Topology-induced containment for general linear systems on weakly connected digraphs

F. Cacace [a] M. Mattioni [b] S. Monaco [b] L. Ricciardi Celsi [b]

[a] *Università Campus Bio-Medico di Roma, Roma, Italy*

[b] *Dipartimento di Ingegneria Informatica, Automatica e Gestionale " Antonio Ruberti," Università di Roma* La Sapienza, *Rome, Italy*

Abstract

The paper deals with topology-induced containment output feedback for ensuring multi-consensus of homogeneous linear systems evolving over a weakly connected communication digraph. Starting from the extension of a recent characterization of multi-consensus, a decentralized static feedback enforcing multi-consensus is designed based on a suitable network-induced decomposition; a neighbourhood state-observer is proposed for completing the design. The results are finally illustrated over a simple simulated example.

Key words: Multi-consensus; containment control; output feedback control; linear systems.

1 Introduction

There are several emerging engineering applications that require the behaviour of single units in a multi-agent system to be differentiated into small subgroups. For example, a formation of a team of robots may require to be split into smaller subformations in order to simultaneously accomplish several tasks. Also, the temperature of a building may have to be controlled so that the rooms of different floors have distinct set points (Andreasson et al., 2014). This scenario is referred to as *multi-consensus* or *cluster consensus* and is characterized by parts of the multi-agent system simultaneously reaching different consensus states (Yu and Wang, 2010; Xiao and Wang, 2006).
In addition, multi-consensus is now momentous in brain science where, thanks to the connectivity and the structure of the brain, each area could perform specific tasks (Schnitzler and Gross, 2005), as well as in other natural systems, e.g., bird flocks or schools of fish splitting into different subgroups for avoiding predation or for foraging. Examples of multi-consensus are also found in social systems, e.g., the dynamics of different coexisting opinions or pattern formation in bacteria colonies (Blondel et al., 2010; You et al., 2009).
Previous works on multi-consensus have focused on criteria and methods to attain the desired state: in Chen et al. (2011), two multi-consensus criteria are derived for multi-agent systems with fixed and switching topology; in Lou and Hong (2012), a distributed containment control approach is devised to target a multiple leader scenario; in Han et al. (2013), the cluster state is guaranteed thanks to the introduction of different inputs to different clusters; in Qin and Yu (2013), cluster consensus in a directed topology via distributed feedback control is achieved; in Chen et al. (2015) the second-order multi-consensus problems for agents with discrete-time dynamics is studied. In particular, in Monaco and Ricciardi-Celsi (2019), with reference to the graph topology introducted in Caughman and Veerman (2006), it is shown how a multi-agent system consisting of interconnected integrators induces as many clusters as the number of reaches of the interconnecting graph plus a series of additional clusters which can be grouped to form the common part of all the reaches.

1.1 Statement of contribution

The aim of this work, with respect to the same digraph topology proposed in Monaco and Ricciardi-Celsi (2019) and introduced in Caughman and Veerman (2006), is

Email addresses: `f.cacace@unicampus.it` (F. Cacace), `mattia.mattioni@uniroma1.it` (M. Mattioni), `salvatore.monaco@uniroma1.it` (S. Monaco), `ricciardicelsi@diag.uniroma1.it` (L. Ricciardi Celsi).

- to recast the multi-consensus problem into the static containment control framework proposed in Li et al. (2015); Liu et al. (2012); Li et al. (2010);
- to extend the result of multi-consensus to the case of interconnected linear systems of order n, thus proposing a *topology-induced containment control law*;
- to extend the proposed topology-induced containment control to the output feedback scenario.

1.2 Notation

Given a square matrix M, $\sigma(M)$ denotes its spectrum. $\Re(s)$ denotes the real part of a complex number s. If $\Re(\lambda) < 0 \; \forall \lambda \in \sigma(M)$ we say that M is Hurwitz. Given a set $\mathcal{S}$, $|\mathcal{S}|$ denotes its cardinality. We denote by row(), col() and diag() the, respectively, horizontal, vertical and diagonal composition of vectors or matrices. $\mathbf{1}_n \in \mathbb{R}^n$ is a vector with entries 1. I_n is the unit matrix in $\mathbb{R}^n$. $\otimes$ denotes the Kronecker product of vectors and matrices.

2 Preliminaries

An *unweighted directed graph* (or digraph) of order N is represented by $\mathcal{G} := (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1, \dots, v_N\}$ is a finite nonempty node set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is an edge set of ordered pairs of nodes, called edges. For two distinct nodes $v_i, v_j \in \mathcal{V}$, $(v_i, v_j) \in \mathcal{E}$ if there is an edge from v_i to v_j with v_i being the *tail* and v_j being the *head* of the edge: hence, $v_i \in \mathcal{N}(v_j)$, the set of *neighbours* of v_j. We write $v_j \rightsquigarrow v_i$ if there exists a directed path from node v_j to node v_i. For $v \in \mathcal{V}$, $\mathcal{R}(v) \subset \mathcal{V}$ denotes the set of reachable nodes from v, including v itself. A digraph $\mathcal{G}$ contains a *spanning tree* $\mathcal{G}' = (\mathcal{V}, \mathcal{E}')$ if $\mathcal{E}' \subseteq \mathcal{E}$ does not contain a directed cycle and there is a *root node* v_{root} such that $\mathcal{R}(v_{\text{root}}) = \mathcal{V}$. The Laplacian matrix $\mathcal{L}$ of $\mathcal{G}$ has entries $L_{ii} = |\mathcal{N}(v_i)|$, $L_{ij} = -1$ if $(v_j, v_i) \in \mathcal{E}$, and $L_{ij} = 0$ otherwise.

We call any subset ρ of $\mathcal{V}$ a *cell* of $\mathcal{V}$. We call a collection of cells, given by $\pi = \{\rho_1, \rho_2, \dots, \rho_k\}$, a *partition* of $\mathcal{V}$ if $\rho_i \cap \rho_j = \emptyset$, whenever $i \neq j$, and $\cup_{i=1}^{k} \rho_i = \mathcal{V}$. The characteristic vector $p(\rho) \in \mathbb{R}^N$ of a cell ρ has entries $p_i(\rho) = 1$ if $v_i \in \rho$ and 0 otherwise. The characteristic matrix $P(\pi)$ of a partition $\pi = \{\rho_i\}$ is defined as $P(\pi) = \text{row}_i(p(\rho_i))$. Finally, $\mathcal{N}(v_i, \rho)$ denotes the set of neighbors of v_i in the cell ρ.

Definition 2.1 *A partition $\pi_{AE} = \{\rho_1, \rho_2, \dots, \rho_k\}$ is said to be an* almost equitable partition (AEP) *of $\mathcal{G}$ if, for each $i, j \in \{1, 2, \dots, k\}$, with $i \neq j$, there exists an integer d_{ij} such that $|\mathcal{N}(v, \rho_j)| = d_{ij}$ for all $v \in \rho_i$, where $|\mathcal{N}|$ denotes the cardinality of the set $\mathcal{N}$.*

In other words, a partition such that each node in ρ_i has the same number of neighbors in ρ_j, for all i, j with $i \neq j$, is an AEP. The property of almost equitability is equivalent to the invariance of the subspaces generated by the characteristic vectors of its cells. In particular, we can give the following equivalent characterization of an AEP π_{AE} (Monaco and Ricciardi-Celsi, 2019; Monshizadeh et al., 2015)

$$LP(\pi_{AE}) = P(\pi_{AE}) L_{\pi_{AE}} \tag{1}$$

where $L_{\pi_{AE}}$ is the Laplacian of the quotient graph of $\mathcal{G}$ over π_{AE}.

In this paper we consider *weakly connected digraphs*, i.e., digraphs whose disoriented version is connected. If $\mathcal{R}(v_i)$, the set of reachable nodes from v_i, is not strictly contained in any other $\mathcal{R}(v_j)$, then $\mathcal{R}(v_i)$ is called a *reach* of $\mathcal{G}$ and denoted with $\mathcal{R}_i$. The set $\mathcal{P}_i = \{v \in \mathcal{R}_i : \mathcal{R}(v) \equiv \mathcal{R}_i\}$ is called the set of *root nodes of* $\mathcal{R}_i$. Clearly, $\mathcal{P}_i \subseteq \mathcal{R}_i$. The multiplicity of $\lambda_0 = 0$ as an eigenvalue of $\mathcal{L}$ is equal to the number μ of distinct reaches of $\mathcal{G}$ (see Theorem 3.2 in Caughman and Veerman (2006)). Let $\mathcal{R}_1, \dots, \mathcal{R}_\mu$ denote the reaches of $\mathcal{G}$. For each reach $\mathcal{R}_i$, we define the *exclusive* part of $\mathcal{R}_i$ to be the set $\mathcal{H}_i = \mathcal{R}_i \setminus \cup_{j \neq i} \mathcal{R}_j$. We denote by $h_i = |\mathcal{H}_i|$ the number of nodes in $\mathcal{H}_i$. Likewise, we define the *common* part of $\mathcal{R}_i$ to be the cell $\rho_i = \mathcal{R}_i \setminus \mathcal{H}_i$. Let $\mathcal{C} = \cup_{i=1}^{\mu} \rho_i$, with cardinality $\delta = |\mathcal{C}|$, denote the union of the common parts of all the reaches. According to Proposition 4 and Theorem 1 of Monaco and Ricciardi-Celsi (2019), $\mathcal{C}$ can be partitioned as $\mathcal{C} = \cup_{j=\mu+1}^{j=\mu+k} \mathcal{C}_j$, $k \leq \delta$, $j \neq l \Rightarrow \mathcal{C}_j \cap \mathcal{C}_l = \emptyset$, such that

$$\pi^* = \{\mathcal{H}_1, \dots, \mathcal{H}_\mu, \mathcal{C}_{\mu+1}, \dots, \mathcal{C}_{\mu+k}\} \tag{2}$$

is the *coarsest AEP* of $\mathcal{G}$, that is, any other AEP is finer than π^*. The property that π^* is an AEP can be immediately verified from (1). Moreover, the computation of π^* can be carried out by means of a suitable modification of the algorithm proposed in Section 4 of Zhang et al. (2013) for the case of $\mu = 1$, as described in Remark 5 of Monaco and Ricciardi-Celsi (2019).

We consider a weakly directed digraph $\mathcal{G}$ with $\mu > 1$ where each node of $\mathcal{G}$ is an agent with the nth-order dynamics

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad i = 1, \dots, N \tag{3}$$

$$u_i(t) = -K \sum_{j=1}^{N} L_{ij} x_j(t), \tag{4}$$

with $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^p$ and $K \in \mathbb{R}^{p \times n}$. The complete state of the multi-agent system $X(t) = \text{col}_{i=1}^{N}(x_i(t))$ obeys the equation

$$\dot{X}(t) = (I_N \otimes A - \mathcal{L} \otimes (BK)) X(t), \tag{5}$$

where $\mathcal{L}$ is the Laplacian matric of $\mathcal{G}$. In order to investigate the dynamical properties of system (5), we perform a reordering of the variables in $X(t)$ by means of the permutation matrix T defined in Monaco and Ricciardi-Celsi (2019), Proposition 3,

$$\bar{X} = (T \otimes I_n)X = \mathrm{col}_{i=1}^{N}(\bar{x}_i(t)).$$

With this transformation, (5) becomes

$$\dot{\bar{X}}(t) = \left(I_N \otimes A - \bar{\mathcal{L}} \otimes (BK)\right) \bar{X}(t) = A_{\bar{X}}\bar{X}(t), \quad (6)$$

where $\bar{\mathcal{L}} = T\mathcal{L}T^{-1}$,

$$\bar{\mathcal{L}} = \begin{pmatrix} \bar{L}_1 & \dots & 0_{h_1 \times h_\mu} & 0_{h_1 \times h_\delta} \\ \vdots & \ddots & \vdots & \vdots \\ 0_{h_\mu \times h_1} & \dots & \bar{L}_\mu & 0_{h_\mu \times h_\delta} \\ M_1 & \dots & M_\mu & M \end{pmatrix}, \; \bar{L}_i = \begin{pmatrix} P_i & 0 \\ Q_{1i} & Q_i \end{pmatrix}. \quad (7)$$

$\bar{L}_i$'s are $h_i \times h_i$ Laplacian matrices associated with the $\mathcal{H}_i$'s, the M_i's are $\delta \times h_i$ matrices, and M is a square matrix of order δ associated with the union of the common parts of all the digraphs reaches (*i.e.* with $\mathcal{C}$). P_i is a square matrix of size $|\mathcal{P}_i|$ that refers to the root nodes $\mathcal{P}_i$ in $\mathcal{H}_i$, and Q_i a square matrix that refers to the remaining (i.e. non root) nodes.

Notice that $T^\top = T^{-1}$ since T is a permutation matrix. T is trivially obtained as the matrix that exchanges the rows and columns of $\mathcal{L}$ to make consecutive the nodes belonging to the same $\mathcal{H}_i$ (and within each $\mathcal{H}_i$ the root and non-root nodes). Clearly, T is not unique since there are in general many ways to perform this ordering.

Lemma 2.1 *The sub-graph associated to $\mathcal{P}_i$ is strongly connected.*

Proof. Any two nodes $v_1, v_2 \in \mathcal{P}_i$ have by definition the same reachable set in $\mathcal{G}$. From $\mathcal{R}(v_1) = \mathcal{R}(v_2)$ it follows $\mathcal{R}(v_1) \cap \mathcal{P}_i = \mathcal{R}(v_2) \cap \mathcal{P}_i$. Since by definition $\mathcal{R}(v)$ includes v, v_1 is in $\mathcal{R}(v_2) \cap \mathcal{P}_i$ and vice-versa. □.
It is also trivial to verify the following properties of the cells $\mathcal{H}_i$ (see also Proposition 3 in Monaco and Ricciardi-Celsi (2019)).

Lemma 2.2 *$\bar{L}_i$ has the following properties.*

(1) Zero is a simple eigenvalue of $\bar{L}_i$ and $\bar{L}_i \mathbf{1}_{h_i} = 0$, that is, $\mathbf{1}_{h_i}$ is the right eigenvector associated to 0.
(2) The left eigenvector $v_{h_i}^\top$ of $\bar{L}_i$ associated to 0 has the structure $v_{h_i}^\top = [v_{|\mathcal{P}_i|}^\top, \; 0_{h_i - |\mathcal{P}_i|}]$, where $v_{|\mathcal{P}_i|}^\top$ is the left eigenvector of P_i associated to the 0 eigenvalue.
(3) All nonzero eigenvalues of $\bar{L}_i$ have positive real parts.

Proof. The first property follows from the fact that M is non-singular (see Theorem 3.2 in Caughman and Veerman (2006)), and each $\bar{L}_i$ has a 0 eigenvalue because the sum of the rows of $\mathcal{L}$, and therefore of each $\bar{L}_i$, is 0. Consequently, the algebraic multiplicity of 0 as an eigenvalue of $\bar{L}_i$ is exactly 1 and $\mathbf{1}_{h_i}$ is the corresponding right eigenvector. The second property descends trivially from the structure of $\bar{L}_i$ in (7). The last part follows from well known properties of graphs that admit a spanning tree (see for example Ren and Beard (2005); Li et al. (2015)).

Lemma 2.3 *For any $\mathcal{H}_i$ the modified subgraph $\widetilde{\mathcal{L}}_i$ obtained by replacing the nodes in $\mathcal{P}_i$ with a unique node $v_{\mathcal{P}_i}$ admits a spanning tree with $v_{\mathcal{P}_i}$ as root.*

3 Consensus dynamics

From (6)–(7) it follows that the spectrum $\sigma(A_{\bar{X}})$ of the multi-agent system can be decomposed as

$$\begin{aligned} \sigma(A_{\bar{X}}) &= \sigma(I_\delta \otimes A - M \otimes (BK)) \\ &\cup \left(\bigcup_{i=1}^{\mu} \sigma(I_{h_i} \otimes A - \bar{L}_i \otimes (BK)) \right), \end{aligned} \quad (8)$$

and the dynamics of the portion $\bar{X}_i \in \mathbb{R}^{nh_i}$, $i = 1, \dots, \mu$ of X that correspond to the cell $\mathcal{H}_i$ is

$$\dot{\bar{X}}_i = \left(I_{h_i} \otimes A - \bar{L}_i \otimes (BK)\right) \bar{X}_i = \bar{A}_i \bar{X}_i(t). \quad (9)$$

It is useful to provide a full characterization of the spectrum of the type of matrices in the right side of (8).

Theorem 3.1 *Let $Q \in \mathbb{R}^{N \times N}$, $\sigma(Q) = \{c_i\}$, $A \in \mathbb{R}^{n \times n}$, B and K such that $BK \in \mathbb{R}^{n \times n}$, and*

$$\bar{A} = I_N \otimes A - Q \otimes (BK).$$

Then, $\sigma(\bar{A}) = \bigcup_i \sigma(A - c_i BK)$.

Proof. Given $c_i \in \sigma(Q)$, let $Qu_i = c_i u_i$ and $(A - c_i BK)x_{ij} = \lambda_{ij} x_{ij}$. We prove that $\lambda_{ij} \in \sigma(\bar{A})$ with right eigenvector $u_i \otimes x_{ij}$.

$$\begin{aligned} \bar{A}(u_i \otimes x_{ij}) =& (I_N \otimes A)(u_i \otimes x_{ij}) - (Q \otimes (BK))(u_i \otimes x_{ij}) \\ =& u_i \otimes (Ax_{ij}) - (Qu_i) \otimes (BKx_{ij}) = \\ =& u_i \otimes (Ax_{ij}) - u_i \otimes (c_i BKx_{ij}) \\ =& u_i \otimes ((A - c_i BK)x_{ij}) = \lambda_{ij}(u_i \otimes x_{ij}). \end{aligned} \quad \square$$

When Q is a Laplacian matrix the result of Theorem 3.1 coincides with the consensus conditions reported in several works (see for example Isidori (2017), Proposition 5.2, Li et al. (2010) or Lemma 1 of Zhang et al. (2011)). However, the statement in Theorem 3.1 is slightly more general and the concise proof reported above, as far as we know, original. From Theorem 3.1 it follows immediately that $\sigma(A) \subset \sigma(\bar{A}_i)$ $i = 1, \ldots, \mu$. The corresponding eigenvectors are easy to characterize.

Corollary 3.1 *The matrices $\bar{A}_i$, $i = 1, \ldots, \mu$ defined in* (9) *are such that $\sigma(A) \subset \sigma(\bar{A}_i)$. The corresponding right and left eigenvectors are $1_{h_i} \otimes u_j$ and $v_{h_i}^\top \otimes v_j^\top$, where u_j and $v_j^\top$ are the right and left eigenvectors of A and $v_{h_i}^\top$ is the left eigenvector of $\bar{L}_i$ associated to $\lambda_0 = 0$.*

Proof. Since $\lambda_0 = 0 \in \sigma(\bar{L}_i)$ (Lemma 2.2), $\sigma(A) \subset \sigma(\bar{A}_i)$ follows by setting $c_i = 0$ in Theorem 3.1. The right eigenvectors of $\bar{A}_i$ for the eigenvalues in $\sigma(A)$ are $\mathbf{1}_{h_i} \otimes u_j$, where u_j is a right eigenvector of A. Analogously,

$$\begin{aligned}(v_{h_i}^\top \otimes v_j^\top)\bar{A}_i =&(v_{h_i}^\top \otimes v_j^\top)\left(I_{h_i} \otimes A - \bar{L}_i \otimes (BK)\right)\\ =&v_{h_i}^\top \otimes (v_j^\top A) = \lambda_{ij}(v_{h_i}^\top \otimes v_j^\top). \quad \square\end{aligned}$$

For each cell $\mathcal{H}_i$ it is possible to determine a linear combination of the states of the agents in the cell whose evolution is determined solely by the matrix A, i.e., it is independent of K. We will show that it is possible to design K so that this linear combination is the consensus trajectory of nodes in $\mathcal{H}_i$ for any i. Let $v_{h_i}^\top$ be the left eigenvector of $\bar{L}_i$ associated to $\lambda_0 = 0$ and such that $v_{h_i}^\top \mathbf{1}_{h_i} = 1$. $v_{h_i}^\top$ has the structure reported in Lemma 2.2. Define the following vector in $\mathbb{R}^n$

$$\bar{x}_i^m = \left(v_{h_i}^\top \otimes I_n\right) \bar{X}_i. \tag{10}$$

Lemma 3.1 *In each cell $\mathcal{H}_i$ and for $t \geq 0$, $\bar{x}_i^m(t)$ evolves according to $\bar{x}_i^m(t) = e^{At}\bar{x}_i^m(0) = e^{At}\left(v_{h_i}^\top \otimes I_n\right)\bar{X}_i(0)$.*

Proof.

$$\begin{aligned}\dot{\bar{x}}_i^m(t) =& \left(v_{h_i}^\top \otimes I_n\right)\left(I_{h_i} \otimes A - \bar{L}_i \otimes (BK)\right)\bar{X}_i(t)\\ =&(v_{h_i}^\top \otimes A)\bar{X}_i(t) = A(v_{h_i}^\top \otimes I_n)\bar{X}_i(t) = A\bar{x}_i^m(t).\end{aligned}$$

$\square$

Remark 3.1 *Notice that, due to the structure of $v_{h_i}^\top$ in Lemma 2.2, $\bar{x}_i^m$ depends only on the state of the root nodes $\mathcal{P}_i$ in $\mathcal{H}_i$ and it may be represented as $\bar{x}_i^m = \left(v_{|\mathcal{P}_i|}^\top \otimes I_{|\mathcal{P}_i|}\right)\mathrm{col}_{j\in\mathcal{P}_i}(x_j)$.*

By resorting to standard results on rooted digraphs it is now easy to determine K so that the $\bar{X}_i(t) \to \mathbf{1}_{h_i} \otimes \bar{x}_i^m(t)$, *i.e.*, each cell $\mathcal{H}_i$ reaches consensus on $\bar{x}_i^m(t)$.

Theorem 3.2 *Given a weakly connected digraph $\mathcal{G}$ with $\mu > 1$ and the agent structure in* (3)–(4) *with (A, B) controllable, and given arbitrary symmetric and positive definite matrices $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{p\times p}$ if the control gain in* (5) *is chosen as $K = cR^{-1}B^\top P$, where P is the unique positive definite solution of the Riccati equation*

$$0 = A^\top P + PA + Q - PBR^{-1}B^\top P, \tag{11}$$

and

$$c \geq \frac{1}{2\min_i \min_{\lambda_{ij}\in\sigma(\bar{L}_i),\lambda_{ij}\neq 0}\{\Re(\lambda_{ij})\}} \tag{12}$$

then in each cell $\mathcal{H}_i$

$$\lim_{t\to\infty} \bar{X}_i(t) - \mathbf{1}_{h_i} \otimes \bar{x}_i^m(t) = 0 \tag{13}$$

$$\bar{x}_i^m(t) = \left(e^{At}\left(v_{|\mathcal{P}_i|}^\top \otimes I_{|\mathcal{P}_i|}\right)\mathrm{col}_{j\in\mathcal{P}_i}(x_j(0))\right). \tag{14}$$

where $v_{|\mathcal{P}_i|}^\top$ is the left eigenvalue of P_i associated to 0 and such that $v_{|\mathcal{P}_i|}^\top \mathbf{1}_{|\mathcal{P}_i|} = 1$.

Proof. Since the sub-graph associated to each $\mathcal{H}_i$ admits a spanning tree it follows from Lemma A.1 in Appendix A that the disagreement error dynamics is asymptotically stable if and only if all the matrices $\bar{A}_{ij} = A - \lambda_{ij}BK$, where $\lambda_{ij} \in \sigma(\bar{L}_i)\setminus\{0\}$ are Hurwitz. The choice $c_i \geq 1/(2\min\Re(\lambda_{ij}))$, $\lambda_{ij} \in \sigma(\bar{L}_i)\setminus\{0\}$ guarantees that this is the case in each cell $\mathcal{H}_i$. To see this it suffices to show that $x^*(P\bar{A}_{ij} + \bar{A}_{ij}^* P)x < 0$ for all $x \neq 0$.

$$\begin{aligned}&x^*(P\bar{A}_{ij} + \bar{A}_{ij}^*P)x\\ =&x^*(A^\top P + PA - 2c\Re(\lambda_{ij})PBR^{-1}B^\top P)x\\ \leq&x^*(A^\top P + PA - PBR^{-1}B^\top P)x = -x^*Qx < 0.\end{aligned}$$

Taking the maximum over all the cells $\mathcal{H}_i$ yields (12). The consensus trajectory (14) follows from Lemma 3.1 and Remark 3.1. $\square$

Remark 3.2 *Theorem 3.2 states that it is possible to choose a uniform K so that in each cell $\mathcal{H}_i$ the consensus trajectory depends only on the initial conditions of the root nodes $\mathcal{P}_i$ of the cell and on the matrix A. Notice that the solution of* (11) *does not depend on the graph structure, whereas c does. However, c can be computed in a fully distributed way by using, for example, the algorithm in Li et al. (2015).*

$\square$

Remark 3.3 *Invoking the results in Jameson and Kreindler (1973), for single-input agents it is always possible to prove that convergence to consensus is guaranteed by all $K = c\hat{K}$ with $c > 0$ verifying (12) and $\hat{K}$ making $A - B\hat{K}$ Hurwitz. In this case, an explicit bound on the convergency rate to the multi-consensus trajectory can be imposed to the network by suitably assigning $\sigma(A - B\hat{K})$. More in general, for multi-input systems (i.e., $u_i \in \mathbb{R}^p$) this holds true provided that $\hat{K}$ satisfies the following conditions: (i) rank $B\hat{K}$ = rank $\hat{K}$; (ii) the matrix $B\hat{K}$ has p independent eigenvectors with all nonpositive eigenvalues.*

Finally, we can state a consensus result for the nodes that belong to the cells $\mathcal{C}_{\mu+i}$.

Theorem 3.3 *In the hypotheses of Theorem 3.2, if P is the solution of* (11) *and $K = \bar{c}R^{-1}B^\top P$, where $\bar{c} =$*

$\max\{c, 1/(2\min_{\lambda\in\sigma(M)}) \Re(\lambda)\}$, *where c is defined in* (12) *and M is the matrix in* (7)*, then the trajectories of the nodes belonging to* $\mathcal{C}_{\mu+i} \subseteq \mathcal{C}$ *converge to a trajectory* $\bar{x}^m_{\mu+i}(t)$ *which is a convex combination of the consensus trajectories* $\bar{x}^m_j$ *of the* $\mathcal{H}_j$*, that is,* $\forall i = 1,\dots,k$*:*

$$\bar{x}^m_{\mu+i}(t) = \sum_{j=1}^{\mu} \alpha_{ij}\bar{x}^m_j(t),\ \alpha_{ij} \in [0,\,1],\ \sum_{j=1}^{\mu}\alpha_{ij} = 1. \quad (15)$$

Proof. Let us denote $\zeta_i = |\mathcal{C}_{\mu+i}|$, $i = 1,\dots,k$. We assume that, by a suitable choice of $\bar{X}$, the vector $\bar{X}_{\mu+i} \in \mathbb{R}^{\zeta_i n}$ collects the state of the nodes belonging to the cell $\mathcal{C}_{\mu+i}$ and that, within $\bar{X}$, $\bar{X}_{\mu+i+1}$ follows $\bar{X}_{\mu+i}$. Let $\bar{X}_\delta(t) = \mathrm{col}^k_{i=1}(\bar{X}_{\mu+i}(t)) \in \mathbb{R}^{\delta n}$ represent the stack of the state of all the nodes in $\mathcal{C}$ in this order, and $\bar{X}_\mu(t) = \mathrm{col}^\mu_{i=1}(\bar{x}^m_i(t)) \in \mathbb{R}^{\mu n}$ the stack of the consensus trajectories of the cells $\mathcal{H}_i$. With the aim of expressing in an equivalent way the thesis (15) with respect to $\bar{X}_\delta(t)$, let

$$\tilde{I}_\zeta = \mathrm{diag}^k_{i=1}(\mathbf{1}_{\zeta_i}) \in \mathbb{R}^{\delta\times k}, \quad \tilde{I}_h = \mathrm{diag}^\mu_{i=1}(\mathbf{1}_{h_i}) \in \mathbb{R}^{\bar{h}\times\mu}.$$

Finally, let $\boldsymbol{\alpha} \in \mathbb{R}^{k\times\mu}$ be the matrix with entries α_{ij}. With these premises, we can reformulate (15) as

$$\begin{aligned}&\lim_{t\to\infty}\bar{X}_\delta(t) - \left(\tilde{I}_\zeta \otimes I_n\right)(\boldsymbol{\alpha}\otimes I_n)\,\bar{X}_\mu(t)\\ &= \lim_{t\to\infty}\bar{X}_\delta(t) - \left((\tilde{I}_\zeta\boldsymbol{\alpha})\otimes I_n\right)\bar{X}_\mu(t) = 0. \end{aligned} \quad (16)$$

From (6)–(7) the dynamics of $\bar{X}_\delta(t)$ can be expressed as

$$\begin{aligned}\dot{\bar{X}}_\delta(t) &= (I_\delta\otimes A - M\otimes(BK))\,\bar{X}_\delta(t)\\ &\quad - \sum_{i=1}^{\mu}(M_i\otimes(BK))\,\bar{X}_i(t).\end{aligned} \quad (17)$$

We can apply the result $\bar{X}_i(t) \to \mathbf{1}_{h_i}\otimes\bar{x}^m_i(t)$ of Theorem 3.2 to derive the asymptotic dynamics

$$\begin{aligned}\dot{\bar{X}}_\delta(t) &= (I_\delta\otimes A - M\otimes(BK))\,\bar{X}_\delta(t)\\ &\quad - \sum_{i=1}^{\mu}(M_i\otimes(BK))\,(\mathbf{1}_{h_i}\otimes\bar{x}^m_i(t)).\end{aligned}$$

that represents the behavior of $\bar{X}_\delta$ when the cells $\mathcal{H}_i$ have reached consensus. Introducing the consensus mismatch

$$\epsilon(t) = \bar{X}_\delta(t) - \left((\tilde{I}_\zeta\boldsymbol{\alpha})\otimes I_n\right)\bar{X}_\mu(t), \quad (18)$$

our task is to prove $\epsilon(t) \to 0$. Let $\bar{M} = \mathrm{row}^\mu_{i=1}(M_i)$. From Lemma 3.1 it descends that $\dot{\bar{x}}^m_i(t) = A\bar{x}^m_i(t)$ and we obtain the asymptotic dynamics

$$\begin{aligned}\dot{\epsilon}(t) &= (I_\delta\otimes A - M\otimes(BK))\,\bar{X}_\delta(t)\\ &\quad - \sum_{i=1}^{\mu}(M_i\otimes(BK))\,(\mathbf{1}_{h_i}\otimes\bar{x}^m_i(t))\\ &\quad - \left((\tilde{I}_\zeta\boldsymbol{\alpha})\otimes I_n\right)(I_\mu\otimes A)\bar{X}_\mu(t)\\ &= (I_\delta\otimes A - M\otimes(BK))\,\bar{X}_\delta(t)\\ &\quad - \left(\bar{M}\otimes(BK)\right)(\tilde{I}_h\otimes I_n)\bar{X}_\mu(t)\\ &\quad - \left((\tilde{I}_\zeta\boldsymbol{\alpha})\otimes A\right)\bar{X}_\mu(t).\end{aligned}$$

By replacing $\bar{X}_\delta = \epsilon + \left((\tilde{I}_\zeta\boldsymbol{\alpha})\otimes I_n\right)\bar{X}_\mu$,

$$\begin{aligned}\dot{\epsilon}(t) &= (I_\delta\otimes A - M\otimes(BK))\,\epsilon(t) + \left((\tilde{I}_\zeta\boldsymbol{\alpha})\otimes A\right)\bar{X}_\mu(t)\\ &\quad - \left((M\tilde{I}_\zeta\boldsymbol{\alpha})\otimes(BK)\right)\bar{X}_\mu(t)\\ &\quad - \left(\bar{M}\otimes(BK)\right)(\tilde{I}_h\otimes I_n)\bar{X}_\mu(t)\\ &\quad - \left((\tilde{I}_\zeta\boldsymbol{\alpha})\otimes A\right)\bar{X}_\mu(t)\\ &= (I_\delta\otimes A - M\otimes(BK))\,\epsilon(t)\\ &\quad - \left((M\tilde{I}_\zeta\boldsymbol{\alpha} + \bar{M}\tilde{I}_h)\otimes(BK)\right)\bar{X}_\mu(t).\end{aligned}$$

We notice that $I_\delta\otimes A - M\otimes(BK)$ is Hurwitz. This descends from Theorem 3.1, the fact that M is nonsingular and the choice of K in the hypotheses. The proof is therefore concluded if we show that $M\tilde{I}_\zeta\boldsymbol{\alpha} + \bar{M}\tilde{I}_h = 0$. To see that this is indeed the case we resort to a result of Monaco and Ricciardi-Celsi (2019) (Proposition 3), that states that the μ right eigenvectors of $\bar{\mathcal{L}}$ associated to the 0 eigenvalue have structure

$$u_i = \begin{pmatrix}0 & \dots & 1^\top_{h_i} & 0 & \dots & (\gamma^i)^\top\end{pmatrix}^\top \quad (19)$$

where the vectors $\gamma_i \in \mathbb{R}^\delta$ are nonnegative, have components in $[0,\,1]$ and $\sum_{i=1}^{\mu}\gamma^i = \mathbf{1}_\delta$. From $\bar{\mathcal{L}}u_i = 0$ it follows that $M_i\mathbf{1}_{h_i} = -M\gamma^i$ and therefore

$$\bar{M}\tilde{I}_h = -M\mathrm{row}^\mu_{i=1}(\gamma^i).$$

Moreover, the components of the γ^i vectors associated to the nodes in the same cell $\mathcal{C}_{\mu+j}$ are identical (Theorem 1 of Monaco and Ricciardi-Celsi (2019)), and consequently $\mathrm{row}^\mu_{i=1}(\gamma^i)$ can be factorized as $\mathrm{row}^\mu_{i=1}(\gamma^i) = \tilde{I}_\zeta\boldsymbol{\alpha}$ where the entries of $\boldsymbol{\alpha}$ are in $[0,1]$ and the sum of each rows is 1. It follows that $\bar{M}\tilde{I}_h = -M\tilde{I}_\zeta\boldsymbol{\alpha}$ and the theorem is proved. □

From Theorem 3.2 and Theorem 3.3 it immediately descends the following result on the dynamics of the consensus on the whole graph $\mathcal{G}$ when $K = cR^{-1}B^\top P$.

Corollary 3.2 *The rate of convergence to* 0 *of the consensus mismatch is given by the smallest real part of the eigenvalues of the matrices* $A - \lambda_{ij} BK$*, for all* $\lambda_{ij} \in \sigma(\bar{L}_1) \cup \cdots \cup \sigma(\bar{L}_\mu) \cup \sigma(M)$.

Corollary 3.3 *In the assumptions of Theorem 3.3 if* A *is Hurwitz then: (i)* $A_{\bar{X}}$ *defined in* (6) *is Hurwitz; (ii)* $X(t)$ *is exponentially stable.*

Proof. Notice that $0 \notin \sigma(M)$ since all the zero eigenvalues of $\bar{\mathcal{L}}$ in (7) are contained in one of the $\bar{L}_i$, $i = 1, \ldots, \mu$. The proof follows trivially from the factorization (8) of the spectrum of $A_{\bar{X}}$ and from Theorem 3.1 by noticing that for $c_i = 0$ A is Hurwitz by hypothesis and for $c_i \neq 0$ the matrices $I_{h_i} \otimes A - \bar{L}_i \otimes (BK)$ and $I_\delta \otimes A - M \otimes (BK)$ are Hurwitz thanks to the choice of K and to the fact that M is non singular.

Remark 3.4 *Theorems 3.2 and 3.3 allow to handle multiple leader-based coordination of multi-agent systems rom a multi-consensus perspective. Assuming a network composed of* μ *leaders, nodes can be sorted so that the corresponding Laplacian gets the form (7) with* $P_i = 0 \in \mathbb{R}$ *for* $i = 1, \ldots, \mu$*. As a consequence, each cell* $\mathcal{H}_i$ *possesses exactly one root (being the leader) and each consensus trajectory is provided by the one of the corresponding leader. Accordingly, consensuses over* $\mathcal{C}$ *can be modulated by suitably assigning the initial condition of each leader to exploit the corresponding influence.*

4 Consensus via output feedback

In many practical applications full state information is not available for feedback design. It is therefore of interest to extend the consensus state feedback described in the previous section to the case of output feedback. Duality results for the cooperative observer design has been presented in previous works, for example Zhang et al. (2011). Following the same ideas, two main approaches can be pursued.

- Each agent is endowed with an observer to estimate its own state from local output information. State estimates are exchanged to generate the control input (*neighborhood controller and local observer*).
- Each agent runs a consensus estimator from the output information of its neighbours and generate the control input from this consensus estimate (*local controller and neighborhood observer*, Li et al. (2010)).

4.1 Neighborhood controller and local observer

The local output information available at each node is

$$y_i(t) = Cx_i(t), \tag{20}$$

where $C \in \mathbb{R}^{q\times n}$. In the hypothesis that the pair (C, A) is observable each agent may estimate its own state via a plain Luenberger observer so that $\|x_i(t) - \hat{x}_i(t)\| \to 0$ with an arbitrary exponential rate. The control input (4) is replaced by

$$u_i(t) = -K \sum_{j=1}^{N} L_{ij} \hat{x}_j(t), \tag{21}$$

that requires the nodes to exchange their own estimates rather then the actual states.

Corollary 4.1 *If the pair* (C, A) *is observable, each agent is endowed with an exponential observer and the input is generated by the controller* (21)*, then the conclusions of Theorem 3.2 and Theorem 3.3 continue to hold.*

The proof follows immediately by noticing that (21) differs from (4) only by an exponentially vanishing term.

4.2 Local controller and neighborhood observer

The results in this section extend those of Li et al. (2010) to the case of multiconsensus. Results similar to Li et al. (2010) have been presented in Zhang et al. (2011) for the case of single leader following and they can be extended in a similar way to the case of multiple leader following. At each agent the output information $y_i(t)$ in (20) is available. Each agent i computes a local consensus variable $v_i(t) \in \mathbb{R}^n$ and exchanges with its neighbors the information $\tilde{y}_i(t) = y_i(t) - Cv_i(t)$. The structure of the controller at each agent are as follows

$$u_i(t) = -Kv_i(t) \tag{22}$$

$$\dot{v}_i(t) = (A - BK)v_i(t) - F \sum_{j=1}^{N} L_{ij}(\tilde{y}_j(t) - \tilde{y}_i(t)) \tag{23}$$

$$\tilde{y}_i(t) = y_i(t) - Cv_i(t). \tag{24}$$

Theorem 4.1 *Given a weakly connected digraph* $\mathcal{G}$ *with* $\mu > 1$ *and the agent structure* (3), (20), (22)–(24)*, if: (i)* (A, B) *controllable and* (C, A) *observable; (ii)the control gain* K *is such that* $A - BK$ *is Hurwitz and the observer gain* F *is chosen as* $F = cPC^\top R_o^{-1}$ *where* P *is the unique positive definite solution of the Riccati equation*

$$0 = AP + PA^\top + Q_o - PC^\top R_o^{-1} CP \tag{25}$$

for an arbitrary choice of the positive definite symmetric matrices $Q_o \in \mathbb{R}^{n\times n}$*,* $R_o \in \mathbb{R}^{q\times q}$*; (iii)* c *satisfies* (12)*; then,* $\forall j \in \mathcal{H}_i$*,* $v_j(t) \to 0$*, and* $x_j(t) \to \bar{x}_i^m(t)$ *where* $\bar{x}_i^m(t)$ *is defined in* (14).

Proof. Let $\bar{X}_i = \mathrm{col}_j(x_j)$, $j \in \mathcal{H}_i$ be the stack of the states of nodes in $\mathcal{H}_i$, $V_i = \mathrm{col}_j(v_j)$, and $\xi_i = \mathrm{col}(\bar{X}_i, V_i)$. From (22)–(24) we obtain, with the change of variables

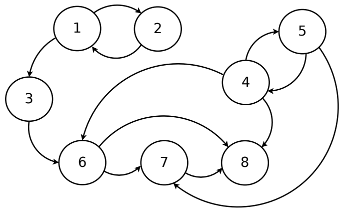

Fig. 1. Graph used in the example.

$\theta_i = \mathrm{col}(\bar{X}_i,\, V_i - \bar{X}_i)$, $\dot{\theta}_i = \tilde{A}_{\theta_i}\theta_i$

$$\tilde{A}_{\theta_i} = \begin{pmatrix} I_{h_i} \otimes (A - BK) & -I_{h_i} \otimes (BK) \\ 0 & I_{h_i} \otimes A - \bar{L}_i \otimes (FC) \end{pmatrix}. \quad (26)$$

Let us introduce the candidate consensus trajectory $\theta_i^m = [\theta_{1,i}^{m\top},\, \theta_{2,i}^{m\top}]^\top = [x_i^{m\top},\, (v_i^m - x_i^m)^\top]^\top \in \mathbb{R}^{2n}$,

$$\dot{\theta}_i^m(t) = \begin{pmatrix} A - BK & -BK \\ 0 & A \end{pmatrix} \theta_i^m(t) = A_i^m \theta_i^m(t). \quad (27)$$

Due to the triangular structure of $\tilde{A}_{\theta_i}$ and A_i^m we can proceed as in Lemma A.1 and Theorem 3.2 to obtain that, with the prescribed choice of F and c, the disagreement dynamics $(V_i - \bar{X}_i) - 1_{h_i} \otimes \theta_{2,i}^m$ is asymptotically stable. Since in (26) the matrix $I_{h_i} \otimes (A - BK)$ is Hurwitz, we also have that $\bar{X}_i \to 1_{h_i} \otimes x_i^m$. This in turn implies $V_i \to 1_{h_i} \otimes v_i^m$ and thus θ_i^m is the consensus trajectory. It is now straightforward to obtain from (27)

$$\begin{aligned} \dot{x}_i^m &= (A - BK)x_i^m - BK\theta_{2,i}^m = Ax_i^m - BKv_i^m \\ \dot{v}_i^m &= (A - BK)v_i^m. \end{aligned}$$

Thus, $v_i^m(t) \to 0$ and $x_i^m(t) \to e^{At}x_i^m(0)$, that is, (14). □

Notice that V_i does not converge to $\bar{X}_i$, i.e., v_i is an estimate of the disagreement and not an observer of x_i.

Theorem 4.2 *In the hypotheses of Theorem 4.1, if* $F = \bar{c}PC^\top R_o^{-1}$, *with* $\bar{c}$ *as in Theorem 3.3, then the conclusions of Theorem 3.3 continue to hold.*

Proof. The proof is analogue to that of Theorem 3.3 by considering the extended state $\xi_i = \mathrm{col}(\bar{X}_i,\, V_i)$.

5 Example

5.1 State feedback multiconsensus

The graph in Fig. 1 contains $|\mathcal{N}| = 8$ nodes and the coarsest equitable partition is $\pi_{AE} = \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{C}_3, \mathcal{C}_4\}$ where $\mathcal{H}_1 = \{1,\, 2,\, 3\}$, $\mathcal{H}_2 = \{4,\, 5\}$, $\mathcal{C}_3 = \{6\}$ and $\mathcal{C}_4 = \{7,\, 8\}$. The multiplicity of 0 as an eigenvalue of $\mathcal{L}$ is $\mu = 2$, thus there are two reaches, whose exclusive parts are $\mathcal{H}_1$ and $\mathcal{H}_2$ with $h_1 = 3$ and $h_2 = 2$. The common part contains $\delta = 3$ nodes and it is the union of $\mathcal{C}_3$ (with $\zeta_1 = 1$) and $\mathcal{C}_4$ (with $\zeta_2 = 2$). Thus the number of cells in the common part is $k = 2$. At each node is associated a linear system (3) of size $n = 3$ with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}. \quad (28)$$

The first two variables compose a harmonic oscillator, while the third one is an integrator. Clearly, $\sigma(A) = \{\pm i, 0\}$ thus the system is not asymptotically stable. In order to reach consensus within each cell we use the consensus state feedback (4) where K is chosen as in Theorem 3.2. Solving (11) with $Q = I_5$ and $R = 1$ yields $R^{-1}B^\top P = [2.6775\ 1.6825\ 2.2361]$. Since $\min\{\sigma(\mathcal{L}) \setminus \{0\}\} = 1$, the bound of Theorem 3.3 is $c \geq 1/2$. If we choose $c = 1$ then $K = R^{-1}B^\top P$. The plots reported in Fig. 2 for a simulation of the distributed system with $t \in [0,\, 25]$ and random initial conditions clearly show that consensus is actually reached within each cell of π_{AE} with 4 consensus trajectories. The consensus trajectories of $\mathcal{H}_1$ and $\mathcal{H}_2$ reported in the plots have been computed as in Theorem 3.2. Since the normalized left eigenvectors of $\mathcal{L}$ associated to 0 are

$$v_1^\top = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ v_2^\top = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}$$

the computation of (14) yields (subscripts refer to node numbers)

$$\begin{aligned} \bar{x}_1^m(t) &= \frac{1}{2}e^{At}\,(x_1(0) + x_2(0)) \\ \bar{x}_2^m(t) &= \frac{1}{2}e^{At}\,(x_4(0) + x_5(0))\,, \end{aligned}$$

plotted in Fig. 2. For the consensus trajectories of $\mathcal{C}_3$ and $\mathcal{C}_4$ we need the right eigenvectors of $\mathcal{L}$ associated to 0,

$$u_1 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}^\top \ u_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & \frac{1}{2} & \frac{3}{4} & \frac{3}{4} \end{pmatrix}^\top$$

Notice that the components of u_j corresponding to nodes in the same $\mathcal{C}_{\mu+i}$ are identical. Since $\gamma^1 = [\frac{1}{2}\ \frac{1}{4}\ \frac{1}{4}]^\top$, $\gamma^2 = [\frac{1}{2}\ \frac{3}{4}\ \frac{3}{4}]^\top$, the matrix $\boldsymbol{\alpha}$ has entries $(\boldsymbol{\alpha})_{11} = (\boldsymbol{\alpha})_{12} = 1/2$, $(\boldsymbol{\alpha})_{21} = 1/4$, $(\boldsymbol{\alpha})_{22} = 3/4$. Therefore,

$$\begin{pmatrix} \bar{x}_3^m(t) \\ \bar{x}_4^m(t) \end{pmatrix} = (\boldsymbol{\alpha} \otimes I_3) \begin{pmatrix} \bar{x}_1^m(t) \\ \bar{x}_2^m(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\bar{x}_1^m(t) + \frac{1}{2}\bar{x}_2^m(t) \\ \frac{1}{4}\bar{x}_1^m(t) + \frac{3}{4}\bar{x}_2^m(t) \end{pmatrix}.$$

The containment property of the consensus trajectories of $\mathcal{C}_3$ and $\mathcal{C}_4$ is clearly highlighted by the plots of Fig. 2.

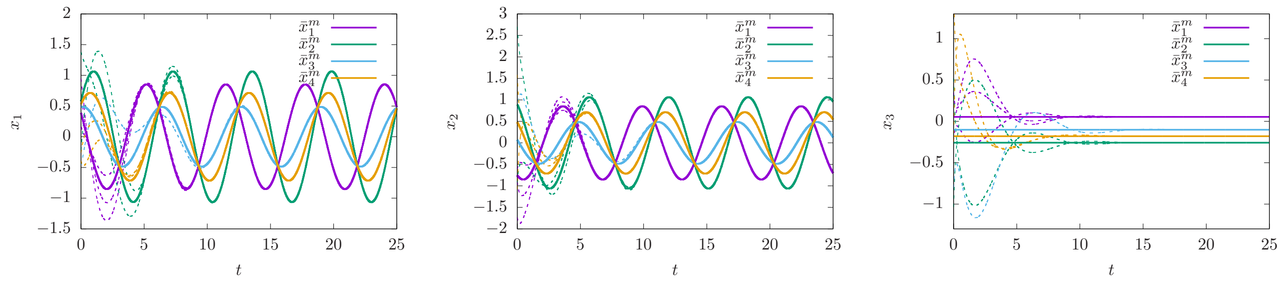

Fig. 2. Plot of the state variables of the agents of the network. Dashed lines (of identical type for nodes in the same cell of π_{AE}) represent single agents while solid lines are the theoretical consensus values.

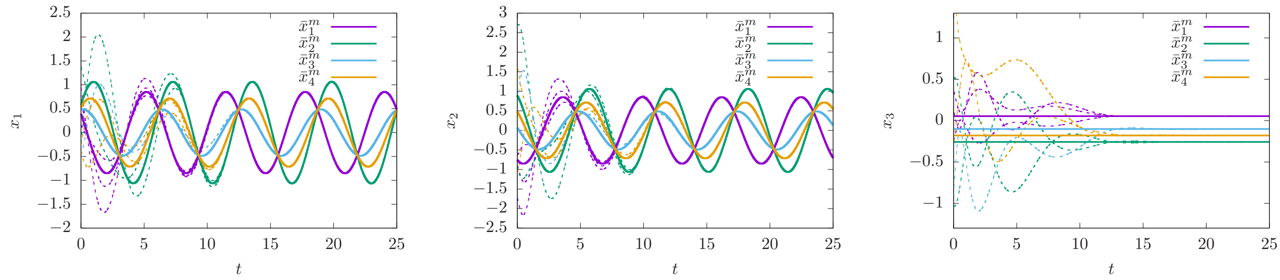

Fig. 3. Plot of the state variables of the agents of the network under local controller and neighborhood observer. Dashed lines represent agents, solid lines are the theoretical consensus.

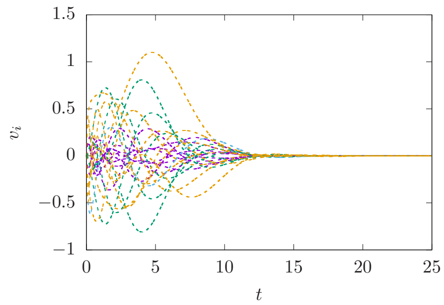

Fig. 4. Plot of the components of v_i for all the agents.

5.2 Output feedback multiconsensus

We apply the approach described in Section 4.2. The control gain $K = [1.75, 2.25, 0.75]$ assigns the eigenvalues $\sigma(A - BK) = \{-0.5, -1.0, -1.5\}$. The "observer" gain F is computed as $F = \bar{c}PC^\top R_o^{-1} = [1.683, 2.678, 2.236]^\top$, where P is the solution of (25) with $Q_o = I_5$ and $R_o = 1$, and $\bar{c} = 1$ as in the state feedback case. All the remaining parameters are as in Section 5.1. The plots in Fig. 3 show that the agents in each cell of the AEP tend to the same consensus trajectories as in the state feedback case. Fig. 4 shows that $v_i \to 0$ for all the agents as predicted.

6 Conclusions

In this paper, the multi-consensus problem is recast into the static containment control framework proposed in Li et al. (2015, 2010), with specific reference to the digraph topology introduced in Caughman and Veerman (2006) and in Monaco and Ricciardi-Celsi (2019).

In particular, we have extended the results on multi-consensus to networks of linear systems of order n and we have shown that this may be equivalently interpreted as a topology-induced containment control law. The proposed topology-induced containment controller is extended to the output feedback scenario.
Future work aims at extending those results to the case of heterogeneous multi-agent context with focus on output consensus. To this end, the concepts of mean-field and emergent dynamics (as introduced in Panteley and Loría (2017) for single consensus) may provide an interesting framework for both the understanding of the network induced behavior and, consequently, the design of decentralized feedback laws.

A A result concerning Theorem 3.2

The following result is used in the proof of Theorem 3.2. The result is already known (see for example Isidori (2017), Section 5.5, Li et al. (2010) Zhang et al. (2011)), but we provide an autonomous proof for completeness.

Lemma A.1 *Consider a network of N agents* (3)–(4) *connected by a graph $\mathcal{G}$ that admits a spanning tree. Then, the disagreement dynamics is asymptotically stable if and only if all the matrices $A - \lambda_i BK$ are Hurwitz where λ_i is any non zero eigenvalue of the Laplacian L of $\mathcal{G}$.*

Proof. Let $X(t) = \mathrm{col}_i(x_i(t)) \in \mathbb{R}^{nN}$ be the state of the network. Let $x^m = (v_0^\top \otimes I_n)X$ as in (10) where $v_0^\top L = 0$ is the left eigenvector of L associated to the 0 eigenvalue and such that $v_0^\top 1_N = 1$, which is unique because $\mathcal{G}$ admits a spanning tree. We shall prove that x^m is the consensus trajectory. Define the disagreement vector $\delta(t) = X(t) - 1_N \otimes x^m(t)$. The components of δ are linearly dependent, since

$$(v_0^\top \otimes I_n)\delta = x^m - x^m = 0. \quad \text{(A.1)}$$

Further, $\dot{\delta} = \bar{A}\delta$, where $\bar{A}$ is defined in (9), because $\dot{\delta} = (I_n \otimes A)\delta - (L \otimes (BK))\delta = \bar{A}\delta$, where we have used the property $(L \otimes (BK))X = (L \otimes (BK))\delta$ that it is easy to check. Notice that $\dot{\delta} = \bar{A}\delta$ and $\bar{A}$ is not in general Hurwitz because $\sigma(A) \subset \sigma(\bar{A})$ (Corollary 3.1). Notwithstanding, asymptotic stability of δ follows from the constraint (A.1). In fact, $\sigma(\bar{A})$ can be partitioned as in Theorem 3.1 in the union of the spectra $\sigma(A - \lambda_i BK)$. Since by hypothesis $A - \lambda_i BK$ are Hurwitz except for $\lambda_0 = 0$, the proof is concluded by showing that both δ and $\bar{A}\delta$ are orthogonal to the eigenspace in $\mathbb{R}^{nN}$ corresponding to $\sigma(A)$. In fact, the left eigenvectors of $\bar{A}$ associated to the eigenvalues of A have the form $v_0^\top \otimes v_j^\top$ (Corollary 3.1), where $v_j^\top A = \lambda_j^A v_j$ is a left eigenvector of A. Now, since $(v_0^\top \otimes v_j^\top)(L \otimes (BK)) = 0$, $(v_0^\top \otimes v_j^\top)\delta = 0$, $(v_0^\top \otimes v_j^\top)\bar{A}\delta = 0$. Thus, the dynamics of δ is influenced only by $\sigma(A - \lambda_i BK)$ for $\lambda_i \neq 0$. □

References

Andreasson, M., Dimarogonas, D., Sandberg, H., Johansson, K., 2014. Distributed control of networked dynamical systems: static feedback, integral action and consensus. IEEE Transactions on Automatic Control 59 (7), 1750–1764.

Blondel, V., Hendrickx, J., , Tsitsiklis, J., 2010. Continuous-time average-preserving opinion dynamics with opinion-dependent communications. SIAM Journal on Control and Optimization 48 (8), 5214–5240.

Caughman, J., Veerman, J., 2006. Kernels of directed graph Laplacians. The Electronic Journal of Combinatorics 13 (1), 1–8.

Chen, J., Guan, Z., He, D., Chi, M., , Cheng, X., 2015. Multi-consensus for second-order multi-agent systems based on sampled-position information. IET Control Theory & Applications 9 (3), 358–366.

Chen, Y., Lu, J., Han, F., , Yu, X., 2011. On the cluster consensus of discrete-time multi-agent systems. Systems & Control Letters 60 (7), 517–523.

Han, Y., Lu, W., , Chen, T., 2013. Cluster consensus in discrete-time networks of multiagents with intercluster nonidentical inputs. IEEE Transactions on Neural Networks and Learning Systems 24 (4), 566–578.

Isidori, A., 2017. Lectures in feedback design for multivariable systems. Springer.

Jameson, A., Kreindler, E., 1973. Inverse problem of linear optimal control. SIAM Journal on Control 11 (1), 1–19.

Li, Z., Duan, Z., Chen, G., Huang, L., 2010. Consensus of multiagent systems and synchronization of complex networks: A unified viewpoint. IEEE Trans. on Circuits and Systems I: Regular Papers 57 (1), 213–224.

Li, Z., Wen, G., Duan, Z., Ren, W., 2015. Designing fully distributed consensus protocols for linear multiagent systems with directed graphs. IEEE Trans. on Automatic Control 60 (4), 1152–1157.

Liu, H., Xie, G., Wang, L., 2012. Containment of linear multi-agent systems under general interaction topologies. Systems & Control Letters 61 (4), 528–534.

Lou, Y., Hong, Y., 2012. Target containment control of multi-agent systems with random switching interconnection topologies. Automatica 48 (5), 879–885.

Monaco, S., Ricciardi-Celsi, L., 2019. On multiconsensus and almost equitable graph partitions. Automatica 103, 53–61.

Monshizadeh, N., Zhang, S., Camlibel, M., 2015. Disturbance decoupling problem for multi-agent systems: A graph topological approach. Systems & Control Letters 76, 35–41.

Panteley, E., Loría, A., 2017. Synchronization and dynamic consensus of heterogeneous networked systems. IEEE Transactions on Automatic Control 62 (8), 3758–3773.

Qin, J., Yu, C., 2013. Cluster consensus control of generic linear multi-agent systmes under directed topology with acyclic partition. Automatica 49 (9), 2898–2905.

Ren, W., Beard, R., 2005. Consensus seeking in multiagent systems under dynamically changing interaction topologies. IEEE Trans. on Automatic Control 50 (5), 655–661.

Schnitzler, A., Gross, J., 2005. Normal and pathological oscillatory communication in the brain. Nature Reviews Neuroscience 6 (4), 285.

Xiao, F., Wang, L., 2006. Consensus behavior of agents in networked systems under general communication topologies. In: 2006 IEEE Conference on Computer Aided Control System Design, 2006 IEEE International Conference on Control Applications, 2006 IEEE International Symposium on Intelligent Control. IEEE, pp. 862–867.

You, S., Kwon, D., Park, Y., Kim, S., Chung, M., , Kim, C., 2009. Collective behaviors of two-component swarms. Journal of Theoretical Biology 261 (3), 494–500.

Yu, J., Wang, L., 2010. Group consensus in multi-agent systems with switching topologies and communication delays. Systems & Control Letters 59 (6), 340–348.

Zhang, H., Lewis, F., Das, A., 2011. Optimal design for synchronization of cooperative systems: state feedback, observer and output feedback. IEEE Trans. on Automatic Control 56 (8), 1948–1952.

Zhang, S., Cao, M., Camlibel, M., 2013. Upper and lower bounds for controllable subspaces of networks of diffusively coupled agents. IEEE Transactions on Automatic control 59 (3), 745–750.