# REGULARIZING EFFECT OF THE INTERPLAY BETWEEN COEFFICIENTS IN SOME NONLINEAR DIRICHLET PROBLEMS WITH DISTRIBUTIONAL DATA

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Abstract. We prove that the solution $u$ of the Dirichlet problem (1.1) below has exponential summability under the only assumption that there exists $R > 0$ such that $|F(x)|^2 \le R\,a(x)$; furthermore we prove the boundedness of $u$ under the slightly stronger assumption that there exists $R > 0$ such that $|F(x)|^p \le R\,a(x)$, $p > 2$.

## 1. Introduction and statement of the results

In this paper, we study the existence of regular (with respect to the summability or boundedness) weak solutions of the problem

$$u \in W_0^{1,2}(\Omega) : -\operatorname{div}(M(x,\nabla u)) + a(x)\,u = -\operatorname{div}(F)\,. \tag{1.1}$$

where $\Omega$ is a bounded set in $\mathbb{R}^N$ and $-\operatorname{div}(M(x,\nabla u))$ is a classical nonlinear differential operator, defined by a Carathéodory function $M(x,\xi)$ satisfying, for some $0 < \alpha \le \beta$, and for almost every $x$ in $\Omega$,

$$\begin{cases} M(x,\xi)\,\xi \ge \alpha|\xi|^2\,, \quad |M(x,\xi)| \le \beta|\xi|\,, & \forall \xi \in \mathbb{R}^N\,, \\ [M(x,\xi) - M(x,\eta)](\xi - \eta) > 0\,, & \forall \xi\,,\eta \in \mathbb{R}^N\,,\ \xi \ne \eta\,. \end{cases} \tag{1.2}$$

Our key assumption is that the function $a(x)$ and the vector-valued function $F(x)$ are such that

$$0 \le a(x) \in L^1(\Omega)\,, \tag{1.3}$$

$$\exists\, R > 0 \text{ such that } |F(x)|^2 \le R\,a(x)\,. \tag{1.4}$$

The existence of bounded solutions for (1.1) can be proved, without assumption (1.4), if $|F|$ belongs to $L^m(\Omega)$, for some $m > N$. This is a consequence of the positivity of $a$ (assumption (1.3)) and of Stampacchia-type estimates (see [12]): see the Appendix. Note that, in our case, since $a$ belongs to $L^1(\Omega)$ it follows from (1.4) that $|F|$ belongs to $L^2(\Omega)$, so that (once again by the fact that $a$ is positive), existence of a solution $u$ in $W_0^{1,2}(\Omega)$ can be proved using classical arguments (see the first part of Theorem 2.1 in Section 2 and the Appendix). We shall prove in the second part of the cited theorem that $e^{qu^2} - 1$ belongs to $W_0^{1,2}(\Omega)$ for every $q < \frac{\alpha}{R}$.

This result must be compared with the corresponding one in [1] where the same problem is studied replacing the term $-\operatorname{div}(F)$ with function source $f(x) \in L^1(\Omega)$. In this case, existence of a bounded solution in $W_0^{1,2}(\Omega)$, and not only an exponentially summable one, is obtained showing the regularizing effect of the term $a(x)\,u$ when condition (1.4) holds true. Recents works studying this effect can be found in [2] (see also [5, 6, 7, 9, 10, 11]).

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In the case of problem (1.1), the boundedness of solutions under assumption (1.4) (as it happens for Lebesgue data in [1]) remains as an open question. We prove in Theorem 3.1 of Section 3 that solutions are bounded under a slightly stronger assumption on $F$, namely that

$$|F(x)|^p \leq R\,a(x)\,, \quad \text{for some } p > 2.$$

On the other hand, in Section 4, working in the radial case, we give an example of a function $a$ and a vector-valued function $F$ satisfying (1.4) such that the corresponding problem (1.1) has a *bounded* solution, and not only an exponentially summable one.

In order to prove the cited theorems we follow an approximation method, that is we approximate the vector-valued function $F$ by a sequence of bounded vector-valued functions $F_n$ for which there is a solution $u_n$ in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ of the problem by the results of the Appendix. Since $u_n$ belongs to $L^\infty(\Omega)$, this will allow us to use either exponential, or power-like, test functions in the problem.

Throughout the paper, we will use the following functions of one real variable, depending on a parameter $k > 0$:

$$T_k(s) = \max(-k, \min(s,k))\,, \qquad G_k(s) = s - T_k(s) = (|s|-k)_+ \operatorname{sgn}(s)\,.$$

## 2. Exponential estimates

THEOREM 2.1. *If assumptions* (1.2), (1.3) *and* (1.4) *hold true, then there exists* $u \in W_0^{1,2}(\Omega)$ *solving* (1.1)*; i.e., satisfying that*

$$\int_\Omega M(x,\nabla u)\nabla\varphi + \int_\Omega a(x)\,u\,\varphi = \int_\Omega F(x)\nabla\varphi\,, \qquad \forall\varphi\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)\,.$$

*Furthermore, for every* $q < \frac{\alpha}{R}$,

$$\mathrm{e}^{q\,u^2} - 1 \in W_0^{1,2}(\Omega)\,. \tag{2.1}$$

PROOF. Let $\{F_n\}$ be a sequence of $L^\infty(\Omega)$ functions, strongly convergent to $F$ in $(L^2(\Omega))^N$, and such that

$$|F_n| \leq |F|\,, \qquad \forall n\in\mathbb{N}\,. \tag{2.2}$$

For instance, it is possible to choose

$$F_n(x) = \begin{cases} F(x), & \text{if } |F(x)|\leq n,\\ n\dfrac{F(x)}{|F(x)|}, & \text{if } |F(x)| > n. \end{cases}$$

Note that, as a consequence of (2.2), if $F$ satisfies (1.4), then every function $F_n$ satisfies also assumption (1.4) with the same constant $R$.

Since $|F_n|$ belongs to $L^\infty(\Omega)$, for every $n$ in $\mathbb{N}$, by the results in the Appendix there exists a solution $u_n$ of the problem

$$u_n \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega) : -\operatorname{div}(M(x,\nabla u_n)) + a(x)\,u_n = -\operatorname{div}(F_n)\,. \tag{2.3}$$

We now choose $u_n$ as test function in (2.3). Using (1.2), and the fact that the sequence $\{F_n\}$ is bounded in $(L^2(\Omega))^N$ thanks to (1.4) (recall that $a$ belongs to $L^1(\Omega)$), we obtain

$$\begin{aligned}\alpha\int_\Omega|\nabla u_n|^2 &\leq \int_\Omega M(x,\nabla u_n)\nabla u_n + \int_\Omega a(x)\,u_n^2\\ &= \int_\Omega F_n(x)\nabla u_n \leq \frac{1}{2\alpha}\int_\Omega|F_n(x)|^2 + \frac{\alpha}{2}\int_\Omega|\nabla u_n|^2\,,\end{aligned}$$

from which it follows that the sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$, as well as that

$$\int_\Omega a(x)\, u_n^2 \le C\,. \tag{2.4}$$

Since the sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$, then, up to subsequences, $u_n$ converges to some function $u$ weakly in $W_0^{1,2}(\Omega)$ and almost everywhere.

A consequence of the almost everywhere convergence of $u_n$ to $u$ is that the sequence $\{a(x)u_n\}$ converges almost everywhere to $a(x)\,u$. Let now $E$ be a measurable subset of $\Omega$; we then have, if $k > 0$, and recalling (2.4),

$$\begin{aligned}\int_E a(x)|u_n| &= \int_{\{|u_n|<k\}\cap E} a(x)|u_n| + \int_{\{|u_n|\ge k\}\cap E} a(x)|u_n| \\ &\le k\int_E a(x) + \frac{1}{k}\int_\Omega a(x)u_n^2 \le k\int_E a(x) + \frac{C}{k}\,.\end{aligned}$$

Let now $\varepsilon > 0$, and let first $k$ large enough so that $\frac{C}{k} \le \frac{\varepsilon}{2}$, and then meas$(E)$ small enough in order to have

$$\int_E a(x) \le \frac{\varepsilon}{2k}\,.$$

We have therefore proved that if meas$(E)$ is small enough, then

$$\int_E a(x)|u_n| \le \varepsilon\,, \qquad \forall n \in \mathbb{N}\,,$$

that is to say, the sequence $\{a(x)|u_n|\}$ is uniformly equi-integrable. Thus, by Vitali theorem, we have that

$$a(x)u_n \quad \text{strongly converges to } a(x)\,u \text{ in } L^1(\Omega).$$

Since the principal part of the differential operator is nonlinear, we need more information on the sequence $\{u_n\}$. It would be possible to use a result on the almost everywhere convergence of $\{\nabla u_n\}$, proved in [3], but we prefer to give below a simple (thanks to our assumptions) self-contained proof.

Let now $\varphi$ be a function in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, and choose $u_n - \varphi$ as test function in (2.3). We obtain

$$\int_\Omega M(x,\nabla u_n)\nabla(u_n-\varphi) + \int_\Omega a(x)u_n\,(u_n-\varphi) = \int_\Omega F_n\nabla(u_n-\varphi)\,.$$

Adding and subtracting the term

$$\int_\Omega M(x,\nabla\varphi)\nabla(u_n-\varphi)\,,$$

we arrive at

$$\begin{aligned}&\int_\Omega [M(x,\nabla u_n) - M(x,\nabla\varphi)]\nabla(u_n-\varphi) + \int_\Omega M(x,\nabla\varphi)\nabla(u_n-\varphi) \\ &\qquad + \int_\Omega a(x)u_n\,(u_n-\varphi) = \int_\Omega F_n\nabla(u_n-\varphi)\,.\end{aligned}$$

Dropping the first term, which is positive thanks to (1.2), and using the weak convergence of $u_n$ to $u$ in $W_0^{1,2}(\Omega)$, the almost everywhere convergence of $u_n$ to $u$, and the strong convergence of $a(x)u_n$ in $L^1(\Omega)$, as well as the strong convergence of $F_n$ to $F$ in $(L^2(\Omega))^N$, we obtain

$$\int_\Omega M(x,\nabla\varphi)\nabla(u-\varphi)+\int_\Omega a(x)u\,(u-\varphi)\leq\int_\Omega F(x)\nabla(u-\varphi)\,,\quad\forall\varphi\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)\,.$$

We now follow [4]: let $k>0$, let $t\neq 0$, let $\psi$ in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, and choose $\varphi=T_k(u)-t\,\psi$ in the above identity. We obtain

$$\int_\Omega M(x,\nabla(T_k(u)-t\,\psi))\nabla(G_k(u)+t\,\psi)+\int_\Omega a(x)u\,(G_k(u)+t\,\psi)\leq\int_\Omega F(x)\nabla(G_k(u)+t\,\psi)\,.$$

Letting $k$ tend to infinity, using that $u$ belongs to $W_0^{1,2}(\Omega)$, that $a(x)u\,G_k(u)\geq 0$, and that $G_k(u)$ tends to zero, we arrive at

$$t\int_\Omega M(x,\nabla(u-t\,\psi))\nabla\psi+t\int_\Omega a(x)u\,\psi\leq t\int_\Omega F(x)\nabla\psi\,.$$

Dividing by $t>0$ and then letting $t$ tend to zero, we obtain that

$$\int_\Omega M(x,\nabla u)\nabla\psi+\int_\Omega a(x)u\,\psi\leq\int_\Omega F(x)\nabla\psi\,,$$

while dividing by $t<0$ and then letting $t$ tend to zero, we obtain

$$\int_\Omega M(x,\nabla u)\nabla\psi+\int_\Omega a(x)u\,\psi\geq\int_\Omega F(x)\nabla\psi\,,$$

so that

$$\int_\Omega M(x,\nabla u)\nabla\psi+\int_\Omega a(x)u\,\psi=\int_\Omega F(x)\nabla\psi\,,\qquad\forall\psi\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)\,,$$

that is, $u$ is a solution of (1.1).

Now, let $t>0$ and choose $(\mathrm{e}^{t\,u_n^2}-1)\,\mathrm{sgn}(u_n)$ as test function in (2.3). Note that such a choice is allowed since $u_n$ belongs to $L^\infty(\Omega)$. We obtain

$$2t\int_\Omega M(x,\nabla u_n)\nabla u_n|u_n|\mathrm{e}^{t\,u_n^2}+\int_\Omega a(x)|u_n|(\mathrm{e}^{t\,u_n^2}-1)=2t\int_\Omega F_n(x)\nabla u_n|u_n|\mathrm{e}^{t\,u_n^2}\,.$$

Using Young inequality in the right hand side, as well as (1.4), we have that, for some $B>0$,

$$\begin{aligned}2t\int_\Omega F_n(x)\nabla u_n|u_n|\mathrm{e}^{t\,u_n^2}&\leq 2tB\int_\Omega|\nabla u_n|^2|u_n|\mathrm{e}^{t\,u_n^2}+\frac{2t}{4B}\int_\Omega|F_n(x)|^2|u_n|\mathrm{e}^{t\,u_n^2}\\&\leq 2tB\int_\Omega|\nabla u_n|^2|u_n|\mathrm{e}^{t\,u_n^2}+\frac{tR}{2B}\int_\Omega a(x)|u_n|\mathrm{e}^{t\,u_n^2}\,.\end{aligned}$$

Using (1.2) in the left hand side we have that

$$2t\int_\Omega M(x,\nabla u_n)\nabla u_n|u_n|\mathrm{e}^{t\,u_n^2}\geq 2t\alpha\int_\Omega|\nabla u_n|^2|u_n|\mathrm{e}^{t\,u_n^2}\,,$$

so that, putting the inequalities together, we have that

$$2t(\alpha-B)\int_\Omega|\nabla u_n|^2|u_n|\mathrm{e}^{t\,u_n^2}+\int_\Omega a(x)\Big[|u_n|(\mathrm{e}^{t\,u_n^2}-1)-\frac{tR}{2B}|u_n|\mathrm{e}^{t\,u_n^2}\Big]\leq 0\,,$$

which can be rewritten as

$$2t(\alpha-B)\int_\Omega|\nabla u_n|^2|u_n|\mathrm{e}^{t\,u_n^2}+\int_\Omega a(x)|u_n|\Big[\mathrm{e}^{t\,u_n^2}\Big(1-\frac{tR}{2B}\Big)-1\Big]\leq 0\,. \tag{2.5}$$

Let now $0<\delta<1$, $B=(1-\delta)\alpha$, and $t$ such that $\frac{tR}{2B}=1-\delta$, that is $t=\frac{2(1-\delta)^2\alpha}{R}$; with this choice of $B$ and $t$, (2.5) becomes

$$\frac{4\delta(1-\delta)^2\alpha^2}{R}\int_\Omega |\nabla u_n|^2|u_n|\mathrm{e}^{t\,u_n^2}+\int_\Omega a(x)|u_n|\Big[\delta\,\mathrm{e}^{t\,u_n^2}-1\Big]\leq 0\,. \tag{2.6}$$

Let now $T=\sqrt{\frac{-\log(\delta)}{t}}$, and note that

$$\delta\,\mathrm{e}^{t\,u_n^2}-1\geq 0 \qquad \text{on the set } \{|u_n|\geq T\}.$$

We therefore have that

$$\frac{4\delta(1-\delta)^2\alpha^2}{R}\int_\Omega |\nabla u_n|^2|u_n|\mathrm{e}^{t\,u_n^2}+\int_{\{|u_n|\geq T\}} a(x)|u_n|\Big[\delta\,\mathrm{e}^{t\,u_n^2}-1\Big]\leq \int_{\{|u_n|<T\}} a(x)|u_n|\Big[1-\delta\,\mathrm{e}^{t\,u_n^2}\Big]\,,$$

which then yields, dropping a positive term,

$$\frac{4\delta(1-\delta)^2\alpha^2}{R}\int_\Omega |\nabla u_n|^2|u_n|\mathrm{e}^{t\,u_n^2}\leq T\int_{\{|u_n|<T\}} a(x)\leq T\int_\Omega a(x)=C(a,\alpha,R,\delta)\,,$$

that is

$$\int_\Omega |\nabla u_n|^2|u_n|\mathrm{e}^{t\,u_n^2}\leq C(a,\alpha,R,\delta)\,. \tag{2.7}$$

Let now $q>0$ such that $2q<t$. Then

$$|s|^2\mathrm{e}^{2qs^2}\leq C(q)|s|\mathrm{e}^{t\,s^2}\,,$$

for every $s$ in $\mathbb{R}$, since the function $s\mapsto |s|\,\exp\big[\big(2q-t\big)s^2\big]$ is bounded, being $2q-t<0$. Thus, we have that

$$\int_\Omega |\nabla u_n|^2|u_n|^2\mathrm{e}^{2q\,u_n^2}\leq C(q)\int_\Omega |\nabla u_n|^2|u_n|\mathrm{e}^{t\,u_n^2}\,,$$

which, together with (2.7), implies that

$$\int_\Omega |\nabla(\mathrm{e}^{q\,u_n^2}-1)|^2\leq C(q)\int_\Omega |\nabla u_n|^2|u_n|\mathrm{e}^{t\,u_n^2}\leq C(q,a,\alpha,R,\delta)\,,$$

thus proving

$$\mathrm{e}^{q\,u_n^2}-1 \quad \text{is bounded in } W_0^{1,2}(\Omega)\,,$$

for every $q<\frac{t}{2}=\frac{(1-\delta)^2\alpha}{R}$. Choosing $\delta$ small enough, we have that $q$ can be chosen any real number smaller than $\frac{\alpha}{R}$. Furthermore, the sequence $\{\mathrm{e}^{q\,u_n^2}-1\}$, which is bounded in $W_0^{1,2}(\Omega)$, weakly converges in the same space to some function $v$; since the function $s\mapsto \mathrm{e}^{q\,s^2}-1$ is continuous, the almost everywhere convergence of $u_n$ to $u$ implies that $v=\mathrm{e}^{q\,u^2}-1$, which then belongs to $W_0^{1,2}(\Omega)$ for every $q<\frac{\alpha}{R}$. □

Remark 2.2. Note that if we improve the assumption on $a$ to $a\in L^{\frac{N}{2}}(\Omega)$, then from (1.4) it follows that $|F|\in L^N(\Omega)$ and the exponential summability of $u$ is proved in [12].

## 3. Bounded solutions

THEOREM 3.1. *If assumptions* (1.2) *and* (1.3) *hold true and there is* $p > 2$ *such that*

$$|F(x)|^p \le R\,a(x)\,, \tag{3.1}$$

*then there exists a solution* $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ *of* (1.1).

PROOF. As in the proof of Theorem 2.1, we consider the sequence $\{u_n\} \subset W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of solutions of (2.3). Recall that we have shown in the mentioned proof that $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$ and that, up to a subsequence, it converges almost everywhere to a solution $u$ of (1.1). Thus, we only need to prove that $\{u_n\}$ is bounded in $L^\infty(\Omega)$.

Since $u_n$ belongs to $L^\infty(\Omega)$, we can choose $\frac{p-2}{p}|G_k(u_n)|^{\frac{2}{p-2}}\,G_k(u_n)$ as test function in (2.3) to obtain, using twice Young inequality (once with exponents 2 and 2, and once with exponents $\frac{p}{2}$ and $\frac{p}{p-2}$) and (2.2):

$$\begin{aligned}
&\int_\Omega M(x,\nabla u_n)\nabla G_k(u_n)|G_k(u_n)|^{\frac{2}{p-2}} + \frac{p-2}{p}\int_\Omega a(x)|G_k(u_n)|^{\frac{2}{p-2}}\,G_k(u_n)\,u_n \\
&\quad= \int_\Omega F_n(x)\nabla G_k(u_n)|G_k(u_n)|^{\frac{2}{p-2}}\,. \\
&\quad\le \int_\Omega |F_n(x)||\nabla G_k(u_n)||G_k(u_n)|^{\frac{2}{p-2}} \\
&\quad= \int_\Omega |\nabla G_k(u_n)||G_k(u_n)|^{\frac{1}{p-2}}|F_n(x)||G_k(u_n)|^{\frac{1}{p-2}} \\
&\quad\le B\int_\Omega |\nabla G_k(u_n)|^2|G_k(u_n)|^{\frac{2}{p-2}} + \frac{1}{4B}\int_\Omega |G_k(u_n)|^{\frac{2}{p-2}}|F_n(x)|^2 \\
&\quad= B\int_\Omega |\nabla G_k(u_n)|^2|G_k(u_n)|^{\frac{2}{p-2}} + \frac{1}{4B}\int_{\{|u_n|\ge k\}} |G_k(u_n)|^{\frac{2}{p-2}}|F_n(x)|^2\cdot 1 \\
&\quad\le B\int_\Omega |\nabla G_k(u_n)|^2|G_k(u_n)|^{\frac{2}{p-2}} + \frac{1}{4B}\int_{\{|u_n|\ge k\}} |G_k(u_n)|^{\frac{p}{p-2}}|F_n(x)|^p + \int_{\{|u_n|\ge k\}} C_{B,p}\,.
\end{aligned}$$

Using now (1.2) and that $G_k(u_n)$ has the same sign as $u_n$, we therefore deduce that

$$\begin{aligned}
(\alpha - B)&\int_\Omega |\nabla G_k(u_n)|^2\,|G_k(u_n)|^{\frac{2}{p-2}} + \frac{p-2}{p}\int_\Omega a(x)|u_n||G_k(u_n)|^{\frac{p}{p-2}} \\
&\le \frac{1}{4B}\int_{\{|u_n|\ge k\}} |G_k(u_n)|^{\frac{p}{p-2}}|F(x)|^p + \int_{\{|u_n|\ge k\}} C_{B,p}\,.
\end{aligned}$$

Choosing $B = \frac{\alpha}{2}$ and using (3.1) we thus obtain

$$\frac{\alpha}{2}\int_\Omega |\nabla G_k(u_n)|^2\,|G_k(u_n)|^{\frac{2}{p-2}} + \int_{\{|u_n|\ge k\}} a(x)\Big[\frac{p-2}{p}|u_n| - \frac{R}{2\alpha}\Big]|G_k(u_n)|^{\frac{p}{p-2}} \le \int_{\{|u_n|\ge k\}} C_{\alpha,p}\,.$$

Note that

$$a(x)\Big[\frac{p-2}{p}|u_n| - \frac{R}{2\alpha}\Big]|G_k(u_n)|^{\frac{p}{p-2}} \ge 0 \quad \text{if} \quad |u_n| \ge \frac{p}{p-2}\frac{R}{2\alpha} = k_0\,.$$

Thus, for $k \ge k_0$, the second integral is positive and we have

$$\Big(\frac{p-2}{p-1}\Big)^2\int_\Omega |\nabla|G_k(u_n)|^{\frac{p-1}{p-2}}|^2 = \int_\Omega |\nabla G_k(u_n)|^2\,|G_k(u_n)|^{\frac{2}{p-2}} \le \int_{\{|u_n|\ge k\}} C_{\alpha,p}\,.$$

Thus, by Sobolev inequality, we have

$$\Big[\int_\Omega |G_k(u_n)|^{\frac{p-1}{p-2}\,2^*}\Big]^{\frac{2}{2^*}} \le \int_{\{|u_n|\ge k\}} C_{\alpha,p}\,, \quad \forall k \ge k_0\,.$$

Let now $h > k$; since $|G_k(u_n)| \geq h - k$ on the set $\{|u_n| \geq h\}$ (which is contained in the set $\{u_n \geq k\}$), we have

$$(h-k)^{\frac{2(p-1)}{p-2}} \operatorname{meas}(\{|u_n| \geq h\})^{\frac{2}{2^*}} \leq \Big[\int_{\{|u_n|\geq h\}} |G_k(u_n)|^{\frac{p-1}{p-2}\,2^*}\Big]^{\frac{2}{2^*}}$$
$$\leq \Big[\int_\Omega |G_k(u_n)|^{\frac{p-1}{p-2}\,2^*}\Big]^{\frac{2}{2^*}} \leq C_{\alpha,p} \operatorname{meas}(\{|u_n| \geq k\})\,,$$

which can be rewritten as

$$\operatorname{meas}(\{|u_n| \geq h\}) \leq \frac{C_{\alpha,p,N}}{(h-k)^{\frac{2^*(p-1)}{p-2}}} \operatorname{meas}(\{|u_n| \geq k\})^{\frac{2^*}{2}}\,, \qquad \forall h > k \geq k_0\,.$$

Since $\frac{2^*}{2} > 1$, a result by Stampacchia (see [12]) yields that there exists $C > k_0$ such that $\operatorname{meas}(\{|u_n| \geq C\}) = 0$; thus, $|u_n| \leq C$ almost everywhere, and so $\{u_n\}$ is bounded in $L^\infty(\Omega)$. By the almost everywhere convergence of $u_n$ to $u$, we conclude that $u$ belongs to $L^\infty(\Omega)$. □

Remark 3.2. Under the assumptions of Theorem 3.1, the proof of the the strong convergence of $a(x)\,u_n$ to $a(x)\,u$ in $L^1(\Omega)$ is easier. Indeed, since $\{u_n\}$ is bounded in $L^\infty(\Omega)$, we can use Lebesgue theorem instead of Vitali theorem.

## 4. The linear radial case - An example

Let us consider the linear equation

$$-\Delta U + \frac{A}{|x|^2}\,U = -\operatorname{div}\Big(-\frac{B(|x|)}{|x|^2}\,x\Big)\,, \qquad |x| < 1. \tag{4.1}$$

Here $A > 0$ and $B$ is a continuous and differentiable function on $[0, 1]$; being continuous, $B$ is bounded, and so there exists $B \geq 0$ such that $|B(|x|)| \leq B$. Thus, one has, if $a(x) = \frac{A}{|x|^2}$, and $F(x) = -\frac{B(|x|)}{|x|^2}\,x$, that

$$|F(x)|^2 \leq R\,a(x)\,,$$

with $R = \frac{B^2}{A}$, so that the data of the problem satisfy the assumptions of Theorem 2.1. Note that $|F|$ does not belong to $L^N(\Omega)$, so that Stampacchia type results cannot be applied.

Passing in radial coordinates, with $r = |x|$, one has that

$$-\operatorname{div}\Big(-\frac{B(|x|)}{|x|^2}\,x\Big) = \frac{(N-2)\,B(r)}{r^2} + \frac{B'(r)}{r}\,.$$

We look for radial solutions $U$; i.e., satisfying

$$-U''(r) - \frac{N-1}{r}U'(r) + \frac{A}{r^2}U(r) = \frac{(N-2)\,B(r)}{r^2} + \frac{B'(r)}{r}\,.$$

Let now $w$ be the solution of

$$-w''(r) - \frac{N-1}{r}w'(r) + \frac{A}{r^2}w(r) = \frac{(N-2)\,B(r)}{r^2}\,,$$

i.e., of the equation

$$-\Delta w + a(x)\,w = \frac{(N-2)\,B(|x|)}{|x|^2} = f(x)\,.$$

Since $|f(x)| \le Q\,a(x)$, with $Q = \frac{B\,(N-2)}{A}$, by the results of [1] one has that $w$ belongs to $L^\infty(\Omega)$, with $|w| \le Q$. On the other hand, $U = u + w$, where, by difference, $u$ is such that

$$-u''(r) - \frac{N-1}{r}u'(r) + \frac{A}{r^2}u(r) = \frac{B'(r)}{r}\,. \tag{4.2}$$

We are going to prove that also $u$ is a bounded function, so that (4.1) will have a bounded solution, and not only an exponentially summable one, as stated by Theorem 2.1. Therefore, the result of Theorem 2.1 may not be sharp; furthermore, we do not have an example of an equation with an unbounded solution.

As for equation (4.2), we look for solutions of the form

$$u(r) = r^\sigma\, z(r)\,, \qquad \sigma = \frac{2 - N + \sqrt{(N-2)^2 + 4A}}{2}\,.$$

Note that $\sigma > 0$ (since $A > 0$), and that

$$\sigma\,(\sigma + N - 2) = A\,.$$

Since

$$u'(r) = \sigma\, r^{\sigma-1}\, z(r) + r^\sigma\, z'(r)\,, \quad u''(r) = \sigma\,(\sigma-1)\, r^{\sigma-2}\, z(r) + 2\sigma\, r^{\sigma-1}\, z'(r) + r^\sigma\, z''(r)\,,$$

substituting in the equation we arrive at

$$-r^\sigma\, z''(r) - \frac{N+2\sigma-1}{r}\, r^\sigma\, z'(r) + \frac{A - \sigma\,(\sigma+N-2)}{r^2}\, r^\sigma\, z(r) = \frac{B'(r)}{r}\,,$$

from which we obtain, since the third term vanishes by the choice of $\sigma$, and dividing by $-r^\sigma$,

$$\frac{1}{r^{N+2\sigma-1}}[r^{N+2\sigma-1}\, z'(r)]' = z''(r) + \frac{N+2\sigma-1}{r}\, z'(r) = -\frac{B'(r)}{r^{\sigma+1}}\,.$$

Multiplying by $r^{N+2\sigma-1}$, and integrating between 0 and $r$, we obtain

$$r^{N+2\sigma-1}\, z'(r) = -\int_0^r \rho^{N+\sigma-2}\, B'(\rho)\, d\rho\,,$$

that is

$$z'(r) = -\frac{1}{r^{N+2\sigma-1}} \int_0^r \rho^{N+\sigma-2}\, B'(\rho)\, d\rho\,.$$

Integrating again, this time between $r$ and 1, yields

$$z(r) = \int_r^1 \frac{1}{t^{N+2\sigma-1}} \left( \int_0^t \rho^{N+\sigma-2}\, B'(\rho)\, d\rho \right) dt\,.$$

Changing the order of integration leads to

$$\begin{aligned} z(r) &= \int_0^1 \rho^{N+\sigma-2}\, B'(\rho) \left( \int_{\max(r,\rho)}^1 \frac{dt}{t^{N+2\sigma-1}} \right) d\rho \\ &= \frac{1}{2-N-2\sigma} \int_0^1 \rho^{N+\sigma-2}\, B'(\rho)\, [1 - \max(r,\rho)^{2-N-2\sigma}]\, d\rho\,. \end{aligned}$$

Recalling the definition of $u$, we therefore have that

$$(N+2\sigma-2)\, u(r) = r^\sigma \int_0^1 \rho^{N+\sigma-2}\, B'(\rho)\, [\max(r,\rho)^{2-N-2\sigma} - 1]\, d\rho = \boxed{\mathrm{I}} + \boxed{\mathrm{II}} - \boxed{\mathrm{III}}\,,$$

where

$$\boxed{\mathrm{I}} = r^\sigma \int_0^r \rho^{N+\sigma-2}\, B'(\rho)\, r^{2-N-2\sigma}\, d\rho = r^{2-N-\sigma} \int_0^r \rho^{N+\sigma-2}\, B'(\rho)\, d\rho\,,$$

$$\boxed{\mathrm{II}} = r^\sigma \int_r^1 \frac{B'(\rho)}{\rho^\sigma}\, d\rho, \qquad \boxed{\mathrm{III}} = r^\sigma \int_0^1 \rho^{N+\sigma-2}\, B'(\rho)\, d\rho.$$

Integrating by parts, we thus have

$$\begin{aligned} \boxed{\mathrm{I}} &= r^{2-N-\sigma}\, \rho^{N+\sigma-2}\, B(\rho)\Big|_0^r - (N+\sigma-2)\, r^{2-N-\sigma} \int_0^r \rho^{N+\sigma-3}\, B(\rho)\, d\rho \\ &= B(r) - (N+\sigma-2)\, r^{2-N-\sigma} \int_0^r \rho^{N+\sigma-3}\, B(\rho)\, d\rho\,. \end{aligned}$$

The first term is bounded by $B$; as for the second, we have

$$\Big|(N+\sigma-2)\, r^{2-N-\sigma} \int_0^r \rho^{N+\sigma-3}\, B(\rho)\, d\rho\Big| \le B\,(N+\sigma-2)\, r^{2-N-\sigma} \int_0^r \rho^{N+\sigma-3}\, d\rho = B\,,$$

so that $\boxed{\mathrm{I}}$ is bounded. Integrating again by parts, we also have

$$\boxed{\mathrm{II}} = r^\sigma \int_r^1 \frac{B'(\rho)}{\rho^\sigma}\, d\rho = r^\sigma\, B(1) - B(r) + \sigma\, r^\sigma \int_r^1 \frac{B(\rho)}{\rho^{\sigma+1}}\, d\rho\,.$$

The first two terms are bounded; as for the third, again by the boundedness of $B$ we have

$$\Big|\sigma\, r^\sigma \int_r^1 \frac{B(\rho)}{\rho^{\sigma+1}}\, d\rho\Big| \le B\, \sigma\, r^\sigma \int_r^1 \frac{d\rho}{\rho^{\sigma+1}} = B\, r^\sigma \Big(\frac{1}{r^\sigma} - 1\Big) \le B\,,$$

so that $\boxed{\mathrm{II}}$ is bounded. Finally, we have, integrating by parts

$$\boxed{\mathrm{III}} = r^\sigma \int_0^1 \rho^{N+\sigma-2}\, B'(\rho)\, d\rho = r^\sigma\, B(1) - (N+\sigma-2)\, r^\sigma \int_0^1 \rho^{N+\sigma-3}\, B(\rho)\, d\rho\,.$$

The first term is bounded since $\sigma > 0$, while for the second we have

$$\Big|(N+\sigma-2)\, r^\sigma \int_0^1 \rho^{N+\sigma-3}\, B(\rho)\, d\rho\Big| \le B\,(N+\sigma-2)\, r^\sigma \int_0^1 \rho^{N+\sigma-3}\, d\rho = B\, r^\sigma\,,$$

which is bounded since $\sigma > 0$. Thus, also $\boxed{\mathrm{III}}$ is bounded.

Summing up, $u$ is bounded, and so $U$ is bounded (by some quantities depending on $B$ and on $A$).

## Appendix: existence for bounded data $F$

We prove here the existence of a solution of (1.1) if $|F|$ is a function in $L^m(\Omega)$, with $m > N (> 2)$. First of all, let $a_n(x) = \min(a(x), n)$. Then, by a straightforward application of the results of [8] (note that the datum $-\operatorname{div}(F)$ belongs to the dual of $W_0^{1,2}(\Omega)$), there exists a solution $u_n$ in $W_0^{1,2}(\Omega)$ of

$$-\operatorname{div}(M(x, \nabla u_n)) + a_n(x)\, u_n = -\operatorname{div}(F)\,.$$

Choosing $u_n$ as test function, and using (1.2), as well as Young inequality, we have that

$$\alpha \int_\Omega |\nabla u_n|^2 + \int_\Omega a_n(x) u_n^2 = \int_\Omega F(x)\, \nabla u_n \le \frac{1}{2\alpha} \int_\Omega |F(x)|^2 + \frac{\alpha}{2} \int_\Omega |\nabla u_n|^2\,,$$

from which it follows that the sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$. Furthermore, using Stampacchia's result (see [12]), and the fact that by (1.3) $a_n \ge 0$, one can prove that the sequence $\{u_n\}$ is also bounded in $L^\infty(\Omega)$. Thus, up to subsequences, $u_n$ converges to some function $u$ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, weakly in $W_0^{1,2}(\Omega)$, $*$-weakly in $L^\infty(\Omega)$ and almost everywhere in $\Omega$. Since $0 \le a_n(x) \le a(x) \in L^1(\Omega)$, the boundedness of $\{u_n\}$ in $L^\infty(\Omega)$, and its almost everywhere convergence to $u$ allow us to apply Lebesgue theorem to prove that

$$a_n(x)\, u_n \quad \text{strongly converges to } a(x)\, u \text{ in } L^1(\Omega).$$

In order to pass to the limit in the approximate equations, we will use Minty's trick: let $\varphi$ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, and choose $u_n - \varphi$ as test function in the equation for $u_n$. We obtain

$$\int_\Omega M(x, \nabla u_n)\nabla(u_n - \varphi) + \int_\Omega a_n(x) u_n\, (u_n - \varphi) = \int_\Omega F \nabla(u_n - \varphi)\,.$$

Adding and subtracting the term

$$\int_\Omega M(x, \nabla\varphi)\nabla(u_n - \varphi)\,,$$

and using the fact that $M(x, \cdot)$ is monotone, we arrive, after passing to the limit, to

$$\int_\Omega M(x, \nabla\varphi)\nabla(u - \varphi) + \int_\Omega a(x) u\, (u - \varphi) \le \int_\Omega F \nabla(u - \varphi)\,.$$

Choosing $\varphi = u - t\,\psi$, with $t \ne 0$ and $\psi$ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, we thus have that

$$t \int_\Omega M(x, \nabla(u - t\psi))\nabla\psi + t \int_\Omega a(x) u\, \psi \le t \int_\Omega F \nabla\psi\,.$$

Dividing by $t > 0$ and letting $t$ tend to zero, we arrive at

$$\int_\Omega M(x, \nabla u)\nabla\psi + \int_\Omega a(x) u\, \psi \le \int_\Omega F \nabla\psi\,,$$

while the reverse inequality can be obtained dividing by $t < 0$, and then letting $t$ tend to zero. Thus, we have proved that

$$\int_\Omega M(x, \nabla u)\nabla\varphi + \int_\Omega a(x)\, u\, \varphi = \int_\Omega F(x) \nabla\varphi\,, \qquad \forall \varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)\,,$$

that is, problem (1.1) has a solution $u$ belonging to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

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## References

[1] D. Arcoya, L. Boccardo. Regularizing effect of the interplay between coefficients in some elliptic equations. *J. Funct. Anal.*, **268** (2015), 1153–1166.

[2] D. Arcoya, L. Boccardo. Regularizing effect of $L^q$ interplay between coefficients in some elliptic equations. *J. Math. Pures Appl. (9)*, **111** (2018), 106–125.

[3] L. Boccardo, F. Murat. Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations. *Nonlinear Anal. TMA*, **19** (1992), 581–597.

[4] L. Boccardo, L. Orsina. Existence results for Dirichlet problems in $L^1$ via Minty's lemma. *Appl. Anal.*, **76** (2000), 309–317.

[5] S. Buccheri. Some nonlinear elliptic problems with $L^1(\Omega)$ coefficients. *Nonlinear Anal.*, **177** (2018), 135–152.

[6] M. Degiovanni, M. Marzocchi. Quasilinear elliptic equations with natural growth and quasilinear elliptic equations with singular drift. *Nonlinear Anal.*, **185** (2019), 206–215.
[7] S. Huang, Q. Tian, J. Wang, J. Mu. Stability for noncoercive elliptic equations. *Electron. J. Differential Equations*, (2016), Paper No. 242, 11.
[8] J. Leray, J.-L. Lions. Quelques résultats de Višik sur les problèmes elliptiques nonlinéaires par les méthodes de Minty-Browder. *Bull. Soc. Math. France*, **93** (1965), 97–107.
[9] Y. Mi, S. Huang, C. Huang. Combined effects of the Hardy potential and lower order terms in fractional Laplacian equations. *Bound. Value Probl.*, (2018), Paper No. 61, 12.
[10] L. Moreno-Merida, M.M. Porzio. Existence and asymptotic behavior of a parabolic equation with $L^1$ data. *Asymptotic Analysis*, **1** (2019), 1-17.
[11] F. Oliva. Regularizing effect of absorption terms in singular problems. *J. Math. Anal. Appl.*, **472** (2019), 1136–1166.
[12] G. Stampacchia. Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. *Ann. Inst. Fourier (Grenoble)*, **15** (1965), 189–258.

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