution cone of $(0, t)$ have been drawn. In the figure $a = \frac{s_1 + s_2}{2}$. Down the same graph of the figure above, transformed via (54) with parameter $b = a$

Fig. 7 Representation of the event $\mathcal{D}_{n,a,t}$



in $Q_{a,t}$ are not coupled. Observe that for every $(x, y) \in Q_{a,t}$ there exists a unique (x', y') in \mathbb{S}_n and vice versa. Recalling that boundaries of $\mathcal{G}_{\mathcal{U}}^f$ are periodic, one can observe from Fig. 7 that, as long as there exists an open path in $\mathcal{G}_{\mathcal{U}}^i$ connecting the top to the bottom of $Q_{a,t}$ and never crossing its diagonal sites (e.g. the path represented by a continuous curve in the figure), there exists also an open path in $\mathcal{G}_{\mathcal{U}}^f$ connecting $[-n, n - 1] \times \{t\}$ to $[-n, n - 1] \times \{0\}$ (i.e. the path represented by a dashed curve in the figure). This implies the statement. \square

Consider now the following change of coordinates,

$$\begin{cases} x' = x - b(t - y) \\ y' = y \end{cases}, \tag{54}$$

under which the graph \mathcal{G}_U is transformed into the new graph $T_b^t \mathcal{G}_U$. We denote by $T_b^t \mathcal{D}_{n,t,a}$ the event $\mathcal{D}_{n,t,a}$, defined for the graph $T_b^t \mathcal{G}_U$, (i. e. replace \mathcal{G}_U with $T_b^t \mathcal{G}_U$ in the definition of the event above). See Fig. 6 for an example. The following equation holds,

$$\mathbb{P}_p(T_b^t \mathcal{D}_{n,t,a}) = \mathbb{P}_p(\mathcal{D}_{n,t,a-b}), \tag{55}$$

as the change of coordinates preserves connection between vertices. Now we introduce the event \mathcal{H}_n ,

$$\begin{aligned} \mathcal{H}_n = \{ & \exists y, y' \text{ s.t. } y \in [[4n, 6n]], y' \in [[0, 2n]] \\ & \text{and } (-n, y) \xrightarrow{\mathcal{G}_U} (n, y') \}, \end{aligned} \tag{56}$$

which is represented in Fig. 8-right. The following proposition is about this event.

Proposition 8 *For any $p > p_c$ there exist positive constants A, b (dependent on p) such that for any $t \in \mathbb{N}$ and for n large enough,*

$$\mathbb{P}_p\left(T_{\frac{\alpha+\beta}{2}}^t \mathcal{H}_n\right) \geq 1 - A \exp(-bn). \tag{57}$$

As before, the event $T_{\frac{\alpha+\beta}{2}}^t \mathcal{H}_n$ denotes the occurrence of \mathcal{H}_n in the graph $T_{\frac{\alpha+\beta}{2}}^t \mathcal{G}_U$. We recall that α and β are defined in Sect. 3.2. We first use Proposition 8 to conclude the proof of the theorem and later we prove the Proposition 8. Define then the new event $\mathcal{F}_{n,t}$, which is represented in Fig. 8. $\mathcal{F}_{n,t}$ occurs iff (a) and (b) hold:

- (a) for every odd $j \in [[0, \frac{t}{2n}]]$ there is a vertex $(-n, y)$, with $y \in [[2nj, 2n(j + 1)]]$, connected to (n, y') by an open path in \mathcal{G}_U , with $y' \in [[2n(j - 2), 2n(j - 1)]]$,
- (b) for any even $j \in [[0, \frac{t}{2n}]]$ there is a vertex (n, y) , with $y \in [[2nj, 2n(j + 1)]]$, connected by an open path in \mathcal{G}_U to $(-n, y')$, with $y' \in [[2n(j - 2), 2n(j - 1)]]$.

Note first that,

$$\mathbb{P}_p\left(T_{\frac{\alpha+\beta}{2}}^t \mathcal{F}_{n,t}\right) \leq \mathbb{P}_p\left(T_{\frac{\alpha+\beta}{2}}^t \mathcal{D}_{n,t,0}\right), \tag{58}$$

because if $\mathcal{F}_{n,t}$ occurs, then the top of the box $2n \times t$ is connected to the bottom by a path that never goes out from the box (compare Fig. 8-left and middle). Secondly, we observe that the event $T_{\frac{\alpha+\beta}{2}}^t \mathcal{F}_{n,t}$ equals the intersection of $\lfloor \frac{t}{n} \rfloor$ events of type $T_{\frac{\alpha+\beta}{2}}^t \mathcal{H}_n$, represented in Fig. 8-right.

As the event \mathcal{H}_n increasing, the FKG inequality is applicable, i.e.

$$\mathbb{P}_p\left(T_{\frac{\alpha+\beta}{2}}^t \mathcal{H}_n\right)^{\lfloor \frac{t}{n} \rfloor} \leq \mathbb{P}_p\left(T_{\frac{\alpha+\beta}{2}}^t \mathcal{F}_{n,t}\right) \tag{59}$$

Then using (57) finally we get,

$$\mathbb{P}_p\left(T_{\frac{\alpha+\beta}{2}}^t \mathcal{F}_{n,t}\right) \geq \left(1 - A \cdot n \exp(-nb)\right)^{\lfloor \frac{t}{n} \rfloor} \tag{60}$$

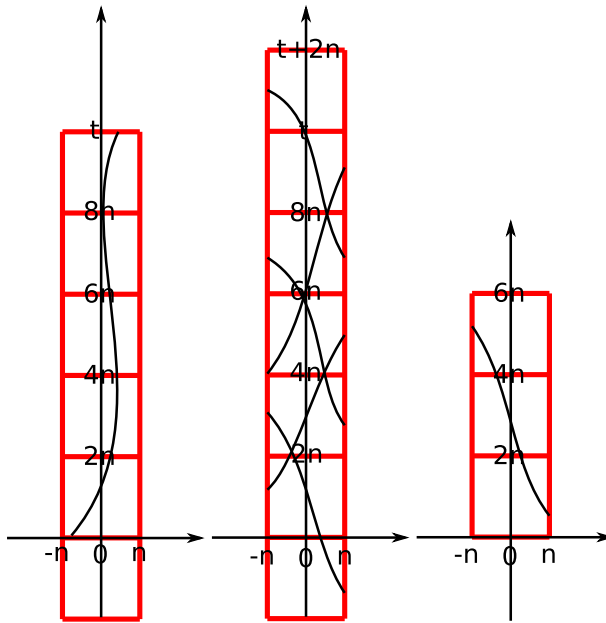


Fig. 8 *Left* representation of the event $\mathcal{D}_{n,t,0}$. *Center* representation of the event $\mathcal{F}_{n,t}$. *Right* representation of the event \mathcal{H}_n . In all figures the details on the structure of the graph have been omitted

Then, from (50) and for n large enough,

$$\begin{aligned}
 \mathbb{E}_{\delta_1}^{(n)}[\tau_n] &\geq \sum_{i=1}^{\infty} \mathbb{P}_p\left(T_{\frac{\alpha+\beta}{2}}^i \mathcal{D}_{n,t,0}\right) \\
 &\geq \sum_{i=1}^{\infty} \mathbb{P}_p\left(T_{\frac{\alpha+\beta}{2}}^i \mathcal{F}_{n,t}\right) \\
 &\geq \sum_{i=1}^{\infty} \left(1 - A \cdot \exp(-bn)\right)^{\lfloor \frac{i}{n} \rfloor} \\
 &\geq j \left(1 - \frac{A \cdot e^{-bn} j}{n}\right),
 \end{aligned} \tag{61}$$

where j is an arbitrary integer. In the previous expression we have used Proposition 3, (53) and (55) for the first inequality, (58) for the second inequality and (60) for the third one. Choosing finally $j = \lfloor \frac{n e^{bn}}{2A} \rfloor$, the part (b) of the theorem follows.

Proof of Proposition 8 We prepare the reader to the proof of the proposition and later we present the proof. We consider two graphs, $T_{\frac{\alpha+\beta}{2}}^t \mathcal{G}_U = (V^1, \mathbf{E}_U^1)$ and $T_{\frac{s_1+s_u}{2}}^t \mathcal{G}_U = (V^2, \mathbf{E}_U^2)$, recalling the definitions of α and β in Sect. 3.2 and the definition of the transformation T^t provided in (54). Observe that vertices of both graphs could take non integer positions. The proof is divided in two parts.

In the **first part** we generalize the dynamic-block argument presented in [8] to the percolation model considered in this article. The idea of the construction is the same of [8], although parameters of the construction have been adapted to the lack of symmetry. The

lack of symmetry involves the structure of the graph \mathcal{G}_U and the slope of \bar{r}_m and $\bar{\ell}_m$, as in general $\alpha \neq -\beta$. Two different spatial transformations have been used in order to recover the symmetric setting and simplify the construction, namely, $T^t_{\frac{s_1+s_2}{2}}$ and $T^t_{\frac{\alpha+\beta}{2}}$.

The argument is based on a coupling between realisations of the graph $T^t_{\frac{s_1+s_u}{2}} \mathcal{G}_U$ and those in $T^t_{\frac{\alpha+\beta}{2}} \mathcal{G}_U$. The construction depends on a rescaling parameter L and it is such that the realisation on $T^t_{\frac{s_1+s_u}{2}}$ is a function of the realisation on $T^t_{\frac{\alpha+\beta}{2}} \mathcal{G}_U$. In $T^t_{\frac{\alpha+\beta}{2}} \mathcal{G}_U$ every site is open with probability p or closed with probability $1 - p$ independently. On the contrary, the states of sites in $T^t_{\frac{s_1+s_u}{2}}$ are not independent. The construction is such that if the event \mathcal{H}_n occurs in the former graph, then the event \mathcal{H}_{Ln} occurs in the latter graph. Secondly, if $p > p_c$, then for every ϵ , by choosing L is large enough, every site in $T^t_{\frac{s_1+s_u}{2}}$ is open with probability larger than $1 - \epsilon$.

The **second part** we define a sub-graph of $T^t_{\frac{s_1+s_u}{2}} \mathcal{G}_U$, that we call \mathcal{L} , for which it is easy to construct a dual graph. We use Peierls argument for the dual graph and we show that $\mathbb{P}_p(\mathcal{H}_n^c) \geq 1 - A \exp(-bn)$. As far as we know, this estimation has not been provided in other works. The dual graph construction can be found in [22]. This implies that $\mathbb{P}_p(T^t_{\frac{s_1+s_u}{2}} \mathcal{H}_n) \geq 1 - A \exp(-bn)$. Recalling the properties of the construction, it follows that $\mathbb{P}_p(T^t_{\frac{\alpha+\beta}{2}} \mathcal{H}_{Ln}) \geq 1 - A \exp(-bn)$. By rearranging the constants, the statement of the proposition follows.

We start now with the proof of the proposition.

Part 1: Dynamic blocks construction We divide $T^t_{\frac{\alpha+\beta}{2}} \mathcal{G}_U$ into *macro-regions* $R_{x,y}$ centred around the point $C_{x,y}$, where $(x, y) \in V^2$ and

$$\begin{aligned}
 C_{x,y} &= \left(x \frac{\gamma}{s_u - s_1} (1 - \delta), yL\right), \\
 R_{x,y} &= C_{x,y} + \left[(-1 - \delta) \frac{\gamma}{2} L, (1 + \delta) \frac{\gamma}{2} L\right] \times \left[0, -(1 + \delta)L\right].
 \end{aligned}
 \tag{62}$$

We recall that from Eq. (33) $\gamma = \alpha - \beta > 0$ for all $p > p_c$. The constants δ and L are positive and have to be properly chosen. In order the argument to work rigorously, $(1 - \delta)\gamma L$ and L should be even integers. To not complicate the exposition here we ignore these details, the same as in [8]. Each vertex $(x, y) \in V^2$ is associated to a random variable $\varphi_{x,y}$ which takes value 1 if a certain event $\mathcal{B}_{x,y}$ occurs in the region $R_{x,y}$ of (V^1, \mathbf{E}_U^1) or 0 otherwise. In order to define such event we introduce the following points in space (see also Fig. 9), for every $s \in \mathcal{U}$,

$$\begin{aligned}
 u &= \left(\frac{\delta\gamma L}{2}, 0\right), \\
 v &= \left(\frac{3\delta\gamma L}{4}, 0\right), \\
 -u &= \left(-\frac{\delta\gamma L}{2}, 0\right), \\
 -v &= \left(-\frac{3\delta\gamma L}{4}, 0\right), \\
 u_s^R &= \left(\frac{\delta\gamma L}{2} + \left(s - \frac{s_1 + s_u}{2}\right) \cdot \frac{(1 - \delta)\gamma L}{s_u - s_1}, -L(1 + \delta)\right),
 \end{aligned}$$

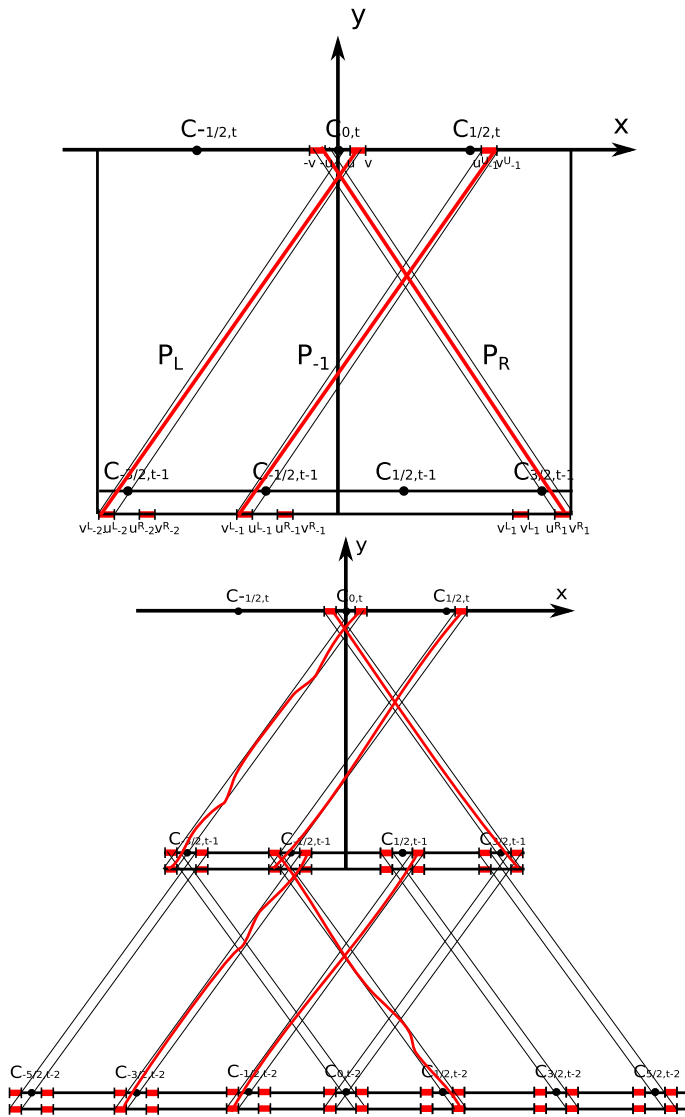


Fig. 9 Up the rectangle in the figure represents the region $R_{0,t}$ of the graph $T^t_{\frac{s_u+s_l}{2}} \mathcal{G}_U$, for some positive integer t , in case $U = \{-2, -1, 1\}$

$$\begin{aligned}
 v_s^R &= \left(\frac{3\delta\gamma L}{4} + \left(s - \frac{s_1 + s_u}{2} \right) \cdot \frac{(1 - \delta)\gamma L}{s_u - s_1}, -L(1 + \delta) \right), \\
 u_s^L &= \left(-\frac{\delta\gamma L}{2} + \left(s - \frac{s_1 + s_u}{2} \right) \cdot \frac{(1 - \delta)\gamma L}{s_u - s_1}, -L(1 + \delta) \right), \\
 v_s^L &= \left(-\frac{3\delta\gamma L}{4} + \left(s - \frac{s_1 + s_u}{2} \right) \cdot \frac{(1 - \delta)\gamma L}{s_u - s_1}, -L(1 + \delta) \right),
 \end{aligned} \tag{63}$$

and for every $s \in \mathcal{U} \setminus \{s_1, s_u\}$,

$$\begin{aligned} u_s^U &= \left(-\frac{\delta\gamma L}{2} + \left(s - \frac{s_1 + s_u}{2} \right) \cdot \frac{(1 - \delta)\gamma L}{s_u - s_1} + \frac{\gamma}{2}(1 + \delta)L, 0 \right), \\ v_s^U &= \left(-\frac{3\delta\gamma L}{4} + \left(s - \frac{s_1 + s_u}{2} \right) \cdot \frac{(1 - \delta)\gamma L}{s_u - s_1} + \frac{\gamma}{2}(1 + \delta)L, 0 \right), \end{aligned} \tag{64}$$

As one can see in the example in Fig. 9, these points identify some *target zones* on the right and on the left of points $C_{x,y}$, $(x, y) \in V^2$. Consider now the parallelograms obtained connecting the following quadruplets of points (see also Fig. 9),

$$\begin{aligned} P_R &= \left(-v, -u, u_{s_u}^R, v_{s_u}^R \right), \\ P_L &= \left(u, v, u_{s_1}^L, v_{s_1}^L \right) \\ P_s &= \left(u_s^L, v_s^L, u_s^U, v_s^U \right), \end{aligned} \tag{65}$$

for all $s \in \mathcal{U} \setminus \{s_1, s_u\}$. Define the translated parallelograms $P_R(x, y) = P_R + C_{x,y}$, $P_L(x, y) = P_L + C_{x,y}$, $P_s(x, y) = P_s + C_{x,y}$ for all $s \in \mathcal{U} \setminus \{s_1, s_u\}$.

Definition 8 The event $\mathcal{B}_{x,y}$ occurs if and only if the top of all parallelograms $P_R(x, y)$, $P_L(x, y)$ and $P_s(x, y)$, for all $s \in \mathcal{U} \setminus \{s_1, s_u\}$, is connected to the bottom side by an open path in $T_{\frac{\alpha+\beta}{2}}^t \mathcal{G}_{\mathcal{U}}$ that remains always inside the parallelogram.

This event is represented in Fig. 9. This construction is such that the following properties are satisfied. Namely,

1. the random variables $\varphi_{x,y}$ are $s_u - s_1$ -dependent. With this we mean that $\varphi_{x,y}$ and $\varphi_{x',y'}$, with $(x, y), (x', y') \in V^2$, are independent if $|x - x'| > s_u - s_1$ or $|y - y'| > 1$.
2. Denote by $z_1 \dots z_m$ the vertices of a path in $T_{\frac{s_1+s_u}{2}}^t \mathcal{G}_{\mathcal{U}}$ and assume that the path is open, i.e. $\varphi_{z_i} = 1$ for all $i \in \{1, 2, \dots, m\}$. Then there exists an open path in $T_{\frac{\alpha+\beta}{2}}^t \mathcal{G}_{\mathcal{U}}$ that connects a vertex in the $C_{z_1} + [-v, v]$ to a vertex in $C_{z_m} + [-v, v]$ and which remains always inside the parallelograms that connect $C_{z_i} + [-v, v]$ to $C_{z_{i+1}} + [-v, v]$, for all $i \in \{1, 2, \dots, m\}$ (note that $C_z + [-v, v]$ denotes the segment $[-v, v]$ translated by C_z).
3. if $\delta, \epsilon > 0$ and $p > p_c$, we can pick L large enough so that for any $(x, y) \in V^2$, $\mathbb{P}_p(\varphi_{x,y} = 1) > 1 - \epsilon$.

Proof of the properties We sketch the proof of the three properties above. The proof can be found also in [8, Section 9] in the case of bond percolation and symmetric neighbourhoods. Property 1 follows from the fact that if $R_{x,y}$ and $R_{x',y'}$ have empty intersection, then the variables $\varphi_{x,y}$ and $\varphi_{x',y'}$ are independent. Property 2 follows by construction (see Fig. 9). In the example in the figure we represent the graph $T_{\frac{s_1+s_u}{2}}^t \mathcal{G}_{\mathcal{U}}$ assuming $\mathcal{U} = \{-2, -1, 1\}$ as a neighbourhood. One should observe that if the events $\mathcal{B}_{0,t}$ and $\mathcal{B}_{-1,t-1}$ occur, then at least one vertex belonging to the interval $C_{0,t} + [-v, v]$ is connected to at least one of the vertices belonging to any of the intervals $C_{-\frac{3}{2},t-2} + [-v, v]$, $C_{-\frac{1}{2},t-2} + [-v, v]$, $C_{\frac{1}{2},t-2} + [-v, v]$.

We now prove the third property. Recall that Proposition 4 implies that in the transformed graph $T_{\frac{\alpha+\beta}{2}}^t \mathcal{G}_{\mathcal{U}}$, $\bar{r}_n/n \xrightarrow{n \rightarrow \infty} \gamma/2$ a.s. and $\bar{\ell}_n/n \xrightarrow{n \rightarrow \infty} -\gamma/2$ a. s. We will prove that $\forall \epsilon > 0$, the probability that in all the parallelograms in the box there is a connection from the top to the bottom that never crosses the diagonal sides is larger than $1 - \epsilon$. Let then e be the

number of parallelograms in the box $R_{0,0}$. This number depends on the neighbourhood \mathcal{U} . We consider the parallelogram P_R and we prove that for every ϵ there exists L large enough such that the probability that there is no such open path in the parallelogram is less than $\frac{\epsilon}{e}$. As this probability is the same for all parallelograms, this implies that the probability that such open path is present in all parallelograms is $> 1 - \epsilon$.

Consider the parallelogram P_R defined above and recall then the definitions provided in Eqs. (40–43). Let then $\tilde{r}_n := \sup \xi_n^{(-\infty, -0.7\delta\gamma L]}$ and observe that $-0.7\delta\gamma L \in [-v, -u]$. Let $\bar{r}_n := \sup \xi_n^{(-\infty, 0]}$ and observe that $\{\tilde{r}_n + 0.7\delta\gamma L : n \geq 0\} =_d \{\bar{r}_n : n \geq 0\}$. As $\bar{r}_n/n \rightarrow \frac{\gamma}{2}$ a.s. in the transformed graph $T_{\frac{\alpha+\beta}{2}}^t$, then we can pick L large enough such that with probability $\geq 1 - \frac{\epsilon}{2e}$ we have that,

$$\begin{aligned} \tilde{r}_{(1+\delta)L} &> -0.7\delta\gamma L + (1 + 0.98\delta)\frac{\gamma}{2}L \\ &= -0.71\delta\gamma L + (1 + \delta)\frac{\gamma}{2}L, \end{aligned} \tag{66}$$

and for all $m \leq (1 + \delta)L$,

$$\tilde{r}_m \leq -0.6\delta\gamma L + m \frac{1 + 1.08\delta}{1 + \delta} \frac{\gamma}{2}. \tag{67}$$

The two previous equations imply that there is an open path path from $(-\infty, -0.7\delta\gamma L) \times \{0\}$ to $[-0.71\delta L\gamma + (1 + \delta)L\frac{\gamma}{2}, -0.56\delta L\gamma + (1 + \delta)L\frac{\gamma}{2}] \times \{-(1 + \delta)L\}$ which does not cross the line $[-u, v_{s_u}^R]$. It remains to show that this path does not cross the line $[-v, u_{s_u}^R]$.

We observe that in order a path to travel from the line $[-v, u_{s_u}^R]$ to $[-0.7\delta L\gamma + \frac{\gamma}{2}(1 + \delta)L, \infty) \times \{-(1 + \delta)L\}$ a path must have an average slope $a > \frac{\gamma}{2}$. Thus recall Eq. (46) and observe that in the transformed graph $T_{\frac{\alpha+\beta}{2}}^t \mathcal{G}\mathcal{U}$,

$$\mathbb{P}_p(\bar{r}_m > am) \leq C e^{-h_2 m}.$$

Consider then M large enough such that,

$$\sum_{m=M}^{\infty} C \exp(-h_2 m) \leq \frac{\epsilon}{4e}.$$

The probability that one of the points on $[-v, u_{s_u}^R]$ with $-(1 + \delta)L + M \leq y \leq 0$ is connected to $[-0.7\delta L\gamma + (1 + \delta)\frac{\gamma}{2}L, \infty) \times \{-(1 + \delta)L\}$ is then $\leq \frac{\epsilon}{4e}$. Furthermore, observe that the number of points on $[-v, u_{s_u}^R]$ with $-(1 + \delta)L \leq y \leq -(1 + \delta)L + M$ does not depend on L and that the distance of any of them from the set $[-0.7\delta L\gamma + (1 + \delta)\frac{\gamma}{2}L, \infty) \times \{-(1 + \delta)L\}$ is proportional to L . Thus we can pick L large enough so that the probability that there exists an open path connecting any of these points to $[-0.7\delta L\gamma + (1 + \delta)\frac{\gamma}{2}L, \infty)$ is less than $\frac{\epsilon}{4e}$. Combining the two estimations, we conclude that the probability that the line $[-v, u_{s_u}^R]$ is connected by an open path to $[-0.7\delta L\gamma + (1 + \delta)\frac{\gamma}{2}L, \infty)$ is less than $\frac{\epsilon}{2e}$.

Summarising, we showed that with probability $\geq 1 - \frac{\epsilon}{e}$, there is an open path from $(-\infty, -0.7\delta\gamma L) \times \{0\}$ to $[-0.71\delta L\gamma + (1 + \delta)L\frac{\gamma}{2}, -0.56\delta L\gamma + (1 + \delta)L\frac{\gamma}{2}] \times \{-(1 + \delta)L\}$ which does not cross the line $[-u, v_{s_u}^R]$ and the line $[-v, u_{s_u}^R]$ is not connected by an open path to $[-0.7\delta L\gamma + (1 + \delta)\frac{\gamma}{2}L, \infty) \times \{-(1 + \delta)L\}$. This implies that the probability that there exists a path joining the top to the bottom of P_R without ever crossing its diagonal lines is $\geq 1 - \frac{\epsilon}{e}$. Repeating the argument for all parallelograms in the box, we conclude that if L is large enough then with probability at least $1 - \epsilon$ the event $\mathcal{B}_{0,0}$ occurs. \square

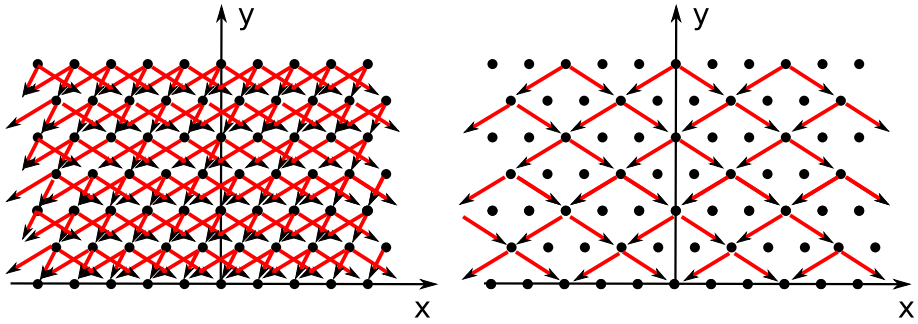


Fig. 10 Left representation of $T_{\frac{s_1+s_u}{2}}^t \mathcal{G}_U$, in case of $U = \{-2, -1, 1\}$. Right points correspond to vertices of $T_{\frac{s_1+s_u}{2}}^t \mathcal{G}_U$, arrows represent edges of \mathcal{L} , points connected by an arrow correspond to vertices of \mathcal{L} . The graph \mathcal{L} , defined in the text, is a subset of $T_{\frac{s_1+s_u}{2}}^t \mathcal{G}_U$

Part 2: Peierls argument Now we use the Peierls argument for the $(s_u - s_1)$ -dependent oriented percolation model to prove that there exists $p_1 > p_c$ and positive constants A', b' (dependent on p) such that for all $p \in (p_1, 1]$,

$$\mathbb{P}_p\left(T_{\frac{s_1+s_u}{2}}^t \mathcal{H}_n\right) \geq 1 - A' \cdot ne^{-b'n}. \tag{68}$$

Let us explain first why this is sufficient to prove the proposition. Later we prove (68).

Recall the third property of the dynamic-block construction presented above and observe that if $p > p_c$, then we can pick L large enough such that, for every (x, y) belonging to the set of vertices of $T_{\frac{s_1+s_2}{2}}^t \mathcal{G}_U$, $\mathbb{P}_p(\mathcal{B}_{x,y}) > p_1$. Recall that the state of sites belonging to $T_{\frac{s_1+s_2}{2}}^t \mathcal{G}_U$ is a function of the realization in the graph $T_{\frac{\alpha+\beta}{2}}^t \mathcal{G}_U$. From the second property of the dynamic-block construction, if such sites are open with probability $> p_1$, then (68) implies that with probability not less than $1 - A' \cdot ne^{-b'n}$ the event \mathcal{H}_n occurs in $T_{\frac{s_1+s_2}{2}}^t$. Hence, from the second property of the dynamic-block construction, the event \mathcal{H}_{Ln} occurs in the graph $T_{\frac{\alpha+\beta}{2}}^t \mathcal{G}_U$ with probability not less than $1 - A' \cdot ne^{-b'n}$. One can rearrange the value of b' getting rid of the factor n , for n large enough. Finally, by defining new constants $A = A'/L$ and $b = b'L$, the statement of Proposition 8 follows.

We start proving (68). We define a new graph \mathcal{L} , that is a sub-graph of $T_{\frac{s_1+s_u}{2}}^t \mathcal{G}_U$, whose vertices (x, y) are,

$$V' = \left\{ (x, y) : x = (s_u - s_1)z - (y - t) \frac{s_u - s_1}{2}, z \in \mathbb{Z}, y \in \mathbb{Z} \right\}, \tag{69}$$

and whose edges connect vertices (x, y) to $(x \pm \frac{s_u-s_1}{2}, y - 1)$. The reason why we introduce \mathcal{L} is that, as every site has only two neighbours, it is easier to construct its dual graph. The new graph \mathcal{L} is represented in the example in Fig. 10 on the right.

As \mathcal{L} is a sub-graph of $T_{\frac{s_1+s_u}{2}}^t \mathcal{G}_U$, the following inequality holds,

$$\mathbb{P}_p\left(\mathcal{H}_n^{\mathcal{L}}\right) \leq \mathbb{P}_p\left(T_{\frac{s_u+s_u}{2}}^t \mathcal{H}_n\right). \tag{70}$$

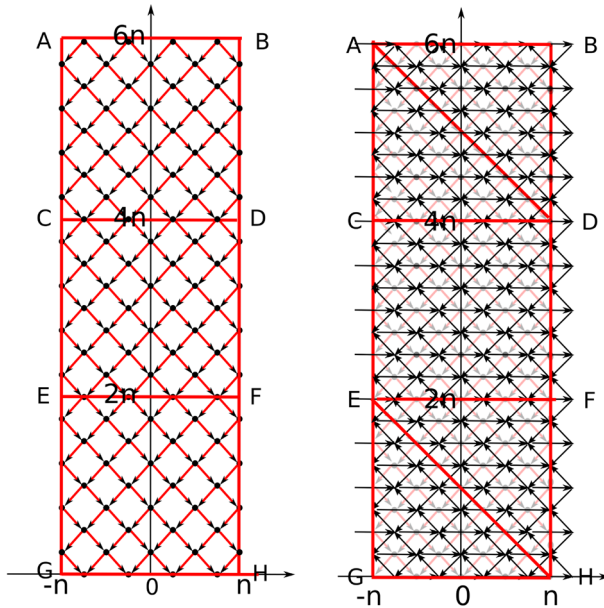


Fig. 11 The horizontal axis has been rescaled by $\frac{s_u - s_l}{2}$ in both graphs. $A = (-n, 6n), B = (n, 6n), C = (-n, 4n), D = (n, 4n), E = (-n, 2n), F = (n, 2n), G = (-n, 0), H = (n, 0)$. Left representation of the graph \mathcal{L} . The event \mathcal{H}_n occurs iff the side AB is connected to the side GH by an open path in \mathcal{L} that does not cross the sides AG and BH . Right representation of the graph \mathcal{L}^D , as defined in the text. The event \mathcal{H}_n does not occur iff one of the sides CE or EH is connected to one of the sides AD or DF by an open path in the dual lattice

In the previous expression, the superscript \mathcal{L} is used to denote that event \mathcal{H}_n , defined in (56), occurs on the graph \mathcal{L} . Call then \mathcal{L}^D the dual graph of \mathcal{L} . The graph is represented on the right of Fig. 11 and its construction is due to [22, 25].

The dual graph is composed of three types of edges, namely, edges pointing down-left, those pointing up-left and those pointing right. Every edge pointing right is positioned over a vertex of the original graph \mathcal{L} . Edges down-left and up-left are always *open*, edges pointing right are *open* if and only if the corresponding vertex of the original graph is *closed*. A path in the dual graph is *open* if and only if all its edges are open. The following proposition connects the occurrence of the event \mathcal{H}_n in \mathcal{L} with the occurrence of a second event on the dual lattice.

Proposition 9 Consider Fig. 11. For every $n \in \mathbb{N}$, there exists an open path in \mathcal{L} connecting AC to FH iff there is no open path in the dual lattice connecting one of the sides CE or EH to one of the sides AD or DF .

Proof We provide a graphical proof. Consider Fig. 11. On the left we have represented the graph \mathcal{L} and on the right we have represented its dual. Consider a realisation in the auxiliary space Ω and recall that if a site is open in \mathcal{L} , then the corresponding horizontal edge is closed in the dual graph and vice versa. The reader should observe that, as long as there is an open path connecting AC to FH in \mathcal{L} , no open path in the dual graph connecting one of the sides CE or EH to one of the sides AD or DF can exist. On the other hand, as long there exists an open path in the dual graph connecting one of the sides CE or EH to one of the sides AD or DF , no open path in \mathcal{L} connecting AC to FH can exist. \square

Both the proposition and the dual construction are analogous to the one presented in [22]. We use this proposition to provide a lower bound for $\mathbb{P}_p(\mathcal{H}_n^{\mathcal{L}})$. Consider then a vertex z on CE or on EH . Call $C_{z,h}$ the set of paths connecting the vertex z to one of the sides AD or DF and having h edges pointing to the right. Call $N_{z,h}$ the total number of such paths. Consider one of these paths and call dl the number of its edges pointing down-left and ul the number of edges pointing up-left. As the last edge of the path cannot be on the left of the first edge, $2h - ul - dl \geq 0$. This implies that for each of these paths $h + ul + dl$ is bounded from above by $3h$. As there are only 3 different types of steps, for any vertex z located on CE or on EH , $N_{z,h} \leq 3^{3h}$. Thus $N_{z,h} \leq 3^{3h}$ for every z . Denote by $\overline{\mathcal{H}_n^{\mathcal{L}}}$ the complementary of $\mathcal{H}_n^{\mathcal{L}}$. Recall Proposition 9 and observe the fact that, in order CG to be connected to AD or to DH , at least $\lfloor \frac{2n}{s_u - s_1} \rfloor$ horizontal steps to the right are needed. Then,

$$\mathbb{P}_p(\overline{\mathcal{H}_n^{\mathcal{L}}}) = \mathbb{P}_p\left(\bigcup_{z \in CE \cup EH} \bigcup_{h=2n}^{\infty} \bigcup_{c \in C_{z,h}} \{c \text{ is open}\}\right).$$

Observe also that, given a path $c \in C_{z,h}$, $\mathbb{P}_p(c \text{ is open}) \leq (1 - p)^{\frac{h}{2}}$, considering only the state of one every two edges to the right, as states of edges located over non-neighbour sites are independent. By using the union bound, we determine an upper bound for $\mathbb{P}_p(\overline{\mathcal{H}_n^{\mathcal{L}}})$,

$$\mathbb{P}_p(\overline{\mathcal{H}_n^{\mathcal{L}}}) \leq \sum_{z \in CE \cup EH} \sum_{h=\lfloor \frac{2n}{s_u - s_1} \rfloor}^{\infty} N_{z,h} (1 - p)^{h/2} \leq A' \cdot n \exp(-b'n), \tag{71}$$

where the second inequality is true with A', b' positive constants if $p > 1 - \frac{1}{3}$. □

4 Critical Probabilities

In this section we prove Theorem 2.2, which provides a lower bound for p_c as a function of the neighbourhood. The proof of Theorem 2.2 requires Lemma 4.1 and Propositions 10 and 11, which are stated in this section.

Proposition 10 *Consider two Percolation PCA in \mathbb{Z} with neighbourhoods respectively \mathcal{U} and \mathcal{U}' , both finite subsets of \mathbb{Z} , such that $\mathcal{U} \subset \mathcal{U}'$. Then*

$$p_c(\mathcal{U}) \geq p_c(\mathcal{U}').$$

Proof From Proposition 2, stated in Sect. 3.1, and as the edge set of the graph $\mathcal{G}_{\mathcal{U}}$ is a subset of the edge set of the graph $\mathcal{G}_{\mathcal{U}'}$, it follows that $\forall x \in \mathbb{Z}, \forall t \in \mathbb{N}_0, \delta_1 \mathcal{P}_{\mathcal{U}}^t(\eta_x = 0) \geq \delta_1 \mathcal{P}_{\mathcal{U}'}^t(\eta_x = 0)$ (we added the subscript to the operator in order to distinguish between the two neighbourhoods). From Definition 4 it follows that,

$$\lim_{t \rightarrow \infty} \delta_1 \mathcal{P}_{\mathcal{U}}^t(\eta_x = 0) < 1 \implies p \geq p_c(\mathcal{U}).$$

Hence, $p_c(\mathcal{U}) \geq p_c(\mathcal{U}')$. □

We introduce some notation.

Definition 9 (*Massif of zeros*) We call a segment of \mathbb{Z} ,

$$\{k, k + 1, \dots, k + \ell - 1\} \subset \mathbb{Z}$$

a *massif of zeros* of length ℓ for a given $\eta \in \Sigma$, if $\eta_{k-1} = \eta_{k+\ell} = 1$ and $\eta_k = \dots = \eta_{k+\ell-1} = 0$.

$$E[\pi_1^1] > E[\xi_1^1] \tag{76}$$

then $\forall \mu \in \mathcal{M}(\Sigma)$,

$$\lim_{t \rightarrow \infty} \mu^{\mathcal{P}^t} = \delta_0. \tag{77}$$

In the statement of the lemma $P(\cdot)$ denotes the probability distribution of the random variables π_k^i or ξ_k^i . Such random variables stochastically dominate from below the change of position of the rightmost and leftmost site of the massif every T steps. We also recall that $\mathcal{E}_{\delta_{\eta^0}}$ has been defined in Definition 1. The proof of the lemma is similar to the proof of Proposition 6.4 in [25].

Proof It is sufficient to prove that $\forall \eta^0 \in \Sigma, \forall x \in \mathbb{Z}, \forall \epsilon > 0, \exists t_0$ such that

$$\forall t > t_0, \quad \delta_{\eta^0} \mathcal{P}^t(\eta_x = 0) \geq 1 - \epsilon, \tag{78}$$

from which condition (77) follows.

We define $c_1 := \frac{3E[\xi_1^1] + E[\eta_1^1]}{4}$ and $c_2 := \frac{E[\xi_1^1] + 3E[\eta_1^1]}{4}$, where $E[\cdot]$ denotes the expectation, and we observe that if (76) holds, then $c_2 > c_1$. Then for every $\eta^0 \in \Sigma, \forall k \in \mathcal{I}, \forall n, m \in \mathbb{Z}^+, \forall i_0, j_0 \in \mathbb{Z}$ such that $j_0 - i_0 \geq T(s_u - s_1) + m + n$, there exists two constants $u, v \in [0, 1)$ such that,

$$\begin{aligned} \mathcal{E}_{\delta_{\eta^0}} \left(\forall i \geq 1, L_k^{iT} \leq c_1(i-1)T + L_k^T + n, R_k^{iT} \right. \\ \left. \geq c_2(i-1)T + R_k^T - m \mid L_k^T = i_0, R_k^T = j_0 \right) \\ \geq P \left(\forall i \geq 1, \sum_{j=1}^i \xi_k^j \leq c_1(i-1)T + i_0 + n, \sum_{j=1}^i \pi_k^j \geq c_2(i-1)T + j_0 - m \right) \\ \geq 1 - u^m - v^n. \end{aligned} \tag{79}$$

In the previous expression $P(\cdot)$ denotes the probability measure defined on the space of outcomes of the sum of the increments ξ_k^t, π_k^t . The first inequality follows from (74) and (75). The second inequality follows from the properties of the one dimensional random walk, observing that by definition $E[\xi_k^t] < c_1$ and $E[\pi_k^t] > c_2$. The two constants u and v depend on the probability distribution of the increments of the random walk.

We observe that if for all t multiple of $T, R_k^t \geq c_2t + j_0 - m$ and $L_k^t \leq c_1t + i_0 + n$ (event in the first expression in (79)), then for all $t \in \mathbb{N}_0, R_k^t \geq c_2t + j_0 - m - s_uT$ and $L_k^t \leq c_1t + i_0 + n - s_1T$. Hence, the state of all sites in the space-time region $Y_{[i_0, j_0]}^{m, n} := \{(x, t) : t \in \mathbb{Z}_+ \text{ and } c_1t + n + i_0 - Ts_1 \leq x \leq c_2t - m + j_0 - Ts_u\}$ is zero. This region is represented in Fig. 13 on the left. This follows from the observation that by definition of transition probability of the Percolation PCA the following property holds, namely,

$$R_k^t - L_k^t \geq T(s_u - s_1) \implies \forall q < T, R_k^{t+q} \geq R_k^t - qs_u, L_k^{t+q} \leq L_k^t - qs_1. \tag{80}$$

Furthermore, we observe that $\forall \eta^0 \in \Sigma, \forall x \in \mathbb{Z}, \forall n, m \in \mathbb{Z}_+$, the measure $\delta_{\eta^0} \mathcal{P}^T$ is such that the probability that there exists a massif of zeros of length $j_0 - i_0 \geq T(s_u - s_1) + n + m$ in $[y, d]$ goes to 1 as $d \rightarrow \infty$. We choose then n and m such that $u^m + v^n < \frac{\epsilon}{2}$ and d large enough such that such probability is larger than $1 - \frac{\epsilon}{2}$ for all y .

Simple geometrical considerations show that for any $y \in \mathbb{Z}, d \in \mathbb{Z}_+$, all regions $Y_{[i_0, j_0]}^{n, m}$, where $[i_0, j_0] \subset [y, y + d]$, have a non empty common region (dark region in Fig. 13-right). We call $U_{[y, d]}^{m, n}$ this region. From (79) and from the previous observations the following property holds,

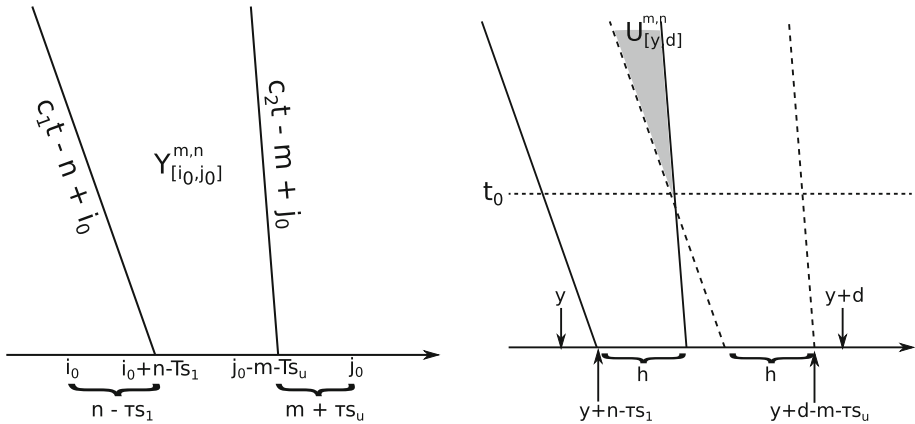


Fig. 13 The variable h on the right is defined as $h := j_0 - i_0 - T(s_u - s_1) - n - m$

$$\mathcal{E}_{\delta, \eta^0} (\forall (x, t) \in U_{[y, d]}, \eta_x^t = 0) > 1 - \epsilon. \tag{81}$$

Choosing y and d such that $(x, t) \in U_{[y, d]}$ implies (78). □

Proof of Theorem 2.2 We provide a lower bound for the critical probability of the Percolation PCA with neighbourhood

$$U := \{s_1, s_1 + 1, \dots, s_u - 1, s_u\}, \tag{82}$$

i.e. all elements between the two extremal ones are present. Our bound is a function of $s_u - s_1$. By Proposition 10 such bound holds also for Percolation PCA with neighbourhood obtained removing some sites from (82).

The proof of the theorem is based on an application of Lemma 4.1. We fix a value $T \in \mathbb{Z}_+$ and by using the monotonicity property of the Percolation PCA, we define the random variables π_k^t and ξ_k^t , whose probability distribution satisfies $\forall p \in [0, 1]$ the conditions (74) and (75) of Lemma 4.1. We define,

$$p_T := \max_{p \in [0, 1]} \left\{ E[\pi_1^1] > E[\xi_1^1] \right\}.$$

From Lemma 4.1, for all $p \geq p_T$ the Percolation PCA is ergodic. From Definitions 4 and 5, $p_T \leq p_c$. We fix first $T = 1$ and we derive p_1 , later we consider $T = 2$ and we derive p_2 . Both p_1 and p_2 appear in the statement of the theorem. Higher is the value of T considered, more challenging is the estimation of p_T , as this involves the characterization of the increments of L_k^t, R_k^t over a larger time interval.

Fix then an integer $T \in \mathbb{Z}^+$ and consider an initial realisation $\eta^0 \in \Sigma$. Enumerate somehow the massifs of zeros having length not smaller than $T(s_u - s_1)$ and recall the definitions of the random variables $R_k^t, L_k^t, t \in \mathbb{N}_0, k \in \mathcal{I}$, provided before the statement of Lemma 4.1. For any $A \subset \mathbb{Z}$, let $\mathbb{1}_A^t : \tilde{\Sigma} \rightarrow \tilde{\Sigma}$ be the function that is equal to 1 if the state of all sites in A at time $t \in \mathbb{N}_0$ is zero and zero otherwise. Let $\mathbb{1}_A : \Sigma \rightarrow \Sigma$ (without the superscript) be the function that is equal to 1 if the state of all sites in A is zero and zero otherwise. Observe that $1 - \mathbb{1}_A^t$ and $1 - \mathbb{1}_A$ are monotone functions. Let also $\rho(x, y) \in \Sigma$ be the realisation having zeros in $[[x, y]]$ and ones everywhere else. Then $\forall \eta^0, \forall \eta \in \Sigma, \forall t \in \mathbb{Z}_+, \forall k \in \mathcal{I}, \forall x, y \in \mathbb{Z}$ such that $y - x \geq T(s_u - s_1), \forall j \in \mathbb{Z}_0$, the following relations hold,

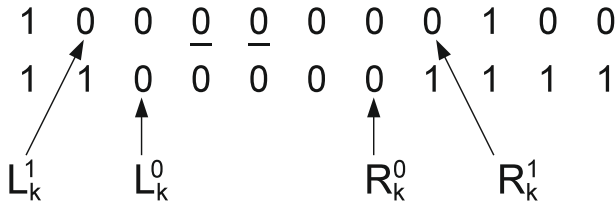


Fig. 14 In this example we consider a Percolation PCA with $\mathcal{U} = \{-1, 0, 1, 2\}$. If the process starts from the realisation represented in the figure (row below), then the state of the sites above the small horizontal ball is almost surely 0 at time 1 (row above)

$$\mathcal{E}_{\delta_{\eta^0}} \left(R_k^{t+T} - R_k^t \geq j \mid R_k^t = y, L_k^t = x, \eta^t = \eta \right) \tag{83}$$

$$= \mathcal{E}_{\delta_{\eta^0}} \left(\mathbb{1}_{[[x-Ts_1, y-Ts_u+j]]} \mid R_k^t = y, L_k^t = x, \eta^t = \eta \right) \tag{84}$$

$$= \delta_{\eta} \mathcal{P}^T \left(\mathbb{1}_{[[x-Ts_1, y-Ts_u+j]]} \right) \tag{85}$$

$$\geq \delta_{\rho(x,y)} \mathcal{P}^T \left(\mathbb{1}_{[[x-Ts_1, y-Ts_u+j]]} \right). \tag{86}$$

Equation (84) follows from the definition of R_k^t , Eq. (85) follows from the Markov property of the probabilistic cellular automaton, inequality (86) follows from the monotonicity property of the Percolation PCA, as any realisation $\eta \in \Sigma$ having a massif of zeros in $[[x, y]]$ is such that $\eta < \rho(x, y)$. Similarly,

$$\mathcal{E}_{\delta_{\eta^0}} \left(L_k^{t+T} - L_k^t \leq -j \mid R_k^t = y, L_k^t = x, \eta^t = \eta \right) \tag{87}$$

$$= \delta_{\eta} \mathcal{P}^T \left(\mathbb{1}_{[[x-Ts_1-j, y-Ts_u]]} \right) \tag{88}$$

$$\geq \delta_{\rho(x,y)} \mathcal{P}^T \left(\mathbb{1}_{[[x-Ts_1-j, y-Ts_u]]} \right). \tag{89}$$

We also observe that from the definition of transition probability of the Percolation PCA, the quantities (86) and (89) do not depend on the sites $x, y \in \mathbb{Z}$, as long as $y - x \geq T(s_u - s_1)$. Thus, we provide the following definitions of the probability distribution of the random variables π_k^t, ξ_k^t . Namely, fix y and x such that $y - x \geq T(s_u - s_1)$ and $\forall k, \forall t \in \mathbb{N}_0$ we define,

$$P(\pi_k^t \geq j) := \delta_{\rho(x,y)} \mathcal{P}^T \left(\mathbb{1}_{[[x-Ts_1, y-Ts_u+j]]} \right) \tag{90}$$

$$P(\xi_k^t \leq -j) := \delta_{\rho(x,y)} \mathcal{P}^T \left(\mathbb{1}_{[[x-Ts_1-j, y-Ts_u]]} \right). \tag{91}$$

With this definition, from (83)–(89) the first two conditions of Lemma 4.1, namely, (74) and (75), are satisfied. The maximum among all $p \in [0, 1]$ such that condition (76) is satisfied is $p_T \leq p_c$.

We fix now $T = 1$ and we provide an estimation for (90) and (91) for any $j \in \mathbb{Z}$. After this we determine which values of p satisfy (76). We consider the Percolation PCA starting from initial realisation $\rho(x, y)$ and we assign the label 1 to the unique massif of zeros, namely, $R_1^0 = y$ and $L_1^0 = x$. We recall that by definition,

$$R_1^t \geq j + R_1^0 - s_u \iff \forall z \in [[L_1^0 - s_1, R_1^0 - s_u + j]], \eta_z^t = 0 \tag{92}$$

(see also Fig. 14). Hence, $\forall j \in \mathbb{N}_0$,

$$\delta_{\rho(x,y)} \mathcal{P}(R_1^t \geq j + R_1^0 - s_u) = (1 - p)^j. \tag{93}$$

This bound is obtained considering that almost surely $\forall z \in [[x - s_1, y - s_u]]$, $\eta_z^1 = 0$ and that $\forall z \in [[y - s_u + 1, y - s_u + j]]$, independently $\delta_{\rho(x,y)}\mathcal{P}(\eta_z = 0) = (1 - p)$. Analogously, $\forall j \in \mathbb{N}_0$,

$$\delta_{\rho(x,y)}\mathcal{P}(L_1^1 \leq -j + L_1^0 - s_1) = (1 - p)^j. \tag{94}$$

Thus for all $j \in \mathbb{N}_0$, we define the probability distributions of π_1^t and ξ_1^t respectively as,

$$P(\pi_k^t \geq j - s_u) := (1 - p)^j \tag{95}$$

$$P(\xi_k^t \leq -j - s_1) := (1 - p)^j. \tag{96}$$

With this definition, from the relations (83)–(89), the relations (74) and (75) are satisfied. It remains to determine for which values of $p \in [0, 1]$ the second condition of Lemma 4.1 holds. By a simple computation,

$$E[\pi_1] = \frac{1 - p}{p} - s_u, \tag{97}$$

$$E[\xi_1] = \frac{1 - p}{p} - s_1. \tag{98}$$

and

$$E[\xi_1] - E[\pi_1] \geq 0 \iff p \geq p_1, \tag{99}$$

where $p_1 := \frac{2}{2+s_u-s_1}$ appears on the statement of the theorem. Thus we proved that $p_c \geq p_1$. \square

We fix now $T = 2$ and we use the same argument. Namely, we consider the Percolation PCA starting from initial realisation $\rho(x, y) \in \Sigma$ such that $y - x \geq 2(s_u - s_1)$ and we assign label 1 to the unique massif of zeros of $\rho(x, y)$. We recall that by definition of R_1^2 ,

$$R_1^2 \geq j + R_1^0 - 2s_u \iff \forall z \in [[L_1^0 - 2s_1, R_1^0 - 2s_u + j]], \eta_z^2 = 0.$$

From definition (3) it follows that

$$\delta_{\rho(x,y)}\mathcal{P}^2(R_1^2 \geq j + y - 2s_u) = \sum_{\eta^1 \in A_R} \delta_{\rho(x,y)}\mathcal{P}(C_{\eta^1}) \prod_{z \in [[y - 2s_u, y - 2s_u + j]]} T(\eta_z^2 = 0 | \eta_{\mathcal{U}(z)}^1), \tag{100}$$

where $A_R := \{0, 1\}^{[[y - 2s_u + s_1, y - s_u + j]]}$ and $C_{\eta^1}^R = \{\eta' \in \Sigma \text{ s.t. } \forall z \in [[y - 2s_u + s_1, y - s_u + j]], \eta'_z = \eta_z^1\}$. The sum is reduced to the elements of $A_R \subset \Sigma$ because the states of the sites in the interval $[[y - 2s_u, y - 2s_u + j]]$ for η^2 depend only on the states of the sites in the finite interval $[[y - 2s_u + s_1, y - s_u + j]]$ for η^1 . A similar expression holds for the random variable L_1^2 ,

$$\delta_{\rho(x,y)}\mathcal{P}^2(L_1^2 \leq -j + x - 2s_1) = \sum_{\eta^1 \in A_L} \delta_{\rho(x,y)}\mathcal{P}(C_{\eta^1}^L) \prod_{z \in [[x - 2s_1 - j, x - 2s_u]]} T(\eta_z^2 = 0 | \eta_{\mathcal{U}(z)}^1), \tag{101}$$

where $A_L := \{0, 1\}^{[[x - s_1 - j, x - 2s_1 + s_u]]}$ and $C_{\eta^1}^L = \{\eta' \in \Sigma \text{ s.t. } \forall z \in [[x - 2s_1 - j, x - 2s_1 + s_u]], \eta'_z = \eta_z^1\}$.

The exact computation of the left hand side of (100) and of (101) for any j is a difficult combinatorial problem, as for each of the 2^j possible realizations one should determine the corresponding product of transition probabilities. We present our estimations in the following proposition.

Proposition 11 *Consider the realisation $\rho(x, y) \in \Sigma$ which has zeros in $[[x, y]]$ and ones everywhere else, where $x, y \in \mathbb{Z}$ are such that $y - x \geq 2(s_u - s_1)$. Assign label 1 to the unique massif of zeros of $\rho(x, y)$ and recall the definition of $(R_1^t)_{t \in \mathbb{N}_0}, (L_1^t)_{t \in \mathbb{N}_0}$. Then,*

$$\delta_{\rho(x,y)} \mathcal{P}^2(R_1^2 \geq R_1^0 - 2s_u) = 1, \tag{102}$$

$$\delta_{\rho(x,y)} \mathcal{P}^2(L_1^2 \leq L_1^0 - 2s_1) = 1, \tag{103}$$

$$\delta_{\rho(x,y)} \mathcal{P}^2(R_1^2 \geq 1 + R_1^0 - 2s_u) \geq 1 - p^2, \tag{104}$$

$$\delta_{\rho(x,y)} \mathcal{P}^2(L_1^2 \leq -1 + L_1^0 - 2s_1) \geq 1 - p^2, \tag{105}$$

$$\delta_{\rho(x,y)} \mathcal{P}^2(R_1^2 \geq 2 + R_1^0 - 2s_u) \geq (1 - p)^2(1 + 2p), \tag{106}$$

$$\delta_{\rho(x,y)} \mathcal{P}^2(L_1^2 \leq -2 + L_1^0 - 2s_1) \geq (1 - p)^2(1 + 2p), \tag{107}$$

for any $3 \leq j \leq s_u - s_1$,

$$\delta_{\rho(x,y)} \mathcal{P}^2(R_1^2 \geq j + R_1^0 - 2s_u) \geq jp(1 - p)^j + (1 - p)^j + (1 - p)^{2j}, \tag{108}$$

$$\delta_{\rho(x,y)} \mathcal{P}^2(L_1^2 \leq -j + L_1^0 - 2s_1) \geq jp(1 - p)^j + (1 - p)^j + (1 - p)^{2j}, \tag{109}$$

and for any $j > s_u - s_1$,

$$\begin{aligned} \delta_{\rho(x,y)} \mathcal{P}^2(R_1^2 \geq j + R_1^0 - 2s_u) &\geq jp(1 - p)^j + (1 - p)^j \\ &\quad + p(1 - p)^{j+s_u-s_1} \left(j - s_u + s_1 - \frac{1}{p} \right) \\ &\quad + 2(1 - p)^{2j}, \end{aligned} \tag{110}$$

$$\begin{aligned} \delta_{\rho(x,y)} \mathcal{P}^2(L_1^2 \leq -j + L_1^0 - 2s_1) &\geq jp(1 - p)^j + (1 - p)^j \\ &\quad + p(1 - p)^{j+s_u-s_1} \left(j - s_u + s_1 - \frac{1}{p} \right) \\ &\quad + 2(1 - p)^{2j}. \end{aligned} \tag{111}$$

We postpone the proof of Proposition 11 to the next paragraph and we conclude the proof of Theorem 2.2. We use the lower bounds provided in the proposition to define the probability distribution of the random variables π_1^k, ξ_1^k . Namely, $\forall j \in \mathbb{N}_0$, we define the probability of the event $\{\pi_1^t \geq j - s_u\}$ (respectively $\{\xi_1^t \leq -j - s_1\}$) as the lower bound of the probability of the event $\{R_1^2 \geq j + R_1^0 - s_u\}$ (respectively $\{L_1^2 \leq -j + L_1^0 - s_1\}$) provided in the proposition. With such definition, the expectation of the random variables ξ_k^t, π_k^t is equal to

$$E[\pi_k^t] = 2 \frac{(1 - p)}{p} - 2s_u + \frac{(1 - p)^6 + (1 - p)^{2s_u-2s_1+2}}{p(2 - p)}, \tag{112}$$

$$E[\xi_k^t] = -2 \frac{(1 - p)}{p} - 2s_1 - \frac{(1 - p)^6 + (1 - p)^{2s_u-2s_1+2}}{p(2 - p)} \tag{113}$$

By simple computations, the maximum $p \in [0, 1]$ such that the inequality $E[\pi_k] - E[\xi_k] \geq 0$ is satisfied (Condition 76 of Lemma 4.1), corresponds to the value p_2 defined in the statement of Theorem 2.1. As the function $E[\pi_k^t] - E[\xi_k^t]$ intersects the line $y = 0$ only in one point of the interval $[0, 1]$, p_2 is the unique solution of $E[\pi_k^t] - E[\xi_k^t] = 0$ that falls in this interval. \square

Proof of Proposition 11 In the proof we present the estimation of the probability $\delta_{\rho(x,y)}\mathcal{P}^2$ of the events $\{R_1^2 \geq j + R_1^0 - 2s_u\}_{j \in \mathbb{N}_0}$. Using the same argument one can estimate the probability of the events $\{L_1^2 \leq -j + L_1^0 - 2s_1\}_{j \in \mathbb{N}_0}$. By definition of R_1^2 ,

$$\{R_1^2 \geq R_1^0 - 2s_u + j\} \iff \forall z \in [[L_1^0 - 2s_1, R_1^0 - 2s_u + j]], \eta_z^2 = 0.$$

As observed previously, the state of the sites in $[[R_1^0 - 2s_u, R_1^0 - 2s_u + j]]$ for η^2 depends only on the state of the sites in $[[R_1^0 - 2s_u + s_1, R_1^0 - s_u + j]]$ for η^1 . Furthermore, we observed that the state of the sites in $[[L_1^0 - 2s_1, R_1^0 - 2s_u]]$ is zero almost surely for η^2 . Hence, from Eq. (100), we obtain the following estimation (see also Fig. 15),

$$\begin{aligned} \delta_{\rho(x,y)}\mathcal{P}^2(R_1^2 \geq 1 + R_1^0 - 2s_u) &= \delta_{\rho(x,y)}\mathcal{P}^2(\forall z \in [[L_1^0 - 2s_1, R_1^0 - 2s_u + 1]], \eta_x = 0) \\ &= \delta_{\rho(x,y)}\mathcal{P}^2(\eta_{R_1^0 - 2s_1 + 1} = 0) \\ &= \delta_{\rho(x,y)}\mathcal{P}(\eta_{R_1^0 - 2s_u + 1} = 0) \\ &\quad + \delta_{\rho(x,y)}\mathcal{P}(\eta_{R_1^0 - 2s_u + 1} = 1)(1 - p) \\ &= (1 - p) + p(1 - p) \\ &= 1 - p^2, \end{aligned} \tag{114}$$

which corresponds to the estimation (104). Similarly we obtain the estimation (106),

$$\begin{aligned} \delta_{\rho(x,y)}\mathcal{P}^2(R_1^2 \geq 2 + R_1^0 - 2s_u) &= \delta_{\rho(x,y)}\mathcal{P}^2(\forall z \in [[L_1^0 - 2s_1, R_1^0 - 2s_u + 2]]) \\ &= \delta_{\rho(x,y)}\mathcal{P}^2(\eta_{R_1^0 - 2s_u + 1} = \eta_{R_1^0 - 2s_u + 2} = 0) \\ &= \delta_{\rho(x,y)}\mathcal{P}^1(\eta_{R_1^0 - s_u + 1} = 0, \eta_{R_1^0 - s_u + 2} = 0) 1 \\ &\quad + \delta_{\rho(x,y)}\mathcal{P}^1(\eta_{R_1^0 - s_u + 1} = 0, \eta_{R_1^0 - s_u + 2} = 1)(1 - p) \\ &\quad + \delta_{\rho(x,y)}\mathcal{P}^1(\eta_{R_1^0 - s_u + 1} = 1, \eta_{R_1^0 - s_u + 2} = 0)(1 - p)^2 \\ &\quad + \delta_{\rho(x,y)}\mathcal{P}^1(\eta_{R_1^0 - s_u + 1} = 1, \eta_{R_1^0 - s_u + 2} = 1)(1 - p)^2 \\ &\geq (1 - p)^2 + p(1 - p)^2 + p(1 - p)^3 + p^2(1 - p)^2. \end{aligned} \tag{115}$$

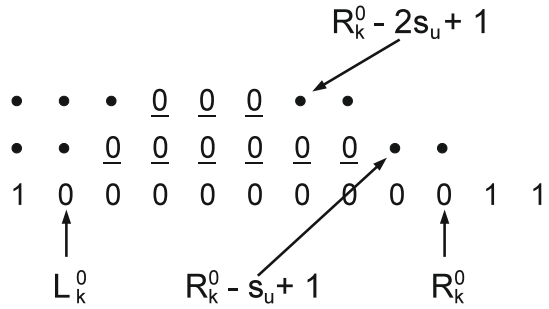
We provide now the estimation (108) considering all $j \geq 3$. We introduce an index $m \in [[0, j - 1]]$, and the mutually disjoint cylinder sets (they will be defined later),

$$\{C^{a,m}\}_{m \in [[0, j - 1]]}, \{C^{b,m}\}_{m \in [[0, j - 2]]}, C^c.$$

We denote by C^d the set of realisations that are not in the sets just introduced, namely,

$$C^d := \Sigma / \bigcup_{m \in [[0, j - 2]]} C^{a,m} \cup C^{b,m} \cup C^c \cup C^{a, j - 1}. \tag{116}$$

Fig. 15 In the figure we consider $\mathcal{U} = \{-1, 0, 1, 2\}$. If the initial realisation of the Percolation PCA is the one represented in the figure (*lowest row*), then almost surely the state of the sites *above* the short *horizontal line* is zero



For every $m \in [[0, j - 2]]$ we estimate $\delta_{\rho(x,y)}\mathcal{P}(C^{a,m})$ and $\delta_{\rho(x,y)}\mathcal{P}(C^{b,m})$, and we also estimate $\delta_{\rho(x,y)}\mathcal{P}(C^{a,j-1})$ and $\delta_{\rho(x,y)}\mathcal{P}(C^c)$. Furthermore, for each of these sets we provide some bounds $B^{a,m}, B^{b,m}, B^c$. Namely, for every $\eta^1 \in C^w$, where w denotes generically (a, m) , (b, m) or c , the following inequality holds,

$$B^w \leq \prod_{z \in [LR_1^0 - 2s_u, \dots, R_1^0 - 2s_u + j]} T(\eta_z^2 = 0 | \eta_{\mathcal{U}(z)}^1), \tag{117}$$

We use such estimations to provide a bound for (100).

We start with the introduction of the cylinder set $C^{a,m} \subset \Sigma$,

$$C^{a,m} := \left\{ \eta \in \Sigma \text{ s.t. } \forall z \in [LR_k^0 - s_u + 1, R_1^0 - s_u + j] / \{R_k^0 - s_u + m + 1\}, \right. \\ \left. \eta_z = 0 \text{ and } \eta_{R_k^0 - s_u + m + 1} = 1 \right\}, \tag{118}$$

(see also Fig. 16a). By a simple computation,

$$\delta_{\rho(x,y)}\mathcal{P}(C^{a,m}) = p(1 - p)^{j-1}. \tag{119}$$

Furthermore, we observe that $\forall \eta \in C^{a,m}$, the product over the transition probabilities of Eq. (100) satisfies the following bound,

$$\prod_{z \in [LR_1^0 - 2s_u, R_1^0 - 2s_u + j]} T(\eta_z^2 = 0 | \eta_{\mathcal{U}(z)}^1) \geq B^{a,m}, \tag{120}$$

where

$$B^{a,m} := \begin{cases} (1 - p)^{s_u - s_1 + 1} & \text{if } 0 \leq m \leq j - (s_u - s_1) - 1 \\ (1 - p)^{j - m} & \text{if } j - (s_u - s_1) \leq m \leq j - 1 \end{cases}. \tag{121}$$

Then we introduce the cylinder sets $C^{b,m} \subset \Sigma$, where $0 \leq m \leq j - 2$.

$$C^{b,m} := \left\{ \eta \in \Sigma \text{ s.t. } \forall z \in [LR_1^0 - s_u + 1, R_1^0 - s_u + m], \eta_z = 0, \eta_{R_1^0 - s_u + m + 1} = 1, \right. \\ \left. \exists y \in [LR_1^0 - s_u + m + 2, R_1^0 - s_u + j] \text{ s.t. } \eta_y = 1 \right\} \tag{122}$$

(see also Fig. 16b). By using the definition of transition probability for the Percolation PCA we estimate the probability measure of this cylinder set

$$\delta_{\rho(x,y)}\mathcal{P}(C^{b,m}) = (1 - p)^m p \left[1 - (1 - p)^{j - m - 1} \right], \tag{123}$$

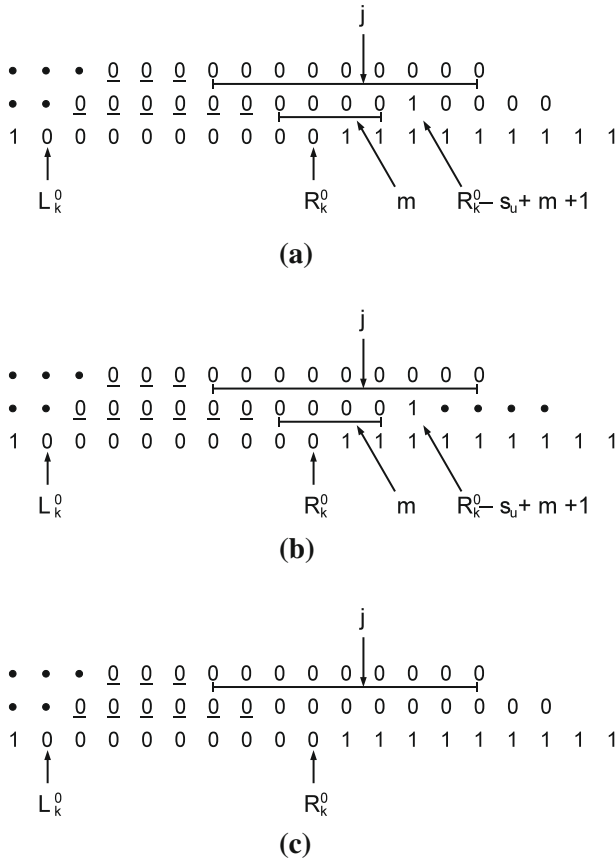


Fig. 16 In the figures above the neighbourhood is assumed to be $\mathcal{U} = \{-1, 0, 1, 2\}$. If the initial realisation of the Percolation PCA (first row) is the one represented in the figure, then the sites underlined by a short line on the second and third row have state zero almost surely. **a** The second row represents a realisation η^2 belonging to the set $C^{a,m}$, $m = 4$, $j = 9$. **b** The second row from below represents a realisation η^2 belonging to the set $C^{b,m}$, $m = 4$, $j = 9$. **c** The second row from below represents a realisation η^2 belonging to the set C^c , $j = 9$

and we observe that $\forall \eta^1 \in C^{b,m}$ the following bound holds

$$\prod_{z \in [[R_1^0 - 2s_u, R_1^0 - 2s_u + j]]} T(\eta_z^2 = 0 | \eta_{\mathcal{U}(z)}^1) \geq (1 - p)^{j-m} \tag{124}$$

Thus we define

$$B^{b,m} := (1 - p)^{j-m}. \tag{125}$$

The bound (124) is obtained considering that $T(\eta_z^2 = 0 | \eta_{\mathcal{U}(z)}^1) = 1$ for all $z \in [[R_1^0 - 2s_u + 1, R_1^0 - 2s_u + m]]$ and $T(\eta_z^2 = 0 | \eta_{\mathcal{U}(z)}^1) \geq (1 - p)$ for all $z \in [[R_1^0 - 2s_u + 1 + m, R_1^0 - 2s_u + j]]$.

Third, we define the cylinder set $C^c \subset \Sigma$,

$$C^c := \left\{ \eta^1 \in \Sigma \text{ s.t. } \eta_z^1 = 0 \quad \forall z \in [[R_1^0 - s_u + 1, R_1^0 - s_u + j]] \right\}. \tag{126}$$

For this set,

$$\delta_{\rho(x,y)}\mathcal{P}(C^c) = (1 - p)^j, \tag{127}$$

and $\forall \eta^1 \in C^c$,

$$\prod_{z \in [R_1^0 - 2s_u, R_1^0 - 2s_u + j]} T(\eta_z^2 = 0 | \eta_{\mathcal{U}(z)}^1) = 1 \tag{128}$$

(see also Fig. 16c). Thus we define

$$B^c := 1. \tag{129}$$

Finally we recall the definition of C^d provided in Eq. (116). We observe that

$$\delta_{\rho(x,y)}\mathcal{P}(C^d) = 1 - \sum_{m=0}^{j-1} \delta_{\rho(x,y)}\mathcal{P}(C^{a,m}) - \sum_{m=0}^{j-2} \delta_{\rho(x,y)}\mathcal{P}(C^{b,m}) - \delta_{\rho(x,y)}\mathcal{P}(C^c) \tag{130}$$

and that $\forall \eta^1 \in \Sigma$,

$$\prod_{x \in [R_1^0 - 2s_u, R_1^0 - 2s_u + j]} T(\eta_x^2 = 0 | \eta_{\mathcal{U}(x)}^1) \geq (1 - p)^j. \tag{131}$$

The inequality is obtained considering that from the definition (2) it follows that $\forall z \in \mathbb{Z}$, $T(\eta_x^2 = 0 | \eta_{\mathcal{U}(x)}^1) \geq (1 - p)$. Thus we define

$$B^d := (1 - p)^j. \tag{132}$$

We finally replace the estimations (119), (121), (123), (125), (127), (129), (132), in the next expression,

$$\begin{aligned} \delta_{\rho(x,y)}\mathcal{P}^2(R_1^2 \geq j + R_1^0 - 2s_u) &\geq \sum_{m \in [0, j-1]} \left[\delta_{\rho(x,y)}\mathcal{P}(C^{a,m}) B^{a,m} \right. \\ &\quad \left. + \delta_{\rho(x,y)}\mathcal{P}(C^{b,m}) B^{b,m} \right] + \delta_{\rho(x,y)}\mathcal{P}(C^c) B^c \\ &\quad + \delta_{\rho(x,y)}\mathcal{P}(C^d), \end{aligned} \tag{133}$$

that follows from the fact that cylinder sets are disjoint and from (117). With a simple computation we derive the bounds (108) and (110). \square

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Appendix: Numerical Simulations

We consider Percolation PCA with space \mathbb{S}_n and periodic boundaries. We divide the interval $[0, 1]$ in smaller intervals of length 0.0002 and for each extremal point p of the smaller intervals we run the process R times and we compute the ratio,

$$P^{R,T,n}(p) := N(R, T, n, p) / R, \tag{134}$$

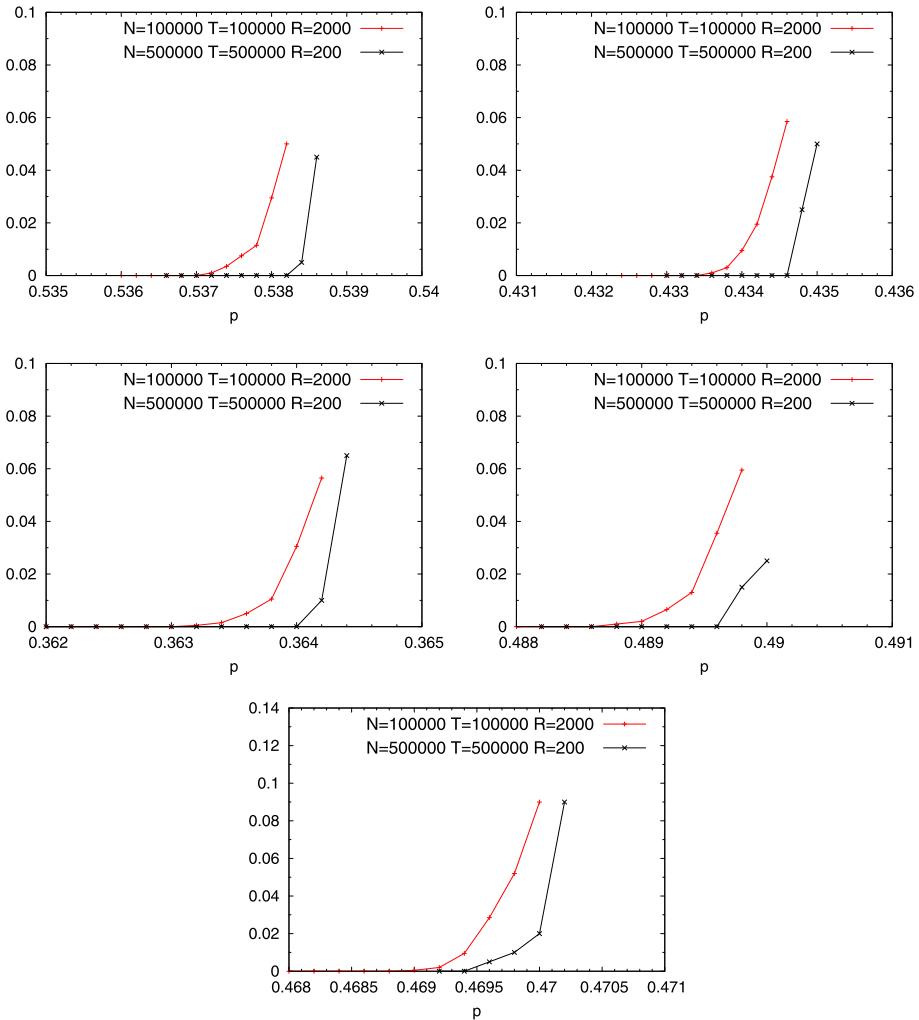


Fig. 17 *Up left* $\mathcal{U} = \{-1, 0, 1\}$. *Up right* $\mathcal{U} = \{-1, 0, 1, 2\}$. *Middle left* $\mathcal{U} = \{-1, 0, 1, 2, 3\}$. *Middle right* $\mathcal{U} = \{-1, 0, 2\}$. *Down* $\mathcal{U} = \{-1, 0, 3\}$

where $N(R, T, n, p)$ is the number of times the origin has state 1 at time T among R computer simulations. As n, T and R are large, $P^{R,T,n}(p)$ converges to the following quantity,

$$\lim_{t \rightarrow \infty} \delta_1 \mathcal{P}_p^t(\eta_0 = 1),$$

which is positive if Percolation PCA is supercritical or zero otherwise. For each extremal point p of the smaller intervals we consider two parameter sets. The first parameter set is $n = 100,000, T = 100,000, R = 2,000$ and the second parameter set is $n = 500,000, T = 500,000$ and $R = 200$ (larger space and less repetitions). For both parameter sets, we plot in Fig. 17 the quantity (134) obtained by means of numerical simulations. The smallest p such that $P^{R,T,n}(p)$ is positive represents our numerical estimation for p_c . This value is

different for the two parameter sets, but the fluctuation is small. The estimation of $p_c(\mathcal{U})$ for several neighbourhoods can be found in the table below.

\mathcal{U}	p_1	p_2	Num. Est.
$\{-1, 0\}$	2/3	0.670	0.705
$\{-1, 0, 1\}$	1/2	0.505	0.538
$\{-1, 0, 1, 2\}$	2/5	0.407	0.435
$\{-1, 0, 2\}$	2/5	0.407	0.490
$\{-1, 0, 1, 2, 3\}$	1/3	0.343	0.364
$\{-1, 0, 3\}$	1/3	0.343	0.470

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