QUANTITATIVE ANALYSIS OF FINITE-DIFFERENCE APPROXIMATIONS OF FREE-DISCONTINUITY PROBLEMS

ANNIKA BACH, ANDREA BRAIDES, AND CATERINA IDA ZEPPIERI

Abstract. Motivated by applications to image reconstruction, in this paper we analyse a *finite-difference discretisation* of the Ambrosio-Tortorelli functional. Denoted by ε the elliptic-approximation parameter and by δ the discretisation step-size, we fully describe the relative impact of ε and δ in terms of Γ-limits for the corresponding discrete functionals, in the three possible scaling regimes. We show, in particular, that when ε and δ are of the same order, the underlying lattice structure affects the Γ-limit which turns out to be an anisotropic free-discontinuity functional.

Keywords: finite-difference discretisation, Ambrosio-Tortorelli functional, Γ-convergence, elliptic approximation, free-discontinuity functionals.

2000 Mathematics Subject Classification: 49M25, 49J45, 68U10, 65M06.

1. Introduction

The detection of objects and object contours in images is a central issue in Image Analysis and Computer Vision. From a mathematical modelling standpoint, a grey-scale image can be described in terms of a scalar function $g : \Omega \to [0, 1]$ (here, $\Omega \subset \mathbb{R}^n$ is a set parameterising the image domain, *e.g.*, a rectangle in the plane), which measures, at every point in Ω, the brightness (or grey-level) of the picture. After a model introduced by Mumford and Shah [52], the relevant information from an input image g can be obtained from a "restored" image described by a function u which solves the minimisation problem

$$\min\Big\{MS(u) + \int_\Omega |u-g|^2 dx \colon u \in SBV(\Omega)\Big\}, \tag{1.1}$$

where

$$MS(u) = \int_\Omega |\nabla u|^2 dx + \mathcal{H}^{n-1}(S_u) \tag{1.2}$$

is the so-called Mumford-Shah functional and $SBV(\Omega)$ denotes the space of special functions of bounded variation in Ω [45], S_u denotes the discontinuity set of u, and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. By solving (1.1) the discontinuous function g is replaced by a function u which is "close" to g and at the same time is smooth outside its discontinuity set S_u. The latter, moreover, having a "minimal" $(n-1)$-dimensional Hausdorff measure will only detect the relevant contours in the input image g. We note that a more complete Mumford-Shah functional would be of the form $\alpha \int_\Omega |\nabla u|^2 dx + \beta\mathcal{H}^{n-1}(S_u)$ with α, β positive "contrast parameters". In the analysis carried out in the present paper it is not restrictive to set $\alpha = \beta = 1$. Although the relevant space dimension for Image Analysis is $n = 2$, we define our problems in a n-dimensional setting for the sake of generality, and also because in the case $n = 3$ the Mumford-Shah functional has an important mechanical interpretation as it coincides with Griffith's fracture energy in the anti-plane case (see [17]). Problem (1.1) is a weak formulation proposed by De Giorgi and Ambrosio of the original minimisation problem proposed by Mumford and Shah, where the minimisation is performed on pairs (u, K), with K piecewise-regular closed set and u smooth function outside K. In the weak formulation (1.1)-(1.2) the set K is replaced by the discontinuity set of u, and a solution of the original problem is obtained by setting $K = \overline{S}_u$ and proving regularity properties of K (see the recent review paper [46]).

The existence of solutions to (1.1) following the direct methods of the Calculus of Variations is by now classical [6]. However, the numerical treatment of (1.1) presents major difficulties which are mainly due to the presence of the surface term $\mathcal{H}^{n-1}(S_u)$. A way to circumvent these difficulties is to replace the Mumford-Shah functional in (1.1) with an elliptic approximation suggested by De Giorgi and studied by Ambrosio-Tortorelli [9, 10], which provides one of the reference approximation argument used in the literature (see *e.g.* [11, 34, 39, 40, 51, 53, 55]). Following the Ambrosio-Tortorelli approximation argument, in place of (1.1) one considers a family of scale-dependent problems

$$\min\Big\{AT_\varepsilon(u,v)+\int_\Omega |u-g|^2dx\colon u,v\in W^{1,2}(\Omega)\Big\}, \tag{1.3}$$

where

$$AT_\varepsilon(u,v)=\int_\Omega (v^2+\eta_\varepsilon)|\nabla u|^2dx+\frac{1}{2}\int_\Omega\Big(\frac{(v-1)^2}{\varepsilon}+\varepsilon|\nabla v|^2\Big)dx. \tag{1.4}$$

Formally, when the approximation parameter $\varepsilon > 0$ is small, the first term in the second integral of (1.4) forces v to be close to the value 1 except on a "small" set, which can be regarded as an approximation of S_u. Additionally, the presence of the term v^2 in the first integral allows u to have a large gradient where v is close to zero. Finally, the optimisation of the singular-perturbation term with $|\nabla v|^2$ produces a transition layer around S_u giving exactly the surface term present in MS. The parameter $\eta_\varepsilon > 0$ is used in the numerical simulations in order to have well-posed minimisation problems in (1.3); it is taken much smaller than ε, but does not intervene in the mathematical analysis. It is interesting to note that the coefficient $v^2 + \eta_\varepsilon$ can be also interpreted as a damage parameter (see e.g. [48]), so that, within Fracture Theory, AT_ε can be seen as an approximation of Griffith's Fracture by concentrated damage. More in general the functionals AT_ε are a prototype of phase-field models for free-discontinuity problems.

Since the functionals in (1.3) are equi-coercive and AT_ε converge to MS in the sense of Γ-convergence [10], solving (1.3) gives pairs $(u_\varepsilon, v_\varepsilon)$, where u_ε approximates a solution u to (1.1) and v_ε provides a diffuse approximation of the corresponding discontinuity set S_u. Moreover, since the functionals AT_ε are elliptic, the difficulties arising in the discretisation of the free-discontinuity set are prevented and *finite-elements* or *finite-difference* schemes for AT_ε can be implemented. From the Γ-convergence of the numerical approximations of AT_ε with mesh size δ (at fixed ε) and the Γ-convergence of AT_ε to MS, one expects that if $\delta \ll \varepsilon$ then the numerical approximations of AT_ε with mesh size δ Γ-converge to MS, as ε and δ go to zero simultaneously. Conversely, note that if $\delta \gg \varepsilon$ then the second integral in (1.4) diverges unless v is uniformly close to 1, which implies that the domain of the Γ-limit of numerical approximations of AT_ε with mesh size δ is with $u \in W^{1,2}(\Omega)$, and hence the Γ-limit is not MS (see also the arguments of Section 6). This separation of scales, however, is not always justified in practice. Indeed, in the numerical implementation of the Ambrosio-Tortorelli approximation scheme the mesh size δ might be comparable with the parameter ε, in which case one expects the underlying lattice structure to affect the Γ-limit.

Discrete approximations of free-discontinuity functionals. In order to illustrate in general the combined effect of δ and ε, in particular when δ is of the same order of ε, we briefly recall some analyses which started from a different discrete approximation scheme for (1.1). In [36] Chambolle considered a *finite-difference* approximation of MS based on an earlier model by Blake and Zissermann [14] (see also [35]): in the case of space-dimension $n = 1, 2$, Chambolle studied the asymptotic behaviour of the discrete functionals given by

$$F_\varepsilon(u) = \frac{1}{2} \sum_{\substack{i,j \in \Omega \cap \varepsilon\mathbb{Z}^n \\ |i-j|=\varepsilon}} \varepsilon^n \min\left\{ \left|\frac{u(i)-u(j)}{\varepsilon}\right|^2, \frac{1}{\varepsilon} \right\}, \tag{1.5}$$

where the energies depend on finite differences through a *truncated quadratic potential* with threshold energy $1/\varepsilon$. If $n = 1$ he showed that the functionals F_ε Γ-converge to MS with respect to an appropriate discrete-to-continuum convergence of lattice functions. In dimension $n = 2$, however, the Γ-limit of F_ε turns out to be anisotropic and given by

$$F(u) = \int_\Omega |\nabla u|^2 dx + \int_{S_u} |\nu_u|_1 d\mathcal{H}^1, \tag{1.6}$$

where ν_u denotes the normal to S_u and $|\nu|_1 = |\nu_1| + |\nu_2|$ is the 1-norm of the vector ν, which appears in the limit due to the specific geometry of the underlying lattice $\varepsilon\mathbb{Z}^2$. Using the lattice energy (1.6) as a model, some continuum approximations of the original isotropic Mumford-Shah energy have been obtained. Notably, a continuum finite-difference approximation was conjectured by De Giorgi and proved by Gobbino [49], while a non-local version involving averages of gradients in place of finite differences was proved by Braides and Dal Maso [23].

Various modifications of F_ε have been studied, many of which in the direction of obtaining more general surface terms in the limit energies. In [37] Chambolle introduced a variant of (1.5) where arbitrary finite differences and truncated energy densities with variable threshold energies are considered. He showed that this new class of functionals provides discrete approximations of image-segmentation functionals where the anisotropy is "reduced" with respect to (1.6) (see also the paper by Braides and Gelli [25]). Braides and Piatnitski [27] examined random mixtures of truncated quadratic and simply quadratic interactions producing surface energies whose anisotropy can be described through percolation results, whereas in the recent paper [54] Ruf shows that the anisotropy in the limit functional can be prevented by considering discrete approximating functionals defined on *statistically isotropic* lattices. The form of the surface energy can be studied separately by examining energies on lattice spin functions (see *e.g.* [33, 3] and [21] and the references therein); in particular patterns of interactions (corresponding to different threshold values in the truncated quadratic potentials) satisfying design constraints and giving arbitrary surface energies has been recently described by Braides and Kreutz [26]. As finite-difference schemes involving energies as in (1.5) are concerned, in [38] Chambolle and Dal Maso show that macroscopic anisotropy can be avoided by considering *alternate finite-elements* of suitable local approximations of the Mumford-Shah functional.

Motivation and main result of the paper. The finite-difference schemes described above suggest that in the numerical implementation of the Ambrosio-Tortorelli approximation, for general values of the mesh-size δ and the parameter ε the anisotropy of the surface term cannot be ruled out as in the case considered in [13]. In terms of Γ-convergence, we may expect that for a general dependence of δ on ε a discretisation of AT_ε with mesh size δ shall not converge to the Mumford-Shah functional but rather to some anisotropic functional of the form

$$E(u) = \int_\Omega |\nabla u|^2 dx + \int_{S_u} \varphi(\nu_u) d\mathcal{H}^{n-1}, \tag{1.7}$$

where the surface integrand φ reflects the geometry of the underlying lattice and may depend on the interaction between δ and ε. These considerations motivate the analysis carried out in the present paper.

In the spirit of a recent paper by Braides and Yip [32] in which the discretisation of the Modica-Mortola functional [50] is analysed, here we propose and analyse a *finite-difference* discretisation of the Ambrosio-Tortorelli functionals; *i.e.,* we consider the functionals defined as

$$E_\varepsilon(u,v) = \frac{1}{2}\Bigg(\sum_{\substack{i,j\in\Omega\cap\delta\mathbb{Z}^n\\ |i-j|=\delta}} \delta^n (v^i)^2 \left|\frac{u^i-u^j}{\delta}\right|^2 + \sum_{i\in\Omega\cap\delta\mathbb{Z}^n} \delta^n \frac{(v^i-1)^2}{\varepsilon} + \frac{1}{2} \sum_{\substack{i,j\in\Omega\cap\delta\mathbb{Z}^n\\ |i-j|=\delta}} \varepsilon\delta^n \left|\frac{v^i-v^j}{\delta}\right|^2 \Bigg) \quad (1.8)$$

and we study their limit behaviour as ε and δ simultaneously tend to zero. In contrast to the continuum formulation (1.4) we omit the additional parameter η_ε in the discrete functionals in (1.8), since it is not needed to ensure existence of solutions to the discrete analogue of (1.3) (see Section 3.3 for more details and a precise formulation of the discrete minimisation problem). For the purpose of this paper it is thus convenient to omit the parameter η_ε also in the definition of AT_ε. Since the discrete functionals in (1.8) are more explicit than the Bellettini-Coscia finite-elements discretisation, we are in a position to perform a rather detailed Γ-convergence analysis for E_ε in all the three possible scaling regimes; *i.e.,* $\delta \ll \varepsilon$ (subcritical regime), $\delta \sim \varepsilon$ (critical regime), and $\delta \gg \varepsilon$ (supercritical regime). More precisely, if $\ell := \lim_\varepsilon \frac{\delta}{\varepsilon}$, in Theorem 2.1 we prove that for every $\ell \in [0,+\infty]$ and for $n = 2$ the functionals E_ε Γ-converge to

$$E_\ell(u) = \int_\Omega |\nabla u|^2 dx + \int_{S_u} \varphi_\ell(\nu_u) d\mathcal{H}^1,$$

for some surface integrand $\varphi_\ell\colon S^1 \to [0,+\infty]$. Furthermore, we show that in the subcritical regime $\varphi_0 \equiv 1$ so that $E_0 = MS$, in the critical regime φ_ℓ explicitly depends on the normal ν (see (1.10), below), and finally in the supercritical regime $\varphi_\infty \equiv +\infty$, so that E_∞ is finite only on the Sobolev space $W^{1,2}(\Omega)$ and it coincides with the Dirichlet functional.

It is worth mentioning that the convergence results in the extreme cases $\ell = 0$ and $\ell = +\infty$ actually hold true in any space dimensions (see Section 4 for the case $\ell = 0$ and Section 6 for the case $\ell = +\infty$), whereas the convergence result in the critical case is explicit only for $n = 1, 2$. In fact, for $\ell \in (0,+\infty)$ the surface integrand φ_ℓ can be explicitly determined only for $n = 1, 2$ (see Theorem 5.11 and Remark 5.6), while for $n > 2$ we can only prove an abstract compactness and integral representation result (see Theorem 5.3 and Theorem 5.5) which, in particular, does not allow us to exclude that the surface energy density may also depend on the jump opening. The main difference between the case $n = 2$ and $n = 3$ (and higher) is related to the problem of describing the structure of the sets of lattice sites where the parameter v is close to 0, which approximates the jump set S_u. In principle, if that discrete set presents "holes" the limit surface energy may depend on the values $u^\pm$ of u on both sides of S_u (see [30]). In two dimensions this is ruled out by showing that such lattice sets can be locally approximated by a continuous line. In dimension $n > 2$ deducing that such set is approximately described by a hypersurface seems more complex and in this case the difficulties are similar to those encountered in some lattice spin problems (*e.g.*, when dealing with dilute lattice spin systems [28]).

Methods of proof. Below we briefly outline the analysis carried out in the present paper, in the three different scaling regimes.

Subcritical regime: $\ell = 0$. In this regime the Γ-limit of the finite-difference discretisation E_ε is the Mumford-Shah functional MS, as in the case of the finite-elements discretisation analysed by Bellettini and Coscia in [13]. Even if the scaling regime is the same as in [13], the proof of the Γ-convergence result for E_ε is substantially different. In particular, the most delicate part in the proof of the Γ-convergence result is to show that the lower-bound estimate holds true. Indeed, in our case the form (and the non-convexity) of the first term in E_ε makes it impossible to have an inequality of the type

$$E_\varepsilon(u,v) \geq AT_\varepsilon(\tilde{u},\tilde{v}) + o(1),$$

where $\tilde{u}$ and $\tilde{v}$ denote suitable continuous interpolations of u and v, respectively. Then, to overcome this difficulty we first prove a non-optimal *asymptotic lower bound* for E_ε which allows us to show that the domain of the Γ-limit is $(G)SBV(\Omega)$ (see Proposition 3.4). Subsequently, we combine this information with a careful blow-up analysis, which eventually provides us with the desired optimal lower bound (see Proposition 4.1). Finally, the upper-bound inequality follows by an explicit construction (see Proposition 4.2).

Critical regime: $\ell \in (0,\infty)$. When the two scales ε and δ are comparable, we appeal to the so-called "direct methods" of Γ-convergence to determine the Γ-limit of E_ε. Namely, we show that E_ε admits a Γ-convergent subsequence whose limit is an integral functional of the form

$$E_\ell(u) = \int_\Omega |\nabla u|^2 dx + \int_{S_u} \phi_\ell([u], \nu_u) d\mathcal{H}^{n-1}, \tag{1.9}$$

for some Borel function $\phi_\ell : \mathbb{R} \times S^{n-1} \to [0,+\infty)$. Here in general the surface integrand ϕ_ℓ depends on the subsequence, on the jump-opening $[u] = u^+ - u^-$, and on the normal ν_u to the jump-set S_u. The delicate part in the convergence result as above is to show that the abstract Γ-limit satisfies the assumptions needed to represent it in an integral form as in (1.9). Specifically, a so-called "fundamental estimate" for the functionals E_ε is needed (see Proposition 5.2).

For $n = 2$, which is the most relevant case for the applications we have in mind, we are able to explicitly characterise the function ϕ_ℓ. In particular we prove that ϕ_ℓ does not depend on the subsequence and on the jump-amplitude $[u]$. Specifically, we show that $\phi_\ell \equiv \varphi_\ell$ where

$$\varphi_\ell(\nu) = \lim_{T\to+\infty} \frac{1}{2T} \inf \Big\{ \ell \sum_{i\in TQ^\nu\cap\mathbb{Z}^2} (v^i - 1)^2 + \frac{1}{2\ell} \sum_{\substack{i,j\in TQ^\nu\cap\mathbb{Z}^2 \\ |i-j|=1}} |v^i - v^j|^2 \colon \exists \text{ channel } \mathscr{C} \text{ in } TQ^\nu \cap \mathbb{Z}^2 \colon$$
$$v = 0 \text{ on } \mathscr{C},\ v = 1 \text{ otherwise near } \partial TQ^\nu \Big\}. \tag{1.10}$$

In (1.10) a channel $\mathscr{C}$ (see Definition 5.10 for a formal definition) is a path on the square lattice $\mathbb{Z}^2$ connecting two opposite sides of the square and can be interpreted as a "discrete approximation" of the discontinuity line $\{x \in \mathbb{R}^2 \colon \langle x,\nu\rangle = 0\}$. When $\nu = e_1, e_2$ we show that the channel $\mathscr{C}$ in (1.10) is actually flat and it coincides with the discrete interface $\{x \in \mathbb{R}^2 \colon \langle x,\nu\rangle = 0\} \cap \mathbb{Z}^2$. As a consequence, the minimisation problem defining φ_ℓ turns out to be one-dimensional (see Remark 5.6).

For $n = 1$ the function ϕ_ℓ is also explicit and equal to a constant; the proof of this fact is a consequence of more elementary one-dimensional arguments and is briefly discussed in Remark 5.6.

Supercritical regime: $\ell = +\infty$. In this scaling regime discontinuities have a cost proportional to $\delta/\varepsilon \gg 1$ and are therefore forbidden. In fact the Γ-limit E_∞ turns out to be finite only in $W^{1,2}(\Omega)$ (see Proposition 3.4) and

$$E_\infty(u) = \int_\Omega |\nabla u|^2 dx.$$

In order to allow for the development of discontinuities in the limit, in the spirit of Braides and Truskinovsky [31], in this case we also analyse the asymptotic behaviour of a suitably rescaled variant of E_ε whose Γ-limit is still a functional of the form (1.7) with $\varphi(\nu) = |\nu|_\infty$ (see Theorem 6.1), so that in this supercritical regime we recover a crystalline surface energy.

Very recently, in [12] the case $\delta = \varepsilon$ has been extended to discretizations of AT_ε via finite-differences on stochastic lattices, while the case $\delta \ll \varepsilon$ has been considered in [43] to approximate Griffith functionals on the space of generalised special functions of bounded deformation.

Plan of the paper. The paper is organised as follows. In Section 2 we introduce a few notation and state the main Γ-convergence result Theorem 2.1. In Section 3 we determine the domain of the Γ-limit in the three scaling regimes, we prove an equicoercivity result for a suitable perturbation of the fuctionals E_ε, and study the convergence of the associated minimisation problems. In Sections 4, 5, and 6 we prove the Γ-convergence result Theorem 2.1, respectively, in the subcritical, critical, and supercritical regime. In Section 6 we also analyse the asymptotic behaviour of a sequence of functionals which is equivalent to E_ε in the sense of Γ-convergence (see [31]). Eventually, in Section 7 we show that for $n = 2$ and $\ell \in (0, +\infty)$, the integrand φ_ℓ interpolates the two extreme regimes $\ell = 0$ and $\ell = +\infty$.

2. Setting of the problem and statement of the main result

Notation. In this section we fix some notation that we will employ in what follows. Given $n \geq 1$, throughout this paper $\Omega \subset \mathbb{R}^n$ is an open bounded set of with Lipschitz boundary. Furthermore, the family of all open subsets of Ω and of all open subsets of Ω with Lipschitz boundary are denoted by $\mathscr{A}(\Omega)$ and $\mathscr{A}_L(\Omega)$, respectively. If $A', A \in \mathscr{A}(\Omega)$ are such that $A' \subset\subset A$, we say that φ is a cut-off function between A' and A if $\varphi \in C_c^\infty(A)$, $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on A'.

The integer part of $t \in \mathbb{R}$ is denoted by $\lfloor t \rfloor$. If $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{R}^n$ the euclidian norm of ν is denoted by $|\nu|$. Moreover, we set $|\nu|_1 := \sum_{k=1}^n |\nu_k|$ and $|\nu|_\infty := \max_{1 \leq k \leq n} |\nu_k|$. We use the notation $\langle \nu, \xi \rangle$ for the scalar product between $\nu, \xi \in \mathbb{R}^n$. We set $S^{n-1} := \{\nu \in \mathbb{R}^n \colon |\nu| = 1\}$ and for every $\nu \in S^{n-1}$ the hyperplane through 0 and orthogonal to ν is denoted by $\Pi_\nu := \{x \in \mathbb{R}^n \colon \langle x, \nu \rangle = 0\}$. Also, the two half spaces defined as $\Pi_\nu^+ := \{x \in \mathbb{R}^n \colon \langle x, \nu \rangle > 0\}$ and $\Pi_\nu^- := \{x \in \mathbb{R}^n \colon \langle x, \nu \rangle \leq 0\}$ are denoted by Π_ν^+ and Π_ν^-, respectively. For every $\nu \in S^{n-1}$ $Q^\nu \subset \mathbb{R}^n$ is a given cube centred at 0 with side length 1 and with one face orthogonal to ν, and for all $x_0 \in \mathbb{R}^n$ and $\rho > 0$ we set $Q_\rho^\nu(x_0) = x_0 + \rho Q^\nu$. If $\{e_1, \dots, e_n\}$ denotes the standard basis in $\mathbb{R}^n$ and $\nu = e_k$ for some $1 \leq k \leq n$ we choose $Q = Q^\nu$ the standard coordinate cube and simply write $Q_\rho(x_0)$.

The Lebesgue measure and the k-dimensional Hausdorff measure in $\mathbb{R}^n$ are denoted by $\mathcal{L}^n$ and $\mathcal{H}^k$, respectively. For $p \in [1, +\infty]$ we use standard notation $L^p(\Omega)$ for the Lebesgue spaces and $W^{1,p}(\Omega)$ for the Sobolev spaces. The space of special functions of bounded variation in Ω is denoted by $SBV(\Omega)$ (for the general theory see *e.g.*, [8, 18]). If $u \in SBV(\Omega)$ ∇udenotes its approximate gradient, S_u the approximate discontinuity set of u, ν_u the generalised outer normal to S_u, and u^+ and u^- are the traces of u on both sides of S_u. We also set $[u] := u^+ - u^-$. Moreover, we consider the larger space $GSBV(\Omega)$, which consists of all functions $u \in L^1(\Omega)$ such that for each $m \in \mathbb{N}$ the truncation of u at level m defined as $u^m := -m \vee (u \wedge m)$ belongs to $SBV(\Omega)$. Furthermore, we set

$$SBV^2(\Omega) := \{u \in SBV(\Omega) : \nabla u \in L^2(\Omega) \text{ and } \mathcal{H}^{n-1}(S_u) < +\infty\}$$

and

$$GSBV^2(\Omega) := \{u \in GSBV(\Omega) : \nabla u \in L^2(\Omega) \text{ and } \mathcal{H}^{n-1}(S_u) < +\infty\}.$$

It can be shown that $SBV^2(\Omega) \cap L^\infty(\Omega) = GSBV^2(\Omega) \cap L^\infty(\Omega)$.

Let u, w be two measurable functions on $\mathbb{R}^n$ and let $A \subset \mathbb{R}^n$ be open, bounded and with Lipschitz boundary; by "$u = w$ near ∂A" we mean that there exists a neighbourhood U of ∂A in $\mathbb{R}^n$ such that $u = w$ $\mathcal{L}^n$-a.e. in $U \cap A$.

Setting. Throughout the paper $\varepsilon > 0$ is a strictly positive parameter and $\delta = \delta(\varepsilon) > 0$ is a strictly increasing function of ε such that such that $\delta(\varepsilon) \to 0$ decreasingly as $\varepsilon \to 0$ decreasingly. Set

$$\ell := \lim_{\varepsilon \to 0} \frac{\delta(\varepsilon)}{\varepsilon}. \tag{2.1}$$

We now introduce the discrete functionals which will be analysed in this paper. To this end let $\Omega \subset \mathbb{R}^n$ be open, bounded, and with Lipschitz boundary. Let $\Omega_\delta := \Omega \cap \delta\mathbb{Z}^n$ denote the portion of the square lattice of mesh-size δ contained in Ω and for every $u : \Omega_\delta \to \mathbb{R}$ set $u^i := u(i)$, for $i \in \Omega_\delta$. It is customary to identify the discrete functions defined on the lattice Ω_δ with their piecewise-constant counterparts belonging to the class

$$\mathcal{A}_\varepsilon(\Omega) := \{u \in L^1(\Omega) : \ u \text{ constant on } i + [0,\delta)^n \text{ for all } i \in \Omega \cap \delta\mathbb{Z}^n\},$$

by simply setting

$$u(x) := u^i \quad \text{for every } x \in i + [0,\delta)^n \text{ and for every } i \in \Omega_\delta. \tag{2.2}$$

If (u_ε) is a sequence of functions defined on the lattice Ω_δ and $u \in L^1(\Omega)$, by $u_\varepsilon \to u$ in $L^1(\Omega)$ we mean that the piecewise-constant interpolation of (u_ε) defined as in (2.2) converges to u in $L^1(\Omega)$.

We define the discrete functionals $E_\varepsilon : L^1(\Omega) \times L^1(\Omega) \to [0,+\infty]$ as

$$E_\varepsilon(u,v) := \begin{cases} \dfrac{1}{2}\Bigg(\displaystyle\sum_{\substack{i,j \in \Omega_\delta \\ |i-j|=\delta}} \delta^n (v^i)^2 \Big|\frac{u^i - u^j}{\delta}\Big|^2 + \sum_{i \in \Omega_\delta} \delta^n \frac{(v^i-1)^2}{\varepsilon} + \frac{1}{2} \sum_{\substack{i,j \in \Omega_\delta \\ |i-j|=\delta}} \varepsilon\delta^n \Big|\frac{v^i - v^j}{\delta}\Big|^2 \Bigg) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } u,v \in \mathcal{A}_\varepsilon(\Omega),\ 0 \le v \le 1, \\ +\infty \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise in } L^1(\Omega) \times L^1(\Omega). \end{cases} \tag{2.3}$$

It is also convenient to consider the functionals F_ε, G_ε given by

$$F_\varepsilon(u,v) := \frac{1}{2} \sum_{\substack{i,j \in \Omega_\delta \\ |i-j|=\delta}} \delta^n (v^i)^2 \Big|\frac{u^i - u^j}{\delta}\Big|^2 \tag{2.4}$$

and

$$G_\varepsilon(v) := \frac{1}{2}\Bigg(\sum_{i \in \Omega_\delta} \delta^n \frac{(v^i-1)^2}{\varepsilon} + \frac{1}{2} \sum_{\substack{i,j \in \Omega_\delta \\ |i-j|=\delta}} \varepsilon\delta^n \Big|\frac{v^i - v^j}{\delta}\Big|^2 \Bigg), \tag{2.5}$$

so that in more compact notation we may write

$$E_\varepsilon(u,v) := \begin{cases} F_\varepsilon(u,v) + G_\varepsilon(v) & \text{if } u,v \in \mathcal{A}_\varepsilon(\Omega),\ 0 \le v \le 1, \\ +\infty & \text{otherwise in } L^1(\Omega) \times L^1(\Omega). \end{cases}$$

In what follows we will also make use of the following equivalent expressions for F_ε and G_ε:

$$F_\varepsilon(u,v) = \frac{1}{2} \sum_{i \in \Omega_\delta} \delta^n (v^i)^2 \Bigg(\sum_{\substack{k=1 \\ i+\delta e_k \in \Omega_\delta}}^{n} \Big|\frac{u^i - u^{i+\delta e_k}}{\delta}\Big|^2 + \sum_{\substack{k=1 \\ i-\delta e_k \in \Omega_\delta}}^{n} \Big|\frac{u^i - u^{i-\delta e_k}}{\delta}\Big|^2 \Bigg)$$

and

$$G_\varepsilon(v) = \frac{1}{2}\Bigg(\sum_{i \in \Omega_\delta} \delta^n \frac{(v^i-1)^2}{\varepsilon} + \sum_{i \in \Omega_\delta} \sum_{\substack{k=1 \\ i+\delta e_k \in \Omega_\delta}}^{n} \varepsilon\delta^n \Big|\frac{v^i - v^{i+\delta e_k}}{\delta}\Big|^2 \Bigg),$$

For $U \in \mathscr{A}(\Omega)$ we will also need to consider the localised versions of F_ε and G_ε; *i.e.,* for every $U \in \mathscr{A}(\Omega)$ we set

$$U_\delta := U \cap \delta\mathbb{Z}^n, \tag{2.6}$$

and

$$F_\varepsilon(u,v,U) := \frac{1}{2}\sum_{i\in U_\delta}\delta^n (v^i)^2 \sum_{\substack{k=1\\ i\pm\delta e_k\in U_\delta}}^{n} \Big|\frac{u^i-u^{i\pm\delta e_k}}{\delta}\Big|^2,$$

$$G_\varepsilon(v,U) := \frac{1}{2}\bigg(\sum_{i\in U_\delta}\delta^n\Big(\frac{(v^i-1)^2}{\varepsilon}+\sum_{\substack{k=1\\ i+\delta e_k\in U_\delta}}^{n}\varepsilon\Big|\frac{v^i-v^{i+\delta e_k}}{\delta}\Big|^2\Big)\bigg),$$

so that finally

$$E_\varepsilon(u,v,U) := \begin{cases} F_\varepsilon(u,v,U)+G_\varepsilon(v,U) & \text{if } u,v\in\mathcal{A}_\varepsilon(\Omega),\ 0\le v\le 1,\\ +\infty & \text{otherwise in } L^1(\Omega)\times L^1(\Omega).\end{cases} \tag{2.7}$$

Sometimes it will be useful to distinguish between points $i\in U_\delta$ such that all their nearest neighbours belong to U and points $i\in U_\delta$ such that $i\pm\delta e_k\notin U$ for some $1\le k\le n$. Then, for a given $U\in\mathcal{A}(\Omega)$ we set

$$\mathring{U}_\delta := \{i\in U_\delta:\ j\in U \text{ for every } j\in\delta\mathbb{Z}^n \text{ s.t. } |i-j|=\delta\},\quad \text{and}\quad \partial U_\delta := U_\delta\setminus \mathring{U}_\delta.$$

With the identification above, we will describe the Γ-limits of energies E_ε with respect to the strong $L^1(\Omega)\times L^1(\Omega)$-topology, in the spirit of recent discrete-to-continuum analyses (see e.g. [4, 29, 5, 22] for some general results in different limit functional settings and [19, 20] for some introductory material).

In all that follows we use the standard notation for the Γ-liminf and Γ-limsup of the functionals E_ε (see [19] Section 1.2); *i.e.,* for every $(u,v)\in L^1(\Omega)\times L^1(\Omega)$ and every $U\in\mathscr{A}(\Omega)$ we set

$$E'_\ell(u,v,U) := \Gamma\text{-}\liminf_{\varepsilon\to0} E_\varepsilon(u,v,U)\quad\text{and}\quad E''_\ell(u,v,U) := \Gamma\text{-}\limsup_{\varepsilon\to0} E_\varepsilon(u,v,U). \tag{2.8}$$

When $U=\Omega$ we simply write $E'_\ell(u,v)$ and $E''_\ell(u,v)$ in place of $E'_\ell(u,v,\Omega)$ and $E''_\ell(u,v,\Omega)$, respectively.

The following Γ-convergence theorem is the main result of this paper.

Theorem 2.1 (Γ-convergence)**.** *Let ℓ be as in* (2.1) *and let E_ε be as in* (2.3)*. Then,*

(i) (Subcritical regime) *If $\ell=0$ the functionals E_ε Γ-converge to E_0 defined as*

$$E_0(u,v) := \begin{cases}\displaystyle\int_\Omega |\nabla u|^2\,dx+\mathcal{H}^{n-1}(S_u\cap\Omega) & \textit{if } u\in GSBV^2(\Omega),\ v=1 \textit{ a.e. in } \Omega,\\ +\infty & \textit{otherwise in } L^1(\Omega)\times L^1(\Omega);\end{cases}$$

(ii) (Critical regime) *If $\ell\in(0,+\infty)$ there exists a subsequence (ε_j) such that the functionals E_{ε_j} Γ-converge to E_ℓ defined as*

$$E_\ell(u,v) := \begin{cases}\displaystyle\int_\Omega |\nabla u|^2\,dx+\int_{S_u\cap\Omega}\phi_\ell([u],\nu_u)\,d\mathcal{H}^{n-1} & \textit{if } u\in GSBV^2(\Omega),\ v=1 \textit{ a.e. in } \Omega,\\ +\infty & \textit{otherwise in } L^1(\Omega)\times L^1(\Omega),\end{cases}$$

for some Borel function $\phi_\ell\colon\mathbb{R}\times S^{n-1}\to[0,+\infty)$ possibly depending on the subsequence (ε_j). If moreover $n=2$ the function ϕ_ℓ does not depend on the subsequence (ε_j). Furthermore, for *every* $(t,\nu) \in \mathbb{R} \times S^{n-1}$ *we have* $\phi_\ell(t,\nu) = \varphi_\ell(\nu)$ *where* $\varphi_\ell \colon S^{n-1} \to [0,+\infty)$ *is given by*

$$\varphi_\ell(\nu) := \lim_{T\to+\infty} \frac{1}{2T} \inf \Big\{ \ell \sum_{i\in TQ^\nu\cap\mathbb{Z}^2} (v^i-1)^2 + \frac{1}{2\ell} \sum_{\substack{i,j\in TQ^\nu\cap\mathbb{Z}^2\\ |i-j|=1}} |v^i - v^j|^2 \colon v \in \mathcal{A}_1(TQ^\nu),$$
$$\exists \text{ channel } \mathscr{C} \text{ in } TQ^\nu \cap \mathbb{Z}^2 \colon v = 0 \text{ on } \mathscr{C},\ v = 1 \text{ otherwise near } \partial TQ^\nu \Big\},$$

(see Definition 5.10 for a precise definition of channel);

(iii) (Supercritical regime) *If* $\ell = +\infty$ *the functionals* E_ε Γ*-converge to* E_∞ *defined as*

$$E_\infty(u,v) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2\,dx & \text{if } u \in W^{1,2}(\Omega),\ v = 1 \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise in } L^1(\Omega)\times L^1(\Omega). \end{cases}$$

The proof of Theorem 2.1 will be divided into a number of intermediate steps and carried out in Sections 4, 5, and 6.

3. Domain of the Γ-limit and compactness

In this section we prove a compactness result for the functionals E_ε. This result is first obtained for $n = 1$ and then extended to the case $n \geq 2$ by means of a slicing-procedure (see [19] Section 15).

The main result of this section is the following.

Theorem 3.1 (Domain of the Γ-limit)**.** *Let* $(u_\varepsilon, v_\varepsilon) \subset L^1(\Omega) \times L^1(\Omega)$ *be such that*

$$(u_\varepsilon, v_\varepsilon) \to (u,v) \quad \text{in} \quad L^1(\Omega)\times L^1(\Omega) \quad \text{and} \quad \sup_{\varepsilon>0} E_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty$$

and let ℓ *be as in* (2.1)*.*

(i) (Subcritical and critical regime) *If* $\ell \in [0,+\infty)$ *then* $u \in GSBV^2(\Omega)$ *and* $v = 1$ *a.e. in* Ω*.*

(ii) (Supercritical regime) *If* $\ell = +\infty$ *then* $u \in W^{1,2}(\Omega)$ *and* $v = 1$ *a.e. in* Ω*.*

The proof of Theorem 3.1 will be carried out in Proposition 3.2 and Proposition 3.4 below.

3.1. **The one-dimensional case.** In this subsection we deal with the case $n = 1$.

In what follows we only consider the case $\Omega = I := (a,b)$ with $a, b \in \mathbb{R}$, $a < b$. The case of a general open set can be treated by repeating the proof below in each connected component of Ω.

Proposition 3.2. *Let* $(u_\varepsilon, v_\varepsilon) \subset L^1(I) \times L^1(I)$ *be such that*

$$(u_\varepsilon, v_\varepsilon) \to (u,v) \quad \text{in} \quad L^1(I)\times L^1(I) \quad \text{and} \quad \sup_{\varepsilon>0} E_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty$$

and let ℓ *be as in* (2.1)*.*

(i) (Subcritical and critical regime) *If* $\ell \in [0,+\infty)$ *then* $u \in SBV^2(I)$ *and* $v = 1$ *a.e. in* I*. Moreover,*

$$E'_\ell(u,v) \geq \int_I (u')^2 dt + \#(S_u).$$

(ii) (Supercritical regime) *If* $\ell = +\infty$ *then* $u \in W^{1,2}(I)$ *and* $v = 1$ *a.e. in* I*. Moreover,*

$$E'_\infty(u,v) \geq \int_I (u')^2 dt.$$

Proof. The proof will be divided into two steps.

Step 1: proof of (i); *i.e.,* the case $\ell \in [0, +\infty)$.

Let $(u_\varepsilon, v_\varepsilon) \subset L^1(I) \times L^1(I)$ be as in the statement. We claim that $E_\varepsilon(u_\varepsilon, v_\varepsilon)$ can be bounded from below by $AT_\varepsilon(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)$ for suitable functions $\tilde{u}_\varepsilon$ and $\tilde{v}_\varepsilon$ with $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon) \to (u, v)$ in $L^1(I) \times L^1(I)$. Then the conclusion follows appealing to the classical Ambrosio and Tortorelli convergence result [10, Theorem 2.1].

For our purposes it is convenient to rewrite E_ε as follows

$$E_\varepsilon(u_\varepsilon, v_\varepsilon) = \sum_{\substack{i \in I_\delta \\ i+\delta \in I}} \delta \frac{(v_\varepsilon^i)^2 + (v_\varepsilon^{i+\delta})^2}{2} \left| \frac{u_\varepsilon^i - u_\varepsilon^{i+\delta}}{\delta} \right|^2 + \frac{1}{2} \Bigg(\sum_{i \in I_\delta} \delta \frac{(v_\varepsilon^i - 1)^2}{\varepsilon} + \sum_{\substack{i \in I_\delta \\ i+\delta \in I}} \varepsilon\delta \left| \frac{v_\varepsilon^i - v_\varepsilon^{i+\delta}}{\delta} \right|^2 \Bigg).$$

We define moreover $\tilde{u}_\varepsilon, \tilde{v}_\varepsilon$ as the piecewise-affine interpolations of u_ε, v_ε on I_δ, respectively; *i.e.,*

$$\tilde{u}_\varepsilon(t) := u_\varepsilon^i + \frac{u_\varepsilon^{i+\delta} - u_\varepsilon^i}{\delta}(t - i) \quad \text{if } t \in [i, i+\delta), \quad i, i+\delta \in I_\delta,$$
$$\tilde{v}_\varepsilon(t) := v_\varepsilon^i + \frac{v_\varepsilon^{i+\delta} - v_\varepsilon^i}{\delta}(t - i) \quad \text{if } t \in [i, i+\delta), \quad i, i+\delta \in I_\delta.$$

We note that $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon) \to (u, v)$ in $L^1(I) \times L^1(I)$.

Let $\eta > 0$ be fixed; for ε sufficiently small we have

$$(a + \eta, b - \eta) \subset \bigcup_{i \in \overset{\circ}{I}_\delta} [i, i + \delta),$$

therefore

$$\sum_{i \in \overset{\circ}{I}_\delta} \varepsilon\delta \left| \frac{v_\varepsilon^i - v_\varepsilon^{i+\delta}}{\delta} \right|^2 = \sum_{i \in \overset{\circ}{I}_\delta} \int_i^{i+\delta} \varepsilon(\tilde{v}_\varepsilon')^2 dt \geq \int_{a+\eta}^{b-\eta} \varepsilon(\tilde{v}_\varepsilon')^2 dt, \tag{3.1}$$

for ε small. Moreover, in view of the definition of $\tilde{v}_\varepsilon$ and the convexity of $z \to (z-1)^2$, for every $i \in \overset{\circ}{I}_\delta$ we get

$$\begin{aligned} \int_i^{i+\delta} \frac{(\tilde{v}_\varepsilon - 1)^2}{\varepsilon} dt &= \frac{1}{\varepsilon} \int_i^{i+\delta} \left(\left(1 - \frac{t-i}{\delta}\right) v_\varepsilon^i + \frac{t-i}{\delta} v_\varepsilon^{i+\delta} - 1 \right)^2 dt \\ &\leq \frac{(v_\varepsilon^i - 1)^2}{\varepsilon} \int_i^{i+\delta} \left(1 - \frac{t-i}{\delta}\right) dt + \frac{(v_\varepsilon^{i+\delta} - 1)^2}{\varepsilon} \int_i^{i+\delta} \frac{t-i}{\delta} dt \\ &= \frac{\delta}{2\varepsilon} \left((v_\varepsilon^i - 1)^2 + (v_\varepsilon^{i+\delta} - 1)^2 \right), \end{aligned}$$

from which we get

$$\sum_{i \in \overset{\circ}{I}_\delta} \delta \frac{(v_\varepsilon^i - 1)^2}{\varepsilon} \geq \int_{a+\eta}^{b-\eta} \frac{(\tilde{v}_\varepsilon - 1)^2}{\varepsilon} dt, \tag{3.2}$$

for ε small. Finally, the definition of $\tilde{u}_\varepsilon$ together with the convexity of $z \to z^2$ yield

$$\begin{aligned}\int_i^{i+\delta} (\tilde{v}_\varepsilon)^2 (\tilde{u}'_\varepsilon)^2 dt &= \left|\frac{u_\varepsilon^i - u_\varepsilon^{i+\delta}}{\delta}\right|^2 \int_i^{i+\delta} \left(\left(1 - \frac{t-i}{\delta}\right) v_\varepsilon^i + \frac{t-i}{\delta} v_\varepsilon^{i+\delta}\right)^2 dt \\ &\leq \left|\frac{u_\varepsilon^i - u_\varepsilon^{i+\delta}}{\delta}\right|^2 \left((v_\varepsilon^i)^2 \int_i^{i+\delta} \left(1 - \frac{t-i}{\delta}\right) dt + (v_\varepsilon^{i+\delta})^2 \int_i^{i+\delta} \frac{t-i}{\delta} dt\right) \\ &= \delta \frac{(v_\varepsilon^i)^2 + (v_\varepsilon^{i+\delta})^2}{2} \left|\frac{u_\varepsilon^i - u_\varepsilon^{i+\delta}}{\delta}\right|^2,\end{aligned}$$

for every $i \in \overset{\circ}{I}_\delta$, and thus

$$\sum_{i \in \overset{\circ}{I}_\delta} \delta \frac{(v_\varepsilon^i)^2 + (v_\varepsilon^{i+\delta})^2}{2} \left|\frac{u_\varepsilon^i - u_\varepsilon^{i+\delta}}{\delta}\right|^2 \geq \int_{a+\eta}^{b-\eta} (\tilde{v}_\varepsilon)^2 (\tilde{u}'_\varepsilon)^2 dt, \tag{3.3}$$

for ε small. Eventually, gathering (3.1)-(3.3) we deduce

$$E_\varepsilon(u_\varepsilon, v_\varepsilon) \geq AT_\varepsilon(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon, (a+\eta, b-\eta)), \tag{3.4}$$

where AT_ε denotes the Ambrosio-Tortorelli functional; *i.e.,*

$$AT_\varepsilon(u, v) := \int_\Omega v^2 |\nabla u|^2 d\,x + \frac{1}{2} \int_\Omega \frac{(v-1)^2}{\varepsilon} + \varepsilon |\nabla v|^2\, dx,$$

for every $(u, v) \in W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ with $0 \leq v \leq 1$. Hence the claim follows first appealing to [10, Theorem 2.1] and then by letting $\eta \to 0$.

Step 2: proof of (ii); *i.e.,* the case $\ell = +\infty$.

Let $(u_\varepsilon, v_\varepsilon) \subset L^1(I) \times L^1(I)$ be as in the statement, then in particular

$$\sup_{\varepsilon > 0} \sum_{i \in I_\delta} \frac{\delta}{\varepsilon} (v_\varepsilon^i - 1)^2 < +\infty.$$

Hence there exists a constant $c > 0$ such that for every $i \in I_\delta$ and for every $\varepsilon > 0$

$$(v_\varepsilon^i - 1)^2 \leq c\, \frac{\varepsilon}{\delta}.$$

Let $\eta \in (0, 1)$ be arbitrary; since by assumption $\varepsilon/\delta \to 0$ there exists $\varepsilon_0 = \varepsilon_0(\eta) > 0$ such that $|v_\varepsilon^i - 1| < \eta$ for every $i \in I_\delta$ and for every $\varepsilon \in (0, \varepsilon_0)$. Then, up to choosing ε small enough, we have

$$\frac{1}{2} \sum_{i \in \overset{\circ}{I}_\delta} \delta (v_\varepsilon^i)^2 \left|\frac{u_\varepsilon^i - u_\varepsilon^{i\pm\delta}}{\delta}\right|^2 \geq (1-\eta)^2 \int_{a+\eta}^{b-\eta} (\tilde{u}'_\varepsilon)^2 dt. \tag{3.5}$$

Since $\tilde{u}_\varepsilon \to u$ in $L^1(I)$, in view of the bound on the energy, from (3.5) we may deduce that $\tilde{u}_\varepsilon \rightharpoonup u$ in $W^{1,2}(a+\eta, b-\eta)$ so that in particular $u \in W^{1,2}(a+\eta, b-\eta)$. Moreover, (3.5) entails

$$\begin{aligned}\liminf_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon, v_\varepsilon) &\geq \liminf_{\varepsilon \to 0} \frac{1}{2} \sum_{i \in \overset{\circ}{I}_\delta} \delta (v_\varepsilon^i)^2 \left|\frac{u_\varepsilon^i - u_\varepsilon^{i\pm\delta}}{\delta}\right|^2 \geq (1-\eta)^2 \liminf_{\varepsilon \to 0} \int_{a+\eta}^{b-\eta} (\tilde{u}'_\varepsilon)^2 dt \\ &\geq (1-\eta)^2 \int_{a+\eta}^{b-\eta} (u')^2 dt,\end{aligned}$$

so that the desired lower bound follows by letting $\eta \to 0$. $\square$

Remark 3.3. Let $(u_\varepsilon, v_\varepsilon) \subset L^1(I) \times L^1(I)$ be a sequence such that $(u_\varepsilon, v_\varepsilon) \to (u, v)$ in $L^1(I) \times L^1(I)$ and $\sup_\varepsilon E_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty$; let moreover $\ell \in [0, +\infty)$. In view of (3.1)-(3.3), arguing as in [10, Lemma 2.1] we note that the two inequalities

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon) \geq \int_a^b (u')^2 dt, \qquad \liminf_{\varepsilon \to 0} G_\varepsilon(v_\varepsilon) \geq \#(S_u)$$

also hold.

3.2. **The n-dimensional case.** In this section we deal with the case $n \geq 2$. The following proposition will be obtained by combining the one-dimensional result in Proposition 3.2 and a slicing procedure in the coordinate directions.

To this end it is convenient to introduce the following notation. For every $k \in \{1, \dots, n\}$ set $\Pi^k := \{x \in \mathbb{R}^n \colon x_k = 0\}$ and let $p^k : \mathbb{R}^n \to \Pi^k$ be the orthogonal projection onto Π^k. For all $y \in \Pi^k$ let

$$\Omega_{k,y} := \{t \in \mathbb{R} : y + te_k \in \Omega\} \tag{3.6}$$

and

$$\Omega_k := \{y \in \Pi^k : \ \Omega_{k,y} \neq \emptyset\}. \tag{3.7}$$

For every $w \colon \Omega \to \mathbb{R}$, $t \in \Omega_{k,y}$, and $y \in \Omega_k$ we set

$$w^{k,y}(t) := w(y + te_k). \tag{3.8}$$

Proposition 3.4. *Let $(u_\varepsilon, v_\varepsilon) \subset L^1(\Omega) \times L^1(\Omega)$ be such that*

$$(u_\varepsilon, v_\varepsilon) \to (u, v) \quad \text{in} \quad L^1(\Omega) \times L^1(\Omega) \quad \text{and} \quad \sup_{\varepsilon > 0} E_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty$$

and let ℓ be as in (2.1).

(i) (Subcritical and critical regime) *If $\ell \in [0, +\infty)$ then $u \in GSBV^2(\Omega)$ and $v = 1$ a.e. in Ω. Moreover,*

$$E'_\ell(u, v) \geq \int_\Omega |\nabla u|^2 dx + \int_{S_u \cap \Omega} |\nu_u|_\infty d\mathcal{H}^{n-1}. \tag{3.9}$$

(ii) (Supercritical regime) *If $\ell = +\infty$ then $u \in W^{1,2}(\Omega)$ and $v = 1$ a.e. in Ω. Moreover,*

$$E'_\infty(u, v) \geq \int_\Omega |\nabla u|^2 \, dx. \tag{3.10}$$

Proof. The proof will be divided into two steps.

Step 1: proof of (i); *i.e.,* the case $\ell \in [0, +\infty)$.

Let $(u_\varepsilon, v_\varepsilon) \subset L^1(\Omega) \times L^1(\Omega)$ be a sequence converging to (u, v) in $L^1(\Omega) \times L^1(\Omega)$ and such that $\sup_\varepsilon E_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty$. Note that $v_\varepsilon \to 1$ in $L^2(\Omega)$, so that $v = 1$ a.e. in Ω.

We now show that $u \in GSBV^2(\Omega)$. To this end let $k \in \{1, \cdots, n\}$ be fixed and for every $y \in \Omega_k$ consider the two sequences of functions $(u_\varepsilon^{k,y})$, $(v_\varepsilon^{k,y})$ defined on $\Omega_{k,y}$ as in (3.8) with w replaced by u_ε and v_ε, respectively. Let $\eta > 0$ be fixed; set $\Omega^\eta := \{x \in \Omega : \ \mathrm{dist}(x, \mathbb{R}^n \setminus \Omega) > \eta\}$ and let $\Omega^\eta_{k,y}$ be as in (3.6) with Ω replaced by Ω^η. Moreover let Ω^η_k be defined according to (3.7).

Set $\Pi^k_\delta := \Pi^k \cap \delta\mathbb{Z}^{n-1}$; a direct computation yields

$$
\begin{aligned}
&E_\varepsilon(u_\varepsilon, v_\varepsilon)\\
&\geq \delta^{n-1}\Bigg(\frac{1}{2}\sum_{\substack{i\in\Omega_\delta\\ i\pm\delta e_k\in\Omega_\delta}}\delta(v^i_\varepsilon)^2\left|\frac{u^i_\varepsilon-u^{i\pm\delta e_k}_\varepsilon}{\delta}\right|^2+\frac{1}{2}\Bigg(\sum_{i\in\Omega_\delta}\delta\frac{(v^i_\varepsilon-1)^2}{\varepsilon}+\sum_{\substack{i\in\Omega_\delta\\ i+\delta e_k\in\Omega_\delta}}\varepsilon\delta\left|\frac{v^i_\varepsilon-v^{i+\delta e_k}_\varepsilon}{\delta}\right|^2\Bigg)\Bigg)\\
&=\delta^{n-1}\sum_{j\in\Pi^k_\delta}\Bigg(\frac{1}{2}\sum_{\substack{i\in\Omega_{k,j}\cap\delta\mathbb{Z}\\ i\pm\delta e_k\in\Omega_{k,j}}}\delta(v^{k,j}_\varepsilon(i))^2\left|\frac{u^{k,j}_\varepsilon(i)-u^{k,j}_\varepsilon(i\pm\delta e_k)}{\delta}\right|^2 \qquad (3.11)\\
&\qquad+\frac{1}{2}\Bigg(\sum_{i\in\Omega_{k,j}\cap\delta\mathbb{Z}}\delta\frac{(v^{k,j}_\varepsilon(i)-1)^2}{\varepsilon}+\sum_{\substack{i\in\Omega_{k,j}\cap\delta\mathbb{Z}\\ i+\delta e_k\in\Omega_{k,j}}}\varepsilon\delta\left|\frac{v^{k,j}_\varepsilon(i)-v^{k,j}_\varepsilon(i+\delta e_k)}{\delta}\right|^2\Bigg)\Bigg)\\
&\geq\int_{\Omega^\eta_k}\Big(F^k_\varepsilon(u^{k,y}_\varepsilon,v^{k,y}_\varepsilon,\Omega_{k,y})+G^k_\varepsilon(v^{k,y}_\varepsilon,\Omega_{k,y})\Big)\,d\mathcal{H}^{n-1}(y),
\end{aligned}
$$

with

$$
F^k_\varepsilon(u^{k,y}_\varepsilon,v^{k,y}_\varepsilon,\Omega_{k,y}):=\frac{1}{2}\sum_{\substack{i\in\Omega_{k,y}\cap\delta\mathbb{Z}\\ i\pm\delta e_k\in\Omega_{k,y}}}\delta(v^{k,y}_\varepsilon(i))^2\left|\frac{u^{k,y}_\varepsilon(i)-u^{k,y}_\varepsilon(i\pm\delta e_k)}{\delta}\right|^2
$$

and

$$
G^k_\varepsilon(v^{k,y}_\varepsilon,\Omega_{k,y}):=\frac{1}{2}\Bigg(\sum_{i\in\Omega_{k,j}\cap\delta\mathbb{Z}}\delta\frac{(v^{k,j}_\varepsilon(i)-1)^2}{\varepsilon}+\sum_{\substack{i\in\Omega_{k,j}\cap\delta\mathbb{Z}\\ i+\delta e_k\in\Omega_{k,y}}}\varepsilon\delta\left|\frac{v^{k,y}_\varepsilon(i)-v^{k,y}_\varepsilon(i+\delta e_k)}{\delta}\right|^2\Bigg),
$$

where u_ε and v_ε are identified with their piecewise-constant interpolations.

Then invoking Fatou's Lemma gives

$$
\liminf_{\varepsilon\to0}E_\varepsilon(u_\varepsilon,v_\varepsilon)\geq\int_{\Omega^\eta_k}\liminf_{\varepsilon\to0}\Big(F^k_\varepsilon(u^{k,y}_\varepsilon,v^{k,y}_\varepsilon,\Omega_{k,y})+G^k_\varepsilon(v^{k,y}_\varepsilon,\Omega_{k,y})\Big)\,d\mathcal{H}^{n-1}(y).
$$

Since for $\mathcal{H}^{n-1}$-a.e $y\in\Omega_k$ there holds $u^{k,y}_\varepsilon\to u^{k,y}$ in $L^1(\Omega_{k,y})$, Proposition 3.2-(i) together with Remark 3.3 yield $u\in SBV^2(\Omega^\eta_{k,y})$ for $\mathcal{H}^{n-1}$-a.e. $y\in\Omega^\eta_k$ and

$$
\begin{aligned}
\liminf_{\varepsilon\to0}E_\varepsilon(u_\varepsilon,v_\varepsilon)&\geq\int_{\Omega^\eta_k}\liminf_{\varepsilon\to0}F_\varepsilon(u^{k,y}_\varepsilon,v^{k,y}_\varepsilon,\Omega_{k,y})+\liminf_{\varepsilon\to0}G_\varepsilon(v^{k,y}_\varepsilon,\Omega_{k,y})d\mathcal{H}^{n-1}(y)\\
&\geq\int_{\Omega^\eta_k}\left(\int_{\Omega^\eta_{k,y}}((u^{k,y})')^2dt+\#\left(S_{u^{k,y}}\cap\Omega^\eta_{k,y}\right)\right)d\mathcal{H}^{n-1}(y). \qquad (3.12)
\end{aligned}
$$

Since (3.12) holds for every $k\in\{1,\dots,n\}$, applying [18, Theorem 4.1 and Remark 4.2] we deduce that $u\in GSBV^2(\Omega^\eta)$.

In order to prove the lower bound (3.9) we notice that for every $k \in \{1, \dots, n\}$ we also have

$$
\begin{aligned}
&\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, v_\varepsilon) \\
&\quad \geq \sum_{l=1}^{n} \int_{\Omega_l^\eta} \liminf_{\varepsilon\to 0} F_\varepsilon^l(u_\varepsilon^{l,y}, v_\varepsilon^{l,y}, \Omega_{l,y}) d\mathcal{H}^{n-1}(y) + \int_{\Omega_k^\eta} \liminf_{\varepsilon\to 0} G_\varepsilon^k(v_\varepsilon^{k,y}, \Omega_{k,y}) d\mathcal{H}^{n-1}(y) \\
&\quad \geq \sum_{l=1}^{n} \int_{\Omega_l^\eta} \left(\int_{\Omega_{l,y}^\eta} ((u^{l,y}(t))')^2 dt \right) d\mathcal{H}^{n-1}(y) + \int_{\Omega_k^\eta} \# \left(S_{u^{k,y}} \cap \Omega_{k,y}^\eta \right) d\mathcal{H}^{n-1}(y) \\
&\quad = \sum_{l=1}^{n} \int_{\Omega^\eta} \left(\frac{\partial u}{\partial x_l} \right)^2 dx + \int_{S_u \cap \Omega^\eta} |\langle \nu_u, e_k \rangle| d\mathcal{H}^{n-1} \\
&\quad = \int_{\Omega^\eta} |\nabla u|^2 dx + \int_{S_u \cap \Omega^\eta} |\langle \nu_u, e_k \rangle| d\mathcal{H}^{n-1}.
\end{aligned}
$$

Then taking the sup on $k \in \{1, \dots, n\}$ and using a standard localization argument (see, *e.g.*, [18, Proposition 1.16]) we get

$$
\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, v_\varepsilon) \geq \int_{\Omega^\eta} |\nabla u|^2 dx + \int_{S_u \cap \Omega^\eta} |\nu_u|_\infty d\mathcal{H}^{n-1}.
$$

Finally, by letting $\eta \to 0$ we both deduce that $u \in GSBV^2(\Omega)$ and (3.9).

Step 2: proof of (ii); *i.e.,* the case $\ell = +\infty$.

Arguing as in Step 1 and now appealing to Proposition 3.2-(ii) yield both $u \in W^{1,2}(\Omega)$ and the lower-bound estimate (3.10). ☐

Remark 3.5. From the proof of (3.9) in Proposition 3.4-(i) we get that if $\ell \in [0, +\infty)$ and $(u_\varepsilon, v_\varepsilon) \subset L^1(\Omega) \times L^1(\Omega)$ is such that $(u_\varepsilon, v_\varepsilon) \to (u, v)$ in $L^1(\Omega) \times L^1(\Omega)$ with $\sup_\varepsilon E_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty$ then the two inequalities

$$
\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon) \geq \int_\Omega |\nabla u|^2 \, dx, \qquad \liminf_{\varepsilon\to 0} G_\varepsilon(v_\varepsilon) \geq \int_{S_u \cap \Omega} |\nu_u|_\infty \, d\mathcal{H}^{n-1}
$$

hold also true.

Remark 3.6. For later use we notice that Proposition 3.4 can be localised in the following sense. Let $\ell \in [0, +\infty)$, $U \in \mathscr{A}_L(\Omega)$ and let E'_ℓ be as in (2.8). Then

$$
E'_\ell(u, 1, U) \geq \int_U |\nabla u|^2 dx + \int_{S_u \cap U} |\nu_u|_\infty d\mathcal{H}^{n-1} \quad \text{for every } \; u \in GSBV^2(\Omega).
$$

3.3. **Convergence of minimisation problems.** On account of the Γ-convergence result Theorem 2.1 in this subsection we establish a convergence result for a class of minimisation problems associated to E_ε. Specifically, we consider a suitable perturbation of E_ε which will also satisfy the needed equi-coercivity property. To this end, having in mind applications to image-segmentation problems, for a given $g \in L^\infty(\Omega)$ we define

$$
g^i := \frac{1}{\delta^n} \int_{i+[0,\delta)^n} g(x) dx
$$

and consider the functionals

$$
E_\varepsilon^g(u, v) := E_\varepsilon(u, v) + \sum_{i \in \Omega_\delta} \delta^n |u^i - g^i|^2. \tag{3.13}
$$

Moreover, we will only focus on the subcritical and critical regimes; *i.e.,* $\ell \in [0, +\infty)$, as these are the only regimes giving rise to a nontrivial Γ-limit.

The following result holds true.

Proposition 3.7 (Equicoercivity). *Let $\ell \in [0,+\infty)$ and $g \in L^\infty(\Omega)$. Then the functionals E^g_ε defined as in* (3.13) *are equi-coercive with respect to the strong $L^1(\Omega) \times L^1(\Omega)$-topology. More precisely, for every sequence $(u_\varepsilon, v_\varepsilon) \subset L^1(\Omega) \times L^1(\Omega)$ satisfying $\sup_\varepsilon E^g_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty$ there exist a subsequence (not relabelled) and a function $u \in GSBV^2(\Omega)$ such that $(u_\varepsilon, v_\varepsilon) \to (u, 1)$ in $L^1(\Omega) \times L^1(\Omega)$.*

Proof. For $n = 1$ the claim directly follows by combining the estimate (3.4) with the equi-coercivity of the (perturbed) Ambrosio-Tortorelli functional [10, Theorem 1.2].

For $n \geq 2$ the proof is also standard and the equi-coercivity of E^g_ε follows from the one-dimensional case invoking *e.g.* [2, Theorem 6.6] (see also [1, Section 3.8]). □

We are now ready to prove the following result on the convergence of the associated minimisation problems.

Corollary 3.8 (Convergence of minimisation problems). *For every fixed $\varepsilon > 0$ the minimisation problem*

$$m_\varepsilon := \min \{E^g_\varepsilon(u, v) : \ (u, v) \in \mathcal{A}_\varepsilon(\Omega) \times \mathcal{A}_\varepsilon(\Omega)\} .$$

admits a solution $(\hat{u}_\varepsilon, \hat{v}_\varepsilon)$.

Let $\ell \in [0,+\infty)$; then, up to subsequences, the pair $(\hat{u}_\varepsilon, \hat{v}_\varepsilon)$ converges in $L^1(\Omega) \times L^1(\Omega)$ to $(\hat{u}, 1)$ with $\hat{u}$ solution to

$$m_\ell := \min \left\{ E_\ell(u, 1) + \int_\Omega |u - g|^2 dx : \ u \in SBV^2(\Omega), \ \|u\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\Omega)} \right\} ; \tag{3.14}$$

moreover, $m_\varepsilon \to m_\ell$ as $\varepsilon \to 0$.

Proof. The existence of a minimising pair $(\hat{u}_\varepsilon, \hat{v}_\varepsilon)$ follows by applying the direct methods. Indeed, let $\varepsilon > 0$ be fixed and let (u_k, v_k) be a minimising sequence for E^g_ε. Then there exists a constant $c > 0$ such that

$$\sum_{i \in \Omega_\delta} \delta^n |u^i_k - g^i|^2 \leq c ,$$

for every $k \in \mathbb{N}$. Since for every $i \in \Omega_\delta$ it holds $|g^i| \leq \|g\|_{L^\infty(\Omega)}$, we deduce that $|u^i_k| \leq c$ for every $i \in \Omega_\delta$ and every $k \in \mathbb{N}$ (where now the constant c possibly depends on ε). Hence, up to subsequences not relabelled, $\lim_k u^i_k = \hat{u}^i_\varepsilon$, for some $\hat{u}^i_\varepsilon \in \mathbb{R}$. Since moreover, $0 \leq v^i_k \leq 1$ for every $k \in \mathbb{N}$ and every $i \in \Omega_\delta$, up to subsequences, we also have $\lim_k v^i_k = \hat{v}^i_\varepsilon$ for some $\hat{v}^i_\varepsilon \in [0, 1]$. Since for fixed $\varepsilon > 0$ the set $\{(u^i_k, v^i_k) : i \in \Omega_\delta\}$ is finite, up to choosing a diagonal sequence we can always assume that

$$\lim_{k \to +\infty} (u^i_k, v^i_k) = (\hat{u}^i_\varepsilon, \hat{v}^i_\varepsilon) \quad \text{for every} \quad i \in \Omega_\delta.$$

Then, to deduce that $(\hat{u}_\varepsilon, \hat{v}_\varepsilon)$ is a minimising pair for E^g_ε it suffices to notice that

$$m_\varepsilon = \liminf_{k \to +\infty} E^g_\varepsilon(u_k, v_k) \geq E^g_\varepsilon(\hat{u}_\varepsilon, \hat{v}_\varepsilon).$$

Since E_ε decreases by truncations in u, by definition of E^g_ε it is not restrictive to assume that $\|\hat{u}_\varepsilon\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\Omega)}$. Moreover, invoking Proposition 3.7 gives the existence of a subsequence (not relabelled) $(\hat{u}_\varepsilon, \hat{v}_\varepsilon)$ and of a function $\hat{u} \in GSBV^2(\Omega)$ such that $(\hat{u}_\varepsilon, \hat{v}_\varepsilon) \to (\hat{u}, 1)$ in $L^1(\Omega) \times L^1(\Omega)$; further, the Dominated Convergence Theorem also yields $\hat{u}_\varepsilon \to \hat{u}$ in $L^2(\Omega)$. Since clearly $\|\hat{u}\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\Omega)}$, we actually deduce that $\hat{u} \in SBV^2(\Omega)$. Then it only remains to show that $\hat{u}$ is a solution to (3.14). To this end, let $g_\varepsilon \in \mathcal{A}_\varepsilon(\Omega)$ be the piecewise-constant function defined by

$$g_\varepsilon(x) := g^i \quad \text{for every } x \in i + [0, \delta)^n, \quad \text{for every } i \in \Omega_\delta.$$

Then $g_\varepsilon \to g$ a.e. in Ω; since moreover $\|g_\varepsilon\|_{L^\infty(\Omega)} \le \|g\|_{L^\infty(\Omega)}$ the Dominated Convergence Theorem guarantees that $g_\varepsilon \to g$ in $L^2(\Omega)$. Therefore Theorem 2.1 gives

$$m_\ell \le E_\ell(\hat{u}, 1) + \int_\Omega |\hat{u} - g|^2\, dx \le \liminf_{\varepsilon\to 0} E^g_\varepsilon(\hat{u}_\varepsilon, \hat{v}_\varepsilon) = \liminf_{\varepsilon\to 0} m_\varepsilon. \tag{3.15}$$

On the other hand, for every $w \in GSBV^2(\Omega) \cap L^\infty(\Omega)$ with $\|w\|_{L^\infty(\Omega)} \le \|g\|_{L^\infty(\Omega)}$ Theorem 2.1 provides us with a sequence $(u_\varepsilon, v_\varepsilon)$ such that $(u_\varepsilon, v_\varepsilon) \to (w, 1)$ in $L^1(\Omega) \times L^1(\Omega)$ and

$$\limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, v_\varepsilon) \le E_\ell(w, 1).$$

Then if $(\bar{u}_\varepsilon)$ is the sequence obtained by truncating (u_ε) at level $\|w\|_{L^\infty(\Omega)}$, we clearly have $\bar{u}_\varepsilon \to w$ in $L^2(\Omega)$ and

$$\limsup_{\varepsilon\to 0} E_\varepsilon(\bar{u}_\varepsilon, v_\varepsilon) \le E_\ell(w, 1),$$

so that

$$\limsup_{\varepsilon\to 0} m_\varepsilon \le \limsup_{\varepsilon\to 0} E^g_\varepsilon(\bar{u}_\varepsilon, v_\varepsilon) \le E_\ell(w, 1) + \int_\Omega |w - g|^2\, dx.$$

Hence by the arbitrariness of w we get

$$\limsup_{\varepsilon\to 0} m_\varepsilon \le m_\ell. \tag{3.16}$$

Eventually, gathering (3.15) and (3.16) yields

$$\lim_{\varepsilon\to 0} m_\varepsilon = m_\ell = E_\ell(\hat{u}, 1) + \int_\Omega |\hat{u} - g|^2\, dx$$

and thus the claim is proved. □

4. Proof of the Γ-convergence result in the subcritical regime $\ell = 0$

In this section we study the Γ-convergence of E_ε in the subcritical regime; *i.e.,* when $\ell = 0$. This regime corresponds to the case where the mesh-size δ is much smaller than the approximation parameter ε. We show that under such assumptions the discreteness of the problem does not play a role in the limit behaviour of the functionals E_ε whose Γ-limit is actually given by the Mumford-Shah functional, as in the continuous case.

We recall the following definition

$$E_0(u, v) := MS(u, v) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2\, dx + \mathcal{H}^{n-1}(S_u \cap \Omega) & \text{if } u \in GSBV^2(\Omega),\ v = 1 \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise in } L^1(\Omega) \times L^1(\Omega). \end{cases}$$

We deal with the lower-bound and upper-bound inequalities separately.

4.1. Lower-bound inequality.

4.1. Lower-bound inequality. In this subsection we establish the liminf inequality for E_ε when $\ell = 0$. The main result of this subsection is as follows.

Proposition 4.1 (Lower-bound for $\ell = 0$)**.** *Let E_ε be as in* (2.3) *and $\ell = 0$. Then for every $(u, v) \in L^1(\Omega)\times L^1(\Omega)$ and every $(u_\varepsilon, v_\varepsilon) \subset L^1(\Omega)\times L^1(\Omega)$ with $(u_\varepsilon, v_\varepsilon) \to (u, v)$ in $L^1(\Omega)\times L^1(\Omega)$ we have*

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, v_\varepsilon) \ge E_0(u, v).$$

Proof. Up to subsequences we can always assume that $\sup_\varepsilon E_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty$, otherwise there is nothing to prove. Then Proposition 3.4-(i) gives $u \in GSBV^2(\Omega)$ and $v = 1$ a.e. in Ω. Thus, it remains to show that

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, v_\varepsilon) \geq \int_\Omega |\nabla u|^2 dx + \mathcal{H}^{n-1}(S_u \cap \Omega) \tag{4.1}$$

holds true for every $u \in GSBV^2(\Omega)$. We first prove (4.1) for $u \in SBV^2(\Omega)$. To this end we use the Fonseca and Müller blow-up procedure [47]. For every $\varepsilon > 0$ we define the discrete Radon measure

$$\mu_\varepsilon := \sum_{i\in\mathring{\Omega}_\delta} \frac{\delta^n}{2}\left((v_\varepsilon^i)^2 \sum_{k=1}^n \left|\frac{u_\varepsilon^i - u_\varepsilon^{i\pm\delta e_k}}{\delta}\right|^2 + \frac{(v_\varepsilon^i - 1)^2}{\varepsilon} + \sum_{k=1}^n \varepsilon\left|\frac{v_\varepsilon^i - v_\varepsilon^{i+\delta e_k}}{\delta}\right|^2\right)\mathbf{1}_i,$$

where $\mathbf{1}_i$ denotes the Dirac delta in i. Since

$$\sup_{\varepsilon>0} \mu_\varepsilon(\Omega) \leq \sup_{\varepsilon>0} E_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty$$

we deduce that, up to subsequences, $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ weakly* in the sense of measures, for some positive finite Radon measure μ. Then, appealing to the Radon-Nikodỳm Theorem we can write μ as the sum of three mutually orthogonal measures

$$\mu = \mu_a \mathcal{L}^n \llcorner \Omega + \mu_J \mathcal{H}^{n-1} \llcorner (S_u \cap \Omega) + \mu_s.$$

Moreover, the Besicovitch Derivation Theorem together with Remark 3.6 imply that

$$\mu_a(x_0) \geq |\nabla u(x_0)|^2 \quad \text{for } \mathcal{L}^n\text{-a.e. } x_0 \in \Omega. \tag{4.2}$$

We claim that there also holds

$$\mu_J(x_0) \geq 1 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x_0 \in S_u. \tag{4.3}$$

Assume for the moment that we can prove (4.3). Then, together with (4.2), by choosing an increasing sequence of cut-off functions $(\varphi_k) \subset C_c^\infty(\Omega)$ such that $0 \leq \varphi_k \leq 1$ and $\sup_k \varphi_k = 1$ we get

$$\begin{aligned}\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, v_\varepsilon) &\geq \liminf_{\varepsilon\to 0} \mu_\varepsilon(\Omega) \geq \liminf_{\varepsilon\to 0} \int_\Omega \varphi_k d\mu_\varepsilon = \int_\Omega \varphi_k d\mu \\ &\geq \int_\Omega \varphi_k d\mu_a + \int_{S_u\cap\Omega} \varphi_k d\mu_J \\ &\geq \int_\Omega |\nabla u|^2 \varphi_k dx + \int_{S_u\cap\Omega} \varphi_k d\mathcal{H}^{n-1},\end{aligned}$$

hence the conclusion follows by letting $k \to +\infty$ and appealing to the Monotone Convergence Theorem.

We now prove (4.3). To this end, let $x_0 \in S_u \cap \Omega$ be such that

$$\mu_J(x_0) = \lim_{\rho\to 0^+} \frac{\mu\left(Q_\rho^\nu(x_0)\right)}{\mathcal{H}^{n-1}\left(Q_\rho^\nu(x_0) \cap S_u\right)} = \lim_{\rho\to 0^+} \frac{\mu(Q_\rho^\nu(x_0))}{\rho^{n-1}} \tag{4.4}$$

and

$$\lim_{\rho\to 0^+} \frac{1}{\rho^n} \int_{(Q_\rho^\nu(x_0))^\pm} |u(x) - u^\pm(x_0)| dx = 0, \tag{4.5}$$

where $\nu := \nu_u(x_0)$ and $(Q_\rho^\nu(x_0))^\pm := \{x \in Q_\rho^\nu(x_0) :\ \pm\langle x - x_0, \nu\rangle > 0\}$. We notice that (4.4) and (4.5) hold true for $\mathcal{H}^{n-1}$-a.e. $x_0 \in S_u \cap \Omega$ thanks to the Besicovitch Derivation Theorem and the definition of approximate jump point, respectively.

Since μ is a finite Radon measure, we can choose a sequence $\rho_m \to 0$ with $\mu(\partial Q^\nu_{\rho_m}(x_0)) = 0$. Thus, from [8, Proposition 1.62(a)] and the convergence $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ we deduce that for every $\varepsilon_j \to 0$ we have

$$\mu_J(x_0) \geq \lim_{m\to+\infty} \limsup_{j\to+\infty} \frac{1}{\rho_m^{n-1}} \mu_{\varepsilon_j}(Q^\nu_{\rho_m}(x_0)),$$

hence we are now led to estimate $\frac{1}{\rho_m^{n-1}}\mu_{\varepsilon_j}(Q^\nu_{\rho_m}(x_0))$ from below. To this end, we notice that for every j and for every m we can find $x_0^j \in \delta_j\mathbb{Z}^n$ and $\rho_{m,j} > 0$ such that $x_0^j \to x_0$, $\rho_{m,j} \to \rho_m$, as $j \to +\infty$ and

$$\delta_j\mathbb{Z}^n \cap Q^\nu_{\rho_{m,j}}(x_0^j) = \delta_j\mathbb{Z}^n \cap Q^\nu_{\rho_m}(x_0)$$

so that

$$\begin{aligned}
&\frac{1}{\rho_m^{n-1}}\mu_{\varepsilon_j}(Q^\nu_{\rho_m}(x_0)) \\
&= \frac{1}{\rho_m^{n-1}} \sum_{i\in\delta_j\mathbb{Z}^n\cap Q^\nu_{\rho_{m,j}}(x_0^j)} \frac{\delta_j^n}{2}\left((v^i_{\varepsilon_j})^2\sum_{k=1}^n \left|\frac{u^i_{\varepsilon_j} - u^{i\pm\delta_j e_k}_{\varepsilon_j}}{\delta_j}\right|^2 + \frac{(v^i_{\varepsilon_j}-1)^2}{\varepsilon_j} + \sum_{k=1}^n \varepsilon_j\left|\frac{v^i_{\varepsilon_j} - v^{i+\delta_j e_k}_{\varepsilon_j}}{\delta_j}\right|^2\right) \\
&\geq \left(\frac{\rho_{m,j}}{\rho_m}\right)^{n-1} E_{\frac{\varepsilon_j}{\rho_{m,j}}}(u_{j,m}, v_{j,m}, Q^\nu),
\end{aligned}$$

where

$$u^l_{j,m} := u^{x_0^j+\rho_{m,j}l}_{\varepsilon_j},\, v^l_{j,m} := v^{x_0^j+\rho_{m,j}l}_{\varepsilon_j} \quad \text{for every} \quad l \in \frac{\delta_j}{\rho_{m,j}}\mathbb{Z}^n \cap Q.$$

Set

$$u_0(x) := \begin{cases} u^+(x_0) & \text{if } \langle x,\nu\rangle \geq 0 \\ u^-(x_0) & \text{if } \langle x,\nu\rangle < 0; \end{cases}$$

in view of (4.5) we deduce that $u_{j,m} \to u_0$ in $L^1(Q^\nu)$ if we first let $j \to +\infty$ and then $m \to +\infty$. Then appealing to a diagonalisation argument we may find a sequence $m_j \to +\infty$ as $j \to +\infty$ such that $\sigma_j := \frac{\varepsilon_j}{\rho_{m_j,j}} \to 0$, $(u_j, v_j) := (u_{j,m_j}, v_{j,m_j}) \to (u_0, 1)$ in $L^1(Q^\nu) \times L^1(Q^\nu)$, as $j \to +\infty$, and

$$\mu_J(x_0) \geq \liminf_{j\to+\infty} E_{\sigma_j}(u_j, v_j, Q^\nu).$$

Therefore to prove (4.3) we now need to show that

$$\liminf_{j\to+\infty} E_{\sigma_j}(u_j, v_j, Q^\nu) \geq 1. \tag{4.6}$$

For the sake of clarity in what follows we only consider the case $n = 2$; the proof for $n > 2$ can be obtained by means of analogous constructions and arguments.

Upon possibly extracting a subsequence, we assume that the liminf in (4.6) is actually a limit.

We now define suitable continuous counterparts of u_j and v_j. To this end, set $\tau_j := \delta_j/\rho_{m_j,j}$ and consider the triangulation $\mathcal{T}_j$ of Q^ν defined as follows: For every $i \in \tau_j\mathbb{Z}^2 \cap Q^\nu$ set

$$T_i^+ := \mathrm{conv}\{i, i+\tau_j e_1, i+\tau_j e_2\} \quad \text{and} \quad T_i^- := \mathrm{conv}\{i, i-\tau_j e_1, i-\tau_j e_2\},$$

and $\mathcal{T}_j := \{T_i^+, T_i^- : i \in \tau_j\mathbb{Z}^2 \cap Q^\nu\}$. Then $\tilde{u}_j$ and $\tilde{v}_j$ denote the piecewise-affine interpolations of u_j and v_j on $\mathcal{T}_j$, respectively. Moreover, we also consider the piecewise-constant function

$$\hat{v}_j(x) := v_j^i \quad \text{if } x \in \overset{\circ}{T_i^+ \cup T_i^-}.$$

We clearly have $\tilde{u}_j \to u_0$, $\hat{v}_j \to 1$, and $\tilde{v}_j \to 1$ in $L^1(Q^\nu)$.

Let $\eta > 0$ be fixed; we now claim that

$$E_{\sigma_j}(u_j, v_j, Q^\nu) \geq \int_{Q^\nu_{1-\eta}} (\hat{v}_j)^2 |\nabla \tilde{u}_j|^2 dx + \frac{1}{2} \int_{Q^\nu_{1-\eta}} \frac{(\tilde{v}_j - 1)^2}{\sigma_j} + \sigma_j |\nabla \tilde{v}_j|^2 \, dx \tag{4.7}$$

for j large.

For j sufficiently large there holds

$$\int_{Q^\nu_{1-\eta}} \sigma_j |\nabla \tilde{v}_j|^2 dx \leq \sum_{i \in \tau_j \mathbb{Z}^2 \cap Q^\nu} \sum_{k=1}^{2} \sigma_j \tau_j^2 \left| \frac{v_j^i - v_j^{i+\tau_j e_k}}{\tau_j} \right|^2. \tag{4.8}$$

Moreover, for every $i \in \tau_j \mathbb{Z}^2 \cap Q^\nu$ there holds

$$\tilde{v}_j(x) = \lambda_0(x) v_j^i + \lambda_1(x) v_j^{i+\tau_h e_1} + \lambda_2(x) v_j^{i+\tau_h e_2} \quad \forall \, x \in T_i^+,$$

for some $\lambda_0(x), \lambda_1(x), \lambda_2(x) \in [0,1]$ satisfying $\lambda_0(x) + \lambda_1(x) + \lambda_2(x) = 1$ for every $x \in T_i^+$ and

$$\int_{T_i^+} \lambda_k(x) dx = \frac{1}{3} \mathcal{L}^2(T_i^+) = \frac{1}{6} \tau_j^2, \quad \text{for every} \quad k = 0, 1, 2.$$

Then, the convexity of $z \to (z-1)^2$ yields

$$\begin{aligned}
&\int_{T_i^+} \frac{(\tilde{v}_j - 1)^2}{\sigma_j} dx = \frac{1}{\sigma_j} \int_{T_i^+} \Big(\lambda_0(x) v_j^i + \lambda_1(x) v_j^{i+\tau_j e_1} + \lambda_2(x) v_j^{i+\tau_j e_2} - 1 \Big)^2 dx \\
&\leq \frac{1}{\sigma_j} \left((v_j^i - 1)^2 \int_{T_i^+} \lambda_0(x) dx + (v_j^{i+\tau_j e_1} - 1)^2 \int_{T_i^+} \lambda_1(x) dx + (v_j^{i+\tau_j e_2} - 1)^2 \int_{T_i^+} \lambda_2(x) dx \right) \\
&= \frac{1}{6} \tau_j^2 \left(\frac{(v_j^i - 1)^2}{\sigma_j} + \frac{(v_j^{i+\tau_j e_1} - 1)^2}{\sigma_j} + \frac{(v_j^{i+\tau_j e_2} - 1)^2}{\sigma_j} \right),
\end{aligned}$$

and analogously for T_i^-. Therefore summing up on all triangles $T_i^+, T_i^- \in \mathcal{T}_j$ yields

$$\int_{Q^\nu_{1-\eta}} \frac{(\tilde{v}_j - 1)^2}{\sigma_j} dx \leq \sum_{i \in \tau_j \mathbb{Z}^2 \cap Q^\nu} \tau_j^2 \frac{(v_j^i - 1)^2}{\sigma_j}, \tag{4.9}$$

for j sufficiently large.

Further, for every $i \in \tau_j \mathbb{Z}^2 \cap Q$ there holds

$$\int_{T_i^+} (\hat{v}_j)^2 |\nabla \tilde{u}_j|^2 dx = \frac{\tau_j^2}{2} (v_j^i)^2 \left(\left| \frac{u_j^i - u_j^{i+\tau_j e_1}}{\tau_j} \right|^2 + \left| \frac{u_j^i - u_j^{i+\tau_j e_2}}{\tau_j} \right|^2 \right)$$

and analogously on T_i^- so that we may deduce

$$\int_{Q^\nu_{1-\eta}} (\hat{v}_j)^2 |\nabla \tilde{u}_j|^2 dx \leq \frac{1}{2} \sum_{i \in \tau_j \mathbb{Z}^2 \cap Q^\nu} \tau_j^2 (v_j^i)^2 \sum_{k=1}^{2} \left| \frac{u_j^i - u_j^{i \pm \tau_j e_k}}{\tau_j} \right|^2 \tag{4.10}$$

for j sufficiently large. Finally, gathering (4.8)-(4.10) entails (4.7).

Now let $\Pi_\nu := \{x \in \mathbb{R}^2 \colon \langle x, \nu \rangle = 0\}$, $y \in Q^\nu \cap \Pi_\nu$ and set

$$\tilde{u}_j^{\nu,y}(t) := \tilde{u}_j(y + t\nu), \quad \hat{v}_j^{\nu,y}(t) := \hat{v}_j(y + t\nu).$$

Clearly, $\tilde{u}_j^{\nu,y} \in W^{1,2}(\frac{-1+\eta}{2}, \frac{1-\eta}{2})$ for $\mathcal{H}^1$-a.e. $y \in Q^\nu_{1-\eta} \cap \Pi_\nu$, moreover there holds

$$\tilde{u}_h^{\nu,y} \to u_0^{\nu,y} \quad \text{in } L^1(-1/2,1/2) \quad \text{with} \quad S_{u_0^{\nu,y}} = \{0\}, \tag{4.11}$$

for $\mathcal{H}^1$-a.e. $y \in Q^\nu \cap \Pi_\nu$. By Fubini's Theorem we have

$$\begin{aligned}\int_{Q^\nu_{1-\eta}} (\hat{v}_j)^2 |\nabla \tilde{u}_j|^2 dx &= \int_{Q^\nu_{1-\eta}\cap\Pi_\nu} \left(\int_{\frac{-1+\eta}{2}}^{\frac{1-\eta}{2}} (\hat{v}_j(y+t\nu))^2 |\nabla \tilde{u}_j(y+t\nu)|^2 dt\right) d\mathcal{H}^1(y) \\ &\geq \int_{Q^\nu_{1-\eta}\cap\Pi_\nu} \left(\int_{\frac{-1+\eta}{2}}^{\frac{1-\eta}{2}} (\hat{v}_j^{\nu,y})^2 ((\tilde{u}_j^{\nu,y})')^2 dt\right) d\mathcal{H}^1(y),\end{aligned}$$

thus, from the bound on the energy we deduce the existence of a set $N \subset \Pi_\nu$ with $\mathcal{H}^1(N) = 0$ such that

$$\sup_j \int_{\frac{-1+\eta}{2}}^{\frac{1-\eta}{2}} (\hat{v}_j^{\nu,y})^2 ((\tilde{u}_j^{\nu,y})')^2 dt < +\infty \tag{4.12}$$

for every $y \in (Q^\nu_{1-\eta}\cap\Pi_\nu)\setminus N$. Further, it is not restrictive to assume that $\tilde{u}_j^{\nu,y} \in W^{1,2}(\frac{-1+\eta}{2}, \frac{1-\eta}{2})$ for every $y \in (Q^\nu_{1-\eta} \cap \Pi_\nu) \setminus N$. Therefore, in view of (4.11)-(4.12), appealing to a classical one-dimensional argument (see *e.g.* the proof of [18, Theorem 3.15]) for every $y \in (Q^\nu_{1-\eta} \cap \Pi_\nu) \setminus N$ we can find a sequence $(s_j^y) \subset (\frac{-1+\eta}{2}, \frac{1-\eta}{2})$ satisfying

$$\hat{v}_j^{\nu,y}(s_j^y) \to 0 \ \text{ as } \ j \to +\infty. \tag{4.13}$$

Now let $\tilde{v}_j^{\nu,y}$ be the one-dimensional slice of $\tilde{v}_j$ in the direction ν; *i.e.*, $\tilde{v}_j^{\nu,y}(t) := \tilde{v}_j(y+t\nu)$. For every $y \in (Q^\nu_{1-\eta} \cap \Pi_\nu) \setminus N$ let s_j^y be as in (4.13) and consider $\tilde{v}_j^{\nu,y}(s_j^y)$. Let moreover $d > 0$ be fixed; we want to exhibit a set $N_j^d \subset \Pi_\nu$ with $\mathcal{H}^1(N_j^d) \to 0$ as $j \to +\infty$ with the following property: for every $y \in (Q^\nu_{1-\eta} \cap \Pi_\nu) \setminus (N \cup N_j^d)$ there exists $j_0 := j_0(d,y) \in \mathbb{N}$ satisfying

$$\tilde{v}_j^{\nu,y}(s_j^y) \leq d \quad \text{for every } j \geq j_0.$$

To this end, for every $i \in \tau_j\mathbb{Z}^2 \cap Q^\nu$ set

$$M_j^i := \max\{|v_j^i - v_j^l| : \ j \in \tau_j\mathbb{Z}^2 \cap Q^\nu, \ |i-l| = \tau_j\}; \tag{4.14}$$

set moreover

$$\mathcal{I}_j^d := \left\{ i \in \tau_j\mathbb{Z}^2 \cap Q^\nu \colon M_j^i \geq \frac{d}{2} \right\}. \tag{4.15}$$

From the energy bound we deduce the existence of a constant $c > 0$ such that

$$c \geq \sum_{i\in\mathcal{I}_j^d} \sum_{\substack{j\in\tau_j\mathbb{Z}^2\cap Q^\nu \\ |i-l|=\tau_j}} \sigma_j\tau_j^2 \left|\frac{v_j^i - v_j^l}{\tau_j}\right|^2 \geq \sum_{i\in\mathcal{I}_j^d} \sigma_j (M_j^i)^2 \geq \#(\mathcal{I}_j^d)\frac{d^2}{4}\sigma_j$$

for every j. Hence, there exists a constant $c(d) > 0$ such that

$$\#(\mathcal{I}_j^d) \leq \frac{c(d)}{\sigma_j} \quad \text{for every } j. \tag{4.16}$$

Let $p^\nu : \mathbb{R}^n \to \Pi_\nu$ be the orthogonal projection onto the hyperplane Π_ν and set

$$N_j^d := \bigcup_{i\in\mathcal{I}_j^d} p^\nu(\overset{\circ}{T_i^+} \cup T_i^-);$$

then in view of (4.16) we have

$$\mathcal{H}^1\big(N_j^d\big) \le 2\sqrt{2}\tau_j\,\#(\mathcal{I}_j^d) \le 2\sqrt{2}\,c(d)\,\frac{\tau_j}{\sigma_j} \to 0 \quad \text{as } j \to +\infty, \tag{4.17}$$

where the convergence to zero comes from the identity $\tau_j/\sigma_j = \delta_j/\varepsilon_j$.

Now let $j \in \mathbb{N}$ be large, let $y \in (Q^\nu_{1-\eta} \cap \Pi_\nu) \setminus (N \cup N_j^d)$, and consider the corresponding s_j^y as in (4.13). By definition of $\hat{v}_j$ we deduce the existence of $i_0 := i_0(y) \in (\tau_j\mathbb{Z}^2 \cap Q^\nu) \setminus \mathcal{I}_j^d$ such that $y + s_j^y\nu \in T^+_{i_0} \cup T^-_{i_0}$ and $v_j^{i_0} \to 0$ as $j \to +\infty$. Therefore for every $d > 0$ and every $y \in (Q^\nu_{1-\eta} \cap \Pi_\nu) \setminus (N \cup N_j^d)$ there exists $j_0 := j_0(d, y) \in \mathbb{N}$ such that $v_j^{i_0} < d/2$ for every $j \ge j_0$. Moreover, since $i_0 \in (\tau_j\mathbb{Z}^2 \cap Q^\nu) \setminus \mathcal{I}_h^d$ we also have

$$v_h^{i_0 \pm \tau_j e_k} < v_j^{i_0} + \frac{d}{2} < d\,, \quad \text{for } k = 1, 2 \text{ and for every } j \ge j_0.$$

Therefore, since $\tilde{v}_j(y + s_j^y\nu)$ is a convex combination of either the triple $(v_j^{i_0}, v_j^{i_0+\tau_je_1}, v_j^{i_0+\tau_je_2})$ or of the triple $(v_j^{i_0}, v_j^{i_0-\tau_je_1}, v_j^{i_0-\tau_je_2})$ we finally get

$$\tilde{v}_j(y + s_j^y\nu) < d\,, \quad \text{for every } j \ge j_0.$$

Since on the other hand (up to a possible extraction) $\tilde{v}_j^{\nu,y} \to 1$ a.e., we can find $r_j^y, \tilde{r}_j^y \in (\frac{-1+\eta}{2}, \frac{1-\eta}{2})$ such that $r_j^y < s_j^y < \tilde{r}_j^y$ and

$$\tilde{v}_j^{\nu,y}(r_j^y) > 1 - d\,, \quad \tilde{v}_j^{\nu,y}(\tilde{r}_j^y) > 1 - d$$

for j sufficiently large.

Hence, for every fixed $y \in (Q^\nu_{1-\eta} \cap \Pi_\nu) \setminus (N \cup N_j^d)$ using the so-called "Modica-Mortola trick" as follows we get that

$$\begin{aligned}\frac{1}{2}\int_{\frac{-1+\eta}{2}}^{\frac{1-\eta}{2}} \frac{(\tilde{v}_j^{\nu,y} - 1)^2}{\sigma_j} + \sigma_j((\tilde{v}_j^{\nu,y})')^2 dt &\ge \int_{r_j^y}^{s_j^y} (1 - \tilde{v}_j^{\nu,y})|(\tilde{v}_j^{\nu,y})'|dt + \int_{s_j^y}^{\tilde{r}_j^y} (1 - \tilde{v}_j^{\nu,y})|(\tilde{v}_j^{\nu,y})'|dt \\ &\ge 2\int_d^{1-d} (1 - z)dz = (1 - 2d)^2,\end{aligned} \tag{4.18}$$

for every $j \ge j_0$. Moreover, by (4.17) we deduce that (up to subsequences)

$$\chi_{(Q^\nu_{1-\eta} \cap \Pi_\nu) \setminus (N \cup N_j^d)} \to 1 \quad \mathcal{H}^1\text{-a.e. in } Q^\nu_{1-\eta} \cap \Pi_\nu$$

so that

$$\liminf_{j \to +\infty} \left(\frac{1}{2}\int_{\frac{-1+\eta}{2}}^{\frac{1-\eta}{2}} \frac{(\tilde{v}_j^{\nu,y} - 1)^2}{\sigma_j} + \sigma_h((\tilde{v}_j^{\nu,y})')^2 dt\right) \chi_{(Q^\nu_{1-\eta} \cap \Pi_\nu) \setminus (N \cup N_j^d)} \ge (1 - 2d)^2 \tag{4.19}$$

for $\mathcal{H}^1$-a.e. y in $Q^\nu_{1-\eta} \cap \Pi_\nu$. Thus, in view of (4.7), the Fatou's Lemma together with (4.19) give

$$\begin{aligned}\lim_{j \to +\infty} E_{\sigma_j}(u_j, v_j, Q^\nu) &\ge \liminf_{j \to +\infty} \int_{Q^\nu_{1-\eta}} \frac{(\tilde{v}_j - 1)^2}{\sigma_j} + \sigma_j|\nabla\tilde{v}_j|^2 dx \\ &\ge \int_{Q^\nu_{1-\eta} \cap \Pi^\nu} \liminf_{j \to +\infty} \left(\int_{\frac{-1+\eta}{2}}^{\frac{1-\eta}{2}} \frac{(\tilde{v}_j^{\nu,y} - 1)^2}{\sigma_j} + \sigma_j((\tilde{v}_j^{\nu,y})')^2 dt\right) \chi_{(Q^\nu_{1-\eta} \cap \Pi_\nu) \setminus (N \cup N_j^d)} d\mathcal{H}^1(y) \\ &\ge (1 - 2d)^2\mathcal{H}^1(Q^\nu_{1-\eta} \cap \Pi_\nu) = (1 - 2d)^2(1 - \eta),\end{aligned}$$

so that we deduce (4.3) by first letting $d \to 0$ and then $\eta \to 0$.

Step 3: extension to the case $u \in GSBV^2(\Omega)$.

Let $u \in GSBV^2(\Omega)$ and for $m \in \mathbb{N}$ let u^m be the truncation of u at level m. Clearly, $u^m \to u$ in $L^1(\Omega)$ as $m \to +\infty$. Therefore, since $E_0 = MS$ is lower semicontinuous with respect to the strong $L^1(\Omega)$-topology and $E_0'(\cdot, v))$ decreases by truncations, we deduce

$$E_0'(u,1) \geq \liminf_{m\to+\infty} E_0'(u^m,1) \geq \liminf_{m\to+\infty} MS(u^m,1) \geq MS(u,1) = E_0(u,1).$$

and hence the claim is proved. $\square$

4.2. **Upper-bound inequality.** In this subsection we prove the limsup inequality for E_ε when $\ell = 0$. To this end we start by recalling some well-known facts about the so-called optimal profile problem for the Ambrosio-Tortorelli functional. We define

$$\mathbf{m} := \min\left\{\int_0^{+\infty} (f-1)^2 + (f')^2 dt : \ f \in W^{1,2}_{\mathrm{loc}}(0,+\infty),\ f(0)=0,\ \lim_{t\to+\infty} f(t) = 1\right\}, \tag{4.20}$$

a straightforward computation shows that $\mathbf{m} = 1$ and that this minimum value is attained at the function $f(t) = 1 - e^{-t}$.

For our purposes it is also convenient to notice that $1 = \mathbf{m} = \tilde{\mathbf{m}}$ where

$$\tilde{\mathbf{m}} := \inf_{T>0} \inf\left\{\int_0^T (f-1)^2 + (f')^2 dt \colon f \in C^2([0,T]),\ f(0)=0,\ f(T)=1, f'(T)=f''(T)=0\right\}. \tag{4.21}$$

We are now ready to prove the following proposition.

Proposition 4.2 (Upper-bound for $\ell = 0$)**.** *Let E_ε be as in* (2.3) *and $\ell = 0$. Then for every $(u,v) \in L^1(\Omega) \times L^1(\Omega)$ there exists $(u_\varepsilon, v_\varepsilon) \subset L^1(\Omega) \times L^1(\Omega)$ with $(u_\varepsilon, v_\varepsilon) \to (u,v)$ in $L^1(\Omega) \times L^1(\Omega)$ such that*

$$\limsup_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon, v_\varepsilon) \leq E_0(u,v).$$

Proof. We can consider only those target functions $u \in GSBV^2(\Omega)$ and $v = 1$ a.e. in Ω, otherwise there is nothing to prove.

By virtue of [41, Theorem 3.9 and Corollary 3.11], using a standard density and diagonalisation argument it suffices to approximate those functions u which belong to the space $\mathcal{W}(\Omega)$ defined as the space of all the $SBV(\Omega)$-functions satisfying the following conditions:

(1) S_u is essentially closed; *i.e.*, $\mathcal{H}^{n-1}(\overline{S_u} \setminus S_u) = 0$,
(2) $\overline{S_u}$ is the intersection of Ω with the union of a finite number of pairwise disjoint closed and convex sets each contained in an $(n-1)$ dimensional hyperplane, and whose (relative) boundaries are C^∞,
(3) $u \in W^{l,\infty}(\Omega \setminus \overline{S_u})$ for all $l \in \mathbb{N}$.

We prove the limsup inequality only in the case $\overline{S_u} = \Omega \cap K$ with $K \subset \Pi_\nu$, K closed and convex, the proof in the general case being analogous.

Let p^ν denote as usual the orthogonal projection onto Π_ν and for $x \in \mathbb{R}^n$ set $d(x) := \mathrm{dist}(x, \Pi_\nu)$. For every $h > 0$ define $K_h := \{x \in \Pi_\nu \colon \mathrm{dist}(x,K) \leq h\}$. Let $\eta > 0$ be fixed; by (4.21) there exist $T_\eta > 0$ and $f_\eta \in C^2([0,T_\eta])$ such that $f_\eta(0) = 0$, $f_\eta(T_\eta) = 1$, $f_\eta'(T_\eta) = f_\eta''(T_\eta) = 0$, and

$$\int_0^{T_\eta} (f_\eta - 1)^2 + (f_\eta')^2 dt \leq 1 + \eta.$$

Clearly, up to setting $f_\eta(t) = 1$ for every $t \geq T_\eta$ we can always assume that $f_\eta \in C^2([0,+\infty))$.

Let $T > T_\eta$ and choose $\xi_\varepsilon > 0$ such that $\xi_\varepsilon/\varepsilon \to 0$ as $\varepsilon \to 0$. We set

$$\begin{aligned}
A_\varepsilon &:= \{x \in \mathbb{R}^n : \ p^\nu(x) \in K_{\varepsilon+\sqrt{n}\delta}, \ d(x) \le \xi_\varepsilon + \sqrt{n}\delta\},\\
B_\varepsilon &:= \{x \in \mathbb{R}^n : \ p^\nu(x) \in K_{2\varepsilon+\sqrt{n}\delta}, \ d(x) \le \xi_\varepsilon + \sqrt{n}\delta + \varepsilon T\},\\
C_\varepsilon &:= \{x \in \mathbb{R}^n : \ p^\nu(x) \in K_{\varepsilon/2}, \ d(x) \le \xi_\varepsilon/2\},\\
D_\varepsilon &:= \{x \in \mathbb{R}^n : \ p^\nu(x) \in K_{\varepsilon}, \ d(x) \le \xi_\varepsilon\},
\end{aligned}$$

and according to (2.6) $A_{\varepsilon,\delta}$, $B_{\varepsilon,\delta}$, $C_{\varepsilon,\delta}$, $D_{\varepsilon,\delta}$ denote the corresponding discretised sets. Let φ_ε be a smooth cut-off function between C_ε and D_ε and set

$$u_\varepsilon(x) := u(x)(1-\varphi_\varepsilon(x)).$$

Since $u \in W^{l,\infty}(\Omega \setminus \overline{S_u})$ for every $l \in \mathbb{N}$, we can assume in particular that $u \in C^2(\Omega \setminus D_\varepsilon)$ so that $u_\varepsilon \in C^2(\Omega)$. We notice that $u_\varepsilon \to u$ in $L^1(\Omega)$ by the Lebesgue Dominated Convergence Theorem. Moreover, choose a smooth cut-off function γ_ε between $K_{\varepsilon+\sqrt{n}\delta}$ and $K_{2\varepsilon+\sqrt{n}\delta}$ and define

$$v_\varepsilon(x) := \gamma_\varepsilon(p^\nu(x))h_\varepsilon(d(x)) + (1-\gamma_\varepsilon(p^\nu(x))),$$

where $h_\varepsilon : [0,+\infty) \to \mathbb{R}$ is given by

$$h_\varepsilon(t) := \begin{cases} 0 & \text{if } t < \xi_\varepsilon + \sqrt{n}\delta,\\ f_\eta\left(\frac{t-\xi_\varepsilon-\sqrt{n}\delta}{\varepsilon}\right) & \text{if } \xi_\varepsilon + \sqrt{n}\delta \le t < \xi_\varepsilon + \sqrt{n}\delta + \varepsilon T,\\ 1 & \text{if } t \ge \xi_\varepsilon + \sqrt{n}\delta + \varepsilon T. \end{cases} \tag{4.22}$$

By construction $v_\varepsilon \in W^{1,\infty}(\Omega) \cap C^0(\Omega) \cap C^2(\Omega \setminus A_\varepsilon)$ and $v_\varepsilon \to 1$ in $L^1(\Omega)$. We then define the recovery sequence $(\bar u_\varepsilon, \bar v_\varepsilon) \subset \mathcal{A}_\varepsilon(\Omega) \times \mathcal{A}_\varepsilon(\Omega)$ by setting

$$\bar u_\varepsilon^i := u_\varepsilon(i), \quad \bar v_\varepsilon^i := 0 \vee (v_\varepsilon(i) \wedge 1) \quad \text{for every} \quad i \in \Omega_\delta.$$

We clearly have $\bar v_\varepsilon \to 1$ in $L^1(\Omega)$ as $\varepsilon \to 0$. Moreover,

$$\begin{aligned}
\|\bar u_\varepsilon - u_\varepsilon\|_{L^1(\Omega)} &\le \sum_{i\in\Omega_\delta} \int_{i+[0,\delta)^n} |u_\varepsilon(i) - u_\varepsilon(x)|dx\\
&= \sum_{\substack{i\in\Omega_\delta\\ i+[0,\delta)^n \subset (\Omega\setminus D_\varepsilon)}} \int_{i+[0,\delta)^n} |u(i)-u(x)|dx + \sum_{\substack{i\in\Omega_\delta\\ i+[0,\delta)^n \cap D_\varepsilon \neq \emptyset}} \int_{i+[0,\delta)^n} |u_\varepsilon(i) - u_\varepsilon(x)|dx\\
&\le c\left(\|\nabla u\|_{L^\infty}\delta + \|u\|_{L^\infty}\xi_\varepsilon\right),
\end{aligned} \tag{4.23}$$

where to estimate the first term in the second line we have used the mean-value Theorem while to estimate the second term we used the fact that

$$\#(\{i \in \Omega_\delta : \ i + [0,\delta)^n \cap D_\varepsilon \neq \emptyset\}) = \mathcal{O}\Big(\frac{\xi_\varepsilon}{\delta^n}\Big).$$

Hence, the convergence $\bar u_\varepsilon \to u$ in $L^1(\Omega)$ follows from (4.23) and the fact that $u_\varepsilon \to u$ in $L^1(\Omega)$.

Thus it remains to prove that

$$\limsup_{\varepsilon\to 0} E_\varepsilon(\bar u_\varepsilon, \bar v_\varepsilon) \le \int_\Omega |\nabla u|^2 dx + \mathcal{H}^{n-1}(S_u \cap \Omega).$$

We clearly have

$$\limsup_{\varepsilon\to 0} E_\varepsilon(\bar u_\varepsilon, \bar v_\varepsilon) \le \limsup_{\varepsilon\to 0} F_\varepsilon(\bar u_\varepsilon, \bar v_\varepsilon) + \limsup_{\varepsilon\to 0} G_\varepsilon(\bar v_\varepsilon). \tag{4.24}$$

We now estimate the two terms in the right-hand-side of (4.24) separately. We start with $F_\varepsilon(\bar u_\varepsilon, \bar v_\varepsilon)$. Since $\bar v_\varepsilon^i = v_\varepsilon(i) = 0$ for all $i \in A_{\varepsilon,\delta}$, we have

$$F_\varepsilon(\bar u_\varepsilon, \bar v_\varepsilon) = \frac{1}{2} \sum_{i\in\Omega_\delta\setminus A_\varepsilon} \delta^n (v_\varepsilon^i)^2 \sum_{\substack{k=1\\ i\pm\delta e_k\in\Omega_\delta}}^{n} \left| \frac{u_\varepsilon(i) - u_\varepsilon(i\pm\delta e_k)}{\delta} \right|^2. \tag{4.25}$$

Let $i \in \Omega_\delta \setminus A_\varepsilon$. By construction $i \pm \delta e_k \in \Omega_\delta \setminus D_\varepsilon$ for every $k \in \{1,\dots,n\}$ and since $u_\varepsilon = u$ on $\Omega \setminus D_\varepsilon$, using Jensen's inequality we deduce that

$$\left| \frac{u_\varepsilon(i\pm\delta e_k) - u_\varepsilon(i)}{\delta} \right|^2 = \left| \frac{1}{\delta} \int_0^\delta \langle \nabla u(i\pm t e_k), e_k\rangle \, dt \right|^2 \le \frac{1}{\delta} \int_0^\delta |\langle \nabla u(i\pm t e_k), e_k\rangle|^2 dt,$$

for every $k \in \{1,\dots,n\}$. Therefore, thanks to the regularity of u and to the fact that $0 \le \bar v_\varepsilon \le 1$, the mean-value Theorem gives

$$\begin{aligned} F_\varepsilon(\bar u_\varepsilon, \bar v_\varepsilon) &\le \sum_{i\in\Omega_\delta\setminus A_\varepsilon} \sum_{k=1}^n \int_{i+[0,\delta)^{n-1}} \left(\int_0^\delta |\langle \nabla u(i\pm t e_k), e_k\rangle|^2 dt \right) d\mathcal{H}^{n-1}(y) \\ &= \sum_{i\in\Omega_\delta\setminus A_\varepsilon} \sum_{k=1}^n \int_{i+[0,\delta)^{n-1}} \left(\int_0^\delta |\langle \nabla u(y\pm t e_k), e_k\rangle|^2 dt \right) d\mathcal{H}^{n-1}(y) + \mathcal{O}(\delta) \\ &\le \int_\Omega |\nabla u(x)|^2 dx + \mathcal{O}(\delta), \end{aligned} \tag{4.26}$$

so that

$$\limsup_{\varepsilon\to 0} F_\varepsilon(\bar u_\varepsilon, \bar v_\varepsilon) \le \int_\Omega |\nabla u|^2 dx. \tag{4.27}$$

We now turn to estimate the term $G_\varepsilon(\bar v_\varepsilon)$. We have

$$\begin{aligned} G_\varepsilon(\bar v_\varepsilon) \le G(v_\varepsilon) = \frac{1}{2} &\sum_{i\in\Omega_\delta\setminus B_\varepsilon} \delta^n \left(\frac{(v_\varepsilon^i - 1)^2}{\varepsilon} + \varepsilon \sum_{k=1}^n \left| \frac{v_\varepsilon^i - v_\varepsilon^{i+\delta e_k}}{\delta} \right|^2 \right) \\ &+ \frac{1}{2} \sum_{i\in B_{\varepsilon,\delta}\setminus A_\varepsilon} \delta^n \left(\frac{(v_\varepsilon^i - 1)^2}{\varepsilon} + \varepsilon \sum_{k=1}^n \left| \frac{v_\varepsilon^i - v_\varepsilon^{i+\delta e_k}}{\delta} \right|^2 \right) \\ &+ \frac{1}{2} \sum_{i\in A_{\varepsilon,\delta}} \delta^n \left(\frac{(v_\varepsilon^i - 1)^2}{\varepsilon} + \varepsilon \sum_{k=1}^n \left| \frac{v_\varepsilon^i - v_\varepsilon^{i+\delta e_k}}{\delta} \right|^2 \right). \end{aligned} \tag{4.28}$$

We start noticing that

$$\sum_{i\in\Omega_\delta\setminus B_\varepsilon} \delta^n \left(\frac{(v_\varepsilon^i - 1)^2}{\varepsilon} + \varepsilon \sum_{k=1}^n \left| \frac{v_\varepsilon^i - v_\varepsilon^{i+\delta e_k}}{\delta} \right|^2 \right) = 0. \tag{4.29}$$

Indeed, for $i \in \Omega_\delta \setminus B_\varepsilon$ we have $d(i) \ge \xi_\varepsilon + \sqrt{n}\delta + \varepsilon T$. Then, since $T > T_\eta$, for ε sufficiently small we deduce that $d(i+\delta e_k) \ge d(i) - \delta \ge \xi_\varepsilon + \sqrt{n}\delta + \varepsilon T_\eta$ for every $k \in \{1,\dots,n\}$ so that by definition of h_ε we get $v_\varepsilon^i = v_\varepsilon^{i+\delta e_k} = 1$, hence (4.29).

We now claim that

$$\sum_{i\in A_{\varepsilon,\delta}} \delta^n \left(\frac{(v_\varepsilon^i - 1)^2}{\varepsilon} + \varepsilon \sum_{k=1}^n \left| \frac{v_\varepsilon^i - v_\varepsilon^{i+\delta e_k}}{\delta} \right|^2 \right) \le c\, \frac{\xi_\varepsilon + \delta}{\varepsilon} \mathcal{H}^{n-1}(K_{\varepsilon+\sqrt{n}\delta}) \to 0, \tag{4.30}$$

as $\varepsilon \to 0$. To prove the claim we observe that $v_\varepsilon^i = v_\varepsilon^{i+\delta e_k} = 0$ for every $i \in A_{\varepsilon,\delta}$ and every $k \in \{1,\dots,n\}$ such that $i+\delta e_k \in A_{\varepsilon,\delta}$. Hence we have

$$\begin{aligned}&\sum_{i\in A_{\varepsilon,\delta}} \delta^n \Bigg(\frac{(v_\varepsilon^i - 1)^2}{\varepsilon} + \varepsilon \sum_{\substack{k=1\\ i+\delta e_k \in A_{\varepsilon,\delta}}}^{n} \left| \frac{v_\varepsilon^i - v_\varepsilon^{i+\delta e_k}}{\delta} \right|^2 \Bigg)\\ &= \sum_{i\in A_{\varepsilon,\delta}} \delta^n \left(\frac{1}{\varepsilon}\right) = \#(A_{\varepsilon,\delta})\,\frac{\delta^n}{\varepsilon}\\ &\le c\,\frac{\xi_\varepsilon + \delta}{\varepsilon}\mathcal{H}^{n-1}(K_{\varepsilon+\sqrt{n}\delta}). \end{aligned} \tag{4.31}$$

Then, it only remains to estimate the energy for those $i \in A_{\varepsilon,\delta}$ and $k \in \{1,\dots,n\}$ such that $i+\delta e_k \in B_{\varepsilon,\delta} \setminus A_{\varepsilon,\delta}$; *i.e.,* to estimate the term

$$\sum_{i\in A_{\varepsilon,\delta}} \delta^n \sum_{\substack{k=1\\ i+\delta e_k \in B_{\varepsilon,\delta}\setminus A_{\varepsilon,\delta}}}^{n} \varepsilon \left| \frac{v_\varepsilon^i - v_\varepsilon^{i+\delta e_k}}{\delta} \right|^2 .$$

To this end, we observe that in general, for every $i \in \mathring{\Omega}_\delta$ and every $k \in \{1,\dots,n\}$, thanks to the regularity of v_ε, by Jensen's inequality we have

$$\left| \frac{v_\varepsilon^i - v_\varepsilon^{i+\delta e_k}}{\delta} \right|^2 \le \left| \frac{v_\varepsilon(i) - v_\varepsilon(i+\delta e_k)}{\delta} \right|^2 \le \frac{1}{\delta}\int_0^\delta |\langle \nabla v_\varepsilon(i+te_k), e_k\rangle|^2 dt. \tag{4.32}$$

Since

$$\nabla v_\varepsilon(x) = (h_\varepsilon(d(x)) - 1)\,\nabla\gamma_\varepsilon(p^\nu(x)) D_p^\nu(x) \pm \gamma_\varepsilon(p^\nu(x)) h_\varepsilon'(d(x))\nu,$$

where $D_p^\nu(x)$ denotes the Jacobian of p^ν evaluated at x, using that $h_\varepsilon \in W^{1,\infty}(0,+\infty)$ satisfies $\|h_\varepsilon'\|_{L^\infty} \le \frac{C}{\varepsilon}$, while $\|\nabla\gamma_\varepsilon\|_{L^\infty} \le \frac{C}{\varepsilon}$ and $\|D_p^\nu\|_{L^\infty} \le 1$, from (4.32) we obtain

$$\varepsilon \left| \frac{v_\varepsilon^i - v_\varepsilon^{i+\delta e_k}}{\delta} \right|^2 = \mathcal{O}\left(\frac{1}{\varepsilon}\right) \quad \text{for every } i \in \mathring{\Omega}_\delta, \quad \text{for every } k \in \{1,\dots,n\} \tag{4.33}$$

thus, consequently

$$\sum_{i\in A_{\varepsilon,\delta}} \delta^n \sum_{\substack{k=1\\ i+\delta e_k \in B_{\varepsilon,\delta}\setminus A_{\varepsilon,\delta}}}^{n} \varepsilon \left| \frac{v_\varepsilon^i - v_\varepsilon^{i+\delta e_k}}{\delta} \right|^2 \le c\#(\partial A_{\varepsilon,\delta})\frac{\delta^n}{\varepsilon} \le c\frac{\delta}{\varepsilon}\mathcal{H}^{n-1}\left(K_{\varepsilon+\sqrt{n}\delta}\right),$$

which together with (4.31) gives (4.30).

We finally come to estimate

$$\frac{1}{2}\sum_{i\in B_{\varepsilon,\delta}\setminus A_{\varepsilon,\delta}} \delta^n \left(\frac{(v_\varepsilon^i - 1)^2}{\varepsilon} + \varepsilon \sum_{k=1}^{n} \left| \frac{v_\varepsilon^i - v_\varepsilon^{i+\delta e_k}}{\delta} \right|^2 \right).$$

To this end, it is convenient to write $B_\varepsilon \setminus A_\varepsilon$ as the union of the pairwise disjoint sets M_ε, V_ε, W_ε defined as:

$$\begin{aligned} M_\varepsilon &:= \{x \in \mathbb{R}^n :\ p^\nu(x) \in K_{\varepsilon+\sqrt{n}\delta},\ \xi_\varepsilon + \sqrt{n}\delta < d(x) \le \xi_\varepsilon + \sqrt{n}\delta + \varepsilon T\},\\ V_\varepsilon &:= \{x \in \mathbb{R}^n :\ p^\nu(x) \in K_{2\varepsilon+\sqrt{n}\delta} \setminus K_{\varepsilon+\sqrt{n}\delta},\ d(x) \le \xi_\varepsilon + \sqrt{n}\delta\},\\ W_\varepsilon &:= \{x \in \mathbb{R}^n :\ p^\nu(x) \in K_{2\varepsilon+\sqrt{n}\delta} \setminus K_{\varepsilon+\sqrt{n}\delta},\ \xi_\varepsilon + \sqrt{n}\delta < d(x) \le \xi_\varepsilon + \sqrt{n}\delta + \varepsilon T\}. \end{aligned}$$

Further, $M_{\varepsilon,\delta}$, $V_{\varepsilon,\delta}$, $W_{\varepsilon,\delta}$ denote their discrete counterparts as in (2.6). We now estimate the energy along the recovery sequence in the three sets as above, separately. To this end we start noticing that $\#(V_{\varepsilon,\delta}) = \mathcal{O}\left(\frac{\varepsilon\xi_\varepsilon}{\delta^n}\right)$ and $\#(W_{\varepsilon,\delta}) = \mathcal{O}\left(\frac{\varepsilon^2}{\delta^n}\right)$. Thus, appealing to (4.33) we deduce that

$$\frac{1}{2}\sum_{i\in V_{\varepsilon,\delta}}\left(\delta^n\frac{(v_\varepsilon^i-1)^2}{\varepsilon}+\sum_{k=1}^n\varepsilon\left|\frac{v_\varepsilon^i-v_\varepsilon^{i+\delta e_k}}{\delta}\right|^2\right)=\mathcal{O}(\xi_\varepsilon)$$

and

$$\frac{1}{2}\sum_{i\in W_{\varepsilon,\delta}}\left(\delta^n\frac{(v_\varepsilon^i-1)^2}{\varepsilon}+\sum_{k=1}^n\varepsilon\left|\frac{v_\varepsilon^i-v_\varepsilon^{i+\delta e_k}}{\delta}\right|^2\right)=\mathcal{O}(\varepsilon).$$

Finally, again using (4.33) we get

$$\begin{aligned}\sum_{i\in M_{\varepsilon,\delta}}\left(\delta^n\frac{(v_\varepsilon^i-1)^2}{\varepsilon}+\sum_{k=1}^n\varepsilon\left|\frac{v_\varepsilon^i-v_\varepsilon^{i+\delta e_k}}{\delta}\right|^2\right)\leq&\sum_{i\in \overset{\circ}{M}_{\varepsilon,\delta}}\left(\delta^n\frac{(v_\varepsilon^i-1)^2}{\varepsilon}+\sum_{k=1}^n\varepsilon\left|\frac{v_\varepsilon^i-v_\varepsilon^{i+\delta e_k}}{\delta}\right|^2\right)\\&+c\frac{\delta^n}{\varepsilon}\#(\partial M_{\varepsilon,\delta});\end{aligned}$$

moreover we notice that

$$\#(\partial M_{\varepsilon,\delta})\leq\frac{c}{\delta^{n-1}}\left(\mathcal{H}^{n-1}(K_{\varepsilon+\sqrt{n}\delta})+\varepsilon\right).$$

Let now $i\in\overset{\circ}{M}_{\varepsilon,\delta}$; then $i+[0,\delta)^n\subset M_\varepsilon$. Hence, by definition of v_ε we have

$$v_\varepsilon(x)=f_\eta\left(\frac{d(x)-\xi_\varepsilon-\sqrt{n}\delta}{\varepsilon}\right)\quad\text{and}\quad\nabla v_\varepsilon(x)=\pm\frac{1}{\varepsilon}f_\eta'\left(\frac{d(x)-\xi_\varepsilon-\sqrt{n}\delta}{\varepsilon}\right)\nu\quad\text{in } i+[0,\delta)^n.$$

Since $f_\eta\in C^2([0,T])$, appealing to the Mean-Value Theorem we deduce that for every $x\in i+[0,\delta)^n$

$$\Big|f_\eta\Big(\frac{d(x)-\xi_\varepsilon-\sqrt{n}\delta}{\varepsilon}\Big)-f_\eta\Big(\frac{d(i)-\xi_\varepsilon-\sqrt{n}\delta}{\varepsilon}\Big)\Big|\leq\frac{\delta}{\varepsilon}\,\|f_\eta'\|_{L^\infty(0,T)},$$

while for every $y\in i+[0,\delta)^{n-1}$, every $t\in(0,\delta)$, and every $k\in\{1,\dots,n\}$

$$\Big|\Big\langle\Big(f_\eta'\Big(\frac{d(y+te_k)-\xi_\varepsilon-\sqrt{n}\delta}{\varepsilon}\Big)-f_\eta'\Big(\frac{d(i+te_k)-\xi_\varepsilon-\sqrt{n}\delta}{\varepsilon}\Big)\Big)\nu,e_k\Big\rangle\Big|\leq\frac{\delta}{\varepsilon}\,\|f_\eta''\|_{L^\infty(0,T)}.$$

So that it follows

$$
\begin{aligned}
&\frac{1}{2}\sum_{i\in M_{\varepsilon,\delta}}\delta^n\left(\frac{(v^i_\varepsilon-1)^2}{\varepsilon}+\sum_{k=1}^n\varepsilon\left|\frac{v^i_\varepsilon-v^{i+\delta e_k}_\varepsilon}{\delta}\right|^2\right)\\
&\leq\frac{1}{2}\sum_{i\in\overset{\circ}{M}_{\varepsilon,\delta}}\left(\int_{i+[0,\delta)^n}\frac{1}{\varepsilon}\left(f_\eta\left(\frac{d(x)-\xi_\varepsilon-\sqrt{n}\delta}{\varepsilon}\right)-1\right)^2dx\right.\\
&\left.+\sum_{k=1}^n\frac{1}{\varepsilon}\int_{i+[0,\delta)^{n-1}}\left(\int_0^\delta\left|\left\langle f'_\eta\Big(\frac{d(y+te_k)-\xi_\varepsilon-\sqrt{n}\delta}{\varepsilon}\Big)\nu,e_k\right\rangle\right|^2dt\right)d\mathcal{H}^{n-1}(y)\right)\\
&+c\frac{\delta}{\varepsilon}\left(\mathcal{H}^{n-1}(K_{\varepsilon+\sqrt{n}\delta})+\varepsilon\right)\\
&\leq\frac{1}{2}\int_{M_\varepsilon}\frac{1}{\varepsilon}\left(\Big(f_\eta\Big(\frac{d(x)-\xi_\varepsilon-\sqrt{n}\delta}{\varepsilon}\Big)-1\Big)^2+\left|f'_\eta\left(\frac{d(x)-\xi_\varepsilon-\sqrt{n}\delta}{\varepsilon}\right)\nu\right|^2\right)dx\\
&+c\frac{\delta}{\varepsilon}\left(\mathcal{H}^{n-1}(K_{\varepsilon+\sqrt{n}\delta})+\varepsilon\right)\\
&=\int_{K_{\varepsilon+\sqrt{n}\delta}}\left(\frac{1}{\varepsilon}\int_{\xi_\varepsilon+\sqrt{n}\delta}^{\xi_\varepsilon+\sqrt{n}\delta+\varepsilon T}\left(f_\eta\left(\frac{t-\xi_\varepsilon-\sqrt{n}\delta}{\varepsilon}\right)-1\right)^2\right.\\
&\left.+\left(f'_\eta\left(\frac{t-\xi_\varepsilon-\sqrt{n}\delta}{\varepsilon}\right)\right)^2dt\right)d\mathcal{H}^{n-1}(y)+c\frac{\delta}{\varepsilon}\left(\mathcal{H}^{n-1}(K_{\varepsilon+\sqrt{n}\delta})+\varepsilon\right)\\
&=\int_{K_{\varepsilon+\sqrt{n}\delta}}\left(\int_0^T(f_\eta(t)-1)^2+(f'_\eta(t))^2dt\right)d\mathcal{H}^{n-1}(y)+c\frac{\delta}{\varepsilon}\left(\mathcal{H}^{n-1}(K_{\varepsilon+\sqrt{n}\delta})+\varepsilon\right)\\
&\leq\left(1+\eta+c\frac{\delta}{\varepsilon}\right)\mathcal{H}^{n-1}(K_{\varepsilon+\sqrt{n}\delta})+c\delta.
\end{aligned}
$$

Thus, gathering (4.24), (4.27)-(4.30) gives

$$
\limsup_{\varepsilon\to0}E_\varepsilon(\bar u_\varepsilon,\bar v_\varepsilon)\leq\int_\Omega|\nabla u|^2dx+(1+\eta)\mathcal{H}^{n-1}(S_u\cap\Omega).
$$

and hence the claim is proved. ☐

Remark 4.3. For later use we notice that the upper-bound inequality in Proposition 4.2 can be easily generalised in the following way: let $\ell\in(0,+\infty)$ then there exists $\alpha_\ell\in(0,+\infty)$ such that

$$
E''_\ell(u,1,U)\leq\int_U|\nabla u|^2dx+(1+\alpha_\ell)\mathcal{H}^{n-1}(S_u\cap U),
$$

for every $u\in GSBV^2(\Omega)$ and for every $U\in\mathscr{A}_L(\Omega)$.

5. Proof of the Γ-convergence result in the critical regime $\ell\in(0,+\infty)$

In this section we study the Γ-limit of the functionals E_ε in the case where ε and δ are of the same order. We show that in this case the presence of the underlying lattice affects the Γ-limit, which in particular turns out to be anisotropic.

The treatment of this case presents some additional difficulties with respect to the subcritical case investigated in Section 4. In particular, when the space-dimension n is larger than two we can only show that the Γ-convergence of E_ε takes place up to a subsequence and that the Γ-limit can be represented as a free-discontinuity functional of the form

$$E_\ell(u) = \int_\Omega |\nabla u|^2 dx + \int_{S_u\cap\Omega} \phi_\ell([u],\nu_u) d\mathcal{H}^{n-1}, \tag{5.1}$$

for some ϕ_ℓ which is not explicit and possibly depends on the amplitude of the jump $[u]$. We observe that this dependence cannot be excluded *a priori* because in general we cannot exclude the possibility for the term F_ε to enter in the definition of ϕ_ℓ (as opposite to the case $\ell = 0$ where the surface energy is only determined by G_ε). When $n = 2$, however, we are able to characterise ϕ_ℓ showing, in particular, that it does not depend on $[u]$ and on the subsequence so that in this case the Γ-convergence of the functionals E_ε takes place for the whole sequence.

The convergence result for E_ε in this critical regime will be achieved by means of the so-called localisation method of Γ-convergence (see *e.g.* [44, Chapters 14-20], [19, Chapters 16]). The latter consists of two main steps: proving the existence of a subsequence (E_{ε_j}) Γ-converging to some abstract functional E when localised to all open subsets of Ω and, subsequently, showing that E can be represented in an integral form as in (5.1).

In order to apply the localisation method we need to consider the functionals E_ε as functions defined on triples $(u, v, U) \in L^1(\Omega)\times L^1(\Omega)\times \mathscr{A}(\Omega)$ using the localised definition of the energies as in (2.7). Moreover, for every $\varepsilon_j \to 0$ we also need to consider $E'_\ell, E''_\ell : L^1(\Omega)\times L^1(\Omega)\times\mathscr{A}(\Omega) \to [0, +\infty]$ the localised versions of the Γ-liminf and Γ-limsup functionals; *i.e.,*

$$E'_\ell(u, v, U) := \Gamma\text{-}\liminf_{j\to+\infty} E_{\varepsilon_j}(u, v, U), \quad E''_\ell(u, v, U) := \Gamma\text{-}\limsup_{j\to+\infty} E_{\varepsilon_j}(u, v, U).$$

Finally, we define the inner regular envelopes of E'_ℓ and E''_ℓ, respectively; *i.e.,*

$$(E'_\ell)_-(\cdot,\cdot,U) := \sup\{E'_\ell(\cdot,\cdot,V) :\ V \subset\subset U,\ V \in \mathscr{A}(\Omega)\},$$
$$(E''_\ell)_-(\cdot,\cdot,U) := \sup\{E''_\ell(\cdot,\cdot,V) :\ V \subset\subset U,\ V \in \mathscr{A}(\Omega)\}.$$

Remark 5.1. The functionals E'_ℓ and E''_ℓ are increasing [44, Proposition 6.7], lower semicontinuous [44, Proposition 6.8], and local [44, Proposition 16.15]. Note that they both decrease by truncation. Moreover, E'_ℓ is also superadditive [44, Proposition 16.12]. Further, the functionals $(E'_\ell)_-$ and $(E''_\ell)_-$ are inner regular by definition, increasing, lower semicontinuous [44, Remark 15.10], local [44, Remark 15.25], and $(E'_\ell)_-$ is superadditive [44, Proposition 16.12].

The next proposition shows that the functionals E_ε satisfy the so-called fundamental estimate, uniformly in ε.

Proposition 5.2 (Fundamental estimate)**.** *For every $\varepsilon > 0$, $\eta > 0$, $U, U', V \in \mathscr{A}(\Omega)$ with $U \subset\subset U'$, and for every $(u, v), (\tilde u, \tilde v) \in L^1(\Omega) \times L^1(\Omega)$, with $0 \le v, \tilde v \le 1$ there exists $(\hat u, \hat v) \in L^1(\Omega) \times L^1(\Omega)$ such that*

$$E_\varepsilon(\hat u, \hat v, U \cup V) \le (1 + \eta)\big(E_\varepsilon(u, v, U') + E_\varepsilon(\tilde u, \tilde v, V)\big) + \sigma_\varepsilon(u, v, \tilde u, \tilde v, U, U', V),$$

where $\sigma_\varepsilon : L^1(\Omega)^4 \times \mathscr{A}(\Omega)^3 \to [0, +\infty)$ depends only on ε and E_ε and is such that

$$\lim_{\varepsilon\to 0} \sigma_\varepsilon(u_\varepsilon, v_\varepsilon, \tilde u_\varepsilon, \tilde v_\varepsilon, U, U', V) = 0, \tag{5.2}$$

for all $U, U', V \in \mathscr{A}(\Omega)$ with $U \subset\subset U'$ and every $(u_\varepsilon, v_\varepsilon)$ and $(\tilde u_\varepsilon, \tilde v_\varepsilon)$ which have the same limit as $\varepsilon \to 0$ in $L^2((U'\setminus U)\cap V) \times L^2((U'\setminus U)\cap V)$ and satisfy

$$\sup_\varepsilon \big(E_\varepsilon(u_\varepsilon, v_\varepsilon, U') + E_\varepsilon(\tilde u_\varepsilon, \tilde v_\varepsilon, V)\big) < +\infty.$$

Proof. Fix $\varepsilon > 0$ and $\eta > 0$ and let $U, U', V \in \mathscr{A}(\Omega)$ with $U \subset\subset U'$. Let $N \in \mathbb{N}$ and choose $U_1, \dots, U_{N+1} \in \mathscr{A}(\Omega)$ such that

$$U_0 := U \subset\subset U_1 \subset\subset \dots \subset\subset U_{N+1} \subset\subset U'.$$

For every $l \in \{2, \dots, N\}$ let φ_l be a cut-off function between U_{l-1} and U_l and set

$$M := \max_{2 \le l \le N} \|\nabla \varphi_l\|_{L^\infty}.$$

Let (u, v) and $(\tilde{u}, \tilde{v})$ be as in the statement and define $w \in \mathcal{A}_\varepsilon(\Omega)$ by setting

$$w^i := \min\{v^i, \tilde{v}^i\},$$

for every $i \in \Omega \cap \delta\mathbb{Z}^n$. Notice that by definition of w^i we have that for every $i \in \Omega \cap \delta\mathbb{Z}^n$

$$(w^i - 1)^2 \le (v^i - 1)^2 + (\tilde{v}^i - 1)^2 \tag{5.3}$$

and

$$\left|\frac{w^i - w^{i+\delta e_k}}{\delta}\right|^2 \le \left|\frac{v^i - v^{i+\delta e_k}}{\delta}\right|^2 + \left|\frac{\tilde{v}^i - \tilde{v}^{i+\delta e_k}}{\delta}\right|^2 \quad \text{for every } k \in \{1, \dots, n\}. \tag{5.4}$$

For every $l \in \{4, \dots, N-2\}$ we define the functions

$$\hat{u}_l^i := \varphi_l(i) u^i + (1 - \varphi_l(i))\tilde{u}^i, \quad i \in \Omega \cap \delta\mathbb{Z}^n,$$

and

$$\hat{v}_l^i := \begin{cases} \varphi_{l-2}(i) v^i + (1 - \varphi_{l-2}(i)) w^i & \text{if } i \in U_{l-2} \cap \delta\mathbb{Z}^n, \\ w^i & \text{if } i \in (U_{l+1} \setminus \overline{U}_{l-2}) \cap \delta\mathbb{Z}^n, \\ \varphi_{l+2}(i) w^i + (1 - \varphi_{l+2}(i))\tilde{v}^i & \text{if } i \in (\Omega \setminus \overline{U}_{l+1}) \cap \delta\mathbb{Z}^n. \end{cases}$$

Then we have

$$\begin{aligned} E_\varepsilon(\hat{u}_l, \hat{v}_l, U \cup V) \le\ & \tilde{E}_\varepsilon(u, v, U_{l-4}) + \tilde{E}_\varepsilon(u, \hat{v}_l, (U_{l-3} \setminus \overline{U}_{l-4}) \cap V) + \tilde{E}_\varepsilon(u, \hat{v}_l, (U_{l-2} \setminus \overline{U}_{l-3}) \cap V) \\ & + \tilde{E}_\varepsilon(\hat{u}_l, \hat{v}_l, (U_{l+1} \setminus \overline{U}_{l-2}) \cap V) + \tilde{E}_\varepsilon(\tilde{u}, \hat{v}_l, (U_{l+2} \setminus \overline{U}_{l+1}) \cap V) \\ & + \tilde{E}_\varepsilon(\tilde{u}, \hat{v}_l (U_{l+3} \setminus \overline{U}_{l+2}) \cap V) + \tilde{E}_\varepsilon(\tilde{u}, \tilde{v}, V \setminus \overline{U}_{l+3}), \end{aligned} \tag{5.5}$$

where to shorten notation for every $(u, v) \in \mathcal{A}_\varepsilon(\Omega) \times \mathcal{A}_\varepsilon(\Omega)$ and $U \in \mathcal{A}(\Omega)$ we set

$$\tilde{E}_\varepsilon(u, v, U) := \frac{1}{2} \sum_{i \in U_\delta} \delta^n \bigg((v^i)^2 \sum_{\substack{k=1 \\ i \pm \delta e_k \in U \cup V}}^n \left|\frac{u^i - u^{i \pm \delta e_k}}{\delta}\right|^2 + \frac{(v^i - 1)^2}{\varepsilon} + \varepsilon \sum_{\substack{k=1 \\ i + \delta e_k \in U \cup V}}^n \left|\frac{v^i - v^{i+\delta e_k}}{\delta}\right|^2 \bigg).$$

Analogously, we set

$$\tilde{F}_\varepsilon(u, v, U) := \frac{1}{2} \sum_{i \in U_\delta} \delta^n (v^i)^2 \sum_{\substack{k=1 \\ i \pm \delta e_k \in U \cup V}}^n \left|\frac{u^i - u^{i \pm \delta e_k}}{\delta}\right|^2$$

and

$$\tilde{G}_\varepsilon(v, U) := \frac{1}{2} \sum_{i \in U_\delta} \delta^n \bigg(\frac{(v^i - 1)^2}{\varepsilon} + \varepsilon \sum_{\substack{k=1 \\ i + \delta e_k \in U \cup V}}^n \left|\frac{v^i - v^{i+\delta e_k}}{\delta}\right|^2 \bigg).$$

We now estimate the different terms on the right hand side of (5.5) separately. Taking into account the definition of $\hat{v}_l$ we have

$$\tilde{E}_\varepsilon(u,\hat{v}_l,(U_{l-3}\setminus\overline{U}_{l-4})\cap V)=\tilde{F}_\varepsilon(u,v,(U_{l-3}\setminus\overline{U}_{l-4})\cap V)+G_\varepsilon(v,(U_{l-3}\setminus\overline{U}_{l-4})\cap V)$$
$$+\sum_{i\in(U_{l-3}\setminus\overline{U}_{l-4})\cap V_\delta}\delta^n\varepsilon\sum_{\substack{k=1\\ i+\delta e_k\in(U_{l-2}\setminus\overline{U}_{l-3})\cap V}}^{n}\left|\frac{\hat{v}_l^i-\hat{v}_l^{i+\delta e_k}}{\delta}\right|^2.$$

On U_{l-2} we have

$$\left|\frac{\hat{v}_l^i-\hat{v}_l^{i+\delta e_k}}{\delta}\right|^2\leq 3\Bigg(\varphi_{l-2}(i+\delta e_k)^2\left|\frac{v^i-v^{i+\delta e_k}}{\delta}\right|^2+(1-\varphi_{l-2}(i))^2\left|\frac{w^i-w^{i+\delta e_k}}{\delta}\right|^2$$
$$+\left|\frac{\varphi_{l-2}(i)-\varphi_{l-2}(i+\delta e_k)}{\delta}\right|^2|v^i-w^i|^2\Bigg)$$
$$\leq 3\left(\left|\frac{v^i-v^{i+\delta e_k}}{\delta}\right|^2+\left|\frac{w^i-w^{i+\delta e_k}}{\delta}\right|^2+2M^2\right). \tag{5.6}$$

Thus, thanks to (5.4) we may also deduce that

$$\tilde{E}_\varepsilon(u,\hat{v}_l,(U_{l-3}\setminus\overline{U}_{l-4})\cap V)\leq\tilde{E}_\varepsilon(u,v,(U_{l-3}\setminus\overline{U}_{l-4})\cap V)$$
$$+6\left(\tilde{G}_\varepsilon(v,(U_{l-3}\setminus\overline{U}_{l-4})\cap V)+\tilde{G}_\varepsilon(\tilde{v},(U_{l-3}\setminus\overline{U}_{l-4}\cap V)\right)$$
$$+3M^2n\delta^n\varepsilon\#((U_{l-3}\setminus\overline{U}_{l-4})\cap V_\delta)). \tag{5.7}$$

Moreover, on U_{l-2} we have

$$(\hat{v}_l^i-1)^2\leq 2\left((v^i-1)^2+(w^i-1)^2\right)$$

and $(\hat{v}_l^i)^2\leq(v^i)^2$, which together with (5.3),(5.4), and (5.6) give

$$\tilde{E}_\varepsilon(u,\hat{v}_l,(U_{l-2}\setminus\overline{U}_{l-3})\cap V)\leq\tilde{F}_\varepsilon(u,v,(U_{l-2}\setminus\overline{U}_{l-3})\cap V)$$
$$+6\left(\tilde{G}_\varepsilon(v,(U_{l-2}\setminus\overline{U}_{l-3})\cap V)+\tilde{G}_\varepsilon(\tilde{v},(U_{l-2}\setminus\overline{U}_{l-3}\cap V)\right)$$
$$+3M^2n\delta^n\varepsilon\#((U_{l-2}\setminus\overline{U}_{l-3})\cap V_\delta)). \tag{5.8}$$

Analogously, we get

$$\tilde{E}_\varepsilon(\tilde{u},\hat{v}_l,(U_{l+2}\setminus\overline{U}_{l+1})\cap V)\leq\tilde{F}_\varepsilon(\tilde{u},\tilde{v},(U_{l+2}\setminus\overline{U}_{l+1})\cap V)$$
$$+6\left(\tilde{G}_\varepsilon(v,(U_{l+2}\setminus\overline{U}_{l+1})\cap V)+\tilde{G}_\varepsilon(\tilde{v},(U_{l+2}\setminus\overline{U}_{l+1}\cap V)\right)$$
$$+3M^2n\delta^n\varepsilon\#((U_{l+2}\setminus\overline{U}_{l+1})\cap V_\delta)) \tag{5.9}$$

and

$$\tilde{E}_\varepsilon(\tilde{u},\hat{v}_l,(U_{l+3}\setminus\overline{U}_{l+2})\cap V)\leq\tilde{E}_\varepsilon(\tilde{u},\tilde{v},(U_{l+3}\setminus\overline{U}_{l+2})\cap V)$$
$$+6\left(\tilde{G}_\varepsilon(v,(U_{l+3}\setminus\overline{U}_{l+2})\cap V)+\tilde{G}_\varepsilon(\tilde{v},(U_{l+3}\setminus\overline{U}_{l+2}\cap V)\right)$$
$$+3M^2n\delta^n\varepsilon\#((U_{l+3}\setminus\overline{U}_{l+2})\cap V_\delta)). \tag{5.10}$$

Then it remains to estimate

$$\tilde{E}_\varepsilon(\hat{u}_l,\hat{v}_l,(U_{l+1}\setminus\overline{U}_{l-2})\cap V)=\tilde{F}_\varepsilon(\hat{u}_l,w,(U_{l+1}\setminus\overline{U}_{l-2})\cap V))+\tilde{G}_\varepsilon(\hat{v}_l,(U_{l+1}\setminus\overline{U}_{l-2})\cap V)).$$

To this end, we observe that by definition of $\hat{v}_l$ we have

$$\begin{aligned}\tilde{G}_\varepsilon(\hat{v}_l,(U_{l+1}\setminus\overline{U}_{l-2})\cap V)) &= G_\varepsilon(w,(U_{l+1}\setminus\overline{U}_{l-2})\cap V)) \\ &\quad+\frac{1}{2}\sum_{i\in(U_{l-1}\setminus\overline{U}_{l-2})\cap V_\delta}\delta^n\varepsilon\sum_{\substack{k=1\\ i+\delta e_k\in U_{l-2}\cap V}}^n\left|\frac{\hat{v}_l^i-\hat{v}_l^{i+\delta e_k}}{\delta}\right|^2 \\ &\quad+\frac{1}{2}\sum_{i\in(U_{l+1}\setminus\overline{U}_l)\cap V_\delta}\delta^n\varepsilon\sum_{\substack{k=1\\ i+\delta e_k\in(U_{l+2}\setminus\overline{U}_{l+1})\cap V}}^n\left|\frac{\hat{v}_l^i-\hat{v}_l^{i+\delta e_k}}{\delta}\right|^2 .\end{aligned}$$

From (5.3)-(5.4) we deduce that

$$G_\varepsilon(w,(U_{l+1}\setminus\overline{U}_{l-2})\cap V))\le G_\varepsilon(v,(U_{l+1}\setminus\overline{U}_{l-2})\cap V))+G_\varepsilon(\tilde{v},U_{l+1}\setminus\overline{U}_{l-2})\cap V)),$$

while the same computations as in (5.6) yield

$$\begin{aligned}&\frac{1}{2}\sum_{i\in(U_{l-1}\setminus\overline{U}_{l-2})\cap V_\delta}\delta^n\varepsilon\sum_{\substack{k=1\\ i+\delta e_k\in U_{l-2}\cap V}}^n\left|\frac{\hat{v}_l^i-\hat{v}_l^{i+\delta e_k}}{\delta}\right|^2 \\ &\qquad\le\frac{3}{2}\sum_{i\in(U_{l-1}\setminus\overline{U}_{l-2})\cap V_\delta}\delta^n\varepsilon\sum_{\substack{k=1\\ i+\delta e_k\in U_{l-2}\cap V}}^n\left(\left|\frac{v^i-v^{i+\delta e_k}}{\delta}\right|^2+\left|\frac{w^i-w^{i+\delta e_k}}{\delta}\right|^2+2M^2\right)\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2}\sum_{i\in(U_{l+1}\setminus\overline{U}_l)\cap V_\delta}\delta^n\varepsilon\sum_{\substack{k=1\\ i+\delta e_k\in(U_{l+2}\setminus\overline{U}_{l+1})\cap V}}^n\left|\frac{\hat{v}_l^i-\hat{v}_l^{i+\delta e_k}}{\delta}\right|^2 \\ &\qquad\le\frac{3}{2}\sum_{i\in(U_{l+1}\setminus\overline{U}_l)\cap V_\delta}\delta^n\varepsilon\sum_{\substack{k=1\\ i+\delta e_k\in(U_{l+2}\setminus\overline{U}_{l+1})\cap V}}^n\left(\left|\frac{\tilde{v}^i-\tilde{v}^{i+\delta e_k}}{\delta}\right|^2+\left|\frac{w^i-w^{i+\delta e_k}}{\delta}\right|^2+2M^2\right).\end{aligned}$$

Hence, again using (5.3) we get

$$\begin{aligned}\tilde{G}_\varepsilon(\hat{v}_l,(U_{l+1}\setminus\overline{U}_{l-2})\cap V)) &\le 6\left(\tilde{G}_\varepsilon(v,(U_{l+1}\setminus\overline{U}_{l-2})\cap V))+\tilde{G}_\varepsilon(\tilde{v},(U_{l+1}\setminus\overline{U}_{l-2})\cap V))\right) \\ &\quad+3M^2n\delta^n\varepsilon\left(\#((U_{l-1}\setminus\overline{U}_{l-2})\cap V_\delta)+\#((U_{l+1}\setminus\overline{U}_l)\cap V_\delta)\right). \quad (5.11)\end{aligned}$$

Finally, arguing as in (5.6) for every $i\in(U_{l+1}\setminus\overline{U}_{l-2})\cap V_\delta$ and for every $k\in\{1,\dots,n\}$ we get

$$\left|\frac{\hat{u}_l^i-\hat{u}_l^{i\pm\delta e_k}}{\delta}\right|^2\le 3\left(\left|\frac{u^i-u^{i\pm\delta e_k}}{\delta}\right|^2+\left|\frac{\tilde{u}^i-\tilde{u}^{i\pm\delta e_k}}{\delta}\right|^2+M^2|u^i-\tilde{u}^i|^2\right),$$

and thus

$$
\begin{aligned}
\tilde{F}_\varepsilon(\hat{u}_l, w, (U_{l+1}\setminus\overline{U}_{l-2})\cap V) &\leq \frac{3}{2}\sum_{i\in(U_{l+1}\setminus\overline{U}_{l-2})\cap V_\delta}\delta^n(w^i)^2\sum_{k=1}^{n}\left(\left|\frac{u^i-u^{i\pm\delta e_k}}{\delta}\right|^2+\left|\frac{\tilde{u}^i-\tilde{u}^{i\pm\delta e_k}}{\delta}\right|^2\right)\\
&\quad+3M^2n\sum_{i\in(U_{l+1}\setminus\overline{U}_{l-2})\cap V_\delta}\delta^n(w^i)^2|u^i-\tilde{u}^i|^2\\
&\leq 3\left(\tilde{F}_\varepsilon(u,v,(U_{l+1}\setminus\overline{U}_{l-2})\cap V)+\tilde{F}_\varepsilon(\tilde{u},\tilde{v},(U_{l+1}\setminus\overline{U}_{l-2})\cap V)\right)\\
&\quad+3M^2n\sum_{i\in(U_{l+1}\setminus\overline{U}_{l-2})\cap V_\delta}\delta^n|u^i-\tilde{u}^i|^2, \qquad (5.12)
\end{aligned}
$$

where in the last estimate we have used the fact that $w\leq v$, $w\leq\tilde{v}$ and $w\leq 1$.

Gathering (5.5) and (5.7)-(5.12), summing up over l we get

$$
\begin{aligned}
\sum_{l=4}^{N-2}E_\varepsilon(\hat{u}_l,\hat{v}_l,U\cup V) &\leq (N-6+42)\big(E_\varepsilon(u,v,U')+E_\varepsilon(\tilde{u},\tilde{v},V)\big)\\
&\quad+21M^2n\delta^n\varepsilon\#((U'\setminus U)\cap V_\delta))\\
&\quad+9M^2n\sum_{i\in(U'\setminus U)\cap V_\delta}\delta^n|u^i-\tilde{u}^i|^2.
\end{aligned}
$$

Hence we can find an index $\hat{l}\in\{4,\dots,N-2\}$ such that

$$
\begin{aligned}
E_\varepsilon(\hat{u}_{\hat{l}},\hat{v}_{\hat{l}},U\cup V) &\leq \frac{1}{N-6}\sum_{l=4}^{N-2}E_\varepsilon(\hat{u}_l,\hat{v}_l,U\cup V)\\
&\leq\left(1+\frac{42}{N-6}\right)\big(E_\varepsilon(u,v,U')+E_\varepsilon(\tilde{u},\tilde{v},V)\big)\\
&\quad+\frac{21M^2n}{N-6}\delta^n\varepsilon\#((U'\setminus U)\cap V_\delta))\\
&\quad+\frac{9M^2n}{N-6}\sum_{i\in(U'\setminus U)\cap V_\delta}\delta^n|u^i-\tilde{u}^i|^2.
\end{aligned}
$$

We now choose N sufficiently large such that $\frac{42}{N-6}\leq\eta$. Then the pair $(\hat{u},\hat{v}):=(\hat{u}_{\hat{l}},\hat{v}_{\hat{l}})$ satisfies

$$
E_\varepsilon(\hat{u},\hat{v},U\cup V)\leq(1+\eta)\big(E_\varepsilon(u,v,U')+E_\varepsilon(\tilde{u},\tilde{v},V)\big)+\sigma_\varepsilon(u,v,\tilde{u},\tilde{v},U,U',V),
$$

where

$$
\sigma_\varepsilon(u,v,\tilde{u},\tilde{v},U,U',V):=\frac{21M^2n}{N-6}\delta^n\varepsilon\#((U'\setminus U)\cap V_\delta))+\frac{9M^2n}{2(N-6)}\sum_{i\in(U'\setminus U)\cap V_\delta}\delta^n|u^i-\tilde{u}^i|^2.
$$

Eventually, note that σ_ε satisfies (5.2). □

On account of Proposition 5.2 we are now in a position to prove the following compactness result for the sequence E_ε.

Theorem 5.3 (Compactness by Γ-convergence and properties of the Γ-limit)**.** *Let E_ε be as in (2.3). Then, for every sequence of positive numbers converging to 0 there exist a subsequence (ε_j) and a functional $E_\ell : L^1(\Omega)\times L^1(\Omega)\times\mathscr{A}(\Omega)\to[0,+\infty]$ such that for all $U\in\mathscr{A}(\Omega)$*

$$
E_\ell(\cdot,1,U)=(E'_\ell)_-(\cdot,1,U)=(E''_\ell)_-(\cdot,1,U)\quad \text{on } GSBV^2(\Omega). \qquad (5.13)
$$

Moreover, E_ℓ satisfies the following properties:

(1) *For every $U \in \mathscr{A}(\Omega)$, the functional $E_\ell(\cdot,1,U)$ is local and lower semicontinuous with respect to the strong $L^1(\Omega)$-topology;*

(2) *for every $u \in GSBV^2(\Omega)$ and every $U \in \mathscr{A}(\Omega)$ there holds*

$$\int_U |\nabla u|^2 dx + \frac{1}{\sqrt{n}}\mathcal{H}^{n-1}(S_u \cap U) \le E_\ell(u,1,U) \le \int_U |\nabla u|^2 dx + (1+\alpha_\ell)\mathcal{H}^{n-1}(S_u \cap U),$$

where $\alpha_\ell \in (0,+\infty)$ is as in Remark 4.3;

(3) *for every $u \in GSBV^2(\Omega)$, the set function $E_\ell(u,1,\cdot)$ is the restriction to $\mathscr{A}(\Omega)$ of a Radon measure;*

(4) *for every $U \in \mathscr{A}_L(\Omega)$ there holds*

$$E_\ell(\cdot,1,U) = E'_\ell(\cdot,1,U) = E''_\ell(\cdot,1,U) \quad \text{on } GSBV^2(\Omega).$$

(5) *E_ℓ is invariant under translations x and in u.*

Proof. The existence of the subsequence (ε_j) and of the functional E_ℓ satisfying (5.13) directly follows from [44, Proposition 16.9]. Further, Remark 5.1 gives (1). The estimates as in (2) are a consequence of Remarks 3.6 and 4.3, and of the inner regularity of $MS(u,1,\cdot)$, where MS denotes the Mumford-Shah functional. The proof of (3) and (4) is standard and appeals to the fundamental estimate Proposition 5.2 and to the De Giorgi and Letta measure property criterion (see *e.g.* [44, Proposition 14.23]). Finally, the proof of (5) is a consequence of the fact that E_ε is invariant under translations in x and u. ☐

For later use we now prove that the abstract Γ-limit E_ℓ provided by Theorem 5.3 is stable under addition of suitable boundary conditions. To this end, for every $(t,\nu) \in \mathbb{R} \times S^{n-1}$ we localise E_ℓ to the oriented unit cube Q^ν and set

$$E^t_\ell(u,1,Q^\nu) := \begin{cases} E_\ell(u,1,Q^\nu) & \text{if } u = u^\nu_t \text{ near } \partial Q^\nu,\ \|u\|_{L^\infty} \le t, \\ +\infty & \text{otherwise in } L^1(Q^\nu), \end{cases}$$

where $u^\nu_t := t\chi_{\{\langle x,\nu\rangle > 0\}}$. Moreover, we define

$$E^t_{\varepsilon_j}(u,v,Q^\nu) := \begin{cases} E_{\varepsilon_j}(u,v,Q^\nu) & \text{if } (u,v) = (\hat{u}_{t,\nu,j}, \hat{v}_{\nu,j}) \text{ near } \partial Q^\nu,\ \|u\|_{L^\infty} \le t, \\ +\infty & \text{otherwise in } L^1(Q^\nu)\times L^1(Q^\nu), \end{cases}$$

where

$$\hat{u}^i_{t,\nu,j} := \begin{cases} t & \text{if } \langle i,\nu\rangle > 0, \\ 0 & \text{if } \langle i,\nu\rangle \le 0, \end{cases} \tag{5.14}$$

and

$$\hat{v}^i_{\nu,j} := \begin{cases} 0 & \text{if } \ i \in S^\nu_j, \\ 1 & \text{otherwise}, \end{cases} \tag{5.15}$$

for every $i \in Q^\nu \cap \delta_j\mathbb{Z}^n$, with

$$S^\nu_j := \{i \in Q^\nu :\ \exists\, l \in Q^\nu \cap \delta_j\mathbb{Z}^n \text{ such that } |i-l| = \delta_j \text{ and } \ \operatorname{sign}\langle i,\nu\rangle \ne \operatorname{sign}\langle l,\nu\rangle\}. \tag{5.16}$$

The following result holds true.

Theorem 5.4 (Γ-convergence with boundary data)**.** *Let (ε_j) and E_ℓ be as in the statement of Theorem 5.3, and let $(t,\nu) \in \mathbb{R}\times S^{n-1}$; then for every and $u \in L^1(Q^\nu)$ there holds*

$$\Gamma\text{-}\lim_{j\to+\infty} E^t_{\varepsilon_j}(u,1,Q^\nu) = E^t_\ell(u,1,Q^\nu).$$

Proof. The liminf inequality is straightforward. Indeed, let $u \in L^1(Q^\nu)$ and let $(u_j, v_j) \subset L^1(Q^\nu) \times L^1(Q^\nu)$ be such that $(u_j, v_j) \to (u, 1)$ in $L^1(Q^\nu) \times L^1(Q^\nu)$ and $\sup_j E^t_{\varepsilon_j}(u_j, v_j, Q^\nu) < +\infty$. Then, in particular, $u_j = \hat{u}_{t,\nu,j}$ in a neighbourhood of ∂Q^ν, hence $u = u^\nu_t$ in a neighbourhood of ∂Q^ν. Since moreover $\|u_j\|_{L^\infty} \leq t$ for every $j \in \mathbb{N}$ we also deduce that $\|u\|_{L^\infty} \leq t$.

To prove the limsup inequality let $u \in GSBV^2(Q^\nu) \cap L^\infty(Q^\nu)$ be such that $u = u^\nu_t$ in $Q^\nu \setminus \overline{Q^\nu_\rho}$ for some $\rho \in (0,1)$ and $\|u\|_{L^\infty} \leq t$. Let $(u_j, v_j) \subset L^1(Q^\nu) \times L^1(Q^\nu)$ be a sequence converging to $(u, 1)$ in $L^1(Q^\nu) \times L^1(Q^\nu)$ and such that

$$\limsup_{j \to +\infty} E_{\varepsilon_j}(u_j, v_j, Q^\nu) \leq E(u, 1, Q^\nu).$$

Since E_{ε_j} decreases by truncation we can always assume that $\|u_j\|_{L^\infty} \leq \|u\|_{L^\infty} \leq t$ so that, in particular, $u_j \to u$ in $L^2(Q^\nu)$. We now modify the sequence (u_j, v_j) in order to attain the boundary condition. To do so we make use of the fundamental estimate Proposition 5.2. We then fix $\rho' \in (\rho, 1)$ and set $U := Q^\nu_\rho$, $U' := Q^\nu_{\rho'}$ and $V := Q^\nu \setminus \overline{Q^\nu_\rho}$. We notice that clearly $u_j \to u^\nu_t$ in $L^2(V)$ and $v_j \to 1$ in $L^2(U')$. We then let $\eta > 0$ be fixed and arbitrary. Hence, invoking Proposition 5.2, with U and V chosen as above, we can find a sequence $(\tilde{u}_j, \tilde{v}_j) \subset L^1(Q^\nu) \times L^1(Q^\nu)$, such that $(\tilde{u}_j, \tilde{v}_j) = (u_j, v_j)$ in Q^ν_ρ, $(\tilde{u}_j, \tilde{v}_j) = (\hat{u}_{t,\nu,j}, \hat{v}_{\nu,j})$ in $Q^\nu \setminus \overline{Q^\nu_{\rho'}}$, $(\tilde{u}_j, \tilde{v}_j) \to (u, 1)$ in $L^2(Q^\nu) \times L^2(Q^\nu)$, $\|u_j\|_{L^\infty} \leq t$, and also satisfying

$$\limsup_{j \to +\infty} E_{\varepsilon_j}(\tilde{u}_j, \tilde{v}_j, Q^\nu) \leq (1+\eta)\Big(\limsup_{j \to +\infty} E_{\varepsilon_j}(u_j, v_j, Q^\nu_{\rho'}) + \limsup_{j \to +\infty} E_{\varepsilon_j}(\hat{u}_{t,\nu,j}, \hat{v}_{\nu,j}, Q^\nu \setminus \overline{Q^\nu_\rho})\Big).$$

Moreover, we notice that $F_{\varepsilon_j}(\hat{u}_{t,\nu,j}, \hat{v}_{\nu,j}, Q^\nu \setminus \overline{Q^\nu_\rho}) = 0$, while

$$\limsup_{j \to +\infty} G_{\varepsilon_j}(\hat{v}_{\nu,j}, Q^\nu \setminus \overline{Q^\nu_\rho}) \leq c\,\mathcal{H}^{n-1}((Q^\nu \setminus \overline{Q^\nu_\rho}) \cap \Pi_\nu),$$

for some $c > 0$. Therefore we obtain

$$\limsup_{j \to +\infty} E_{\varepsilon_j}(\tilde{u}_j, \tilde{v}_j, Q^\nu) \leq (1+\eta)\left(E(u, 1, Q^\nu) + c\,\mathcal{H}^{n-1}((Q^\nu \setminus \overline{Q^\nu_\rho}) \cap \Pi_\nu)\right).$$

Then we conclude by the arbitrariness of $\eta > 0$ and $\rho \in (0,1)$. □

We are now ready to prove the following integral-representation result for the Γ-limit E_ℓ.

Theorem 5.5 (Integral representation)**.** *Let E_ℓ be as in Theorem 5.3, then there exists a Borel function $\phi_\ell : \mathbb{R} \times S^{n-1} \to [0, +\infty)$ such that*

$$E_\ell(u, v) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 dx + \int_{S_u \cap \Omega} \phi_\ell([u], \nu_u) d\mathcal{H}^{n-1} & \text{if } u \in GSBV^2(\Omega) \text{ and } v = 1 \text{ a.e. in } \Omega, \\ +\infty & \text{otherwise in } L^1(\Omega) \times L^1(\Omega). \end{cases}$$

Moreover ϕ_ℓ is given by the following asymptotic formula

$$\phi_\ell(t, \nu) = \limsup_{\rho \to 0^+} \frac{1}{\rho^{n-1}} \lim_{j \to +\infty} \inf\{E_{\varepsilon_j}(u, v, Q^\nu_\rho) \colon (u, v) = (\hat{u}_{t,\nu,j}, \hat{v}_{\nu,j}) \text{ near } \partial Q^\nu_\rho\}, \tag{5.17}$$

where $\hat{u}_{t,\nu,j}$ and $\hat{v}_{\nu,j}$ are as in (5.14) *and* (5.15)*, respectively.*

Proof. We first observe that Theorem 3.1-(i) together with Remark 4.3 ensure that the domain of E_ℓ coincides with $GSBV^2(\Omega) \times \{1\}$. Moreover, in view of Theorem 5.3 E_ℓ satisfies all the assumptions of [15, Theorem 1] except for the lower-bound estimate which can though be recovered by using a standard perturbation argument. That is, let $\sigma > 0$ and for every $u \in SBV^2(\Omega)$ set

$$E^\sigma_\ell(u, 1, U) := E_\ell(u, 1, U) + \sigma \int_{S_u \cap U} |[u]| d\mathcal{H}^{n-1}.$$

Then, for every $\sigma > 0$, E_ℓ^σ satisfies all the assumptions of Theorem [15, Theorem 1] which ensures the existence of two Borel functions $f_\ell^\sigma : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, +\infty)$ and $g_\ell^\sigma : \Omega \times \mathbb{R} \times \mathbb{R} \times S^{n-1} \to [0, +\infty)$ such that

$$E_\ell^\sigma(u, 1, U) = \int_U f_\ell^\sigma(x, u, \nabla u)dx + \int_{S_u \cap U} g_\ell^\sigma(x, u^+, u^-, \nu_u)d\mathcal{H}^{n-1},$$

for every $u \in SBV^2(\Omega)$, $U \in \mathscr{A}(\Omega)$. Since E_ℓ is invariant under translations in u and in x, by (2) and (3) in [15, Theorem 1] we have that f_ℓ^σ does not depend on x and u and that g_ℓ^σ is independent of x and depends on u^+ and u^- only through their difference $[u]$; *i.e.*, $g_\ell^\sigma(x, a, b, \nu) = \phi_\ell^\sigma(a - b, \nu)$, for some $\phi_\ell^\sigma : \mathbb{R} \times S^{n-1} \to [0, +\infty)$. Moreover, appealing to Theorem 5.3-(2) gives $f_\ell^\sigma(\xi) = |\xi|^2$ for every $\sigma > 0$ and every $\xi \in \mathbb{R}^n$. Further, formula (3) in [15, Theorem 1] implies that ϕ_ℓ^σ is decreasing as σ decreases. Thus, setting $\phi_\ell(t, \nu) := \lim_{\sigma \to 0^+} \phi_\ell^\sigma(t, \nu) = \inf_{\sigma > 0} \phi_\ell^\sigma(t, \nu)$, letting $\sigma \to 0^+$, by the pointwise convergence of E_ℓ^σ to E_ℓ and the Monotone Convergence Theorem we get

$$E_\ell(u, 1, U) = \int_U |\nabla u|^2 dx + \int_{S_u \cap U} \phi_\ell([u], \nu_u)d\mathcal{H}^{n-1}, \tag{5.18}$$

for every $u \in SBV^2(\Omega)$, $U \in \mathscr{A}_L(\Omega)$. Hence, choosing $U = \Omega$ gives the desired integral representation on $SBV^2(\Omega)$.

We now show that the integral representation (5.18) also extends to the case $u \in GSBV^2(\Omega)$. To this end, for every $u \in GSBV^2(\Omega)$ set

$$\tilde{E}_\ell(u, 1) := \int_\Omega |\nabla u|^2 dx + \int_{S_u \cap \Omega} \phi_\ell([u], \nu)d\mathcal{H}^{n-1}.$$

Since $\tilde{E}_\ell$ coincides with E_ℓ on $GSBV^2(\Omega) \cap L^\infty(\Omega)$ we can deduce that $\tilde{E}_\ell$ is lower semicontinuous with respect to the strong $L^1(\Omega)$-convergence on $GSBV^2(\Omega) \cap L^\infty(\Omega)$. Hence, in particular, $\tilde{E}_\ell$ is lower semicontinuous on the space of finite partitions. Then, necessarily, ϕ_ℓ is subadditive in the first variable; moreover, for every $t \in \mathbb{R}$ the 1-homogeneous extension of $\phi_\ell(t, \cdot)$ to $\mathbb{R}^n$ is convex (see [7]). Therefore, in view of the lower bound $\phi_\ell \geq 1/\sqrt{n}$ we can apply [8, Theorem 5.22] to deduce that $\tilde{E}_\ell$ is lower semicontinuous with respect to the strong $L^1(\Omega)$-convergence on the whole $GSBV^2(\Omega)$. Since both E_ℓ and $\tilde{E}_\ell$ decrease by truncation, by virtue of their lower semicontinuity we can actually deduce that they are continuous by truncations. Therefore, if u^m denotes the truncation of $u \in GSBV^2(\Omega)$ at level $m \in \mathbb{N}$ we have

$$E_\ell(u, 1) = \lim_{m \to +\infty} E_\ell(u^m, 1) = \lim_{m \to +\infty} \tilde{E}_\ell(u^m, 1) = \tilde{E}_\ell(u, 1),$$

and thus the desired extension.

We now prove (5.17). We start observing that combining the definition of ϕ_ℓ with formula (3) of [15, Theorem 1], for every $(t, \nu) \in \mathbb{R} \times S^{n-1}$ we have

$$\phi_\ell(t, \nu) = \inf_{\sigma > 0} \limsup_{\rho \to 0^+} \frac{1}{\rho^{n-1}} \inf \Big\{ E_\ell(u, 1, Q_\rho^\nu) + \sigma \int_{S_u \cap Q_\rho^\nu} |[u]| d\mathcal{H}^{n-1} : \\ u \in SBV^2(Q_\rho^\nu), u = u_t^\nu \text{ near } \partial Q_\rho^\nu \Big\}.$$

For $(t, \nu) \in \mathbb{R} \times S^{n-1}$ set

$$\tilde{\phi}_\ell(t, \nu) := \limsup_{\rho \to 0^+} \frac{1}{\rho^{n-1}} \inf \left\{ E_\ell(u, 1, Q_\rho^\nu) : u \in SBV^2(Q_\rho^\nu), u = u_t^\nu \text{ near } \partial Q_\rho^\nu \right\};$$

we now claim that

$$\phi_\ell(t, \nu) = \tilde{\phi}_\ell(t, \nu) \quad \text{for every } (t, \nu) \in \mathbb{R} \times S^{n-1}.$$

Since we trivially have $\phi_\ell \geq \tilde{\phi}_\ell$, we only need to prove the opposite inequality. To this end, we notice that up to a truncation argument, in the definition of $\phi_\ell(t,\nu)$ and $\tilde{\phi}_\ell(t,\nu)$ we can restrict to test functions u satisfying $\|u\|_{L^\infty} \leq t$.

Let $\rho > 0$ be fixed; for every $(t,\nu) \in \mathbb{R} \times S^{n-1}$ set

$$\tilde{\phi}_{\ell,\rho}(t,\nu) := \frac{1}{\rho^{n-1}} \inf\left\{E_\ell(u,1,Q^\nu_\rho)\colon u \in SBV^2(Q^\nu_\rho), u = u^\nu_t \text{ near } \partial Q^\nu_\rho,\ \|u\|_{L^\infty} \leq t\right\},$$

and

$$\begin{aligned}\phi^\sigma_{\ell,\rho}(t,\nu) := \frac{1}{\rho^{n-1}} \inf\Bigg\{E_\ell(u,1,Q^\nu_\rho) + \sigma \int_{S_u \cap Q^\nu_\rho} |[u]| d\mathcal{H}^{n-1}\colon \\ u \in SBV^2(Q^\nu_\rho), u = u^\nu_t \text{ near } \partial Q^\nu_\rho,\ \|u\|_{L^\infty} \leq t\Bigg\}.\end{aligned}$$

For $\sigma > 0$ and $\rho > 0$ fixed let $u_{\rho,\sigma} \in SBV^2(Q^\nu_\rho)$ be such that $u_{\rho,\sigma} = u^\nu_t$ in a neighbourhood of ∂Q^ν_ρ, $\|u_{\rho,\sigma}\|_{L^\infty} \leq t$ and such that

$$E_\ell(u_{\rho,\sigma},1,Q^\nu_\rho) \leq \rho^{n-1}(\tilde{\phi}_{\ell,\rho}(t,\nu) + \sigma). \tag{5.19}$$

We then have

$$\begin{aligned}\phi^\sigma_{\ell,\rho}(t,\nu) &\leq \frac{1}{\rho^{n-1}}\left(E_\ell(u_{\rho,\sigma},1,Q^\nu_\rho) + \sigma\int_{S_{u_{\rho,\sigma}}} |[u_{\rho,\sigma}]| d\mathcal{H}^{n-1}\right) \\ &\leq \frac{1}{\rho^{n-1}}\left(\rho^{n-1}(\tilde{\phi}_{\ell,\rho}(t,\nu)+\sigma) + 2\sigma|t|\mathcal{H}^{n-1}(S_{u_{\rho,\sigma}} \cap Q^\nu_\rho)\right). \end{aligned}\tag{5.20}$$

By Theorem 5.3-(2) we also get

$$\mathcal{H}^{n-1}(S_{u_{\rho,\sigma}} \cap Q^\nu_\rho) \leq \sqrt{n}\, E_\ell(u_{\rho,\sigma},1,Q^\nu_\rho),$$

hence combining (5.19) and (5.20) gives

$$\phi^\sigma_{\ell,\rho}(t,\nu) \leq (1 + 2\sigma|t|\sqrt{n})\,(\tilde{\phi}_{\ell,\rho}(t,\nu) + \sigma).$$

Thus, passing first to the limsup in ρ and then to the limit in σ we deduce that $\phi_\ell \leq \tilde{\phi}_\ell$ and hence the claim.

Eventually, to deduce the asymptotic formula (5.17) we notice that also in (5.17) we can restrict the minimisation to those functions u also satisfying the bound $\|u\|_{L^\infty} \leq t$. Therefore, (5.17) follows from Theorem 5.4 and the fundamental property of Γ-convergence once noticed that Theorem 3.1 and the constraint $\|u\|_{L^\infty} \leq t$ ensure the needed equi-coercivity. ☐

Remark 5.6. If $n = 1$ the proof of the Γ-convergence result in this critical regime simplifies. In fact, by adapting carefully the arguments in [10, Lemma 2.1] to the discrete setting, one can explicitly compute the Γ-limit as

$$E_\ell(u,1) = \int_\Omega (u')^2 dt + c_\ell \#(S_u),$$

where

$$c_\ell := \min\left\{\sum_{i\in\mathbb{N}}\left(\ell(v^i-1)^2 + \frac{1}{\ell}(v^{i+1}-v^i)^2\right)\colon\ v^0 = 0,\ \lim_{i\to+\infty} v^i = 1\right\}. \tag{5.21}$$

A solution $\bar{v}$ to (5.21) can be explicitly computed by solving the associated Euler-Lagrange equation, whose solution is

$$\bar{v}^i = 1 - \left(\frac{\ell}{2}\left(\sqrt{\ell^2+4}+\ell\right) - 1\right)^{-i},$$

and corresponding energy

$$c_\ell = \ell + \frac{4+\left(\sqrt{\ell^2+4}+\ell\right)^2}{\ell\left(\sqrt{\ell^2+4}+\ell\right)^2+4\sqrt{\ell^2+4}+4\ell}.$$

Now let $n>1$; using a slicing argument it can be also shown that

$$\phi_\ell(t,\pm e_k)=c_\ell, \quad \text{for every } t\in\mathbb{R}, \quad \text{for every } k\in\{1,\dots,n\},$$

that is, in the coordinate directions, there is a one-dimensional optimal profile.

Eventually, if $\nu\in S^{n-1}$ is of the form

$$\nu=\frac{1}{\sqrt{n}}\sum_{k=1}^n a_k e_k, \quad \text{with } a_k\in\{-1,1\};$$

i.e., ν is a symmetry axis of the underlying lattice, then, again by slicing, it can be shown that

$$\phi_\ell(t,\nu)=\sqrt{n}c_{\ell,n}, \quad \text{for every } t\in\mathbb{R},$$

where

$$c_{\ell,n}:=\min\left\{\sum_{i\in\mathbb{N}}\left(\frac{\ell}{n}(v^i-1)^2+\frac{1}{\ell}(v^{i+1}-v^i)^2\right):\ v^0=0,\ \lim_{i\to+\infty}v^i=1\right\}=\frac{1}{\sqrt{n}}c_{\frac{\ell}{\sqrt{n}}},$$

with $c_{\frac{\ell}{\sqrt{n}}}$ defined according to (5.21). In particular, $\phi_\ell(t,\nu)=c_{\frac{\ell}{\sqrt{n}}}$.

5.1. **Characterisation of ϕ_ℓ for $n=2$.** In this subsection we characterise the surface energy density ϕ_ℓ when $n=2$. In particular we prove that in this case the function ϕ_ℓ does not depend on t; *i.e.*, for every $(t,\nu)\in\mathbb{R}\times S^1$ we have $\phi_\ell(t,\nu)=\varphi_\ell(\nu)$ where φ_ℓ is given by the following formula

$$\begin{aligned}\varphi_\ell(\nu):=\lim_{T\to+\infty}\frac{1}{2T}\inf\Bigg\{&\ell\sum_{i\in TQ^\nu\cap\mathbb{Z}^2}(v^i-1)^2+\frac{1}{2\ell}\sum_{\substack{i,j\in TQ^\nu\cap\mathbb{Z}^2\\|i-j|=1}}|v^i-v^j|^2\colon v\in\mathcal{A}_1(TQ^\nu),\\&\exists \text{ channel } \mathscr{C} \text{ in } TQ^\nu\cap\mathbb{Z}^2\colon v=0 \text{ on } \mathscr{C},\ v=1 \text{ otherwise near } \partial TQ^\nu\Bigg\},\end{aligned} \tag{5.22}$$

where we refer to Definition 5.10 below for the precise definition of a channel $\mathscr{C}$. An immediate consequence of (5.22) is that for $n=2$ the Γ-limit does not depend on the subsequence (ε_j) so that the whole sequence (E_ε) Γ-converges to E_ℓ.

We note that the definition of ϕ_ℓ provided by the general asymptotic formula (5.17) involves the whole functional E_{ε_j}, whereas the minimisation problem in (5.22) depends only on a scaled version of the functional G_{ε_j}, the latter being as in (2.5). Then, to prove that for $n=2$ (5.17) and (5.22) actually coincide the idea is to show that for every $(t,\nu)\in\mathbb{R}\times S^1$ a test pair (u,v) for $\phi_\ell(t,\nu)$ can be suitably modified in a way such that $F_{\varepsilon_j}(u,v,Q^\nu_\rho)=0$ (with F_{ε_j} as in (2.4)). Since in (5.17) a test function u has to satisfy $u^i=u^\nu_t(i)$ in a neighbourhood of ∂Q^ν_ρ, intuitively one would like to choose $u\equiv u^\nu_t$ in Q^ν_ρ, which would carry no energy contribution far from the line Π_ν. On the other hand, close to Π_ν this choice would lead to a diverging F_{ε_j}. Therefore to find a pair (u,v) satisfying $F_{\varepsilon_j}(u,v,Q^\nu_\rho)=0$ we are going to show that we can modify v in a way such that the discrete level set $\{v=0\}$ contains a "path" which is "close" to Π_ν and connects the two opposite sides of Q^ν_ρ which are parallel to ν. Then, along this path, the function u can be discontinuous (jumping from the value 0 to the value t) and at the same time, when coupled with v, it can satisfy the desired equality $F_{\varepsilon_j}(u,v,Q^\nu_\rho)=0$.

Note that this last argument can be generalised to any space dimension, so that an analog of formula (5.22) always provides an upper bound (see Step 3 in the proof of Theorem 5.11), upon suitably defining an n-dimensional analog of a channel, which will be a "discrete hypersurface" disconnecting TQ^ν. We do not make this remark precise further for the sake of brevity.

In what follows we make precise the heuristic idea as above. To this end we need to introduce some further notation.

Definition 5.7. *A sequence $p := \{i_1, \ldots, i_{l_p}\} \subset \delta\mathbb{Z}^2$ is called a* path of cardinality l_p *if*

$$|i_{k+1} - i_k| = \delta \quad \textit{for every } k \in \{1, \ldots, l_p - 1\}.$$

We say that two paths $p = \{i_1, \ldots, i_{l_p}\}$ and $p' = \{i'_1, \ldots, i'_{l_{p'}}\}$ are disjoint *if either $p \cap p' = \emptyset$ or if for every $k \in \{1, \ldots, l_p - 1\}$ and for every $j \in \{1, \ldots, l_{p'} - 1\}$ there holds*

$$i_k = i'_j \qquad \Longrightarrow \qquad \begin{cases} i_{k+1} \neq i'_{j+1} \\ i_{k+1} \neq i'_{j-1} \\ i_{k-1} \neq i'_{j+1}, \end{cases}$$

i.e., if $i_k \in p \cap p'$, then the vertical and the horizontal δ-segments departing from i_k can only "belong" to one of the two paths (see Figure 1a).

The definition of disjoint paths as above is motivated by the following fact: if $p, p' \subset Q^\nu_\rho$ are two disjoint paths we have

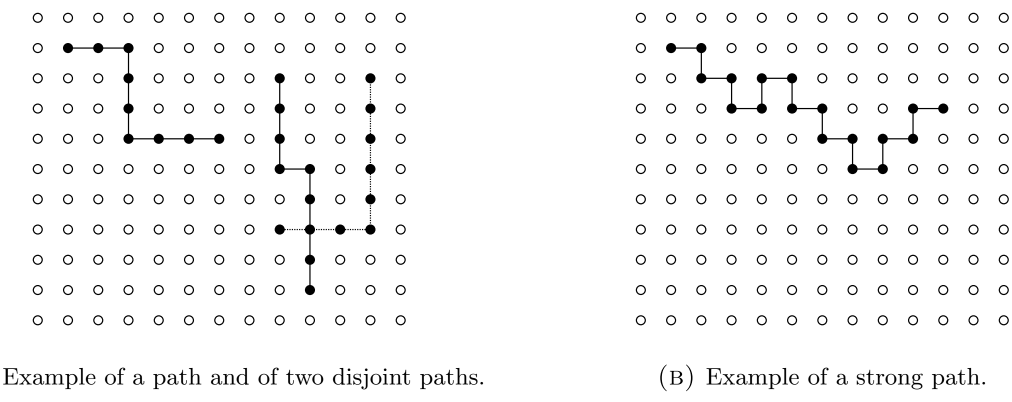
$$F_\varepsilon(u, v, Q^\nu_\rho) \geq F_\varepsilon(u, v, p) + F_\varepsilon(u, v, p').$$

Definition 5.8. *We say that a discrete set $A \subset \delta\mathbb{Z}^2$ is* path connected *if for every pair $(i, j) \in A \times A$ there exists a path $p = \{i_1, \ldots, i_{l_p}\} \subset A$ with $i_1 = i$ and $i_{l_p} = j$.*

We say that p is a strong path *if for every $k \in \{2, \ldots, l_p - 1\}$ there holds*

$$p \cap \{i_k - \delta e_1, i_k + \delta e_1\} \neq \emptyset \quad \textit{and} \quad p \cap \{i_k - \delta e_2, i_k + \delta e_2\} \neq \emptyset,$$

i.e., for each $i_k \in p$ (except for i_1 and i_{l_p}) at least one horizontal and one vertical nearest neighbouring points belong to p (See Figure 1b).



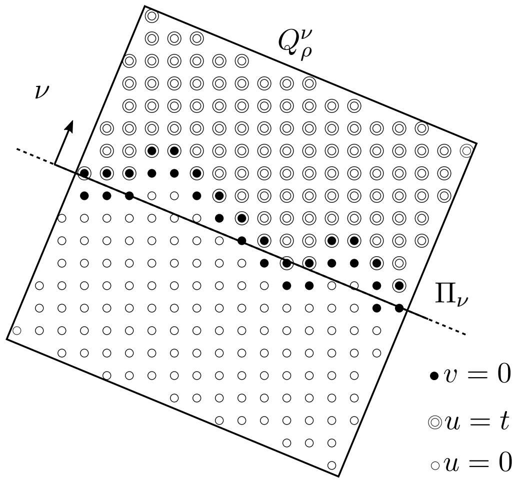
(A) Example of a path and of two disjoint paths. (B) Example of a strong path.

FIGURE 1. Examples of a path and of a strong path.

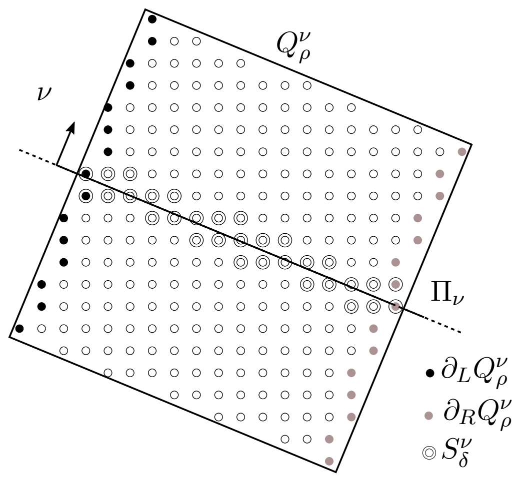
In analogy to the definition of path-connected set, we say that a discrete set $A \subset \delta\mathbb{Z}^2$ is strongly connected *if for every pair $(i, j) \in A \times A$ there exists a strong path $p = \{i_1, \ldots, i_{l_p}\} \subset A$ with $i_1 = i$ and $i_{l_p} = j$.*

Remark 5.9. *If $A \subset \delta\mathbb{Z}^2$ is strongly connected and $\mathcal{J} := \{i \in \delta\mathbb{Z}^2 \colon \exists\, j \in A \text{ s.t. } |i-j| = \delta\}$, then also $A \cup \mathcal{J}$ is strongly connected. In fact, let $(i,j) \in (A \cup \mathcal{J}) \times (A \cup \mathcal{J})$ and suppose that $i \in \mathcal{J} \setminus A$, $j \in A$. By definition there exists $i' \in A$ such that $|i - i'| = \delta$. It is not restrictive to assume that $i = i' + \delta e_1$. Let $p = \{i_1, \dots, i_{l_p}\} \subset A$ be a strong path with $i_1 = i'$, $i_{l_p} = j$. If $p \cap \{i_1 - \delta e_2, i_1 + \delta e_2\} \neq \emptyset$, then $p' := \{i, i_1, \dots, i_{l_p}\}$ is a strong path connecting i and j. If instead $p \cap \{i_1 - \delta e_2, i_1 + \delta e_2\} = \emptyset$, then necessarily $i_2 = i_1 - \delta e_1$. Moreover, $p \cap \{i_2 - \delta e_2, i_2 + \delta e_2\} \neq \emptyset$. Assume without loss of generality that $i_2 + \delta e_2 \in p$. Then $p' := \{i, i_1, i_1 + \delta e_2, i_1, \dots, i_{l_p}\} \subset A \cup \mathcal{J}$ is a strong path connecting i and j. The case where $j \in \mathcal{J} \setminus A$ follows analogously by choosing $j' \in A$ with $|j - j'| = \delta$ and a strong path connecting i' and j'.*

We note that the notion of strong path is motivated by the form of our energy and in particular by the form of the term F_ε. Indeed, if p is a strong path connecting the two opposite sides of Q_ρ^ν which are parallel to ν and $v \in \mathcal{A}_\varepsilon(\Omega)$ is such that $v^i = 0$ for all $i \in p$, then it is possible to construct a function $u : Q_\rho^\nu \cap \delta\mathbb{Z}^2 \to \{0, t\}$ which is discontinuous across p and satisfies $F_\varepsilon(u, v, Q_\rho^\nu) = 0$ (see Figure 2a). In order to make the above construction precise, we



(A) Example of a strong path connecting the left and the right side of a cube Q_ρ^ν and a pair (u,v) with $F(u,v,Q_\rho^\nu) = 0$.



(B) $\partial_L^\delta Q_\rho^\nu$, $\partial_R^\delta Q_\rho^\nu$ and S_δ^ν.

FIGURE 2. Example of a strong path and discretisation of the "left" and "right" boundary.

finally introduce the notion of channel.

Definition 5.10. *We say that $\mathscr{C} \subset Q_\rho^\nu$ is a* channel *in $Q_\rho^\nu \cap \delta\mathbb{Z}^2$ if $\mathscr{C}$ is a strong path connecting $S_\delta^\nu \cap \partial_L^\delta Q_\rho^\nu$ and $S_\delta^\nu \cap \partial_R^\delta Q_\rho^\nu$, where S_δ^ν is as in (5.16) with δ_j replaced by δ and*

$$\partial_L^\delta Q_\rho^\nu := \Big\{ i \in Q_\rho^\nu \cap \delta\mathbb{Z}^2 :\ \exists\, j \in \delta\mathbb{Z}^2 \text{ such that } |i-j| = \delta \text{ and } \langle j, \nu^\perp \rangle \leq -\rho/2 \Big\},$$

$$\partial_R^\delta Q_\rho^\nu := \Big\{ i \in Q_\rho^\nu \cap \delta\mathbb{Z}^2 :\ \exists\, j \in \delta\mathbb{Z}^2 \text{ such that } |i-j| = \delta \text{ and } \langle j, \nu^\perp \rangle \geq \rho/2 \Big\},$$

i.e., $\partial_L^\delta Q_\rho^\nu$ and $\partial_R^\delta Q_\rho^\nu$ are the discretised "left" and "right" boundary of Q_ρ^ν, respectively (see Figure 2b). We also set

$$\partial^\delta Q_\rho^\nu := \big\{ i \in Q_\rho^\nu \cap \delta\mathbb{Z}^2 :\ \exists\, j \in \delta\mathbb{Z}^2 \setminus Q_\rho^\nu \text{ such that } |i-j| = \delta \big\}.$$

We are now ready to state the main result of this section.

Theorem 5.11. *Let $n=2$; let $(t,\nu)\in\mathbb{R}\times S^1$, $\phi_\ell(t,\nu)$ be as in (5.17), and $\varphi_\ell(\nu)$ be as in (5.22). Then φ_ℓ is well-defined, moreover $\phi_\ell(t,\nu)=\varphi_\ell(\nu)$ for every $(t,\nu)\in\mathbb{R}\times S^1$.*

Proof. We divide the proof into three main steps.

Step 1: φ_ℓ is well-defined and continuous.

For $T>0$ and $\nu\in S^1$ set

$$\varphi_\ell^T(\nu):=\frac{1}{2T}\inf\Big\{\ell\sum_{i\in TQ^\nu\cap\mathbb{Z}^2}(v^i-1)^2+\frac{1}{2\ell}\sum_{\substack{i,j\in TQ^\nu\cap\mathbb{Z}^2\\|i-j|=1}}|v^i-v^j|^2: v\in\mathcal{A}_1(TQ^\nu),$$
$$\exists\text{ strong channel }\mathscr{C}\subset TQ^\nu\cap\mathbb{Z}^2\colon v=0\text{ on }\mathscr{C},\ v=1\text{ otherwise near }\partial TQ^\nu\Big\}. \tag{5.23}$$

Let Ξ denote the set of unit rational vectors; *i.e.*, $\Xi:=\{\nu\in S^1\colon\exists\lambda\in\mathbb{R}\text{ s.t. }\lambda\nu\in\mathbb{Q}^2\}$.

Arguing as in [24, Proposition 14.4] we obtain that $\lim_{T\to+\infty}\varphi_\ell^T(\nu)$ exists for every $\nu\in\Xi$.

We now claim that for every $\eta\in(0,1)$ there exists $\sigma=\sigma(\eta)>0$ such that for every $\nu,\nu'\in S^1$ satisfying $|\nu-\nu'|\le\sigma$ there hold

$$\big|\liminf_{T\to+\infty}\varphi_\ell^T(\nu)-\liminf_{T\to+\infty}\varphi_\ell^T(\nu')\big|\le c\,\eta \tag{5.24}$$

and

$$\big|\limsup_{T\to+\infty}\varphi_\ell^T(\nu)-\limsup_{T\to+\infty}\varphi_\ell^T(\nu')\big|\le c\,\eta, \tag{5.25}$$

for some constant $c>0$ independent of η,σ,ν,ν'; hence, in particular, both the functions $\nu\mapsto\liminf_T\varphi_\ell^T(\nu)$ and $\nu\mapsto\limsup_T\varphi_\ell^T(\nu)$ are continuous.

To prove the claim we start observing that for fixed $\eta\in(0,1)$ we can find $\sigma\in(0,\eta)$ such that for all $\nu,\nu'\in S^1$ satisfying $|\nu-\nu'|\le\sigma$ there holds

(1) $(1-2\eta)TQ^\nu\subset(1-\eta)TQ^{\nu'}\subset TQ^\nu$;

(2) $\mathcal{H}^1\Big(\Pi_{\nu'}\cap\Big(TQ^\nu\setminus(1-\eta)TQ^{\nu'}\Big)\Big)\le cT\eta$;

(3) $\mathcal{H}^1\big(\partial TQ^\nu\cap(\Pi_\nu^+\Delta\Pi_{\nu'}^+)\big)\le cT\eta$;

for some $c>0$ independent of σ,η,ν,ν'.

Let now v_T be a test function for $\varphi_\ell^{(1-\eta)T}(\nu')$ with

$$\ell\sum_{i\in(1-\eta)TQ^{\nu'}\cap\mathbb{Z}^2}(v_T^i-1)^2+\frac{1}{2\ell}\sum_{\substack{i,j\in(1-\eta)TQ^{\nu'}\cap\mathbb{Z}^2\\|i-j|=1}}|v_T^i-v_T^j|^2\le 2(1-\eta)T\varphi_\ell^{(1-\eta)T}(\nu')+1. \tag{5.26}$$

We suitably modify v_T to obtain a test function $\tilde v_T$ for $\varphi_\ell^T(\nu)$. By definition there exists a strong channel in $(1-\eta)TQ^{\nu'}\cap\mathbb{Z}^2$ along which $v_T=0$. Moreover, in view of (1)-(3) we may check that there exist a strong path $p_L\subset\mathbb{Z}^2$ of cardinality l_L connecting

$$S_1^{\nu'}\cap\partial_L^1(1-\eta)TQ^{\nu'}\quad\text{and}\quad S_1^\nu\cap\partial_L^1TQ^\nu$$

and a strong path $p_R\subset\mathbb{Z}^2$ of cardinality l_R connecting

$$S_1^{\nu'}\cap\partial_R^1(1-\eta)TQ^{\nu'}\quad\text{and}\quad S_1^\nu\cap\partial_R^1TQ^\nu$$

such that

$$l_L,l_R\le c\Big(\mathcal{H}^1\Big(\Pi_{\nu'}\cap\Big(TQ^\nu\setminus(1-\eta)TQ^{\nu'}\Big)\Big)+\mathcal{H}^1\big(\partial TQ^\nu\cap(\Pi_\nu^+\Delta\Pi_{\nu'}^+)\big)\Big)\le cT\eta.$$

Then we define the function $\tilde v_T$ as follows

$$\tilde v_T^i := \begin{cases} v_T^i & \text{if } i \in (1-\eta)TQ^{\nu'}, \\ 0 & \text{if } i \in p_L \cup p_R, \\ 1 & \text{otherwise in } TQ^{\nu} \setminus (1-\eta)TQ^{\nu'}. \end{cases}$$

Clearly $\tilde v_T$ is admissible for $\varphi_\ell^T(\nu)$; therefore in view of (5.26) we have

$$\begin{aligned} \varphi_\ell^T(\nu) &\leq \frac{1}{2T}\Bigg(\ell \sum_{i\in TQ^\nu\cap\mathbb{Z}^2} (\tilde v_T^i - 1)^2 + \frac{1}{2\ell} \sum_{\substack{i,j\in TQ^\nu\cap\mathbb{Z}^2\\ |i-j|=1}} |\tilde v_T^i - \tilde v_T^j|^2\Bigg) \\ &= \frac{1}{2T}\Bigg(\ell \sum_{i\in(1-\eta)TQ^{\nu'}\cap\mathbb{Z}^2} (v_T^i - 1)^2 + \frac{1}{2\ell} \sum_{\substack{i,j\in(1-\eta)TQ^{\nu'}\cap\mathbb{Z}^2\\ |i-j|=1}} |v_T^i - v_T^j|^2\Bigg) \\ &+ \frac{1}{2T}\Bigg(\ell \sum_{i\in(TQ^\nu\setminus(1-\eta)TQ^{\nu'})\cap\mathbb{Z}^2} (\tilde v_T^i - 1)^2 + \frac{1}{2\ell} \sum_{\substack{i,j\in(TQ^\nu\setminus(1-\eta)TQ^{\nu'})\cap\mathbb{Z}^2\\ |i-j|=1}} |\tilde v_T^i - \tilde v_T^j|^2\Bigg) \\ &\leq (1-\eta)\varphi_\ell^{(1-\eta)T}(\nu') + \frac{1}{2T} + \frac{1}{T}c(l_L + l_R) \\ &\leq \varphi_\ell^{(1-\eta)T}(\nu') + \frac{1}{2T} + c\eta. \end{aligned}$$

Thus, passing to the liminf as $T \to +\infty$ we get

$$\liminf_{T\to+\infty} \varphi_\ell^T(\nu) - c\eta \leq \liminf_{T\to+\infty} \varphi_\ell^T(\nu'),$$

hence by exchanging the role of ν and ν' we obtain (5.24). Then, an analogous argument also gives (5.25).

By combining (5.24) and (5.25) we obtain that $\varphi_\ell(\nu)$ is well-defined for every $\nu \in S^1$. Indeed let $\nu \in S^1$, since Ξ is dense in S^1, for every $\sigma > 0$ we can find $\nu_\sigma \in \Xi$ such that $|\nu - \nu_\sigma| \leq \sigma$. Then, by virtue of (5.24) and (5.25) and in view of the equality $\lim_T \varphi_\ell^T(\nu_\sigma) = \varphi_\ell(\nu_\sigma)$ we get

$$-c\eta + \limsup_{T\to+\infty} \varphi_\ell^T(\nu) \leq \varphi_\ell(\nu_\sigma) \leq \liminf_{T\to+\infty} \varphi_\ell^T(\nu) + c\eta,$$

hence the existence of $\lim_T \varphi_\ell^T(\nu)$ for every $\nu \in S^1$ follows by the arbitrariness of η.

We eventually notice that the existence of $\lim_T \varphi_\ell^T(\nu)$ together with (5.24) and (5.25) yields the continuity of $\nu \mapsto \varphi_\ell(\nu)$.

We now turn to the proof of the equality $\phi_\ell = \varphi_\ell$. This proof will be carried out in two steps.

To simplify the notation, in what follows E_ε denotes the Γ-converging subsequence provided by Theorem 5.3.

Let $(t,\nu) \in \mathbb{R} \times S^1$; for $\varepsilon, \rho > 0$ set

$$\phi_{\varepsilon,\rho}(t,\nu) := \frac{1}{\rho} \inf\{E_\varepsilon(u,v,Q_\rho^\nu) : \ (u,v) = (\hat u_{t,\nu,\varepsilon}, \hat v_{\nu,\varepsilon}) \text{ in a neighbourhood of } \partial Q_\rho^\nu\},$$

where $\hat u_{t,\nu,\varepsilon}$ and $\hat v_{\nu,\varepsilon}$ are given respectively by (5.14) and (5.15) with ε_j replaced by ε.

To simplify the proof we only consider the special case $\delta = \ell\varepsilon$, the general case $\delta/\varepsilon \to \ell$ being a straightforward consequence of this special one.

Under the assumption $\delta=\ell\varepsilon$ we then have

$$\frac{1}{\rho}E_\varepsilon(u,v,Q^\nu_\rho)=\frac{1}{2\rho}\sum_{\substack{i,j\in Q^\nu_\rho\cap\delta\mathbb{Z}^2\\|i-j|=\delta}}(v^i)^2|u^i-u^j|^2+\frac{\delta}{2\rho}\bigg(\ell\sum_{i\in Q^\nu_\rho\cap\delta\mathbb{Z}^2}(v^i-1)^2+\frac{1}{2\ell}\sum_{\substack{i,j\in Q^\nu_\rho\cap\delta\mathbb{Z}^2\\|i-j|=\delta}}|v^i-v^j|^2\bigg).$$

Step 2: $\phi_\ell(t,\nu)\geq\varphi_\ell(\nu)$ for every $(t,\nu)\in\mathbb{R}\times S^1$.

Let $(t,\nu)\in\mathbb{R}\times S^1$ be such that $\phi_\ell(t,\nu)<+\infty$, otherwise there is nothing to prove. Since by definition

$$\phi_\ell(t,\nu):=\limsup_{\rho\to 0^+}\lim_{\varepsilon\to 0}\phi_{\varepsilon,\rho}(t,\nu),$$

up to a subsequence we can assume that

$$\sup_{\varepsilon,\rho>0}\phi_{\varepsilon,\rho}(t,\nu)<+\infty. \tag{5.27}$$

Let $N\in\mathbb{N}$ be fixed and let $(u_{\varepsilon,\rho},v_{\varepsilon,\rho})$ be a test pair for $\phi_{\varepsilon,\rho}(t,\nu)$ such that

$$\frac{1}{\rho}E_\varepsilon(u_{\varepsilon,\rho},v_{\varepsilon,\rho},Q^\nu_\rho)\leq\phi_{\varepsilon,\rho}(t,\nu)+\frac{1}{N}. \tag{5.28}$$

We now claim that we can replace $v_{\varepsilon,\rho}$ with a function $\tilde v_{\varepsilon,\rho}$ which is equal to zero along a channel in $Q^\nu_\rho\cap\delta\mathbb{Z}^2$ without essentially increasing the energy. To this end, fix $\eta\in(0,1)$ and set

$$\mathcal{I}^\eta_\varepsilon:=\{i\in Q^\nu_\rho\cap\delta\mathbb{Z}^2:\ v^i_{\varepsilon,\rho}<\eta\}.$$

We notice that thanks to the boundary conditions satisfied by $v_{\varepsilon,\rho}$ we have $\mathcal{I}^\eta_\varepsilon\neq\emptyset$. Moreover, in view of (5.27) and (5.28) we get

$$\#(\mathcal{I}^\eta_\varepsilon)\leq\frac{c}{(1-\eta)^2}\frac{\rho}{\delta}, \tag{5.29}$$

for some $c>0$ independent of ε, ρ, and η.

The general strategy to construct $\tilde v_{\varepsilon,\rho}$ is as follows: For every $i\in\mathcal{I}^\eta_\varepsilon$ we set $\tilde v^i_{\varepsilon,\rho}:=0$. Thanks to (5.29) this modification to $v_{\varepsilon,\rho}$ increases the energy only by an error proportional to η. Then, if the discrete set $\mathcal{I}^\eta_\varepsilon$ is strongly connected we are done. If instead $\mathcal{I}^\eta_\varepsilon$ consists of more than one strongly connected component we show that we can connect the strongly connected components of $\mathcal{I}^\eta_\varepsilon$ to one another in a way such that if we replace $v_{\varepsilon,\rho}$ by zero along the "channel" obtained in this way, we essentially do not increase the energy.

We now illustrate in detail the multi-step strategy leading to the construction of the channel as above. To this end set

$$\mathcal{C}^\eta_\varepsilon:=\{C\subset\mathcal{I}^\eta_\varepsilon:\ C\text{ cluster in }\mathcal{I}^\eta_\varepsilon\},$$

where by "cluster" in $\mathcal{I}^\eta_\varepsilon$ we mean a maximal strongly connected component in $\mathcal{I}^\eta_\varepsilon$. Thanks to the boundary condition satisfied by $v_{\varepsilon,\rho}$ we can find clusters $C',C''\in\mathcal{C}^\eta_\varepsilon$ such that $C'\cap\partial^\delta_L Q^\nu_\rho\cap S^\nu_\delta\neq\emptyset$ and $C''\cap\partial^\delta_R Q^\nu_\rho\cap S^\nu_\delta\neq\emptyset$. Let now

$$\mathcal{J}_0:=\{i\in Q^\nu_\rho\cap\delta\mathbb{Z}^2:\ \exists\ j\in C'\text{ s.t. }|i-j|=\delta\}$$

denote the set of all lattice points in Q^ν_ρ which have distance δ from C'. Then, by the maximality of C' there exists a path $p_0=\{i_1,\dots,i_{l_0}\}\subset C'\cup\mathcal{J}_0$ connecting $\partial^\delta Q^\nu_\rho\cap\Pi^+_\nu$ and $\partial^\delta Q^\nu_\rho\cap\Pi^-_\nu$ such that

$$\frac{\big(v^{i_k}_{\varepsilon,\rho}\big)^2+\big(v^{i_{k+1}}_{\varepsilon,\rho}\big)^2}{2}\geq\frac{\eta^2}{2}\quad\text{for every }k\in\{1,\dots,l_0-1\}. \tag{5.30}$$

In fact, since by assumption $\mathcal{I}^\eta_\varepsilon$ consists of more than one strongly connected component, p_0 can be chosen in a way such that for every $k\in\{1,\dots,l_0-1\}$ either i_k or i_{k+1} belongs to $\mathcal{J}_0\setminus I^\eta_\varepsilon$, which already gives (5.30). Then, due to the boundary conditions satisfied by $u_{\varepsilon,\rho}$ there holds $u^{i_1}_{\varepsilon,\rho} = t$ and $u^{i_{l_0}}_{\varepsilon,\rho} = 0$. Thus, using Jensen's inequality from (5.30) we deduce

$$\frac{1}{\rho}F_\varepsilon(u_{\varepsilon,\rho}, v_{\varepsilon,\rho}, p_0) = \frac{1}{\rho}\sum_{k=1}^{l_0-1}\frac{\big(v^{i_k}_{\varepsilon,\rho}\big)^2 + \big(v^{i_{k+1}}_{\varepsilon,\rho}\big)^2}{2}\left|u^{i_k}_{\varepsilon,\rho} - u^{i_{k+1}}_{\varepsilon,\rho}\right|^2 \geq \frac{\eta^2}{2}\frac{t^2}{\rho(l_0-1)}. \tag{5.31}$$

We now define

$$\mathcal{C}^{\eta,0}_\varepsilon := \big\{C \in \mathcal{C}^\eta_\varepsilon :\ C' \cup \mathcal{J}_0 \cup C \text{ is strongly connected}\big\},$$

which might be empty. If $C'' \in \mathcal{C}^{\eta,0}_\varepsilon$, we can find two points $i^1_0, i^2_0 \in \mathcal{J}_0$ such that $C' \cup \{i^1_0, i^2_0\} \cup C''$ is the desired channel $\mathscr{C}$ in $Q^\nu_\rho \cap \delta\mathbb{Z}^2$. If instead this is not the case, we proceed as follows. We set

$$C_1 := C' \cup \mathcal{J}_0 \cup \big\{i \in C :\ C \in \mathcal{C}^{\eta,0}_\varepsilon\big\},$$

and we notice that C_1 is strongly connected. Indeed, if there exists $C \in \mathcal{C}^{\eta,0}_\varepsilon$, then C_1 is strongly connected by definition. If instead $\mathcal{C}^{\eta,0}_\varepsilon$ is empty, then $C_1 = C' \cup \mathcal{J}_0$ is strongly connected in view of Remark 5.9. Moreover, we set

$$\mathcal{J}_1 := \{i \in Q^\nu_\rho \cap \delta\mathbb{Z}^2 :\ \exists\, j \in C_1 \text{ s.t. } |i-j| = \delta\}.$$

Since by assumption $C'' \notin \mathcal{C}^{\eta,0}_\varepsilon$, arguing as above we can find a path $p_1 = \{i_1, \dots i_{l_1}\} \subset C_1 \cup \mathcal{J}_1$ connecting $\partial^\delta Q^\nu_\rho \cap \Pi^+_\nu$ and $\partial^\delta Q^\nu_\rho \cap \Pi^-_\nu$ such that (5.30) and (5.31) are satisfied with p_1, l_1 in place of p_0, l_0. Moreover, since $p_0 \subset C' \cup \mathcal{J}_0$ and $C_1 \cup \mathcal{J}_1$ contains at least all lattice points that have distance δ from $C' \cup \mathcal{J}_0$, p_1 can be chosen in such a way that it is disjoint from p_0 in the sense of Definition 5.7. If now C'' is such that $C_1 \cup \mathcal{J}_1 \cup C''$ is strongly connected, we can find two points $i^1_0, i^2_0 \in \mathcal{J}_0$ and two points $i^1_1, i^2_1 \in \mathcal{J}_1$ such that $C' \cup \bigcup \mathcal{C}^{\eta,0}_\varepsilon \cup \{i^1_0, i^2_0, i^1_1, i^2_1\} \cup C''$ contains a channel $\mathscr{C}$ in $Q^\nu_\rho \cap \delta\mathbb{Z}^2$. If this is not the case we can iterate the procedure as above to define $\mathcal{C}^{\eta,m}_\varepsilon$, C_{m+1}, and $\mathcal{J}_{m+1}$ where for every $m \geq 1$ we have

$$\mathcal{C}^{\eta,m}_\varepsilon := \{C \in \mathcal{C}^\eta_\varepsilon :\ C_m \cup \mathcal{J}_m \cup C \text{ is strongly connected}\},$$

$$C_{m+1} := C_m \cup \mathcal{J}_m \cup \bigcup \mathcal{C}^{\eta,m}_\varepsilon,$$

and

$$\mathcal{J}_{m+1} := \{i \in Q^\nu_\rho \cap \delta\mathbb{Z}^2 :\ \exists\, j \in C_{m+1} \text{ s.t. } |i-j| = \delta\}.$$

Let now $M \in \mathbb{N}$ be such that $C'' \in \mathcal{C}^{\eta,M}_\varepsilon$. Then there exist $2M$ points $i^1_m, i^2_m \in \mathcal{J}_m$ for $0 \leq m \leq M-1$ such that $C' \cup \{i^1_0, i^2_0, \dots, i^1_{M-1}, i^2_{M-1}\} \cup \bigcup \mathcal{C}^{\eta,M-1}_\varepsilon \cup C''$ contains a channel $\mathscr{C}$ in $Q^\nu_\rho \cap \delta\mathbb{Z}^2$. We now define $\tilde{v}_{\varepsilon,\rho}$ by setting

$$\tilde{v}^i_{\varepsilon,\rho} := \begin{cases} 0 & \text{if } i \in \mathcal{I}^\eta_\varepsilon \text{ or } i = i^l_m \text{ for some } m \in \{0, \dots, M-1\},\ l = 1, 2, \\ v^i_{\varepsilon,\rho} & \text{otherwise in } Q^\nu_\rho \cap \delta\mathbb{Z}^2. \end{cases}$$

Then by definition $\tilde{v}_{\varepsilon,\rho} = 0$ along a channel $\mathscr{C}$ in $Q^\nu_\rho \cap \delta\mathbb{Z}^2$. Moreover, we have

$$\frac{1}{\rho}G_\varepsilon(u_{\varepsilon,\rho}, v_{\varepsilon,\rho}, Q^\nu_\rho) \geq \frac{1}{\rho}G_\varepsilon(u_{\varepsilon,\rho}, \tilde{v}_{\varepsilon,\rho}, Q^\nu_\rho) - c\frac{\eta}{(1-\eta)^2} - cM\frac{\delta}{\rho}, \tag{5.32}$$

for some constant $c = c(\ell) > 0$ independent of ε, ρ and η. Indeed, to prove (5.32) we observe that (5.29) gives

$$\frac{\delta}{2\rho}\sum_{i\in\mathcal{I}^\eta_\varepsilon}\bigg(\ell(v^i_{\varepsilon,\rho}-1)^2+\frac{1}{\ell}\sum_{k=1}^{2}|v^i_{\varepsilon,\rho}-v^{i+\delta e_k}_{\varepsilon,\rho}|^2\bigg)$$
$$\geq\frac{\delta}{2\rho}\sum_{i\in\mathcal{I}^\eta_\varepsilon}\bigg(\ell(\tilde v^i_{\varepsilon,\rho}-1)^2+\frac{1}{\ell}\sum_{k=1}^{2}|\tilde v^i_{\varepsilon,\rho}-\tilde v^{i+\delta e_k}_{\varepsilon,\rho}|^2\bigg)-c\left(\ell+\frac{3}{\ell}\right)\frac{\eta}{(1-\eta)^2}.$$

Then, since $0\leq\tilde v_{\varepsilon,\rho}\leq 1$ we get

$$\frac{1}{\rho}G_\varepsilon(v_{\varepsilon,\rho},Q^\nu_\rho)\geq\frac{1}{\rho}G_\varepsilon(\tilde v_{\varepsilon,\rho},Q^\nu_\rho)-c\left(\ell+\frac{3}{\ell}\right)\frac{\eta}{(1-\eta)^2}-\left(\ell+\frac{2}{\ell}\right)M\frac{\delta}{\rho},$$

and hence (5.32).

Therefore, to conclude the proof of this step it only remains to estimate M in (5.32). To this end, we notice that by construction there exist M pairwise disjoint paths $p_m\subset C_m\cup\mathcal{J}_m$, $0\leq m\leq M-1$ of length l_m (where we have set $C_0:=C'$) connecting $\partial^\delta Q^\nu_\rho\cap\Pi^+_\nu$ and $\partial^\delta Q^\nu_\rho\cap\Pi^-_\nu$ such that (5.30) and (5.31) are satisfied with p_0, l_0 replaced by p_m, l_m. Further, in view of (5.29) there exists $c>0$ such that $l_m\leq c\rho/\delta$, for every $0\leq m\leq M-1$. Then, since the paths p_m are pairwise disjoint, summing up (5.31) over m yields

$$\frac{1}{\rho}F_\varepsilon(u_{\varepsilon,\rho},v_{\varepsilon,\rho},Q^\nu_\rho)\geq\frac{1}{\rho}\sum_{m=0}^{M-1}F_\varepsilon(u_{\varepsilon,\rho},v_{\varepsilon,\rho},p_m)\geq\frac{\eta^2}{2c}\frac{t^2\delta}{\rho^2}M,$$

which in view of (5.27) implies that there exists $c>0$ such that

$$M\leq\frac{c}{\eta^2t^2}\frac{\rho^2}{\delta}. \tag{5.33}$$

Thus, thanks to (5.33) estimate (5.32) becomes

$$\frac{1}{\rho}G_\varepsilon(v_{\varepsilon,\rho},Q^\nu_\rho)\geq\frac{1}{\rho}G_\varepsilon(\tilde v_{\varepsilon,\rho},Q^\nu_\rho)-c\frac{\eta}{(1-\eta)^2}-\frac{c}{\eta^2t^2}\rho.$$

Finally, setting $T(\varepsilon):=\frac{\rho}{\delta}$ and $w^i_{\varepsilon,\rho}:=\tilde v^{\delta i}_{\varepsilon,\rho}$ for all $i\in T(\varepsilon)Q^\nu\cap\mathbb{Z}^2$ from the above inequality we deduce that

$$\begin{aligned}&\frac{1}{\rho}E_\varepsilon(u_{\varepsilon,\rho},v_{\varepsilon,\rho},Q^\nu_\rho)\\&\geq\frac{\delta}{2\rho}\Big(\ell\sum_{i\in Q^\nu_\rho\cap\delta\mathbb{Z}^2}(\tilde v^i_{\varepsilon,\rho}-1)^2+\frac{1}{2\ell}\sum_{\substack{i,j\in Q^\nu_\rho\cap\delta\mathbb{Z}^2\\|i-j|=\delta}}\big|\tilde v^i_{\varepsilon,\rho}-\tilde v^j_{\varepsilon,\rho}\big|^2\Big)-c\frac{\eta}{(1-\eta)^2}-\frac{c}{\eta^2t^2}\rho\\&=\frac{1}{2T(\varepsilon)}\Big(\ell\sum_{i\in T(\varepsilon)Q^\nu\cap\mathbb{Z}^2}(w^i_{\varepsilon,\rho}-1)^2+\frac{1}{2\ell}\sum_{\substack{i,j\in T(\varepsilon)Q^\nu\cap\mathbb{Z}^2\\|i-j|=1}}\big|w^i_{\varepsilon,\rho}-w^j_{\varepsilon,\rho}\big|^2\Big)-c\frac{\eta}{(1-\eta)^2}-\frac{c}{\eta^2t^2}\rho.\end{aligned} \tag{5.34}$$

Then, since $w_{\varepsilon,\rho}$ is a competitor for $\varphi_\ell(\nu)$, gathering (5.28) and (5.34) and passing to the limit as $\varepsilon\to 0$, we deduce

$$\lim_{\varepsilon\to0}\phi_{\varepsilon,\rho}(t,\nu)\geq\varphi_\ell(\nu)-c\frac{\eta}{(1-\eta)^2}-\frac{c}{\eta^2t^2}\rho-\frac{1}{N}.$$

Eventually, letting $\rho \to 0^+$ we get

$$\phi_\ell(t,\nu) \geq \varphi_\ell(\nu) - c\frac{\eta}{(1-\eta)^2} - \frac{1}{N},$$

hence the desired inequality follows by first letting $\eta \to 0$ and then $N \to +\infty$.

Step 3: $\phi_\ell(t,\nu) \leq \varphi_\ell(\nu)$ for every $(t,\nu) \in \mathbb{R} \times S^1$.

Since φ_ℓ is continuous by Step 1 and for every $t \in \mathbb{R}$ the function $\nu \mapsto \phi_\ell(t,\nu)$ is continuous by lower semicontinuity arguments, it suffices to prove the desired inequality for $\nu \in \Xi$. To simplify the exposition we prove this inequality only in the special case $\nu = e_2$.

Let $T \in \mathbb{N}$ and let v_T be a test function for $\varphi_\ell^T(e_2)$ such that

$$\ell \sum_{i\in TQ\cap\mathbb{Z}^2} (v_T^i - 1)^2 + \frac{1}{2\ell} \sum_{\substack{i,j\in TQ\cap\mathbb{Z}^2 \\ |i-j|=1}} |v_T^i - v_T^j|^2 \leq 2T\varphi_\ell^T(e_2) + 1, \tag{5.35}$$

where $\varphi_\ell^T(e_2)$ is as in (5.23). Let now $\mathscr{C}$ be a channel in $TQ \cap \mathbb{Z}^2$ along which $v_T = 0$. Then $\mathscr{C}$ divides TQ into two parts defined as

$$TQ^+ := \{i \in TQ \cap \mathbb{Z}^2 :\ \exists\, l \in \mathbb{N} \text{ such that } i - le_2 \in \mathscr{C}\} \quad \text{and} \quad TQ^- := (TQ \cap \mathbb{Z}^2) \setminus TQ^+.$$

Now we define a discrete function u_T in $TQ \cap \mathbb{Z}^2$ by setting

$$u_T^i := \begin{cases} t & \text{if } i \in TQ^+, \\ 0 & \text{if } i \in TQ^-. \end{cases}$$

We extend the pair (u_T, v_T) to $\mathbb{Z}^2$ by periodicity in direction e_1 and by setting $(u_T^i, v_T^i) := (u_t^{e_2}(i), 1)$ outside the discrete strip $\{x \in \mathbb{R}^2 \colon \langle x, e_2\rangle < T/2\} \cap \mathbb{Z}^2$. For every $\varepsilon, \rho > 0$ set

$$I_{\varepsilon,\rho} := \{z \in \mathbb{Z} \times \{0\} :\ (\delta TQ + \delta Tz) \subset Q_\rho\}$$

We now define a pair of competitors $(u_{\varepsilon,\rho}, v_{\varepsilon,\rho})$ for $\phi_{\varepsilon,\rho}(t, e_2)$ by setting

$$u_{\varepsilon,\rho}^i := \begin{cases} u_T^{\frac{i}{\delta}} & \text{if } i \in \bigcup_{z\in I_{\varepsilon,\rho}} (\delta TQ + \delta Tz), \\ u_t^{e_2}(i) & \text{otherwise in } Q_\rho \cap \delta\mathbb{Z}^2, \end{cases}$$

$$v_{\varepsilon,\rho}^i := \begin{cases} v_T^{\frac{i}{\delta}} & \text{if } i \in \bigcup_{z\in I_{\varepsilon,\rho}} (\delta TQ + \delta Tz), \\ \hat{v}_{e_2,\varepsilon}^i & \text{otherwise in } Q_\rho \cap \delta\mathbb{Z}^2, \end{cases}$$

where $\hat{v}_{e_2,\varepsilon}^i$ is as in (5.15) with ε_j replaced by ε. Then the pair $(u_{\varepsilon,\rho}, v_{\varepsilon,\rho})$ is admissible for $\phi_{\varepsilon,\rho}$ and by construction we have

$$F_\varepsilon(u_{\varepsilon,\rho}, v_{\varepsilon,\rho}, Q_\rho) = 0. \tag{5.36}$$

Moreover, since $\#(I_{\varepsilon,\rho}) \leq \rho/\lfloor \delta T \rfloor$, in view of the periodicity of (u_T, v_T) we get

$$\begin{aligned} \frac{\delta}{\rho} \sum_{i\in Q_\rho\cap\delta\mathbb{Z}^2} (v_{\varepsilon,\rho}^i - 1)^2 &= \#(I_{\varepsilon,\rho}) \frac{\delta}{\rho} \sum_{i\in TQ\cap\mathbb{Z}^2} (v_T^i - 1)^2 + \frac{\delta}{\rho} \sum_{i\in(Q_\rho\cap\delta\mathbb{Z}^2)\setminus\bigcup_{z\in I_{\varepsilon,\rho}}(\delta TQ+\delta Tz)} (v_{\varepsilon,\rho}^i - 1)^2 \\ &\leq \frac{\delta}{\lfloor \delta T \rfloor} \sum_{i\in TQ\cap\mathbb{Z}^2} (v_T^i - 1)^2 + \frac{\delta}{\rho} \#\Big(\Big\{ i \in \Big(Q_\rho \setminus \bigcup_{z\in I_{\varepsilon,\rho}} (\delta TQ + \delta Tz) \Big) \cap S_\delta(e_2) \Big\} \Big) \\ &\leq \frac{\delta}{\lfloor \delta T \rfloor} \sum_{i\in TQ\cap\mathbb{Z}^2} (v_T^i - 1)^2 + c\frac{\delta}{\rho} T, \end{aligned} \tag{5.37}$$

and analogously

$$\frac{\delta}{\rho}\sum_{\substack{i,j\in Q_\rho\cap\delta\mathbb{Z}^2\\|i-j|=\delta}}|v^i_{\varepsilon,\rho}-v^j_{\varepsilon,\rho}|^2\le\frac{\delta}{\lfloor\delta T\rfloor}\sum_{\substack{i,j\in TQ\cap\mathbb{Z}^2\\|i-j|=1}}|v^i_T-v^j_T|^2+c\frac{\delta}{\rho}T. \tag{5.38}$$

Hence gathering (5.37) and (5.38) gives

$$\frac{1}{\rho}G_\varepsilon(v_{\varepsilon,\rho},Q_\rho)\le\frac{\delta}{2\lfloor\delta T\rfloor}\Big(\ell\sum_{i\in TQ\cap\mathbb{Z}^2}(v^i_T-1)^2+\frac{1}{2\ell}\sum_{\substack{i,j\in TQ\cap\mathbb{Z}^2\\|i-j|=1}}|v^i_T-v^j_T|^2\Big)+c\frac{\delta}{\rho}T.$$

Combining the latter estimate with (5.36) and (5.35) implies

$$\begin{aligned}\phi_{\varepsilon,\rho}(t,e_2)&\le\frac{1}{\rho}E_\varepsilon(u_{\varepsilon,\rho},v_{\varepsilon,\rho},Q_\rho)\\&\le\frac{\delta}{2\lfloor\delta T\rfloor}\Big(\ell\sum_{i\in TQ\cap\mathbb{Z}^2}(v^i_T-1)^2+\frac{1}{2\ell}\sum_{\substack{i,j\in TQ\cap\mathbb{Z}^2\\|i-j|=1}}|v^i_T-v^j_T|^2\Big)+c\frac{\delta}{\rho}T\\&\le\frac{\delta T}{\lfloor\delta T\rfloor}\Big(\varphi^T_\ell(e_2)+\frac{1}{2T}\Big)+c\frac{\delta}{\rho}T.\end{aligned}$$

Therefore letting first ε and then ρ go to zero we get

$$\phi_\ell(t,e_2)\le\varphi^T_\ell(e_2)+\frac{1}{2T},$$

thus finally the desired inequality follows by letting $T\to+\infty$. □

6. Proof of the Γ-convergence result in the supercritical regime $\ell=+\infty$

In this section we study the asymptotic behaviour of the functionals E_ε when ε is much smaller than δ.

We start recalling that Proposition 3.4-(ii) gives that in this case the domain of the Γ-limit is $W^{1,2}(\Omega)\times\{1\}$ and that for every $u\in W^{1,2}(\Omega)$ and for every $(u_\varepsilon,v_\varepsilon)\to(u,1)$ in $L^1(\Omega)\times L^1(\Omega)$ we have

$$\liminf_{\varepsilon\to0}E_\varepsilon(u_\varepsilon,v_\varepsilon)\ge\int_\Omega|\nabla u|^2\,dx. \tag{6.1}$$

On the other hand, the upper-bound inequality is also straightforward. Indeed if $u\in C^\infty(\overline{\Omega})$, then a recovery sequence is simply given by

$$u^i_\varepsilon:=u(i),\quad v^i_\varepsilon:=1\quad\text{for every}\quad i\in\Omega_\delta.$$

In fact, $u_\varepsilon\to u$ in $L^1(\Omega)$ and $G_\varepsilon(v_\varepsilon)=0$, while Jensen's inequality together with the mean-value theorem gives

$$\limsup_{\varepsilon\to0}E_\varepsilon(u_\varepsilon,v_\varepsilon)=\limsup_{\varepsilon\to0}F_\varepsilon(u_\varepsilon,v_\varepsilon)\le\int_\Omega|\nabla u|^2dx \tag{6.2}$$

as in the proof of (4.26) in Proposition 4.2. Then, the general case $u\in W^{1,2}(\Omega)$ follows by a standard density argument.

Therefore gathering (6.1) and (6.2) proves that for $\ell=+\infty$ the functionals E_ε Γ-converge to

$$E_\infty(u,v)=\begin{cases}\displaystyle\int_\Omega|\nabla u|^2dx & \text{if } u\in W^{1,2}(\Omega),\ v=1\text{ a.e. in }\Omega,\\+\infty & \text{otherwise in } L^1(\Omega)\times L^1(\Omega).\end{cases}$$

The Γ-convergence result as above implies, in particular, that if the pair $(u_\varepsilon, v_\varepsilon)$ converges in $L^1(\Omega)\times L^1(\Omega)$ to a pair $(u,1)$ for some $u\in GSBV^2(\Omega)\setminus W^{1,2}(\Omega)$ then $E_\varepsilon(u_\varepsilon,v_\varepsilon)\to+\infty$. Then, in the spirit of [31], the purpose of the following subsection is to study the asymptotic behaviour of a suitable scaling of E_ε leading to a limit functional which is finite on the whole $GSBV^2(\Omega)$.

6.1. **Formulation of an equivalent energy.** We start noticing that the analysis performed in the scaling regimes $\ell=0$ and $\ell\in(0,+\infty)$ suggests that the development of a discontinuity for u is penalised by a factor proportional to δ/ε, which is divergent in this supercritical regime. This observation leads us to consider the following functionals:

$$
\begin{aligned}
H_\varepsilon(w,v) &:= \frac{1}{2}\sum_{i\in\Omega_\delta}\delta^n(v^i)^2\sum_{\substack{k=1\\ i\pm\delta e_k\in\Omega_\delta}}^{n}\left|\frac{w^i-w^{i\pm\delta e_k}}{\delta}\right|^2\\
&\quad+\frac{1}{2}\bigg(\sum_{i\in\Omega_\delta}\delta^{n-1}(v^i-1)^2+\sum_{i\in\Omega_\delta}\sum_{\substack{k=1\\ i+\delta e_k\in\Omega_\delta}}^{n}\delta^{n-1}\frac{\varepsilon^2}{\delta^2}\left|v^i-v^{i+\delta e_k}\right|^2\bigg).
\end{aligned}
$$

We notice that H_ε is obtained by E_ε by scaling at the same time the energy and the variable u; in fact we have

$$
H_\varepsilon(w,v)=\frac{\varepsilon}{\delta}E_\varepsilon(\sqrt{\delta/\varepsilon}\,w,v).
$$

With the following theorem we establish a Γ-convergence result for the scaled functionals H_ε.

Theorem 6.1. *Let $\ell=+\infty$ and let $H_\varepsilon\colon L^1(\Omega)\times L^1(\Omega)\longrightarrow[0,+\infty]$ be defined as*

$$
H_\varepsilon(w,v):=\begin{cases}\dfrac{\varepsilon}{\delta}\,E_\varepsilon\Big(\sqrt{\tfrac{\delta}{\varepsilon}}\,w,v\Big) & \text{if } w,v\in\mathcal{A}_\varepsilon(\Omega),\\ +\infty & \text{otherwise in } L^1(\Omega)\times L^1(\Omega).\end{cases}
$$

Then the functionals H_ε Γ-converge to $H\colon L^1(\Omega)\times L^1(\Omega)\longrightarrow[0,+\infty]$ defined as

$$
H(w,v):=\begin{cases}\displaystyle\int_\Omega|\nabla w|^2dx+\int_{S_w\cap\Omega}|\nu_w|_\infty d\mathcal{H}^{n-1} & \text{if } w\in GSBV^2(\Omega),\ v=1 \text{ a.e. in } \Omega,\\ +\infty & \text{otherwise in } L^1(\Omega)\times L^1(\Omega).\end{cases}\tag{6.3}
$$

Proof. The proof is divided into two main steps.

Step 1: liminf inequality.

The liminf inequality is proven in two substeps first considering the case $n=1$ and then the case $n\geq 2$.

Substep 1.1: the case $n=1$.

Let $\Omega=I:=(a,b)$ be an open bounded interval. We start showing that in the one-dimensional case the domain of the Γ-limit is $SBV^2(I)\times\{1\}$. To this end let $(w,v)\in L^1(I)\times L^1(I)$ and let $(w_\varepsilon,v_\varepsilon)\subset L^1(I)\times L^1(I)$ be a sequence such that $(w_\varepsilon,v_\varepsilon)\to(w,v)$ strongly in $L^1(I)\times L^1(I)$ and satisfying

$$
\sup_\varepsilon H_\varepsilon(w_\varepsilon,v_\varepsilon)<+\infty.\tag{6.4}
$$

Let $\tilde v_\varepsilon$ and $\tilde w_\varepsilon$ be the piecewise-affine interpolations of v_ε, w_ε on $I_\delta:=I\cap\delta\mathbb{Z}$, respectively. From (6.4) we deduce that $\tilde v_\varepsilon\to 1$ in $L^2(I)$; hence $v=1$ a.e. in I.

We now prove that $w\in SBV^2(I)$. To this end we consider the discrete set

$$
J_\varepsilon:=\{i\in I_\delta:\ v_\varepsilon^i<1/2\}.
$$

Again appealing to (6.4) we deduce that

$$c \geq \sum_{i\in J_\varepsilon}(v_\varepsilon - 1)^2 > \frac{1}{4}\#(J_\varepsilon).$$

for some $c>0$, uniformly in ε. Thus, we deduce that there exists $N\in\mathbb{N}$ such that

$$\#(J_\varepsilon)\leq N \quad \text{for every } \varepsilon>0.$$

Without loss of generality we then write

$$J_\varepsilon = \{i_1(\varepsilon),\dots,i_N(\varepsilon)\}, \quad a< i_1(\varepsilon)\leq i_2(\varepsilon)\leq\ldots\leq i_N(\varepsilon)<b,$$

where N is independent of ε. Then for every $1\leq j\leq N$ the sequence $i_j(\varepsilon)$ is bounded and thus there exists $t\in[a,b]$ such that (up to subsequences) $i_j(\varepsilon)\to t$. Let

$$J := \{t\in[a,b]:\ \exists\, j\in\{1,\dots,N\} \text{ s.t. } t=\lim_{\varepsilon\to 0} i_j(\varepsilon)\}$$

denote the set of these limit points. Then we may write

$$J=\{t_1,\dots,t_M\}, \quad a\leq t_1<t_2<\ldots<t_M\leq b, \quad M\leq N.$$

We set $d_0 := \min\{t_{l+1}-t_l:\ 1\leq l\leq M-1\}$. Let $\eta\in(0,d_0)$ and $j\in\{1,\dots,N\}$ be arbitrary. By definition of J there exist $l\in\{1,\dots,M\}$ and $\varepsilon(j)>0$ such that

$$i_j(\varepsilon)\in(t_l-\eta/2, t_l+\eta/2) \quad \text{for every } \varepsilon\in(0,\varepsilon(j)).$$

By the arbitrariness of $j\in\{1,\dots,N\}$, setting $\varepsilon_0 := \min_{1\leq j\leq N}\varepsilon(j)$ we thus deduce that

$$v_\varepsilon^i \geq \frac{1}{2} \quad \forall\, i\in I_\delta\setminus(J+[-\eta/2,\eta/2]) \quad \text{for every } \varepsilon\in(0,\varepsilon_0).$$

Hence, for $\varepsilon\leq\varepsilon_0$ we get

$$\sum_{i\in I_\delta\setminus(J+[-\eta/2,\eta/2])}\delta(v_\varepsilon^i)^2\left|\frac{w_\varepsilon^i-w_\varepsilon^{i\pm\delta}}{\delta}\right|^2 \geq \frac{1}{4}\sum_{i\in I_\delta\setminus(J+[-\eta/2,\eta/2])}\delta\left|\frac{w_\varepsilon^i-w_\varepsilon^{i\pm\delta}}{\delta}\right|^2 \geq \frac{1}{2}\int_{(a+\eta,b-\eta)\setminus(J+[-\eta,\eta])}(\tilde{w}_\varepsilon')^2dt,$$

thus in view of (6.4) we deduce that the $L^2((a+\eta,b-\eta)\setminus(J+[-\eta,\eta]))$-norm of $\tilde{w}_\varepsilon'$ is equibounded. Therefore, since $w_\varepsilon\to w$ in $L^1(I)$ the Poincaré-Wirtinger inequality implies that $\tilde{w}_\varepsilon$ is equibounded in $W^{1,2}((a+\eta,b-\eta))\setminus(J+[-\eta,\eta]))$ and $\tilde{w}_\varepsilon\rightharpoonup w$ weakly in $W^{1,2}((a+\eta,b-\eta))\setminus(J+[-\eta,\eta])$. Moreover, since $\tilde{v}_\varepsilon\to 1$ in $L^2(I)$, we have $\tilde{v}_\varepsilon\tilde{w}_\varepsilon'\rightharpoonup w'$ weakly in $L^1((a+\eta,b-\eta)\setminus(J+[-\eta,\eta]))$. Hence, using estimate (3.3) in the proof of Proposition 3.2 entails

$$\begin{aligned}\liminf_{\varepsilon\to 0}\frac{1}{2}\sum_{i\in \overset{\circ}{I}_\delta}\delta(v_\varepsilon^i)^2\left|\frac{w_\varepsilon^i-w_\varepsilon^{i\pm\delta}}{\delta}\right|^2 &\geq \liminf_{\varepsilon\to 0}\int_{(a+\eta,b-\eta)\setminus(J+[-\eta,\eta])}(\tilde{v}_\varepsilon)^2(\tilde{w}_\varepsilon')^2dt\\ &\geq \int_{(a+\eta,b-\eta)\setminus(J+[-\eta,\eta])}(w')^2dt. \end{aligned}\tag{6.5}$$

Then, by the arbitrariness of $\eta\in(0,d_0)$ we deduce both that $w\in SBV^2(I)$ and $S_w\cap I\subset J$. We therefore set

$$S_w\cap I := \{t_1,\dots,t_L\}, \quad L\leq M.$$

It only remains to prove that

$$\liminf_{\varepsilon\to 0}\frac{1}{2}\left(\sum_{i\in I_\delta}(v_\varepsilon^i-1)^2+\sum_{i\in \overset{\circ}{I}_\delta}\frac{\varepsilon^2}{\delta^2}|v_\varepsilon^i-v_\varepsilon^{i+\delta}|^2\right)\geq L. \tag{6.6}$$

To this end, we show that the number of lattice points $i \in I_\delta$ such that $v^i_\varepsilon \to 0$ is at least $2L$. Let $1 \le l \le L$, set $I'_l := (t_l - \eta/2, t_l + \eta/2)$ and define

$$m_l := \liminf_{\varepsilon\to 0} \inf\left\{ \frac{(v^{i+\delta}_\varepsilon)^2 + (v^i_\varepsilon)^2}{2} \,:\, i \in (I'_l + (-\delta,\delta)) \cap \delta\mathbb{Z} \right\}.$$

We claim that $m_l = 0$. We assume by contradiction that $m_l > 0$. Then there exists a subsequence ε_j such that for j sufficiently large we have

$$\frac{1}{m_l}\frac{(v^{i+\delta_j}_{\varepsilon_j})^2 + (v^i_{\varepsilon_j})^2}{2} \ge 1 \quad \text{for every } i \in (I'_l + (-\delta_j,\delta_j)) \cap \delta_j\mathbb{Z}.$$

Therefore, we get

$$\begin{aligned}\int_{I'_l} (\tilde w'_{\varepsilon_j})^2 dt &\le \sum_{i\in(I'_l+(-\delta_j,\delta_j))\cap\delta_j\mathbb{Z}} \delta_j \left| \frac{w^i_{\varepsilon_j} - w^{i+\delta_j}_{\varepsilon_j}}{\delta_j}\right|^2 \\ &\le \frac{1}{m_l} \sum_{i\in(I'_l+(-\delta_j,\delta_j))\cap\delta_j\mathbb{Z}} \delta_j \frac{(v^{i+\delta_j}_{\varepsilon_j})^2 + (v^i_{\varepsilon_j})^2}{2} \left| \frac{w^i_{\varepsilon_j} - w^{i+\delta_j}_{\varepsilon_j}}{\delta_j}\right|^2 \le c\end{aligned}$$

uniformly in j. Thus, since $(\tilde w_{\varepsilon_j}) \subset W^{1,2}(I)$ and $\tilde w_{\varepsilon_j} \to w$ in $L^1(I)$ as $j \to +\infty$ we would deduce that $\tilde w_{\varepsilon_j} \rightharpoonup w$ in $W^{1,2}(I'_l)$ and hence $w \in W^{1,2}(I'_l)$, which contradicts the fact $t_l \in I'_l$. Hence, we may deduce that $m_l = 0$. Consequently we can find a sequence of lattice points $i_l(\varepsilon) \in (I'_l + (-\delta,\delta)) \cap \delta\mathbb{Z}$ such that

$$\frac{(v^{i_l(\varepsilon)}_\varepsilon)^2 + (v^{i_l(\varepsilon)+\delta}_\varepsilon)^2}{2} \to 0 \quad \text{as } \varepsilon \to 0,$$

the latter implies

$$v^{i_l(\varepsilon)}_\varepsilon, v^{i_l(\varepsilon)+\delta}_\varepsilon \to 0 \quad \text{as } \varepsilon \to 0,$$

which in its turn gives

$$\liminf_{\varepsilon\to 0} \sum_{i\in I_\delta\cap(t_l-\eta,t_l+\eta)} (v^i_\varepsilon - 1)^2 \ge 2.$$

Thus, since $\eta < d_0$ we get

$$\liminf_{\varepsilon\to 0} \sum_{i\in I_\delta} (v^i_\varepsilon - 1)^2 \ge \sum_{l=1}^{L} \liminf_{\varepsilon\to 0} \sum_{i\in I_\delta\cap(t_l-\eta,t_l+\eta)} (v^i_\varepsilon - 1)^2 \ge 2L,$$

from which we deduce (6.6). Gathering (6.5) and (6.6) finally yields the liminf-inequality by the arbitrariness of $\eta > 0$.

Substep 1.2: the case $n \ge 2$.

In this case the proof of the liminf inequality directly follows from the previous substep arguing as in the proof of Proposition 3.4.

Step 2: limsup inequality.

It is enough to show that

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} H_\varepsilon(w,1) \le H(w,1) \quad \text{for every } w \in \mathcal{W}(\Omega), \tag{6.7}$$

where $\mathcal{W}(\Omega)$ is the space of functions introduced in the proof of Proposition 4.2. Indeed, if (6.7) holds than [42, Theorem 3.1, Remark 3.2 and Remark 3.3] allow us to apply a standard density and lower-semicontinuity argument to deduce that

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} H_\varepsilon(w,1) \le H(w,1) \quad \text{for every } w \in SBV^2(\Omega)\cap L^\infty(\Omega).$$

Finally, the case $w \in GSBV^2(\Omega)$ follows by a standard truncation argument.

Therefore we now turn to the proof of (6.7). Let $w \in \mathcal{W}(\Omega)$. To simplify the argument we only discuss the case $\overline{S_w} = K \cap \Omega$, where K is a closed and convex set contained in Π_ν, with $\nu := (\nu_1,\dots,\nu_n) \in S^{n-1}$; then the general case follows as in the proof of Proposition 4.2, Step 2. For every $x, y \in \mathbb{R}^n$ let $\mathcal{S}_{(x,y)}$ denote the open segment joining x and y; moreover, for every $h \ge 0$ set $K_h := \{x \in \Pi_\nu :\ \mathrm{dist}(x,K) \le h\}$. Without loss of generality we suppose that $|\nu|_\infty = |\nu_n| = \nu_n$. Further, upon considering a shifted lattice $\delta\mathbb{Z}^n + \xi_\varepsilon$ for a suitable sequence $(\xi_\varepsilon) \subset \mathbb{R}^n$ converging to 0 as $\varepsilon \to 0$ we may assume that

$$\overline{S_w} \cap \delta\mathbb{Z}^n = \emptyset \quad \text{for every } \varepsilon > 0$$

and we define

$$w_\varepsilon^i := w(i) \qquad \text{for every } \ i \in \Omega_\delta.$$

Finally, we set

$$v_\varepsilon^i := \begin{cases} 0 & \text{if } \mathcal{S}_{(i-\delta e_n, i+\delta e_n)} \cap K_{\sqrt{n}\delta} \neq \emptyset, \\ 1 & \text{otherwise in } \Omega_\delta. \end{cases}$$

We clearly have $(w_\varepsilon, v_\varepsilon) \to (w,1)$ in $L^1(\Omega)\times L^1(\Omega)$. Moreover, we claim that there holds

$$\limsup_{\varepsilon\to 0} \frac{1}{2}\bigg(\sum_{i\in\Omega_\delta} \delta^{n-1}(v_\varepsilon^i - 1)^2 + \sum_{i\in\Omega_\delta} \sum_{\substack{k=1\\ i+\delta e_k\in\Omega_\delta}}^{n} \delta^{n-1}\frac{\varepsilon^2}{\delta^2}|v_\varepsilon^i - v_\varepsilon^{i+\delta e_k}|^2\bigg) \le \mathcal{H}^{n-1}(S_w\cap\Omega)|\nu|_\infty. \tag{6.8}$$

Indeed, since $\#\{i \in \Omega_\delta :\ v_\varepsilon^i = 0\} = \mathcal{O}\left(\frac{1}{\delta^{n-1}}\right)$, we get

$$\sum_{i\in\Omega_\delta} \sum_{\substack{k=1\\ i+\delta e_k\in\Omega_\delta}}^{n} \delta^{n-1}\frac{\varepsilon^2}{\delta^2}|v_\varepsilon^i - v_\varepsilon^{i+\delta e_k}|^2 \le n\,\#\{i\in\Omega_\delta :\ v_\varepsilon^i = 0\}\delta^{n-1}\frac{\varepsilon^2}{\delta^2} \to 0 \quad \text{as } \varepsilon \to 0,$$

while the fact that

$$\#\{i \in \Omega_\delta :\ \mathcal{S}_{(i-\delta e_n, i+\delta e_n)} \cap K_{\sqrt{n}\delta} \neq \emptyset\} = 2\left\lfloor \frac{\mathcal{H}^{n-1}(\Omega\cap K_{\sqrt{n}\delta})\,\langle \nu, e_n\rangle}{\delta^{n-1}} \right\rfloor$$

yields

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \frac{1}{2}\sum_{i\in\Omega_\delta} \delta^{n-1}(v_\varepsilon^i - 1)^2 &= \limsup_{\varepsilon\to 0} \frac{\delta^{n-1}}{2}\#\{i\in\Omega_\delta :\ \mathcal{S}_{(i-\delta e_n, i+\delta e_n)} \cap K_{\sqrt{n}\delta} \neq \emptyset\} \\
&= \limsup_{\varepsilon\to 0} \delta^{n-1}\left\lfloor \frac{\mathcal{H}^{n-1}(\Omega\cap K_{\sqrt{n}\delta})|\nu_n|}{\delta^{n-1}} \right\rfloor \\
&= \mathcal{H}^{n-1}(S_w\cap\Omega)|\nu_n| = \mathcal{H}^{n-1}(S_w\cap\Omega)|\nu|_\infty.
\end{aligned}$$

Then, it remains to show that

$$\limsup_{\varepsilon\to 0} \frac{1}{2}\sum_{i\in\Omega_\delta}(v_\varepsilon^i)^2 \sum_{\substack{k=1\\ i\pm\delta e_k}}^{n} \delta^n \left|\frac{w_\varepsilon^i - w_\varepsilon^{i\pm\delta e_k}}{\delta}\right|^2 \le \int_\Omega |\nabla w|^2 dx. \tag{6.9}$$

To do so, we first notice that for $i \in \Omega_\delta$ and $k \in \{1, \dots, n\}$ with $\mathcal{S}_{(i,i+\delta e_k)} \cap \overline{S_w} = \emptyset$ by Jensen's inequality we have

$$\left| \frac{w_\varepsilon^i - w_\varepsilon^{i+\delta e_k}}{\delta} \right|^2 \leq \frac{1}{\delta} \int_0^\delta |\langle \nabla w(i + te_k), e_k \rangle|^2 dt \quad \forall\, 1 \leq k \leq n.$$

Hence, thanks to the mean-value theorem, using Fubini's Theorem we get

$$\sum_{i \in \Omega_\delta} \delta^n \sum_{\substack{k=1 \\ i+\delta e_k \in \Omega_\delta \\ \mathcal{S}_{(i,i+\delta e_k)} \cap \overline{S_w} = \emptyset}}^n \frac{(v_\varepsilon^i)^2 + (v_\varepsilon^{i+\delta e_k})^2}{2} \left| \frac{w_\varepsilon^i - w_\varepsilon^{i+\delta e_k}}{\delta} \right|^2 \leq \int_\Omega |\nabla w(x)|^2 dx + \mathcal{O}(\delta). \tag{6.10}$$

We now claim that

$$\sum_{i \in \Omega_\delta} \delta^n \sum_{\substack{k=1 \\ i+\delta e_k \in \Omega_\delta \\ \mathcal{S}_{(i,i+\delta e_k)} \cap \overline{S_w} \neq \emptyset}}^n \frac{(v_\varepsilon^i)^2 + (v_\varepsilon^{i+\delta e_k})^2}{2} \left| \frac{w_\varepsilon^i - w_\varepsilon^{i+\delta e_k}}{\delta} \right|^2 = 0, \tag{6.11}$$

so that combining (6.10) and (6.11) entails (6.9).

To prove the claim let $i \in \mathring{\Omega}_\delta$ and $1 \leq k \leq n$ be such that $\mathcal{S}_{(i,i+\delta e_k)} \cap \overline{S}_w \neq \emptyset$. This implies $\langle \nu, e_k \rangle = \nu_k \neq 0$. Without loss of generality we assume that $\nu_k > 0$. We now show that

$$v_\varepsilon^i = v_\varepsilon^{i+\delta e_k} = 0. \tag{6.12}$$

If $k = n$ then (6.12) follows directly from the definition of the sequence (v_ε). We now prove (6.12) when $k \in \{1, \dots, n-1\}$. We have to show that

$$\mathcal{S}_{(i-\delta e_n, i+\delta e_n)} \cap K_{\sqrt{n}\delta} \neq \emptyset \quad \text{and} \quad \mathcal{S}_{(i+\delta e_k - \delta e_n, i+\delta e_k + \delta e_n)} \cap K_{\sqrt{n}\delta} \neq \emptyset.$$

To do so, we choose $t \in (0, \delta)$ such that $i + te_k \in \overline{S_w}$. Then

$$i + t \frac{\sin \alpha_k}{\sin \alpha_n} e_n \in K_{\sqrt{n}\delta},$$

where α_k and α_n denote the angle between e_k and Π_ν and between e_n and Π_ν, respectively; we note that

$$\sin \alpha_k = \langle \nu, e_k \rangle = \nu_k \quad \text{and} \quad \sin \alpha_n = \langle \nu, e_n \rangle = \nu_n.$$

Thus, since $\nu_n = |\nu|_\infty \geq \nu_k > 0$ we deduce that $t \frac{\sin \alpha_k}{\sin \alpha_n} \in (0, \delta)$ and hence

$$i + t \frac{\sin \alpha_k}{\sin \alpha_n} e_n \in \mathcal{S}_{(i, i+\delta e_n)},$$

which yields $\mathcal{S}_{(i,i+\delta e_n)} \cap K_{\sqrt{n}\delta} \neq \emptyset$ and then $v_\varepsilon^i = 0$ by definition. Moreover, we have

$$(i + \delta e_k) + (t - \delta) \frac{\sin \alpha_k}{\sin \alpha_n} e_n \in K_{\sqrt{n}\delta}.$$

Since $(t - \delta) \frac{\sin \alpha_k}{\sin \alpha_n} \in (-\delta, 0)$, we get

$$(i + \delta) e_k + (t - \delta) \frac{\sin \alpha_k}{\sin \alpha_n} e_n \in \mathcal{S}_{(i+\delta e_k - \delta e_n, i+\delta e_k)},$$

from which we deduce that $\mathcal{S}_{(i+\delta e_k - \delta e_n, i+\delta e_k)} \cap K_{\sqrt{n}\delta} \neq \emptyset$ and thus $v_\varepsilon^{i+\delta e_k} = 0$. The latter in its turn implies (6.11) and eventually (6.9). Thus gathering (6.8) and (6.9) gives the limsup inequality. □

Remark 6.2 (a Γ-expansion of E_ε)**.** *For every fixed $w \in GSBV^2(\Omega)$ set $u := \sqrt{\frac{\delta}{\varepsilon}}\, w$, then using the fact that $S_u = S_w$ and $\nu_u = \nu_w$ we get*

$$\frac{\delta}{\varepsilon} H(w,1) = \int_\Omega |\nabla u|^2 dx + \frac{\delta}{\varepsilon} \int_{S_u \cap \Omega} |\nu_u|_\infty d\mathcal{H}^{n-1}$$

which for $\ell = +\infty$ is Γ-equivalent to $E_\varepsilon(u,v)$ in the sense of [31]*.*

7. Interpolation properties of φ_ℓ

In this last section we show that when $n = 2$ the surface energy density φ_ℓ satisfies the following interpolation properties.

Proposition 7.1. *Let $\ell \in (0,+\infty)$ and for $\nu \in S^1$ let $\varphi_\ell(\nu)$ be as in* (5.22)*. Then, we have*

$$\lim_{\ell \to +\infty} \varphi_\ell(\nu) = +\infty \quad \textit{and} \quad \lim_{\ell \to 0} \varphi_\ell(\nu) = 1,$$

for every $\nu \in S^1$. Moreover, there holds

$$\lim_{\ell \to +\infty} \frac{\varphi_\ell(\nu)}{\ell} = |\nu|_\infty,$$

for every $\nu \in S^1$.

Proof. Let $\nu \in S^1$ and for every $T > 0$ let φ_ℓ^T be the auxiliary function as in (5.23). We start showing that $\varphi_\ell(\nu) \to +\infty$ as $\ell \to +\infty$. Indeed, for every $T > 0$ and for every function v_T that is admissible for $\varphi_\ell^T(\nu)$ there exists a channel $\mathscr{C}$ in $TQ^\nu \cap \mathbb{Z}^2$ along which $v_T = 0$. Then it suffices to notice that $\#(\mathscr{C}) \geq T$ to deduce that for every $T > 0$ we have

$$\varphi_\ell^T(\nu) \geq \frac{\ell}{2T} \sum_{i \in TQ^\nu \cap \mathbb{Z}^2} (v_T^i - 1)^2 \geq \frac{\ell}{2}.$$

Hence passing to the limit as $T \to +\infty$ we get $\varphi_\ell(\nu) \geq \ell/2$ and thus the claim.

We now prove that $\varphi_\ell(\nu) \to 1$ as $\ell \to 0$. To this end we first show that

$$\liminf_{\ell \to 0} \varphi_\ell(\nu) \geq 1. \tag{7.1}$$

Let $\nu \in S^1$, $T > 0$, and let v_T be an arbitrary test function for $\varphi_\ell^T(\nu)$. Then in particular $v_T = 1$ in a neighbourhood of the two opposite sides of Q_T^ν perpendicular to ν. For our purposes it is convenient to extend v_T to 1 to the discrete stripe $S_T^\nu \cap \mathbb{Z}^2$, where

$$S_T^\nu := \{x \in \mathbb{R}^2 \colon -T/2 \leq \langle x, \nu^\perp \rangle \leq T/2\}.$$

With a little abuse of notation v_T still denotes such an extension. Moreover, we recall that by definition of v_T there exists a channel $\mathscr{C}$ in $TQ^\nu \cap \mathbb{Z}^2$ such that $v_T = 0$ on $\mathscr{C}$. Since by definition $\mathscr{C}$ is a strong path (see Figure 1b), we can find a triangulation $\mathcal{T}$ of S_T^ν with vertices in $\mathbb{Z}^2$ such that if a triangle $\tau \in \mathcal{T}$ has one vertex in $\mathscr{C}$ then all its vertices belong to $\mathscr{C}$. Let $\tilde{v}_T$ denote the piecewise-affine interpolation of v_T on the triangulation $\mathcal{T}$. We have

$$
\begin{aligned}
&\frac{1}{2T}\Bigg(\ell \sum_{i\in TQ^\nu\cap\mathbb{Z}^2}(v_T^i-1)^2+\frac{1}{2\ell}\sum_{\substack{i,j\in TQ^\nu\cap\mathbb{Z}^2\\|i-j|=1}}|v_T^i-v_T^j|^2\Bigg)\\
=&\frac{1}{2T}\Bigg(\ell \sum_{i\in S_T^\nu\cap\mathbb{Z}^2}(v_T^i-1)^2+\frac{1}{2\ell}\sum_{\substack{i,j\in S_T^\nu\cap\mathbb{Z}^2\\|i-j|=1}}|v_T^i-v_T^j|^2\Bigg)\\
&\geq \frac{1}{2T}\int_{S^\nu_{T-\sqrt{2}}}\ell\,(\tilde v_T-1)^2+\frac{1}{\ell}|\nabla\tilde v_T|^2dx\\
&=\frac{1}{2T}\int_{\Pi_\nu\cap Q^\nu_{T-\sqrt{2}}}\Bigg(\int_{-(T+\sqrt{2})/2}^{(T+\sqrt{2})/2}\ell\,(\tilde v_T^{y,\nu}(t)-1)^2+\frac{1}{\ell}((\tilde v_T^{y,\nu})'(t))^2dt\Bigg)\,d\mathcal{H}^1(y),
\end{aligned}
\tag{7.2}
$$

where $\tilde v_T^{y,\nu}(t):=\tilde v_T(y+t\nu)$, for every $t\in(-(T+\sqrt{2})/2,(T+\sqrt{2})/2)$. Thus, by definition of v_T for every $y\in\Pi_\nu\cap Q^\nu_{T-\sqrt{2}}$ we have $\tilde v_T^{y,\nu}((T+\sqrt{2})/2)=\tilde v_T^{y,\nu}(-(T+\sqrt{2})/2)=1$. Moreover, thanks to the choice of our triangulation $\mathcal{T}$, for every $y\in\Pi_\nu\cap Q^\nu_{T-\sqrt{2}}$ there exists $t_y\in(-(T+\sqrt{2})/2,(T+\sqrt{2})/2)$ such that $\tilde v_T^{y,\nu}(t_y)=0$. Thus, for every $y\in\Pi_\nu\cap Q^\nu_{T-\sqrt{2}}$ we get

$$
\begin{aligned}
&\frac{1}{2}\int_{-(T+\sqrt{2})/2}^{(T+\sqrt{2})/2}\ell(\tilde v_T^{y,\nu}(t)-1)^2+\frac{1}{\ell}((\tilde v_T^{y,\nu})'(t))^2dt\\
&\geq\int_{t_y}^{(T+\sqrt{2})/2}(1-\tilde v_T^{y,\nu}(t))|(\tilde v_T^{y,\nu})'(t)|dt+\int_{-(T+\sqrt{2})/2}^{t_y}(1-\tilde v_T^{y,\nu}(t))|(\tilde v_T^{y,\nu})'(t)|dt\\
&=2\int_0^1(1-z)dz=1.
\end{aligned}
$$

Therefore in view of (7.2) we deduce

$$
\frac{1}{2T}\Bigg(\ell \sum_{i\in TQ^\nu\cap\mathbb{Z}^2}(v_T^i-1)^2+\frac{1}{2\ell}\sum_{\substack{i,j\in TQ^\nu\cap\mathbb{Z}^2\\|i-j|=1}}|v_T^i-v_T^j|^2\Bigg)\geq\frac{T-\sqrt{2}}{T},
$$

hence, by the arbitrariness of v_T we get $\varphi_\ell^T(\nu)\geq\frac{T-\sqrt{2}}{T}$ for every $T>0$ and every $\ell>0$. Then, letting $T\to+\infty$ gives $\varphi_\ell(\nu)\geq1$ for all $\ell>0$ and thus (7.1).

Now it remains to show that

$$
\limsup_{\ell\to0}\varphi_\ell(\nu)\leq1. \tag{7.3}
$$

To prove the above upper-bound inequality (7.3) we construct a suitable test function v_T for $\varphi_\ell(\nu)$. To this end let $\eta>0$ be fixed; let $T_\eta>0$ and $f\in C^2([0,T_\eta])$ be such that $f(0)=0$, $f(T_\eta)=1$, $f'(T_\eta)=f''(T_\eta)=0$, and

$$
\int_0^{T_\eta}(f-1)^2+(f')^2dt\leq1+\eta.
$$

Clearly, up to setting $f(t) = 1$ for every $t \geq T_\eta$ we can always assume that $f \in C^2([0,+\infty))$. Let $T > 0$ and set $T' := T - \sqrt{2}$. Denoting by $d(x)$ the distance of x from Π_ν we set

$$v_T(x) := \begin{cases} 0 & \text{if } d(x) \leq \sqrt{2}, \\ f(\ell(d(x) - \sqrt{2})) & \text{if } x \in T'Q^\nu,\ d(x) > \sqrt{2}, \\ 1 & \text{otherwise,} \end{cases}$$

which is well-defined for $T > \frac{T_\eta}{\ell} + 2\sqrt{2}$. We now define the sets

$$\begin{aligned} A &:= \{i \in TQ^\nu \cap \mathbb{Z}^2 :\ d(i) \leq \sqrt{2}\}, \\ B &:= \{i \in T'Q^\nu \cap \mathbb{Z}^2 :\ \sqrt{2} < d(i) < T_\eta/\ell + 2\sqrt{2}\}, \\ C &:= \{i \in (TQ^\nu \setminus T'Q^\nu) \cap \mathbb{Z}^2 : d(i) > \sqrt{2}\}. \end{aligned}$$

We notice that the set A contains a channel $\mathscr{C}$ along which $v_T = 0$. In particular, v_T is admissible for $\varphi_\ell^T(\nu)$. Thus we obtain

$$\begin{aligned} \varphi_\ell^T(\nu) &\leq \frac{1}{2T}\Bigg(\ell \sum_{i \in TQ^\nu \cap \mathbb{Z}^2} (v_T^i - 1)^2 + \frac{1}{2\ell} \sum_{\substack{i,j \in TQ^\nu \cap \mathbb{Z}^2 \\ |i-j|=1}} |v_T^i - v_T^j|^2\Bigg) \\ &= \frac{1}{2T}\Bigg(\sum_{i \in A} \ell\,(v_T^i - 1)^2 + \frac{1}{\ell}\sum_{i \in C} \sum_{\substack{k=1 \\ i+e_k \in TQ^\nu \setminus C}}^{2} |v_T^i - v_T^{i+e_k}|^2 \\ &\quad + \sum_{i \in B}\Bigg(\ell\,(v_T^i - 1)^2 + \frac{1}{\ell}\sum_{\substack{k=1 \\ i+e_k \in B}}^{2} |v_T^i - v_T^{i+e_k}|^2\Bigg) + \frac{1}{\ell}\sum_{i \in B} \sum_{\substack{k=1 \\ i+e_k \in A}}^{2} |v_T^i|^2\Bigg). \end{aligned} \tag{7.4}$$

We estimate the terms on the right hand side of (7.4) separately. First notice that $\#(A) \leq cT$, while

$$\#(\{i \in C :\ i + e_k \in TQ^\nu \setminus C \text{ for some } k = 1,2\}) \leq c/\ell\,,$$

for some $c > 0$. Thus we get

$$\frac{1}{2T}\sum_{i \in A} \ell\,(v_T^i - 1)^2 = \frac{\ell}{2T}\#(A) \leq c\,\ell \tag{7.5}$$

and

$$\frac{1}{2T\ell}\sum_{i \in C} \sum_{\substack{k=1 \\ i+e_k \in TQ^\nu \setminus C}}^{2} |v_T^i - v_T^{i+e_k}|^2 \leq \frac{c}{T\ell^2}. \tag{7.6}$$

Moreover, if $i \in B$ is such that $i + e_k \in A$ for some $k \in \{1,2\}$ then $|v^i|^2 \leq c\,\ell^2$ for some $c > 0$ and $\#(\{i \in B :\ i + e_k \in A \text{ for some } k = 1,2\}) \leq cT$. Then

$$\frac{1}{2T\ell}\sum_{i \in B} \sum_{\substack{k=1 \\ i+e_k \in A}}^{2} |v_T^i|^2 \leq c\,\ell. \tag{7.7}$$

Finally, arguing as in the proof of Proposition 4.2 yields

$$\begin{aligned}
&\frac{1}{2T}\sum_{i\in B}\Big(\ell\,(v_T^i-1)^2+\frac{1}{\ell}\sum_{\substack{k=1\\ i+e_k\in B}}^{2}|v_T^i-v_T^{i+e_k}|^2\Big)\\
&\le\frac{1}{2T}\sum_{i\in B}\left(\int_{i+[0,1)^2}\ell\,(v_T(x)-1)^2+\frac{1}{\ell}|\nabla v_T(x)|^2dx+c\,\ell^2\right)\\
&\le\frac{1}{T}\int_{\Pi_\nu\cap TQ^\nu}\left(\int_0^{T_\eta}(f-1)^2+(f')^2dt\right)d\mathcal{H}^1+c\,\ell\le(1+\eta)+c\,\ell.
\end{aligned}\tag{7.8}$$

Gathering (7.4)-(7.7) we then obtain

$$\varphi_\ell^T(\nu)\le 1+\eta+c\Big(\ell+\frac{1}{T\ell^2}\Big).$$

Passing first to the limit as $T\to+\infty$ and then letting $\ell\to0$, (7.3) follows by the arbitrariness of $\eta>0$.

We now show that $\varphi_\ell(\nu)/\ell\to|\nu|_\infty$ as $\ell\to+\infty$. To this end let $\nu=(\nu_1,\nu_2)\in S^1$; without loss of generality we may assume that $|\nu|_\infty=|\nu_2|$. Let $p_2:\mathbb{R}^2\to\Pi_{e_2}$ be the orthogonal projection onto Π_{e_2} and for every $j\in\Pi_{e_2}\cap\mathbb{Z}^2$ let $R_j:=\{k\in\mathbb{Z}:\ j+ke_2\in TQ^\nu\}$. Let $T>0$ and suppose that v_T is a test function for $\varphi_\ell^T(\nu)$. Let $\mathscr{C}$ be a channel in $TQ^\nu\cap\mathbb{Z}^2$ along which $v_T=0$. Since $\mathscr{C}$ is a strong path we deduce that for every $j\in p_2(\Pi_\nu)\cap\mathbb{Z}^2$ there exist at least two points $k_1,k_2\in R_j$ such that $v_T^{j+k_1e_2}=v_T^{j+k_2e_2}=0$. This yields

$$\frac{1}{2T}\sum_{i\in TQ^\nu\cap\mathbb{Z}^2}(v_T^i-1)^2\ge\frac{1}{2T}\sum_{j\in p_2(\Pi_\nu)\cap\mathbb{Z}^2}\sum_{k\in R_j}(v_T^{j+ke_2}-1)^2\ge\frac{\lfloor T\rfloor\,|\nu_2|}{T}.$$

Letting $T\to+\infty$ we then obtain

$$\frac{\varphi^\ell(\nu)}{\ell}\ge|\nu_2|=|\nu|_\infty,\tag{7.9}$$

for every $\ell>0$.

To prove that, up to a small error, the reverse inequality also holds, we construct a suitable test function v_T for $\varphi_\ell^T(\nu)$. To this end we set

$$v_T^i:=\begin{cases}0 & \text{if } \mathcal{S}_{(i-e_2,i+e_2]}\cap\Pi_\nu\neq\emptyset,\\ 1 & \text{otherwise in } TQ^\nu\cap\mathbb{Z}^2.\end{cases}$$

Then, arguing as in the proof of Theorem 6.1, Step 2 one can show that the set $\{v_T=0\}$ is a channel in $TQ^\nu\cap\mathbb{Z}^2$. In particular, v_T is admissible for $\varphi_\ell^T(\nu)$. Moreover, a direct computation gives

$$\begin{aligned}
\frac{1}{\ell}\varphi_\ell^T(\nu)&\le\frac{1}{2T}\left(\sum_{i\in TQ^\nu\cap\mathbb{Z}^2}(v_T^i-1)^2+\frac{1}{2\ell^2}\sum_{\substack{i,j\in TQ^\nu\cap\mathbb{Z}^2\\ |i-j|=1}}|v_T^i-v_T^j|^2\right)\\
&\le\frac{1}{T}\lfloor T|\nu_2|\rfloor\left(1+\frac{2}{\ell^2}\right).
\end{aligned}$$

Thus, letting $T\to+\infty$, for every $\ell>0$ we get

$$\frac{1}{\ell}\varphi_\ell(\nu)\le|\nu|_\infty\left(1+\frac{2}{\ell^2}\right),$$

which together with (7.9) proves the claim. □

Acknowledgments.

Andrea Braides acknowledges the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006. Caterina Zeppieri was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under the Germany Excellence Strategy EXC 2044 390685587, Mathematics Münster: Dynamics - Geometry - Structure.

References

[1] Alberti, G., Variational models for phase transitions, an approach via Γ-convergence, in *Calculus of Variations and Partial Differential Equations* (Ambrosio, L. and Dancer, N., eds.), Springer, Berlin, 2000, 95–114.

[2] Alberti, G., Bouchittè, G. and Seppecher, P., Phase transition with a line-tension effect, *Arch. Rational Mech. Anal.* **144** (1999), 1–46.

[3] Alicandro, R., Braides, A. and Cicalese, M.. Phase and anti-phase boundaries in binary discrete systems: a variational viewpoint. *Netw. Heterog. Media* **1** (2006), 85–107.

[4] Alicandro, R. and Cicalese, M.. Representation result for continuum limits of discrete energies with superlinear growth. *SIAM J. Math Anal.* **36** (2004), 1–37.

[5] Alicandro, R. and Gelli, M. S., Local and non local continuum limits of Ising type energies for spin systems, *SIAM J. Math. Anal.* **48** (2016), 895–931.

[6] Ambrosio, L., A compactness theorem for a new class of functions of bounded variation. *Boll. Un. Mat. Ital.* **3-B** (1989), 857–881.

[7] Ambrosio, L. and Braides, A., Functionals defined on partitions of sets of finite perimeter II: semicontinuity, relaxation and homogenization, *J. Math. Pures. Appl.* **69** (1990), 307–333.

[8] Ambrosio, L., Fusco, N. and Pallara, D., *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[9] Ambrosio, L. and Tortorelli, V. M., Approximation of functionals depending on jumps by elliptic functionals via Γ-convergence, *Comm. Pure Appl. Math.* **43**, no. 8 (1990), 999–1036.

[10] Ambrosio, L. and Tortorelli, V. M., On the approximation of free-discontinuity problems, *Boll. Un. Mat. Ital.* (7) **6-B** (1992), 105–123.

[11] Aubert, G. and Kornprobst, P., *Mathematical Problems in Image Processing: Partial Differential Equations and the Calculus of Variations*, Springer Science & Business Media, Berlin, 2006.

[12] Bach, A., Cicalese, M. and Ruf, M., Random finite-difference discretizations of the Ambrosio-Tortorelli functional with optimal mesh size, *preprint* (2019) arXiv:1902.08437.

[13] Bellettini, G. and Coscia, A., Discrete approximation of a free discontinuity problem, *Numer. Funct. Anal. and Optimiz.* **15** (1994), 201–224.

[14] Blake, A. and Zisserman, A., *Visual reconstruction*, MIT Press Series in Artificial Intelligence. MIT Press, Cambridge, MA, 1987.

[15] Bouchittè, G. Fonseca, I., Leoni, G. and Mascarenhas, L., A global method for relaxation in $W^{1,p}$ and SBV_p, *Arch. Ratinal Mech. Anal.* **165** (2002), 187–242.

[16] Bourdin, B., Image segmentation with a finite element method, *Math. Model. Numer. Anal.* **33**, no. 2 (1999), 229–244.

[17] Bourdin, B., Francfort, G. and Marigo, J.-J., The variational approach to fracture, *J. Elasticity* **91** (2008), 5–148.

[18] Braides, A., *Approximation of Free-discontinuity Problems*, Lecture Notes in Mathematics 1694, Springer Verlag, Berlin, 1998.

[19] Braides, A., Γ-*convergence for Beginners*, Oxford University Press, Oxford, 2002.

[20] Braides, A., A handbook of Γ-convergence, in *Handbook of Differential Equations. Stationary Partial Differential Equations, Volume* 3 (Chipot, M. and Quittner, P., eds.), Elsevier, Amsterdam, 2006, 101–213.

[21] Braides, A., Discrete-to-continuum variational methods for lattice systems, *Proceedings of the International Congress of Mathematicians August* 13–21, 2014*, Seoul, Korea* (Jang, S., Kim, Y., Lee, D. and Yie, I., eds.), Kyung Moon Sa, Seoul, 2014, Vol. IV, 997–1015.

[22] Braides, A. and Cicalese, M., Interfaces, modulated phases and textures in lattice systems, *Arch. Ration. Mech. Anal.***223** (2017), 977–1017.

[23] Braides, A. and Dal Maso, G., Non-local approximation of the Mumford-Shah functional, *Calc. Var. PDE* **5** (1997), 293–322.

[24] Braides, A. and Defranceschi, A., *Homogenization of Multiple Integrals*, Oxford University Press, Oxford, 1998.

[25] Braides, A. and Gelli, M. S., Limits of discrete systems with long-range interactions, *J. Convex Anal.* **9** (2002), 363–399.
[26] Braides, A. and Kreutz, L., Design of lattice surface energies, *Calc Var. PDE*, **57** (2018).
[27] Braides, A. and Piatnitski, A., Overall properties of a discrete membrane with randomly distributed defects. *Arch. Ration. Mech. Anal.* **189** (2008), 301–323.
[28] Braides, A. and Piatnitski, A., Variational problems with percolation: dilute spin systems at zero temperature, *J. Stat. Phys.* **149** (2012), 846–864.
[29] Braides, A. and Piatnitski, A., Homogenization of surface and length energies for spin systems, *J. Funct. Anal.* **264** (2013), 1296–1328.
[30] Braides, A. and Sigalotti, L., Models of defects in atomistic systems, *Calc. Var. and PDE* **41** (2011), 71–109.
[31] Braides, A. and L. Truskinovsky, L., Asymptotic expansions by Gamma-convergence, *Contin. Mech. Thermodyn.* **20** (2008), 21–62.
[32] Braides, A. and Yip, A., A quantitative description of mesh dependence for the discretization of singularly perturbed nonconvex problems, *SIAM J. Numer. Anal.* **50**, no. 4 (2012), 1883–1898.
[33] Caffarelli, L. A. and de la Llave, R., Interfaces of ground states in Ising models with periodic coefficients, *J. Stat Phys.* **118** (2005), 687–719.
[34] Carriero, M., Farina, A. and Sgura, I., Image Segmentation in the Framework of Free Discontinuity Problems, in *Calculus of Variations: topics from the mathematical heritage of Ennio De Giorgi* (Pallara, D. ed.) Quaderni di Matematica **14** (2014), 85–133.
[35] Carriero, M., Leaci, A., Tomarelli, F., Image inpainting via variational approximation of a Dirichlet problem with free discontinuity, *Adv. Calc. Var.* **7** no.3 (2014), 267–295.
[36] Chambolle, A., Image segmentation by variational methods: Mumford and Shah functional and the discrete approximations, *SIAM J. Appl. Math.* **55**, no. 3 (1995), 827–863.
[37] Chambolle, A., Finite-differences discretizations of the Mumford-Shah functional, *Math. Model. Numer. Anal.* **33**, no. 2 (1999), 261–288.
[38] Chambolle, A. and Dal Maso, G., Discrete approximation of the Mumford-Shah functional in dimension two, *Math. Model. Numer. Anal.* **33**, no. 4 (1999), 651–672.
[39] Chan, T. F. and Shen, J. J., *Image Processing and Analysis: variational, PDE, wavelet, and stochastic methods*, SIAM, 2005.
[40] Chan, T. F. Esedoglu, S. and Nikolova, M., Algorithms for finding global minimizers of image segmentation and denoising models, *SIAM J. Appl. Math* **66** (2006), 1632–1648.
[41] Cortesani, G., Strong approximation of $GSBV$ functions by piecewise smooth functions, *Ann. Univ. Ferrara Sez VII (N.S.)* **43** (1997), 27–49.
[42] Cortesani, G. and Toader, R., A density result in SBV with respect to non-isotropic energies, *Nonlinear Analysis* **38** (1999), 585–604.
[43] Crismale, V., Scilla, G. and Solombrino, F., A derivation of Griffith functionals from discrete finite-difference models, *preprint* (2020) arXiv:2001.00480.
[44] Dal Maso, G., *An Introduction to* Γ*-convergence*, Birkhäuser, Boston, 1993.
[45] De Giorgi, E. and Ambrosio, L., Un nuovo funzionale del calcolo delle variazioni, *Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **82** (1988), 199–210.
[46] Focardi, M., Fine regularity results for Mumford-Shah minimizers: porosity, higher integrability and the Mumford-Shah conjecture, in *Free Discontinuity Problems* (Fusco, N. and Pratelli, A., eds.), Publications of the Scuola Normale Superiore, vol 19. Edizioni della Normale, Pisa, 2016, 1–68.
[47] Fonseca, I. and Müller, S., Quasiconvex integrands and lower semicontinuity in L^1, *SIAM J. Math. Anal.* **23** (1992), 1081–1098.
[48] Francfort, G. and Marigo, J.-J., Stable damage evolution in a brittle continuous medium, *Eur. J. Mech., A/Solids* **12** (1993), 149–189.
[49] Gobbino, M., Finite difference approximation of the Mumford-Shah functional, *Comm. Pure Appl. Math.* **51** (1998), 197–228.
[50] Modica, L. and Mortola, S., Un esempio di Γ-convergenza, *Boll. Un. Mat. Ital. B (5)* **14** (1977), 285–299.
[51] Morel, J. M. and Solimini, S., *Variational Methods in Image Segmentation: with seven image processing experiments*, Springer Science & Business Media, Berlin, 2012.
[52] Mumford, D. and Shah, J., Optimal approximation by piecewise smooth functions and associated variational problems, *Comm. Pure Appl. Math.* **42** (1989), 577–685.
[53] Osher, S. and Paragios, N., (Eds.), *Geometric Level Set Methods in Imaging, Vision, and Graphics*, Springer Science & Business Media, Berlin, 2003.
[54] Ruf, M., Discrete stochastic approximation of the Mumford-Shah functional, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **36**, no. 4 (2019), 887–937.
[55] Vese, L. A. and Chan, T. F., A multiphase level set framework for image segmentation using the Mumford and Shah modelInternational *J. Computer Vision* **50** (2002), 271–293.

(Annika Bach) Zentrum Mathematik, Technische Universität München, Boltzmannstr. 3, 85747 Garching bei München, Germany
E-mail address: annika.bach@ma.tum.de

(Caterina Ida Zeppieri) Angewandte Mathematik, Universität Münster, Einsteinstr. 62, 48149 Münster, Germany
E-mail address: caterina.zeppieri@uni-muenster.de

(Andrea Braides) Dipartimento di Matematica, Università di Roma "Tor Vergata", via della Ricerca Scientifica 1, 00133 Rome, Italy
E-mail address: braides@mat.uniroma2.it