A QUASILINEAR SINGULAR ELLIPTIC PROBLEM RELATED TO THE KARDAR-PARISI-ZHANG EQUATION

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Abstract. In this paper we will study existence and properties of solutions for the nonlinear Dirichet problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) = b(x)\,\dfrac{|\nabla u|}{u^\theta} + f(x), & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $0 < \theta \le 1$, M is a uniformly elliptic and bounded matrix, and b, f are functions in some Lebesgue spaces.

1. Introduction

The approach of Kardar-Parisi-Zhang in the study of the equation

$$-\Delta v = \sqrt{1 + |\nabla v|^2} + f(x) \tag{1.1}$$

hinges on the Taylor expansion

$$\sqrt{1 + |\nabla v|^2} \approx 1 + \frac{|\nabla v|^2}{2} \tag{1.2}$$

when the gradient is small. The equation

$$w_t - \Delta w = |\nabla w|^2 + f$$

is known as the *KPZ equation* and it is relevant to define a new *universality class* in Statistical Mechanics. See [5] and [13]. Even if with this change the physical discussion becomes more relevant, from the PDE's point of view this starting point is disadvantageous since a nonlinear term of order two growth takes the place of a nonlinear term of order one. However quasilinear elliptic Dirichlet having lower order terms with quadratic growth with respect to the gradient have been studied in several papers; see e.g. [11].

Notice that in the modelization we have the growth, which is produced in the direction to the interface and the diffusion according with the medium. For instance, in the paper [6] a physical model with porous media diffusion is proposed in one spatial dimension.

Here, we consider an anisotropic extension of the stationary model of the Kardar-Parisi-Zhang equation with a porous media type diffusion, that is

$$-\Delta v^s = \sqrt{1 + |\nabla v|^2} + f(x), \quad s > 1. \tag{1.3}$$

If v is a positive solution (which is true if, for example, $f \ge 0$), setting $u = v^s$ we obtain

$$-\Delta u = \sqrt{1 + \frac{1}{s^2}\Big(\frac{|\nabla u|}{u^{1-\frac{1}{s}}}\Big)^2} + f(x), \tag{1.4}$$

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which is a singular equation if $s > 1$. It is worthy to point out that if $0 < s < 1$ the equation (1.3) represents the so called *fast diffusion* model and in this case (1.2) becomes not singular. We will restrict ourselves to the case $s > 1$.

For the equation (1.4) we will not use the Taylor expansion approach, which leads to the singular quadratic equation

$$-\Delta u = \frac{1}{2\,s^2}\Big(\frac{|\nabla u|}{u^{1-\frac{1}{s}}}\Big)^2 + g(x),$$

but we will consider the actual growth of the lower order term, which leads to the (still) singular equation

$$-\Delta u = \frac{|\nabla u|}{s\,u^{1-\frac{1}{s}}} + g(x), \tag{1.5}$$

having a linear growth with respect to the gradient. Thus in this paper we study the boundary value problem (2.5) below.

In order to prove existence of weak or distributional solutions, our method consists in approximating the singular equation with a nonsingular one, and in proving *a priori* estimates on a sequence of approximating solutions. Then, since the equation has a nonlinear lower order term, some extra work is needed in order to prove the compactness of the sequence of approximating solutions and of the lower order terms. A further difficulty is due to the fact that, if $s > 2$ in (1.5) (i.e., if the exponent of the function u is larger than $\frac{1}{2}$), then we are only able to prove *local* estimates on the lower order term. Note furthermore that we do dot use comparison methods.

The case $s = 1$ corresponds to linear diffusion, the equation (1.5) is not singular, and the above problem is strongly related to some existence results proved in [8] and [17]. However, the homogeneity of the principal part and of the lower order term are equal (one and one), while if $s \neq 1$ we have that the principal part has a larger growth with respect to the lower order term: one, versus $1 - (1 - \frac{1}{s}) = \frac{1}{s}$. In this paper we wil not deal with the case $s = 1$, corresponding to $\theta = 0$ below.

Some results for a problem related to (1.5) and with lower order term $\frac{|\nabla u|^q}{u^\theta}$, $q > 1$, have been obtained in the paper [1], even for the fast diffusion equation, by using comparison and *a priori* estimates (see also [4], [7] and [12] if the singular lower order term does not depend on the gradient). The parabolic equation with this type of nonlinearity has been studied in the above quoted papers [2] and [3].

2. Setting and approximation

Let Ω be a bounded, open subset of $\mathbb{R}^N$, α, β in $\mathbb{R}^+$ and $M : \Omega \times \mathbb{R} \to \mathbb{R}^{N^2}$, be a bounded and measurable matrix-valued function such that

$$\alpha|\xi|^2 \leq M(x)\,\xi\,\xi\,, \qquad |M(x)| \leq \beta\,, \quad \text{a.e. } x \in \Omega\,,\ \forall\,\xi \in \mathbb{R}^N\,. \tag{2.1}$$

We assume that

$$0 \leq\ b(x) \in L^\infty(\Omega), \tag{2.2}$$

$$0 \leq f(x) \in L^m(\Omega),\ m > 1, \tag{2.3}$$

and

$$0 < \theta \leq 1. \tag{2.4}$$

In this paper we study the existence of positive solutions of the boundary value problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) = b(x)\,\dfrac{|\nabla u|}{u^\theta} + f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

that is: u belongs to some Sobolev space $W_0^{1,q}(\Omega)$, $1 < q \le 2$, $\frac{|\nabla u|}{u^\theta}$ belongs (either locally, or globally) to some Lebesgue space $L^r(\Omega)$ and

$$\int_\Omega M(x)\nabla u\nabla\varphi = \int_\Omega b(x)\,\frac{|\nabla u|}{u^\theta}\varphi + \int_\Omega f(x)\varphi(x), \qquad \forall\,\varphi\in C_c^1(\Omega). \tag{2.6}$$

We will prove the existence of a solution by approximation. In order to do that, let $f_n(x) = \frac{f(x)}{1+\frac{1}{n}f(x)}$ and let u_n be a weak solution of the Dirichlet problem

$$u_n \in W_0^{1,2}(\Omega):\ -\operatorname{div}(M(x)\nabla u_n) = b(x)\,\frac{|\nabla u_n|}{(1+\frac{1}{n}\,|\nabla u_n|)(\frac{1}{n}+|u_n|)^\theta} + f_n(x).$$

Since, for every $n\in\mathbb{N}$, $g_n(s,\xi) = \frac{|\xi|}{(1+\frac{1}{n}|\xi|)(\frac{1}{n}+s)^\theta}$ and $f_n(x)$ are bounded functions, the existence of a weak solution $u_n \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ is a consequence of the Schauder theorem (see also the general results of [16]). Moreover $u_n \ge 0$, since the right hand side is positive. Thus we can write the above Dirichlet problem as

$$u_n \in W_0^{1,2}(\Omega):\ -\operatorname{div}(M(x)\nabla u_n) = b(x)\,\frac{|\nabla u_n|}{(1+\frac{1}{n}\,|\nabla u_n|)(\frac{1}{n}+u_n)^\theta} + f_n(x) \tag{2.7}$$

Now, let w be the unique weak bounded solution of the Dirichlet problem

$$w \in W_0^{1,2}(\Omega):\ -\operatorname{div}(M(x)\nabla w) = \frac{f(x)}{1+f(x)}\,.$$

Since $\frac{f(x)}{1+f(x)} \ge 0$, the strong maximum principle (see [14]) implies that $w > 0$ in Ω, in the sense that for every $\omega \subset\subset \Omega$ there exists $m_\omega > 0$ such that

$$w(x) \ge m_\omega \quad \forall\,x\in\omega\,.$$

Then we have

$$-\operatorname{div}(M(x)\nabla u_n) = b(x)\,\frac{|\nabla u_n|}{(1+\frac{1}{n}|\nabla u_n|)(\frac{1}{n}+u_n)^\theta} + f_n \ge \frac{f}{1+\frac{1}{n}f} \ge \frac{f}{1+f}\,,$$

so that $u_n \ge w$. Therefore,

(2.8) for every $\omega\subset\subset\Omega$ there exists $m_\omega > 0$ such that $u_n \ge m_\omega > 0$ in ω.

3. A priori estimates

We recall the definition of the Sobolev conjugate exponent $m^* = \frac{Nm}{N-m}$ and of

$$T_k(s) = \begin{cases} s, & \text{if } |s|\le k; \\ k\frac{s}{|s|}, & \text{if } |s|>k, \end{cases} \qquad G_k(s) = s - T_k(s).$$

Our first result deals with *a priori* estimates on the sequence $\{u_n\}$ of solutions of (2.7).

Lemma 3.1. *We assume (2.1), (2.2), (2.3) and (2.4). Then, if* $1 < m < \frac{N}{2}$,

$$\text{the sequence } \{u_n\} \text{ is bounded in } L^{m^{**}}(\Omega) \text{ and } W_0^{1,q}(\Omega), \text{ with } q = \min(m^*, 2), \tag{3.1}$$

and

$$\text{the sequence } \left\{ \frac{|\nabla u_n|}{u_n^\theta} \right\} \text{ is bounded in } L^q_{\mathrm{loc}}(\Omega). \tag{3.2}$$

If $m > \frac{N}{2}$ *then*

$$\text{the sequence } \{u_n\} \text{ is bounded in } L^\infty(\Omega) \text{ and } W_0^{1,2}(\Omega), \tag{3.3}$$

and

$$\text{the sequence } \left\{ \frac{|\nabla u_n|}{u_n^\theta} \right\} \text{ is bounded in } L^2_{\mathrm{loc}}(\Omega). \tag{3.4}$$

Remark 3.2. If $m = \frac{N}{2}$, a consequence of the above result and of the well-known inclusions of Lebesgue spaces is that the sequence $\{u_n\}$ is bounded in $L^p(\Omega)$ for every $p > 1$. We will not prove results of exponential summability for the sequence $\{u_n\}$ (which are standard for elliptic equations with data in $L^m(\Omega)$, $m = \frac{N}{2}$).

Proof.

Here, and in the following, we will denote by C various constants whose values depend on the data of the problem (typically, Ω, N and m), but never on n.

First case: $1 < m < \frac{N}{2}$.

Define, for $h > 0$ and $s > 0$,

$$\psi_h(s) = \frac{1}{h} T_h(G_1(s)) = \begin{cases} 0, & \text{if } 0 \le s < 1; \\ \dfrac{s-1}{h}, & \text{if } 1 \le s < 1+h; \\ 1, & \text{if } s \ge 1+h. \end{cases}$$

Let $\lambda > \frac{1}{2}$, and choose $v_n = u_n^{2\lambda-1}\psi_h(u_n)$ as test function in the weak formulation of (2.7); this choice is possible since every u_n belongs to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. We have, dropping a first positive term, and since $f_n \le f$,

$$\alpha(2\lambda-1)\int_\Omega |\nabla u_n|^2\, u_n^{2\lambda-2}\psi_h(u_n) \le \int_\Omega b(x)\frac{|\nabla u_n|}{(1+\frac{1}{n}|\nabla u_n|)(\frac{1}{n}+u_n)^\theta}\, u_n^{2\lambda-1}\,\psi_h(u_n)$$
$$+\int_\Omega f_n\, u_n^{2\lambda-1}\psi_h(u_n) \le \|b\|_{L^\infty(\Omega)}\int_\Omega |\nabla u_n| u_n^{\lambda-1}\psi_h(u_n) u_n^{\lambda-\theta} + \int_\Omega f\, u_n^{2\lambda-1}\psi_h(u_n).$$

Using Young inequality, with $0 < B < \alpha(2\lambda-1)$, we get

$$[\alpha(2\lambda-1)-B]\int_\Omega |\nabla u_n|^2 u_n^{2\lambda-2}\psi_h(u_n) \le \frac{1}{4B}\|b\|^2_{L^\infty(\Omega)}\int_\Omega \psi_h(u_n) u_n^{2(\lambda-\theta)} + \int_\Omega f u_n^{2\lambda-1}\psi_h(u_n).$$

Letting h tend to zero, and using Fatou lemma in the left hand side and Lebesgue theorem in the right hand side, we deduce that

$$C\int_{\{u_n\ge1\}} |\nabla u_n|^2\, u_n^{2\lambda-2} \le \int_{\{u_n\ge1\}} u_n^{2(\lambda-\theta)} + \int_{\{u_n\ge1\}} f\, u_n^{2\lambda-1}. \tag{3.5}$$

We now remark that for every $s \ge 1$ and $\delta > 0$, there exists $C_\delta > 0$ such that

$$s^{2(\lambda-\theta)} \le \delta s^{2\lambda} + C_\delta.$$

The inequality is trivially true if $\theta \geq \lambda$, while is a consequence of Young inequality if $\lambda > \theta$. Therefore, from (3.5) we deduce that

$$\int_{\{u_n\geq 1\}} |\nabla u_n|^2\, u_n^{2\lambda-2} \leq \delta \int_{\{u_n\geq 1\}} u_n^{2\lambda} + C_\delta + \int_{\{u_n\geq 1\}} f\, u_n^{2\lambda-1}. \tag{3.6}$$

Thanks to the fact that $0 \leq u_n = T_1(u_n) + G_1(u_n) \leq 1 + G_1(u_n)$, and to Poincaré inequality, we thus have that

$$\begin{aligned} C\int_\Omega |\nabla G_1(u_n)^\lambda|^2 &\leq \delta \int_\Omega G_1(u_n)^{2\lambda} + C + \int_\Omega f\, G_1(u_n)^{2\lambda-1} \\ &\leq \frac{\delta}{\lambda_1}\int_\Omega |\nabla G_1(u_n)^\lambda|^2 + C + \int_\Omega f\, G_1(u_n)^{2\lambda-1}, \end{aligned}$$

where λ_1 is the Poincaré constant for Ω (i.e., the first eigenvalue of the Laplacian with homogeneous Dirichlet boundary conditions). Choosing δ small enough, we thus have

$$\int_\Omega |\nabla G_1(u_n)^\lambda|^2 \leq C + C\int_\Omega f\, G_1(u_n)^{2\lambda-1}.$$

We now follow [9], and choose $\lambda = \frac{m^{**}}{2^*}$, which is admissible since $m > 1$ implies $\lambda = \frac{m^{**}}{2^*} > \frac{N}{N-2}\,\frac{N-2}{2N} = \frac{1}{2}$. Note that with such a choice, we have that $\lambda\, 2^* = m^{**}$, and $(2\lambda-1)m' = m^{**}$. By Sobolev and Hölder inequalities, we thus have

$$\begin{aligned} \mathcal{S}\Big(\int_\Omega G_1(u_n)^{m^{**}}\Big)^{\frac{2}{2^*}} &\leq \int_\Omega |\nabla G_1(u_n)^\lambda|^2 \leq C + C\int_\Omega f\, G_1(u_n)^{2\lambda-1} \\ &\leq C + C\,\|f\|_{L^m(\Omega)}\Big(\int_\Omega G_1(u_n)^{m^{**}}\Big)^{\frac{1}{m'}}. \end{aligned}$$

Since $\frac{2}{2^*} > \frac{1}{m'}$ being $m < \frac{N}{2}$, from the previous inequality we deduce that

$$\|G_1(u_n)\|_{L^{m^{**}}(\Omega)} \leq C(\|f\|_{L^m(\Omega)}). \tag{3.7}$$

Note that from the boundedness of $\{G_1(u_n)\}$ in $L^{m^{**}}(\Omega)$ it trivially follows the boundedness of $\{u_n\}$ in $L^{m^{**}}(\Omega)$ since, as before, $0 \leq u_n \leq 1 + G_1(u_n)$.

Suppose now that $m \geq \frac{2N}{N+2}$, so that $\lambda \geq 1$. From (3.6) and (3.7) (note that the right hand side is bounded) we have that

$$\int_\Omega |\nabla G_1(u_n)|^2 \leq \int_{\{u_n\geq 1\}} |\nabla u_n|^2\, u_n^{2\lambda-2} \leq C\,,$$

so that the sequence $\{G_1(u_n)\}$ is bounded in $W_0^{1,2}(\Omega)$. If on the other hand $1 < m < \frac{2N}{N+2}$, then $\lambda < 1$ and we have to proceed differently. If $q < 2$ we have, by Hölder inequality,

$$\int_\Omega |\nabla G_1(u_n)|^q = \int_\Omega \frac{|\nabla G_1(u_n)|^q}{u_n^{q(\lambda-1)}}\, u_n^{q(\lambda-1)} \leq \Big(\int_{\{u_n\geq 1\}} |\nabla u_n|^2\, u_n^{2\lambda-2}\Big)^{\frac{q}{2}} \Big(\int_{\{u_n\geq 1\}} u_n^{\frac{2q(\lambda-1)}{2-q}}\Big)^{\frac{2-q}{2}}.$$

Choosing $q = m^*$ we have that $\frac{2q(\lambda-1)}{2-q} = m^{**}$, so that the above inequality becomes, thanks to (3.6) and (3.7),

$$\int_\Omega |\nabla G_1(u_n)|^{m^*} \leq C.$$

Summing up, we have therefore proved that

(3.8) the sequence $\{G_1(u_n)\}$ is bounded in $L^{m^{**}}(\Omega)$ and in $W_0^{1,q}(\Omega)$, $q = \min(m^*, 2)$.

On the other hand, the use of $T_1(u_n)$ as test function in (2.7) gives (here we use that $\theta \le 1$)

$$\begin{aligned}\alpha\int_\Omega |\nabla T_1(u_n)|^2 &\le \|b\|_{L^\infty(\Omega)}\int_\Omega \frac{|\nabla T_1(u_n)|}{(\frac{1}{n}+u_n)^\theta}\,T_1(u_n) + \|b\|_{L^\infty(\Omega)}\int_\Omega |\nabla G_1(u_n)| + \int_\Omega f \\ &\le \|b\|_{L^\infty(\Omega)}\int_\Omega |\nabla T_1(u_n)| + \|b\|_{L^\infty(\Omega)}\int_\Omega |\nabla G_1(u_n)| + \int_\Omega f,\end{aligned}$$

which implies (thanks to (3.8)) that the the sequence $\{T_1(u_n)\}$ is bounded in $W_0^{1,2}(\Omega)$. This estimate and the estimate (3.8) give (3.1).

The proof of (3.2) is then a simple consequence of (2.8) and (3.1): if $\omega \subset\subset \Omega$, then

$$\int_\omega \left(\frac{|\nabla u_n|}{u_n^\theta}\right)^q \le \frac{1}{m_\omega^\theta}\int_\omega |\nabla u_n|^q \le \frac{1}{m_\omega^\theta}\int_\Omega |\nabla u_n|^q \le C_\omega .$$

Second case: $m > \frac{N}{2}$.

Let $k > 1$, and choose $v_n = G_k(u_n)$ as test function in (2.7). We obtain, using (2.1) and (2.2),

$$\begin{aligned}\alpha\int_{\{u_n\ge k\}} |\nabla u_n|^2 &\le \|b\|_{L^\infty(\Omega)}\int_{\{u_n\ge k\}} |\nabla u_n|\frac{G_k(u_n)}{u_n^\theta} + \int_{\{u_n\ge k\}} f\,G_k(u_n) \\ &\le \frac{1}{k^\theta}\|b\|_{L^\infty(\Omega)}\int_{\{u_n\ge k\}} |\nabla u_n|\,G_k(u_n) + \int_{\{u_n\ge k\}} f(x)\,G_k(u_n).\end{aligned}$$

By Young and Poincaré inequalities, we have

$$\int_{\{u_n\ge k\}} |\nabla u_n|\,G_k(u_n) \le \frac{1}{2}\int_{\{u_n\ge k\}} |\nabla u_n|^2 + \frac{1}{2}\int_{\{u_n\ge k\}} G_k(u_n)^2 \le \frac{1+\lambda_1}{2\lambda_1}\int_{\{u_n\ge k\}} |\nabla u_n|^2 .$$

Therefore,

$$\left(\alpha - \frac{1}{k^\theta}\frac{\|b\|_{L^\infty(\Omega)}(1+\lambda_1)}{2\lambda_1}\right)\int_{\{u_n\ge k\}} |\nabla u_n|^2 \le \int_{\{u_n\ge k\}} f\,G_k(u_n).$$

Thus, if $k > k_0$, with

$$k_0^\theta = \frac{\|b\|_{L^\infty(\Omega)}(1+\lambda_1)}{\alpha\lambda_1},$$

we have

$$\frac{\alpha}{2}\int_{\{u_n\ge k\}} |\nabla u_n|^2 \le \int_{\{u_n\ge k\}} f\,G_k(u_n).$$

From this inequality one can follow the proof of [18], Théorème 4.1, to prove that the sequence $\{u_n\}$ is bounded in $L^\infty(\Omega)$, as desired. $\square$

If $0 < \theta < \frac{1}{2}$, the estimates on the right hand side $\frac{|\nabla u_n|}{u_n^\theta}$ are not only *local*, but also *global*.

Lemma 3.3. *Let $0 < \theta < \frac{1}{2}$, and suppose that (2.1), (2.2), (2.3) hold true. Then, if*

$$r = \frac{Nm}{N(1-\theta) - m(1-2\theta)},$$

we have that

$$\text{the sequence } \left\{\frac{|\nabla u_n|}{u_n^\theta}\right\} \text{ is bounded } \begin{cases} \text{in } L^r(\Omega) \text{ if } 1 < m < \frac{2N(1-\theta)}{N+2-4\theta}, \\ \text{in } L^2(\Omega) \text{ if } m \ge \frac{2N(1-\theta)}{N+2-4\theta}. \end{cases} \tag{3.9}$$

Proof. We fix $\lambda > \frac{1}{2}$, let $0 < \varepsilon < \frac{1}{n}$, and choose $v_n = (u_n+\varepsilon)^{2\lambda-1} - \varepsilon^{2\lambda-1}$ as test function in (2.7); this choice is possible since every u_n belongs to $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$. We obtain, dropping some negative terms, and since $\varepsilon < \frac{1}{n}$,

$$\alpha(2\lambda-1)\int_\Omega |\nabla u_n|^2 (u_n+\varepsilon)^{2\lambda-2} \le \int_\Omega b(x)\frac{|\nabla u_n|}{(1+\frac{1}{n}|\nabla u_n|)(\frac{1}{n}+u_n)^\theta}(u_n+\varepsilon)^{2\lambda-1}$$
$$+\int_\Omega f_n\,(u_n+\varepsilon)^{2\lambda-1} \le \|b\|_{L^\infty(\Omega)}\int_\Omega |\nabla u_n|(u_n+\varepsilon)^{(\lambda-1)+(\lambda-\theta)} + \int_\Omega f\,(u_n+\varepsilon)^{2\lambda-1},$$

where in the last passage we have used that $0 \le f_n \le f$. Using Young inequality, we thus obtain

$$\frac{\alpha(2\lambda-1)}{2}\int_\Omega |\nabla u_n|^2 (u_n+\varepsilon)^{2\lambda-2} \le C\int_\Omega (u_n+\varepsilon)^{2(\lambda-\theta)} + C\int_\Omega f\,(u_n+\varepsilon)^{2\lambda-1}.$$

Letting ε tend to zero, and using Fatou lemma (in the left hand side) and Lebesgue theorem (in the right one, recall that u_n is in $L^\infty(\Omega)$), we arrive at

$$\int_\Omega |\nabla u_n|^2\, u_n^{2\lambda-2} \le C\int_\Omega u_n^{2(\lambda-\theta)} + C\int_\Omega f\,u_n^{2\lambda-1}.$$

Since now our assumption is $0 < \theta < \frac{1}{2}$ and $\lambda > \frac{1}{2}$, we have that $\lambda > \theta$; therefore, by Young inequality we have that, for $\delta > 0$,

$$\int_\Omega |\nabla u_n|^2\, u_n^{2\lambda-2} \le \delta\int_\Omega u_n^{2\lambda} + C_\delta + C\int_\Omega f\,u_n^{2\lambda-1} \le \frac{\delta}{\lambda_1}\int_\Omega |\nabla u_n|^2 u_n^{2\lambda-2} + C_\delta + C\int_\Omega f\,u_n^{2\lambda-1},$$

where in the last inequality we have used Poincaré inequality. Thus, if δ is small enough, we have

$$\int_\Omega |\nabla u_n|^2\, u_n^{2\lambda-2} \le C + C\int_\Omega f\,u_n^{2\lambda-1}.$$

If $1 < s < \frac{2N}{N+2}$, the choice $\lambda(s) = \frac{s^{**}}{2^*}$ implies $\frac{1}{2} < \lambda(s) < 1$ and (reasoning as in the proof of Lemma 3.1)

$$\int_\Omega \frac{|\nabla u_n|^2}{u_n^{2(1-\lambda(s))}} \le C(\|f\|_{L^s(\Omega)}). \tag{3.10}$$

To conclude, we have to distinguish among two cases.

First case: $m \ge \frac{2N(1-\theta)}{N+2-4\theta}$.

If $s = \overline{m} = \frac{2N(1-\theta)}{N+2-4\theta}$, we have that $\lambda(s) = 1-\theta$, and so (3.10) becomes

$$\int_\Omega \Big(\frac{|\nabla u_n|}{u_n^\theta}\Big)^2 \le C(\|f\|_{L^{\overline{m}}(\Omega)}), \tag{3.11}$$

which is (3.9) if $m = \overline{m}$. Since Ω has finite measure, if $m > \overline{m}$ and if f belongs to $L^m(\Omega)$, then it is also in $L^{\overline{m}}(\Omega)$, so that (3.11) still holds for these values of m.

Second case: $1 < m < \frac{2N(1-\theta)}{N+2-4\theta}$.

In this case, since $\theta > 0$ and $N > 2$, we have $m < \frac{2N}{N+2}$. Let $1 < r < 2$; then, by Hölder inequality with exponents $\frac{2}{r}$ and $\frac{2}{2-r}$, we have

$$\int_\Omega \frac{|\nabla u_n|^r}{u_n^{r\theta}} = \int_\Omega \frac{|\nabla u_n|^r}{u_n^{r(1-\lambda(m))}}\, u_n^{r(1-\lambda(m)-\theta)} \le \Big(\int_\Omega \frac{|\nabla u_n|^2}{u_n^{2(1-\lambda(m))}}\Big)^{\frac{r}{2}} \Big(\int_\Omega u_n^{\frac{2r(1-\lambda(m)-\theta)}{2-r}}\Big)^{\frac{2-r}{2}}.$$

Using (3.10) with $s = m$, which is admissible since $m < \frac{2N}{N+2}$, we thus obtain

$$\int_\Omega \frac{|\nabla u_n|^r}{u_n^{r\theta}} \leq C(\|f\|_{L^m(\Omega)}) \left(\int_\Omega u_n^{\frac{2r(1-\lambda(m)-\theta)}{2-r}} \right)^{\frac{2-r}{2}}. \tag{3.12}$$

We now choose $r = r(m)$ such that $\frac{2r(m)(1-\lambda(m)-\theta)}{2-r(m)} = m^{**}$, that is $r(m) = \frac{Nm}{N(1-\theta)-m(1-2\theta)}$; the assumptions on m, and the fact that $r(m)$ is increasing, imply that

$$1 < \frac{N}{N(1-\theta)-(1-2\theta)} = r(1) < r(m) < r\left(\frac{2N(1-\theta)}{N+2-4\theta}\right) = 2,$$

so that from (3.12) we obtain that

$$\int_\Omega \left(\frac{|\nabla u_n|}{u_n^\theta}\right)^r \leq C(\|f\|_{L^m(\Omega)}),$$

as desired. $\square$

4. Existence results

In this final section we are going to prove the existence of a distributional solution for (2.5).

Theorem 4.1. *Assume* (2.1), (2.2), (2.3) *and* (2.4). *Then there exists a distributional solution u of* (2.5), *with*

$$u \quad \text{in} \begin{cases} L^{m^{**}}(\Omega) & \text{if } 1 < m < \frac{N}{2}, \\ L^\infty(\Omega) & \text{if } m > \frac{N}{2}, \end{cases} \qquad |\nabla u| \text{ in} \begin{cases} L^{m^*}(\Omega) & \text{if } 1 < m < \frac{2N}{N+2}, \\ L^2(\Omega) & \text{if } m \geq \frac{2N}{N+2}, \end{cases}$$

and

$$\frac{|\nabla u|}{u^\theta} \text{ in} \begin{cases} L^{m^*}_{\mathrm{loc}}(\Omega) & \text{if } 1 < m < \frac{2N}{N+2}, \\ L^2_{\mathrm{loc}}(\Omega) & \text{if } m \geq \frac{2N}{N+2}. \end{cases}$$

Furthermore, if $0 < \theta < \frac{1}{2}$, and $r = \frac{Nm}{N(1-\theta)-m(1-2\theta)}$, then

$$\frac{|\nabla u|}{u^\theta} \quad \text{belongs to} \quad \begin{cases} L^r(\Omega) & \text{if } 1 < m < \frac{2N(1-\theta)}{N+2-4\theta}, \\ L^2(\Omega) & \text{if } m \geq \frac{2N(1-\theta)}{N+2-4\theta}. \end{cases}$$

Proof. Thanks to (3.1) (or (3.3)), the sequence $\{u_n\}$ of solutions of (2.7) is bounded in $W_0^{1,q}(\Omega)$, with $q = \min(m^*, 2)$. Thus, up to subsequences, u_n weakly converges to some function u in $W_0^{1,q}(\Omega)$, with q as above and therefore u satisfies the boundary condition. However, due to the nonlinear nature of the lower order term, the weak convergence of u_n is not enough to pass to the limit in the distributional formulation of (2.7). In order to proceed, we use the fact that, thanks to (3.2) (or (3.4)), we have that the right hand side

$$b(x)\,\frac{|\nabla u_n|}{(1+\frac{1}{n}|\nabla u_n|)(\frac{1}{n}+u_n)^\theta} + f_n \quad \text{is bounded in (at least) } L^1_{\mathrm{loc}}(\Omega).$$

Therefore, thanks to Remark 2.2 after Theorem 2.1 of [10], we have that $\nabla u_n(x)$ almost everywhere converges to $\nabla u(x)$ in Ω; this implies that

$$\lim_{n\to+\infty} \frac{|\nabla u_n|}{(1+\frac{1}{n}|\nabla u_n|)(\frac{1}{n}+u_n)^\theta} = \frac{|\nabla u|}{u^\theta} \quad \text{almost everywhere in } \Omega.$$

This almost everywhere convergence, and the local boundedness of the sequence in $L^q(\Omega)$, with $q = \min(m^*, 2)$, yield that

$$\lim_{n\to+\infty} \frac{|\nabla u_n|}{(1+\frac{1}{n}|\nabla u_n|)(\frac{1}{n}+u_n)^\theta} = \frac{|\nabla u|}{u^\theta} \quad \text{locally weakly in } L^q(\Omega).$$

We now take φ in $C_c^1(\Omega)$ as test function in (2.7), to have that

$$\int_\Omega M(x)\nabla u_n \cdot \nabla\varphi = \int_\Omega b(x)\,\frac{|\nabla u_n|}{(1+\frac{1}{n}|\nabla u_n|)(\frac{1}{n}+u_n)^\theta}\,\varphi + \int_\Omega f_n\,\varphi\,.$$

Passing to the limit in n we obtain

$$\int_\Omega M(x)\nabla u \cdot \nabla\varphi = \int_\Omega b(x)\,\frac{|\nabla u|}{u^\theta}\,\varphi + \int_\Omega f\,\varphi,$$

for every φ in $C_c^1(\Omega)$, so that u is a solution in the sense of distributions. □

Remark 4.2. If $0 < \theta < \frac{1}{2}$ and $m \geq \frac{2N}{N+2}$, the results of Lemma 3.3 yield that u is a *weak* solution of (2.5), and not only a distributional one.

Remark 4.3. Due to the nonlinear lower order term, the *a priori* estimates alone were not sufficient to prove the existence results above, and we needed the almost everywhere convergence of the gradients of u_n to conclude the proof. Since the results of [10] hold also for equations with a nonlinear principal part, the same proof above yields the existence of a (weak or distributional) positive solution u of

$$-\operatorname{div}(a(x,\nabla u)) = b(x)\frac{|\nabla u|}{u^\theta} + f(x)\,,$$

with $a : \Omega\times\mathbb{R}^N \to \mathbb{R}^N$ a Carathéodory function such as

$$a(x,\xi)\cdot\xi \geq |\xi|^2\,, \quad |a(x,\xi)| \leq \beta|\xi|, \qquad \forall\xi\in\mathbb{R}^N, \text{ a.e. in } \Omega,$$

for some $0 < \alpha \leq \beta$, and

$$(a(x,\xi)-a(x,\eta))\cdot(\xi-\eta) > 0\,, \qquad \forall\xi\neq\eta\in\mathbb{R}^N \text{ a.e. in } \Omega.$$

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