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GABRIEL'S THEOREM AND BIRATIONAL GEOMETRY

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ABSTRACT. Extending work of Meinhardt and Partsch, we prove that two varieties are isomorphic away from a subset of a given dimension if and only if certain quotients of their categories of coherent sheaves are equivalent. This result interpolates between Gabriel's reconstruction theorem and the fact that two varieties are birational if and only if they have the same function field.

CONTENTS

It is a well-known fact that two varieties (i.e., irreducible and reduced schemes of finite-type over a field $\mathbf{k}$) X and Y are birational if and only if their function fields are isomorphic. At the same time, a theorem of Gabriel says that X and Y are isomorphic if and only if their categories of coherent sheaves are equivalent (as $\mathbf{k}$-linear categories). In this article we show that these results are actually related: they are the two extreme cases of our main theorem.

Before giving a precise statement, we introduce some notation. For an integer k, we write $\operatorname{Coh}_{\leq k}(X) \subset \operatorname{Coh}(X)$ for the subcategory of sheaves supported in dimension at most k. There is a robust theory of quotients of abelian categories, and we define $\mathcal{C}_k(X) := \operatorname{Coh}(X)/\operatorname{Coh}_{\leq k-1}(X)$. It is often convenient to re-index these categories by codimension, defining $\mathcal{C}^c(X) := \mathcal{C}_{\dim X - c}(X)$. We have $\mathcal{C}^{\dim X}(X) = \operatorname{Coh}(X)$, and one shows that $\mathcal{C}^0(X)$ is equivalent to finite-dimensional vector spaces over the function field of X. Finally, recall that two schemes X and Y of finite-type over a field are *isomorphic in codimension c* (resp., outside of dimension $k-1$) if there exist open subsets $U \subset X$, $V \subset Y$ such that U is isomorphic to V, and the codimensions of $X \smallsetminus U$ and $Y \smallsetminus V$ are at least $c+1$ (resp., dimension at most $k-1$). In particular, two varieties X and Y are birational if and only if they are isomorphic in codimension zero.

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Theorem. *Two schemes of finite-type over a field X, Y are isomorphic in codimension c if and only if the categories $\mathcal{C}^c(X)$, $\mathcal{C}^c(Y)$ are equivalent.*

This is proven in Theorem 3.7. Note that we do not need to assume our schemes to be reduced or irreducible.

As in Gabriel's theorem for $\mathrm{Coh}(X)$, we can also characterize the group of autoequivalences. Let $\mathrm{Aut}^{\leq c}(X)$ (resp., $\mathrm{Aut}_{\geq k}(X)$) denote the group of birational self-maps which are defined away from a closed subset of codimension at least c (resp., dimension at most $k-1$).

Let $\mathrm{Pic}^{\leq c}(X)$ (resp., $\mathrm{Pic}_{\geq k}(X)$) denote the group (under tensor products) of sheaves which are invertible away from a closed subset of codimension at least $c+1$ (resp., dimension at most $k-1$). The group $\mathrm{Aut}^{\leq c}(X)$ acts on $\mathrm{Pic}^{\leq c}(X)$ by $(f, L) \mapsto (f^{-1})^* L$. We may thus form the semi-direct product $\mathrm{Aut}^{\leq c}(X) \ltimes \mathrm{Pic}^{\leq c}(X)$.

Theorem. *Let X be a scheme of finite-type over a field. There is an isomorphism between* $\mathrm{Aut}(\mathcal{C}^c(X))$*, the group of autoequivalences, and* $\mathrm{Aut}^{\leq c}(X) \ltimes \mathrm{Pic}^{\leq c}(X)$*.*

This is proven in Theorem 4.1.

Dimension vs. codimension. Although the theorems here are stated in terms of the codimension $c = \dim(X) - k$, which the authors consider more elegant and intuitive, the rest of the paper is written in terms of the dimension k. The latter is much better behaved for our purposes, especially in the context of disconnected or reducible spaces. For example, if a scheme X is reducible it can have an open subset U of positive codimension r, which means that the restriction of an element $F \in \mathcal{C}^c(X)$ would be in $\mathcal{C}^{c-r}(U)$.

Previous work. Gabriel originally proved his reconstruction theorem in [8]. This was later generalized considerably by various people [3, 4, 12]. In [10], Meinhardt and Partsch were interested in constructing stability conditions on the (derived categories of the) quotient categories $\mathcal{C}^c(X)$. Along the way they showed the $c = 0$ and $c = 1$ cases of our main result, when X, Y are smooth and projective over an algebraically closed field.

Future work. It is of course tempting to speculate about possible future directions. While we do not expect our work to immediately be useful in constructing more stability conditions on the derived category of the quotients $\mathrm{D}(\mathcal{C}^c(X))$, we can certainly see our main theorem being used to reduce the problem to simpler cases. Indeed, if X and Y are isomorphic in codimension c, then constructing stability conditions on $\mathrm{D}(\mathcal{C}^c(X))$ is just as hard as constructing them on $\mathrm{D}(\mathcal{C}^c(Y))$.

Diverging a little from [10], finding the precise relationship between the derived category $\mathrm{D}(X)$ and the birational geometry of X has been an active area of research for quite some time (see [5] for an excellent overview). We wonder if the study of the derived category of the quotients $\mathrm{D}(\mathcal{C}^c(X))$, or some intermediate version of them, might help make some progress in the matter.

Going in a different direction, we will point out later in the paper that the construction of the quotients $\mathcal{C}_k(X)$ is completely intrinsic to the abelian category $\mathrm{Coh}(X)$. We wonder if this observation could be useful in the study of noncommutative birational geometry. See Section 2.5 for more details.

Conventions. For the entirety of this paper we fix a base (commutative) noetherian ring **k**. All algebras, schemes, categories, morphisms, and functors are implicitly assumed to be over **k**. Starting from Section 3, all schemes will be assumed to be of finite-type over **k**. If X is a scheme, $U \subset X$ is open, and F is a sheaf on X, we will write F_U for the restriction (i.e., pullback) of F to U. If R is a ring, we will write $\mathrm{mod}(R)$ for the category of finitely generated R-modules.

1. Quotient categories

We begin by briefly reviewing some standard notions and later introduce some notation. We refer to [7, 8] for a thorough treatment of quotients of abelian categories. Let $\mathcal{A}$ be an abelian category. A subcategory $\mathcal{S} \subset \mathcal{A}$ is *Serre* if, given an exact sequence $A \to B \to C$, then $B \in \mathcal{S}$ if and only if $A, C \in \mathcal{S}$. Given a Serre subcategory $\mathcal{S} \subset \mathcal{A}$, we may form the quotient $\Phi\colon \mathcal{A} \to \mathcal{A}/\mathcal{S}$, and the projection Φ is exact. This category is initial among all abelian categories $\mathcal{B}$ equipped with an exact functor $\Psi\colon \mathcal{A} \to \mathcal{B}$ such that $\Psi(S) = 0$ for all $S \in \mathcal{S}$. Recall that the *kernel* of a functor Φ is the full subcategory whose objects satisfy $\Phi(M) = 0$. The Serre subcategory $\mathcal{S}$ is precisely the kernel of $\mathcal{A} \to \mathcal{A}/\mathcal{S}$.

Lemma 1.1. *Let $\mathcal{A}/\mathcal{S}$ be a quotient category.*

- $P \equiv 0$ *if and only if* $P \in \mathcal{S}$.
- *If* $0 \rightharpoonup A \rightharpoonup B \rightharpoonup C \rightharpoonup 0$ *is short exact in* $\mathcal{A}/\mathcal{S}$*, then there exist objects* A', B', C' *and isomorphisms* (*in* $\mathcal{A}/\mathcal{S}$) $A \rightharpoonup A'$, $B \rightharpoonup B'$, $C \rightharpoonup C'$, *and a short exact sequence* $0 \to A' \to B' \to C' \to 0$ *in* $\mathcal{A}$*, such that the diagram*

$$\begin{tikzcd}
0 \arrow[r, harpoon] & A' \arrow[r, harpoon] \arrow[d, harpoon] & B' \arrow[r, harpoon] \arrow[d, harpoon] & C' \arrow[r, harpoon] \arrow[d, harpoon] & 0 \\
0 \arrow[r, harpoon] & A \arrow[r, harpoon] & B \arrow[r, harpoon] & C \arrow[r, harpoon] & 0
\end{tikzcd}$$

commutes in $\mathcal{A}/\mathcal{S}$.

Proof. The first claim follows from [9, Prop 7.1.20 (ii)], the second is [7, Cor 15.8]. □

Lemma 1.2. *Let $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathcal{A}$ be two Serre subcategories of the abelian category $\mathcal{A}$. Then $\mathcal{S}_2/\mathcal{S}_1$ is a Serre subcategory of $\mathcal{A}/\mathcal{S}_1$ and the quotient $\mathcal{A}/\mathcal{S}_2$ is naturally equivalent to the iterated quotient* $(\mathcal{A}/\mathcal{S}_1)\,/\,(\mathcal{S}_2/\mathcal{S}_1)$.

Proof. The first assertion follows from the definitions, while the second is a consequence of the universal property. □

Let $\mathcal{A}/\mathcal{S}$ be a quotient category. Let $P, Q \in \mathcal{A}$. To distinguish arrows in $\mathcal{A}$ from arrows in $\mathcal{A}/\mathcal{S}$, we will write $P \to Q$ for an arrow in $\mathcal{A}$ and $P \rightharpoonup Q$ for its image in $\mathcal{A}/\mathcal{S}$. If two objects $P, Q \in \mathcal{A}$ become isomorphic in $\mathcal{A}/\mathcal{S}$, we will write $P \equiv Q$. Finally, a morphism $f\colon P \to Q$ such that $\ker f, \operatorname{coker} f \in \mathcal{S}$ will be called a *weak equivalence*.

The quotient category $\mathcal{A}/\mathcal{S}$ may be concretely built as follows [9, Exercise 8.12]. The objects of $\mathcal{A}/\mathcal{S}$ are the same as the objects of $\mathcal{A}$. A morphism $P \rightharpoonup Q$ is an equivalence class[1] of "roofs", i.e., diagrams

$$\begin{tikzcd}
 & U \arrow[dl] \arrow[dr] & \\
P & & Q
\end{tikzcd}$$

with $U \to P$ a weak equivalence.

1.1. Dimension.

Let $\mathcal{A}$ be an abelian category. We say an object P is *minimal* if it has no non-trivial sub-objects.[2] Note that for us the zero object is also minimal (this is slightly non-standard). We let $\mathcal{S}_0$ be the smallest Serre subcategory containing all minimal objects of $\mathcal{A}$. Let $\mathcal{A}_1 = \mathcal{A}/\mathcal{S}_0$. Since $\mathcal{A}_1$ is also an abelian category, we may repeat the process. Let $\mathcal{S}_1'$ be the smallest Serre subcategory containing all minimal objects of $\mathcal{A}_1$ and let $\mathcal{A}_2 = \mathcal{A}_1/\mathcal{S}_1'$. We define $\mathcal{S}_1$ to be the kernel of $\mathcal{A} \to \mathcal{A}_1 \to \mathcal{A}_2$. By iterating we obtain a sequence of quotients

$$\mathcal{A} = \mathcal{A}_0 \twoheadrightarrow \mathcal{A}_1 \twoheadrightarrow \mathcal{A}_2 \twoheadrightarrow \mathcal{A}_3 \twoheadrightarrow \cdots \twoheadrightarrow \{0\} \tag{1.1}$$

and, by taking kernels, we find a nested family of Serre subcategories

$$\{0\} \subset \mathcal{S}_0 \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathcal{S}_3 \subset \cdots \subset \mathcal{A}. \tag{1.2}$$

Of course, it need not be the case that there is a k such that $\mathcal{A}_k = \{0\}$ or that $\mathcal{S}_k = \mathcal{A}$.

Definition 1.3. The *Krull dimension* of $\mathcal{A}$ is

$$\dim \mathcal{A} := \inf\{k \mid \mathcal{A}_{k+1} = 0\} = \inf\{k \mid \mathcal{S}_k = \mathcal{A}\}.$$

We also define

$$\dim M := \inf\{k \mid M = 0 \text{ in } \mathcal{A}_{k+1}\} = \inf\{k \mid M \in \mathcal{S}_k\}$$

for any object $M \in \mathcal{A}$.

These definitions are of course justified by the algebro-geometric context.

1.2. The geometric case.

In this subsection, X denotes an arbitrary noetherian scheme. Let $\mathcal{C} = \mathrm{Coh}(X)$ be its category of coherent sheaves. As before, we write $\mathcal{S}_k = \mathcal{S}_k(X)$ for the sequence of kernels as in (1.2) (here $\mathcal{C}$ has taken the role of $\mathcal{A}$). Denote by $\mathrm{Coh}_{\le k}(X)$ the category of sheaves supported in dimension at most k.

Proposition 1.4. *We have* $\mathcal{S}_k(X) = \mathrm{Coh}_{\le k}(X)$.

Before we prove this result, we need an alternative way to compute morphisms in our quotients. We define $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{C}_k = \mathcal{C}/\mathcal{S}_{k-1}$ for all other k.

Let Σ be a *family of supports*, i.e., a collection of closed subsets of X such that

- $\varnothing \in \Sigma$,
- if $Z \in \Sigma$, $V \subset Z$ is closed, then $V \in \Sigma$,
- if $Z_1, Z_2 \in \Sigma$, then $Z_1 \cup Z_2 \in \Sigma$.

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[1] An equivalence of roofs is a commutative diagram as in Lemma 1.5.

[2] What we call minimal objects are often called *simple* objects. However, by a *simple sheaf* one typically means one that has only scalar endomorphisms.

For example, Σ might be the collection of all closed subsets of X of dimension at most k. Write $\mathrm{Coh}_\Sigma(X) \subset \mathrm{Coh}(X)$ for the subcategory of sheaves F such that $\mathrm{supp}\, F \in \Sigma$. Because of our assumptions, $\mathrm{Coh}_\Sigma(X)$ is a Serre subcategory. We write $\mathcal{Q} = \mathrm{Coh}(X)/\mathrm{Coh}_\Sigma(X)$ for the quotient.

Lemma 1.5. *Let $F, G \in \mathrm{Coh}(X)$. Then*

$$\mathrm{Hom}_{\mathcal{Q}}(F,G) = \varinjlim_{\substack{\varnothing \subset U \subset X \\ X \setminus U \in \Sigma}} \mathrm{Hom}_U(F_U, G_U),$$

where F_U, G_U denote the restrictions to U.

Proof. We will define mutually inverse maps, following [10, Lemma 3.6]. We start by constructing a map from right to left. Any element of the right hand side may be represented as a pair (U, f), where $X \setminus U \in \Sigma$ and $f\colon F_U \to G_U$. Write $\Gamma_f \subset F_U \oplus G_U$ for the graph of f. Let $E \subset F \oplus G$ be any coherent sheaf such that $E_U = \Gamma_f$. Then $E \xrightarrow{\mathrm{pr}_1} F$ is a weak equivalence in $\mathcal{Q}$, so the roof $F \xleftarrow{\mathrm{pr}_1} E \xrightarrow{\mathrm{pr}_2} G$ represents an element $f \in \mathrm{Hom}_{\mathcal{Q}}(F,G)$.

Suppose now $f'\colon F_{U'} \to G_{U'}$ represents the same element as (U, f). As above, we construct a roof $F \xleftarrow{\mathrm{pr}_1'} E' \xrightarrow{\mathrm{pr}_2'} G$. Since (U, f) and (U', f') must eventually agree, there exists an open subset U'' with $f_{U''} = f'_{U''}$ and such that $E'' = E \cap E'$ is equivalent to both E and E' through the inclusion map. The commutative diagram

$$\begin{tikzcd}
& & E'' \arrow[dl, "i"'] \arrow[dr, "j"] & & \\
& E \arrow[dl, "\mathrm{pr}_1"'] \arrow[drrr, "\mathrm{pr}_2" near end] & & E' \arrow[dr, "\mathrm{pr}_2'"] \arrow[dlll, "\mathrm{pr}_1'"' near end] & \\
F & & & & G
\end{tikzcd}$$

shows that the two roofs are equivalent precisely as in [10, Lemma 3.6].

Consider now a morphism $f \in \mathrm{Hom}_{\mathcal{Q}}(F,G)$, represented by the roof $F \xleftarrow{s} E \xrightarrow{t} G$. Since s is a weak equivalence (by definition of roof), we have $\mathrm{supp}(\ker(s)) \cup \mathrm{supp}(\mathrm{coker}(s)) \in \Sigma$. Let U be its complement. Once we restrict to U, s becomes an isomorphism, hence the map $t \circ s^{-1}\colon F_U \to G_U$ makes sense. To make sure the function $f \mapsto t \circ s^{-1}$ is well defined, we use that two different roofs representing the same morphism must be equivalent. Indeed, up to restricting to a smaller subset, we will obtain the same map. Finally, since we are allowed to choose ad hoc representatives, one checks the two maps just defined compose to the respective identities. ☐

Lemma 1.6. *An object $P \in \mathrm{Coh}(X)/\mathrm{Coh}_{\leq k-1}(X)$ is minimal if and only if either $P \equiv 0$ or $P \equiv \mathcal{O}_Z$ for $Z \hookrightarrow X$ an integral closed subscheme of dimension k.*

Proof. Let us write $\mathcal{Q} = \mathrm{Coh}(X)/\mathrm{Coh}_{\leq k-1}(X)$. We begin by showing that $\mathcal{O}_Z$ is minimal for $Z \hookrightarrow X$ an integral closed subscheme. Suppose $\mathcal{O}_Z$ sits in the middle of a short exact sequence in $\mathcal{Q}$. By Lemma 1.1, this means that, up to passing to an equivalent object $B \equiv \mathcal{O}_Z$, there is a short exact sequence in $\mathrm{Coh}(X)$

$$0 \to A \to B \to C \to 0.$$

The goal is to show that either $C \equiv 0$ or that $B \to C$ is a weak equivalence.

Let η be the generic point of Z and let $R = \mathcal{O}_{X,\eta}$ be the corresponding local ring with maximal ideal $\mathfrak{m}$. By taking germs, we get a short exact sequence

$$0 \to A_\eta \to B_\eta \to C_\eta \to 0$$

of R-modules. Since B and $\mathcal{O}_Z$ are isomorphic up to $\operatorname{Coh}_{\leq k-1}(X)$, we have $B_\eta \simeq \mathcal{O}_{Z,\eta} = R/\mathfrak{m}$. It follows that either $C_\eta = 0$ or $C_\eta = B_\eta$.

If $C_\eta = 0$, then $\dim\operatorname{supp} C < k$, which implies $C \equiv 0$. If $C_\eta = B_\eta$, then $\dim\operatorname{supp} A < k$, which implies $C \equiv B \equiv \mathcal{O}_Z$.

To prove the other direction, let $F \in \operatorname{Coh}(X)$ represent a minimal element in $\mathcal{Q}$. Let Z be the scheme-theoretic support of F, i.e., $Z = V(\operatorname{Ann}(F))$. If $\dim Z < k$, then $F \equiv 0$. If not, without loss of generality we may assume Z is irreducible, reduced, and of dimension k. This previous claim follows by minimality: we always have a surjection $F \twoheadrightarrow F/IF$ for I the ideal sheaf of a closed subscheme of Z.

By the argument above, we already know $\mathcal{O}_Z$ is minimal in $\mathcal{Q}$. Our goal is to construct a non-zero map $\mathcal{O}_Z \rightharpoonup F$, proving $\mathcal{O}_Z \equiv F$.

Write $i\colon Z \hookrightarrow X$ for the inclusion. We know $F = i_* F'$, where $F' = i^* F \in \operatorname{Coh}(Z)$. Let $U \subset Z$ be an affine open. We must have a non-torsion element $a \in F'_U$, otherwise the whole F' would be torsion and $\dim Z < k$. Now view a as a non-zero map $\mathcal{O}_U \to F'$.

Let $W = Z \setminus U$. Since Z is irreducible, we must have $\dim W < k$. Let $\tilde{U} = X \setminus W$. We have

$$\operatorname{Hom}_{\tilde{U}}\left(\mathcal{O}_{\tilde{U}}, F_{\tilde{U}}\right) = \operatorname{Hom}_{\tilde{U}}\left(\mathcal{O}_{\tilde{U}}, i_{U*}\left(F'_U\right)\right) = \operatorname{Hom}_U\left(\mathcal{O}_U, F'_U\right).$$

Since the map $\mathcal{O}_U \to F'_U$ is compatible with further restricting $\tilde{U}$, by Lemma 1.5 we have cooked up a non-zero map $\mathcal{O}_Z \rightharpoonup F$. The claim follows. ☐

Lemma 1.7. *The subcategory* $\operatorname{Coh}_{\leq k}(X) \subset \operatorname{Coh}(X)$ *is the smallest Serre subcategory containing* $\operatorname{Coh}_{\leq k-1}(X)$ *and all the sheaves* $\mathcal{O}_Z$*, with* $Z \hookrightarrow X$ *integral closed subscheme and* $\dim Z = k$.

Proof. It is easier to work in the quotient $\mathcal{Q} = \operatorname{Coh}(X)/\operatorname{Coh}_{\leq k-1}(X)$. The claim is equivalent to showing the following. Suppose $\mathcal{S} \subset \mathcal{Q}$ is a Serre subcategory containing all $\mathcal{O}_Z$, for $Z \hookrightarrow X$ an integral closed subscheme with $\dim Z = k$, and suppose $F \in \operatorname{Coh}_{\leq k}(X)$. Then $F \in \mathcal{S}$.

If $F = \mathcal{O}_Z$, with Z integral, then $F \in \mathcal{S}$ by assumption. If $F = \mathcal{O}_Y$, with Y reduced but possibly reducible, let $Y_1, \dots, Y_r$ be its irreducible components, equipped with the reduced scheme structure. The sheaves $\mathcal{O}_{Y_i}$ all belong to $\mathcal{S}$ by assumption. There is a map $\mathcal{O}_Y \to \mathcal{O}_{Y_1} \oplus \cdots \oplus \mathcal{O}_{Y_r}$. Since the intersection of all minimal primes in a reduced ring is zero, this map is injective. As $\mathcal{S}$ is a Serre category, it follows that $\mathcal{O}_Y \in \mathcal{S}$.

Suppose now $F \in \operatorname{Coh}_{\leq k}(X)$ satisfies the following: there exists a reduced Y, with ideal sheaf I, $\dim Y \leq k$, and $IF = 0$. Let $i\colon Y \hookrightarrow X$ be the inclusion. By assumption there is an $F' \in \operatorname{Coh}(Y)$ such that $i_* F' = F$. Let now $W \subset Y$ be a closed subset, with $\dim W < k$, and such that $U = Y \setminus W$ is affine. Let $\tilde{U} = X \setminus W$. The restriction F'_U is globally generated, hence there is a surjective map $\mathcal{O}_U^{\oplus r} \twoheadrightarrow F'_U$. By pushing forward, there is a surjection $\mathcal{O}_{\tilde{U}}^{\oplus r} \twoheadrightarrow F_{\tilde{U}}$. Using Lemma 1.5, this induces a surjection $\mathcal{O}_Y^{\oplus r} \twoheadrightarrow F$ in the category $\mathcal{Q}$. Since $\mathcal{O}_Y \in \mathcal{S}$, we see that $F \in \mathcal{S}$.

Suppose now $F \in \operatorname{Coh}_{\leq k}(X)$ is arbitrary. Let Y be its scheme-theoretic support. Let $I \subset \mathcal{O}_X$ be the ideal sheaf defining Y_{red}. By noetherianity, there exists an m

such that $I^{m+1}F = 0$. We have a string of short exact sequences

$$\begin{aligned} 0 \to IF \to F \to F/IF \to 0 \\ 0 \to I^2F \to IF \to IF/I^2F \to 0 \\ \dots \\ 0 \to I^mF \to I^{m-1}F \to I^{m-1}F/I^mF \to 0. \end{aligned}$$

Now, $I(I^aF/I^{a+1}F) = 0$ for any a, hence they belong to $\mathcal{S}$. Using induction starting from $I(I^mF) = 0$, and using as always that $\mathcal{S}$ is Serre, we see that $F \in \mathcal{S}$. □

Proof of Proposition 1.4. We proceed by induction on k. By definition, $\mathcal{S}_0$ is the smallest Serre subcategory containing all the minimal objects of $\operatorname{Coh}(X)$. By Lemma 1.6, these are precisely the skyscraper sheaves. By Lemma 1.7, $\mathcal{S}_0 = \operatorname{Coh}_{\le 0}(X)$.

Suppose the theorem is true for $k-1$, let

$$\mathcal{Q} = \operatorname{Coh}(X)/\operatorname{Coh}_{\le k-1}(X) = \operatorname{Coh}(X)/\mathcal{S}_{k-1}.$$

Let $\mathcal{S}$ be the smallest Serre subcategory of $\mathcal{Q}$, containing all minimal objects of $\mathcal{Q}$. By definition, $\mathcal{S}_k$ is the kernel of $\operatorname{Coh}(X) \to \mathcal{Q} \to \mathcal{Q}/\mathcal{S}$. By Lemma 1.6, the minimal objects of $\mathcal{Q}$ are precisely the $\mathcal{O}_Z$ with Z integral of dimension k. Combined with Lemma 1.7, we see that $\mathcal{S}$ is the image of $\operatorname{Coh}_{\le k}(X)$. Hence, $\mathcal{S}_k = \operatorname{Coh}_{\le k}(X)$. □

The following result follows from Lemma 1.5, and will be useful later.

Lemma 1.8. *Let P be a non-zero minimal object in $\mathcal{C}_k$, corresponding to a (non-necessarily closed) point $x \in X$. We have $\operatorname{Hom}_{\mathcal{C}_k}(P,P) = \kappa(x)$, where $\kappa(x)$ is the residue field of the point x.*

From now on we will write $\mathcal{C}_k = \mathcal{C}_k(X) = \operatorname{Coh}(X)/\operatorname{Coh}_{\le k-1}(X)$.

2. A LOCALLY RINGED SPACE

For this section, let X be a finite-dimensional noetherian scheme, which (as per our blanket conventions) is, moreover, defined over our base ring $\mathbf{k}$. Our present goal is to define an auxiliary locally ringed space $\Theta_k = \Theta_k(X,L)$, depending on X, an integer k and a sheaf L. This space will control the isomorphism type of X up to subsets of dimension $k-1$. A posteriori it will be obvious that Θ does not depend on the sheaf L. The reason we initially insist on the dependency on L will become clear in the next section: an equivalence $\mathcal{C}_k(X) \simeq \mathcal{C}_k(Y)$ need not send $\mathcal{O}_X$ to $\mathcal{O}_Y$.

We say a point $x \in X$ has *dimension* k if $\dim \overline{\{x\}} = k$. Put differently, x is the generic point of a subvariety of dimension k, which when X is equidimensional and catenary is in turn equivalent to $\dim \mathcal{O}_{X,x} = \dim X - k$. We write $X_{\ge k} \subset X$ for the subset of points of dimension at least k. In particular, $X_{\ge k}$ always contains the generic points of the irreducible components of dimension at least k of X. We endow $X_{\ge k}$ with the subspace topology. If $i\colon X_{\ge k} \to X$ denotes the inclusion, we may view the former as a locally ringed space with structure sheaf $i^{-1}\mathcal{O}_X$. This space will turn out to be isomorphic to the space Θ_k we are about to define. Using this, we will show that an isomorphism $\mathcal{C}_k(X) \simeq \mathcal{C}_k(Y)$ induces an isomorphism $X_{\ge k} \simeq X_{\ge k}$, a fundamental step in proving our theorem.

Before we proceed, recall that $i^{-1}\mathcal{O}_X$ is (by definition) obtained by sheafifying the presheaf $i^{+}\mathcal{O}_X$, given by

$$V \mapsto \varinjlim_{U\supset V} \mathcal{O}_X(U),$$

where the limit ranges over all $U \subseteq X$ open and containing $V \subseteq X_{\geq k}$.

Lemma 2.1. *The presheaf* $i^{+}\mathcal{O}_X$ *is already a sheaf, hence*

$$i^{-1}\mathcal{O}_X(V) = \varinjlim_{U\supset V} \mathcal{O}_X(U).$$

Proof. The locality of $i^{+}\mathcal{O}_X$ is a simple consequence of $\mathcal{O}_X$ being a sheaf and is left to the reader. It remains to prove that $i^{+}\mathcal{O}_X$ satisfies the gluing condition.

Let $V \subseteq X_{\geq k}$ be open. A section $[\alpha]$ in $i^{+}\mathcal{O}_X(V)$ can be represented by an $\alpha \in \mathcal{O}_X(U)$, for $U \subseteq X$ open and $V \subset U$. Let $\{V_i\}_{i\in I}$ be an open cover of V. We may assume the set I to be finite.

Let $[\alpha_i] \in i^{+}\mathcal{O}_X(V_i)$ be a collection of sections such that $[\alpha_i]|V_i \cap V_j = [\alpha_j]|V_i \cap V_j$. For each i, suppose $\alpha_i \in \mathcal{O}_X(U_i)$ for $V_i \subset U_i \subset X$. Up to removing a closed subset, we can assume that $U_i \cap X_{\geq k} = V_i$. Now, for each pair ij, we may choose an open subset $V_i \cap V_j \subset U_{ij} \subset U_i \cap U_j$, such that $\alpha_i|U_{ij} = \alpha_j|U_{ij}$. Note that $W_{ij} = U_i \cap U_j \smallsetminus U_{ij}$ has dimension at most $k-1$ as the k-dimensional points of $U_i \cap U_j$ belong to $V_i \cap V_j$. Consider the open subset $\tilde{U} = \bigcup_i U_i \smallsetminus \bigcup_{i,j} \overline{W_{ij}}$, which is open and contains $X_{\geq k}$, and replace the open subsets U_i with $U_i' = U_i \cap \tilde{U}$.

Since $\mathcal{O}_{\tilde{U}}$ is a sheaf, and $U_i' \cap U_j' \subset U_{ij}$, there exists $\alpha \in U$ restricting to α_i on U_i', which gives the desired section as the complement of $\tilde{U}$ in X has dimension at most $k-1$. ☐

2.1. **The topological space Θ_k.** We now come to the definition of Θ_k. Let $k \geq 0$ be an integer. As a set, Θ_k consists of isomorphism classes of (non-zero) minimal objects of $\mathcal{C}_d(X)$, where d ranges between k and $\dim(X)$:

$$\Theta_k := \bigcup_{d\geq k} \{0 \neq P \in \mathcal{C}_d(X) \mid P \text{ minimal}\}_{/\text{iso}}$$

We define a topology on Θ_k, by declaring what the closure of a point ought to be. Let $P \in \mathrm{Coh}(X)$ represent a non-zero minimal object in $\mathcal{C}_d(X)$, with $d \geq k$.

Intuitively, if $P \in \mathcal{C}_d(X)$ represents the structure sheaf of an integral closed irreducible subscheme Z of X, we want its closure in Θ_k to contain all the structure sheaves of closed subschemes $Q \subset Z$ of dimension k or higher. These structure sheaves will live in the (larger) categories $\mathcal{C}_j$, where j ranges between k and d. A necessary and sufficient condition for the support of such a Q to belong to Z is that all possible inverse images P' of P in $\mathcal{C}_j$ surject to it.[3]

Concisely, we can define

$$Z_P := \bigcup_{d\geq j\geq k} \quad \bigcap_{\substack{P'\in\mathcal{C}_j(X)\\ P \text{ is the image of } P'}} \quad \{0 \neq Q \in \mathcal{C}_j(X) \mid Q \text{ is minimal, } Q \text{ is a quotient of } P'\}_{/\text{iso}}$$

We endow Θ with the coarsest topology containing the sets $\Theta \smallsetminus Z_P$ for all $P \in \mathrm{Coh}(X)$.

---

[3] Note that we have to necessarily take all possible inverse images of P': for example a point $P \in \mathcal{C}_2(\mathbb{A}^3)$ representing the structure sheaf of a plane H in $\mathbb{A}^3$ will have among its inverse images in $\mathcal{C}_1(\mathbb{A}^3)$ the structure sheaf of $H \cup l$, where l is any line, and of $H \cup p$, where p is a point, and we have no a priori way to tell them apart.

Lemma 2.2. *Consider the map of topological spaces* $X_{\geq k} \to \Theta_k$ *given by*

$$x \mapsto \mathcal{O}_{\overline{\{x\}}},$$

where $\overline{\{x\}}$ *is equipped with the reduced scheme structure. This map is a homeomorphism.*

Proof. Since X is noetherian, the subsets $X \setminus \overline{\{x\}}$, as x varies in X, form a basis of the Zariski topology, and consequently of the induced topology on $X_{\geq k}$. Proposition 1.4 and Lemma 1.6 allow us to conclude. ☐

2.2. **A Picard group.** We introduce now the analogue of $\mathrm{Pic}(X)$ for the category $\mathcal{C}_k(X)$. We define $\mathrm{Pic}_{\geq k}(X)$ to be the group (under tensor product) of isomorphism classes of objects $L_1 \in \mathcal{C}_k(X)$ for which there exists a representative $L \in \mathrm{Coh}(X)$ satisfying the following property: there exists an open subset $U \subset X$, such that $X_{\geq k} \subset U$, and the restriction L_U is an invertible sheaf. Put concisely: $\mathrm{Pic}_{\geq k}(X)$ consists of (the $\mathcal{C}_k(X)$-isomorphism classes of) those L which are line bundles away from a closed subset of dimension at most $k-1$. We will see in Proposition 2.7 that this definition is intrinsic to the category $\mathcal{C}_k(X)$.

2.3. **The locally ringed space Θ_k.** Fix now $L \in \mathrm{Pic}_{\geq k}(X)$. We will now define a sheaf of rings $\mathcal{O}_{\Theta_k}$ on Θ_k, depending on L. For $V \subset \Theta_k$ an arbitrary open subset, we define

$$\mathcal{O}_{\Theta}(V) := \varinjlim_{U \supset V} \mathrm{Hom}_U(L_U, L_U),$$

where $U \supset V$ runs over all open subsets of X containing V.

Proposition 2.3. *The assignment* $V \mapsto \mathcal{O}_{\Theta_k}(V)$ *defines a sheaf of rings on* Θ_k, *making* $(\Theta_k, \mathcal{O}_{\Theta_k})$ *into a locally ringed space. Moreover, the homeomorphism* $X_{\geq k} \to \Theta_k$ *from Lemma* 2.2 *induces an isomorphism of sheaves between* $\mathcal{O}_{\Theta_k}$ *and* $i^*\mathcal{O}_X$, *making it an isomorphism of locally ringed spaces over* $\mathrm{Spec}\,\mathbf{k}$.

Proof. Suppose first $L = \mathcal{O}_X$. Then Lemmas 2.2 and 2.1 imply the claim. Let L be general. There exists then an open subset U, with $X_{\geq k} \subset U$, such that L becomes invertible once restricted to U. But then the natural map $\mathcal{O}_X \to \underline{\mathrm{Hom}}_X(L, L)$ also becomes an isomorphism once restricted to U. The claim then follows from the case where $L = \mathcal{O}_X$. ☐

2.4. **Intermezzo.** To proceed with the final proof of this section, we have to introduce one last category. Suppose $0 \neq P \in \mathcal{C}_k(X)$ is minimal, and denote by $x \in X_{\geq k}$ the corresponding point. In particular, if $Z = \overline{\{x\}}$, the sheaf $\mathcal{O}_Z$ represents P. The goal is to define a category $\mathcal{C}_P$, recovering the local ring $\mathcal{O}_{X,x}$ at x.

For $F \in \mathrm{Coh}(X)$, we define its *topological k-support* to be $\mathrm{supp}_k F := X_{\geq k} \cap \mathrm{supp}\, F$. Using the homeomorphism of Lemma 2.2, we may view $\mathrm{supp}_k F$ as a closed subset of Θ. Similarly, if $G \in \mathcal{C}_d(X)$ with $d \leq k$, we define $\mathrm{supp}_k G = \mathrm{supp}_k \tilde{G}$, where $\tilde{G} \in \mathrm{Coh}(X)$ is any representative of G.

Lemma 2.4. *Given an element* $F \in \mathrm{Coh}(X)$, *the subset* $\mathrm{supp}_k(F) \subset X_{\geq k}$ *only depends on the class of* F *in* $\mathcal{C}_k$.

Proof. Let P be a minimal object in $\mathcal{C}_k$. Then by Lemma 1.5 the corresponding point $P \in X_{\geq k}$ belongs to $\mathrm{supp}_k(F)$ if and only if $\mathrm{Hom}_{\mathcal{C}_k}(F, P) \neq 0$. ☐

Consider the collection of $E \in \Theta_k$ such that $P \notin \operatorname{supp}_k E$, and let $\mathcal{S}_P \subset \mathcal{C}_k(X)$ be the smallest Serre subcategory containing them. We define $\mathcal{C}_P$ to be the quotient $\mathcal{C}_k(X)/\mathcal{S}_P$. Furthermore, write E_P for the class of E in $\mathcal{C}_P$.

Lemma 2.5. *Let $L \in \operatorname{Pic}_{\geq k}(X)$. We may identify the ring $\operatorname{End}_{\mathcal{C}_P}(L)$ with the local ring $\mathcal{O}_{X,x}$. Moreover, the functor $\mathcal{C}_P \to \operatorname{mod}(\mathcal{O}_{X,x})$ sending E to $\operatorname{Hom}_{\mathcal{C}_P}(L, E)$ is an equivalence.*

In particular we may identify the stalk $E_x \in \operatorname{mod}(\mathcal{O}_{X,x})$ with the object $E_P \in \mathcal{C}_P$.

Proof. Let Σ_x be the collection of closed subsets of X not containing x. From the definition of $\mathcal{S}_P$, we see that $\operatorname{Coh}(X)/\mathcal{S}_{\Sigma_x} = \mathcal{C}_P$. Using Lemma 1.5, we see

$$\operatorname{Hom}_{\mathcal{C}_P}(E, F) = \varinjlim_{x \in U \subset X} \operatorname{Hom}_U(E_U, F_U) = \operatorname{Hom}_{\mathcal{O}_{X,x}}(E_x, F_x).$$

Since $L_x = \mathcal{O}_{X,x}$, we have

$$\mathcal{O}_{X,x} = \operatorname{End}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}) = \operatorname{End}_{\mathcal{C}_P}(L), \quad \operatorname{Hom}_{\mathcal{C}_P}(L, E) = \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}, E_x) = E_x.$$

Hence the functor is fully faithful. To prove it is essentially surjective, it suffices to observe that, given a finitely generated $\mathcal{O}_{X,x}$-module M, there exists a coherent sheaf $E \in \operatorname{Coh}(X)$ such that $E_x = M$. ☐

Remark 2.6. When X in the above lemma is integral, and x is its generic point, then $\mathcal{O}_{X,x} = K(X)$ is its function field. It then follows that $\mathcal{C}_{\dim X}(X) = \mathcal{C}_{\mathcal{O}_X}$, and the functor $\mathcal{C}_{\mathcal{O}_X} \to \operatorname{mod}(K(X))$ is an equivalence. Hence $\mathcal{C}_{\dim X}(X)$ captures the birational type of X.

2.5. A non-commutative remark.

Following up Remark 2.6, we briefly discuss a non-commutative avenue. One perspective on non-commutative geometry is that a non-commutative space should be given by an abelian category $\mathcal{A}$, satisfying some niceness properties. Much work has been devoted to this point of view; see for example [1, 2, 13, 16]. A basic question is then whether there exists a non-commutative analogue of birational geometry. For example in [14] one finds a candidate for a function field of a general $\mathcal{A}$, and in [11] there are interesting examples of birational non-commutative surfaces.

However, a complete and satisfactory theory of non-commutative birational geometry does not seem to exist presently. For instance, to capture higher dimensional phenomena (such as flops and flips), one needs to know not just when two spaces are merely birational, but also when they are isomorphic in a certain codimension.

On the other hand, as we discussed in the previous section, the categories $\operatorname{Coh}_{\leq k}(X)$ are *intrinsic* to the category $\operatorname{Coh}(X)$. Indeed, given an abelian category $\mathcal{A}$, we may form our sequence of quotients $\mathcal{A}_k$ as in (1.1). In light of Remark 2.6, if there exists an n such that $\mathcal{A}_n \neq 0$ but $\mathcal{A}_{n+1} = 0$, we would view $\mathcal{A}_n$ as the *non-commutative function field* of $\mathcal{A}$. It would be interesting to see the relation between this and Smith's function field [14].

In general, given the main theorem of this paper, the quotient categories $\mathcal{A}_k$ (for $k \geq n$) could be seen as capturing the non-commutative space $\mathcal{A}$, up to a certain codimension.

2.6. Equivalences.

We conclude this section by showing that equivalences of the quotient categories induce an isomorphism of the locally ringed spaces.

Proposition 2.7. *Let X, Y be schemes of finite-type over* $\mathbf{k}$*. A* $\mathbf{k}$*-linear equivalence* $\Phi: \mathcal{C}_k(X) \to \mathcal{C}_k(Y)$ *induces an isomorphism of locally ringed spaces* $\phi: X_{\geq k} \to Y_{\geq k}$*.*

Proof. Let $L \in \mathrm{Coh}(Y)$ be a representative for $\Phi(\mathcal{O}_X)$. We claim that L is a line bundle away from a small enough closed subset. It follows from the claim that the equivalence Φ induces an isomorphism of locally ringed spaces $\Theta_k(X, \mathcal{O}_X) \simeq \Theta_k(Y, \Phi(\mathcal{O}_X))$. The proposition then follows: the former is isomorphic to $X_{\geq k}$, and the latter to $Y_{\geq k}$, as locally ringed spaces.

Notice first that $\mathcal{O}_X$ satisfies the following property: if P is any non-zero minimal object of $\mathcal{C}_k(X)$, then $\mathrm{Hom}_{\mathcal{C}_k(X)}(\mathcal{O}_X, P)$ is a one-dimensional $\mathrm{End}_{\mathcal{C}_k(X)}(P)$ vector space. Moreover, $\mathcal{O}_X$ is *locally maximal* with respect to this property: given F such that $\mathrm{Hom}_{\mathcal{C}_k(X)}(F, P)$ is one-dimensional, any surjection $F \to \mathcal{O}_X$ in $\mathcal{C}_P$ must be an isomorphism. Since these two properties are categorical (i.e., intrinsic to $\mathcal{C}_k(X)$ and the minimal object P) they will be satisfied by L as well.

We claim that, for any sheaf L satisfying the two properties above, there exists an open subset $U \subset Y$, with $Y_{\geq k} \subset U$, such that the restriction L_U is a line bundle. Indeed, consider the function $\beta: Y \to \mathbf{N}$, taking $y \in Y$ to the rank of the fiber $\beta(y) = \dim_{\kappa(y)} L \otimes \kappa(y)$. This function is upper semicontinuous. The locus $\{\beta = 0\}$ is therefore open. Let Z be the union of the irreducible components of dimension strictly less than k. We see that $\{\beta = 0\} \cap (Y \setminus Z)$ is empty. Therefore, the locus $U = \{\beta = 1\} \cap (Y \setminus Z)$ is open, and (by assumption) contains all points of dimension k. By Nakayama, the $\mathcal{O}_{Y,y}$-module L_y has rank 1 at all points y in U. Using Lemma 2.5, and the local maximality property of L, we deduce that $L_y = \mathcal{O}_{Y,y}$ at all points of dimension k, as there is by definition a surjection $\mathcal{O}_{Y,y} \to L_y$, which must be an isomorphism by maximality. Finally, the set of points y such that L_y is free is open. Since it contains all points of dimension k it follows that L_y is free of rank 1 at all points in a subset U' containing all points of dimension k. ☐

3. Gabriel's theorem

As usual, let $\mathbf{k}$ be our base ring, and let X, Y be schemes over it. Recall that we write $\mathcal{C}_k(X)$ for the quotient $\mathrm{Coh}(X)/\mathrm{Coh}_{\leq k-1}(X)$, with analogous notation for Y. To prove our main theorem we will deal with the two directions separately.

3.1. **The "hard" direction.**

Suppose we have an equivalence of categories $\mathcal{C}_k(X) \simeq \mathcal{C}_k(Y)$, which by default is assumed linear over $\mathbf{k}$. Proposition 2.7 implies there is an isomorphism of locally ringed spaces $X_{\geq k} \simeq Y_{\geq k}$ over $\mathrm{Spec}\,\mathbf{k}$.

Proposition 3.1. *Let* $\phi: X_{\geq k} \to Y_{\geq k}$ *be an isomorphism of locally ringed spaces over* $\mathrm{Spec}\,\mathbf{k}$*. There exist open subsets* $U \subset X, V \subset Y$ *containing all points of dimension* $\geq k$*, and an isomorphism* $f: U \to V$ *of* $\mathbf{k}$*-schemes, which restricts to* ϕ*.*

Let us start with a lemma.

Lemma 3.2. *Let X, Y be schemes of finite-type over* $\mathbf{k}$*. Assume there is an isomorphism of* $\mathbf{k}$*-algebras between local rings* $\mathcal{O}_{X,p} \simeq \mathcal{O}_{Y,q}$ *where* $p \in X$, $q \in Y$ *are not necessarily closed. Then there are open subschemes* $p \in U \subset X$, $q \in V \subset Y$ *such that* $U \simeq V$ *as* $\mathbf{k}$*-schemes.*

Proof. Fix open neighborhoods $p \in \mathrm{Spec}(A) \subset X, q \in \mathrm{Spec}(B) \subset Y$, and consider the composition $B \to B_q \to A_p$. Given a finite set of generators $y_1, \ldots, y_n$ for B as a $\mathbf{k}$-algebra, their images will be in the form $x_1/s_1, \ldots, x_n/s_n$ where $x_1, \ldots, x_n$

belong to A and $s_1, \dots, s_n$ belong to $A \smallsetminus I(p)$. Here $I(p) \subset A$ denotes the prime ideal corresponding to the point $p \in \operatorname{Spec} A$.

The image of B is thus contained in $S^{-1}A$ where S is the multiplicative set $\{s_1^{i_1} \dots s_n^{i_n}\}$. The inclusion $\operatorname{Spec}(S^{-1}A) \subset \operatorname{Spec}(A)$ is open and contains the point p. Hence we just constructed a map $W \to \operatorname{Spec}(B)$ where W is a neighborhood of p, sending p to q. By restricting to open subsets $p \in U \subset \operatorname{Spec}(A)$ and $q \in V \subset \operatorname{Spec}(B)$, we may further assume this map to be unramified, quasi-finite, and of degree one at all points, i.e., an isomorphism. ☐

Lemma 3.3. *Let* $U = \operatorname{Spec}(A), V = \operatorname{Spec}(B)$ *be affine schemes of finite-type over* $\mathbf{k}$*, and let* $f, g\colon U \to V$ *be two morphisms. Suppose further that there is a* $p \in U$ *such that* $f(p) = g(p) = q$*, and* $f^*_{|\mathcal{O}_{V,q}} = g^*_{|\mathcal{O}_{V,q}}$*. Then there is an open subscheme* $p \in U' \subset U$ *such that* f *and* g *are equal when restricted to* U'*.*

Proof. Let $y_1, \dots, y_n$ be a set of generators for B as a $\mathbf{k}$-algebra. We know $f(y_i) = g(y_i)$ in the localization $A_{I(p)}$ for all i. After inverting a finite number of elements in $A \smallsetminus I(p)$ we will have $f(y_i) = g(y_i)$ in $S^{-1}A$, for some finitely generated multiplicative set $S \subset (A \smallsetminus I(p))$. To conclude, set $U' = \operatorname{Spec}(S^{-1}A) \subset \operatorname{Spec}(A)$. ☐

Proof of Proposition 3.1. Let $V \subset Y$ be an affine open subset. As Y is of finite-type over $\mathbf{k}$, the algebra $\mathcal{O}_Y(V)$ is generated by a finite set $y_1, \dots, y_n$ of generators. Consider $y_1, \dots, y_n$ as elements of $\mathcal{O}_{Y_{\geq k}}(V \cap Y_{\geq k})$. There exists an open subset $U \subset X$, containing the inverse image of $V \cap Y_{\geq k}$, such that the elements $\phi^*(y_1), \dots, \phi^*(y_n)$ are represented by elements in $\mathcal{O}_X(U)$. This induces a morphism $\phi_U\colon U \to V$.

The morphism ϕ_U is an isomorphism at all the local rings of points $p \in V \cap Y_{\geq k}$. By Lemmas 3.2, and 3.3 given any point $p \in U \cap Y_{\geq k}$ there is a neighborhood $U_p \subset U$ such that $(\phi_U)_{|U_p}$ is an isomorphism. Up to restricting U and V to smaller open subsets containing all points of $X_{\geq k} \cap U$ and $Y_{\geq k} \cap V$, we may then assume that ϕ_U is an isomorphism.

Note that, following the construction above, the morphism ϕ_U coincides with the original ϕ on all local rings in $U \cap X_{\geq k}$. Explicitly, for each $p \in X_{\geq k}$ we have $\mathcal{O}_{X_{\geq k},p} = \mathcal{O}_{X,p}$, and $\phi_{U,p} = \phi_p$.

Suppose now we apply the same construction to two different open subsets, obtaining isomorphisms $\phi_{U_i}\colon U_i \to V_i, \phi_{U_j}\colon U_j \to V_j$. The two maps must agree on local rings for all $p \in U_i \cap U_j \cap Y_{\geq k}$. Lemma 3.3 informs us that, up to removing a closed subset of dimension at most $k-1$ from U_i and U_j and their images, the two maps must agree. Pick now a finite open affine cover $U_1, \dots, U_n$ of X. We may iterate the procedure above to all m-fold intersections, with $m \leq n$, refining our open cover and obtaining the claim. ☐

3.2. **The "easy" direction.**

The converse direction, i.e., that a birational map induces an equivalence of categories, is actually false in the generality we have maintained so far.

Remark 3.4. Let X be the spectrum of a DVR R, with closed point x and open point η. Let $K = \mathcal{O}_{X,\eta}$ be the field of fractions of R. Let Y be $\operatorname{Spec} K$, and write y for the unique point of Y. Then X and Y are birational, in the sense that $\mathcal{O}_{Y,y} \simeq \mathcal{O}_{X,\eta}$, but $X_{\geq 1} \not\simeq Y_{\geq 1}$. Indeed, the former consists of just η, while the latter is empty. At the categorical level, $\mathcal{C}_1(X) \simeq \operatorname{mod}(K)$ while $\mathcal{C}_1(Y) = 0$. The culprit is the fact that the singleton $\{\eta\}$ is *open* in X, but has smaller dimension than X.

However, over a field everything works just fine.

Proposition 3.5. *Suppose our base ring* $\mathbf{k}$ *is a finite type algebra over a field. Let* $j{:}U \hookrightarrow X$ *be an open immersion. Suppose* $\dim X \setminus U < k$. *Then* j^* *induces an equivalence* $\mathcal{C}_k(X) \simeq \mathcal{C}_k(U)$.

Proof. Note that every open subset of U whose complement has dimension lower than k is also an open subset of X with the same property (this fails in the example above for the empty set and $k = 1$). Then the statement is a consequence of [15, Tag 01PI], which assures us that the pullback functor is essentially surjective, and Lemma 1.5, which tells us that it is fully faithful. ☐

Remark 3.6. The hypotheses in the previous proposition can be weakened. The statement holds true as long as we always have that $U_{\geq k} = X_{\geq k} \cap U$, which by [6, 10.6.2] is implied by X being Jacobson, universally catenary and every irreducible component of X being equicodimensional.

3.3. **The main theorem.** Assembling our previous results together, we arrive at our main result. We say two schemes X, Y are *isomorphic outside of dimension* $k-1$ if there exist isomorphic open subsets $U \subset X, V \subset Y$, such that $X_{\geq k} \subset U$, and $Y_{\geq k} \subset V$.

Theorem 3.7. *Assume our base ring* $\mathbf{k}$ *is a finite type algebra over a field. Let* X, Y *be schemes of finite-type over* $\mathbf{k}$. *Then* X *and* Y *are isomorphic* (*over* $\mathbf{k}$) *outside of dimension* $k-1$ *if and only if* $\mathcal{C}_k(X) \simeq \mathcal{C}_k(Y)$.

Proof. By Proposition 2.7 the equivalence $\mathcal{C}_k(X) \simeq \mathcal{C}_k(Y)$ induces an isomorphism $X_{\geq k} \simeq Y_{\geq k}$. Proposition 3.1 allows us to extend the isomorphism to an open subset, completing the proof. ☐

4. The group of autoequivalences

Gabriel originally also characterized the autoequivalences of the category of coherent sheaves. Indeed, $\mathrm{Aut}(\mathrm{Coh}(X)) = \mathrm{Aut}(X) \ltimes \mathrm{Pic}(X)$, where to form the semi-direct product we use the standard (left) action of automorphisms on line bundles: $(f, L) \mapsto f_* L = (f^{-1})^* L$. We want to now describe what happens when passing to the quotients.

For this section, assume $\mathbf{k}$ is a finite type algebra over a field. Let $\mathrm{Aut}_{\geq k}(X)$ be the group of birational self-maps $X \dashrightarrow X$ which are defined on an open subset containing $X_{\geq k}$, modulo the obvious equivalence relation: $f \sim g$ if the two are equal on an open subset containing all points of dimension at least k. Recall the definition of $\mathrm{Pic}_{\geq k}(X)$ from Lemma 2.2.

Theorem 4.1. *There is an isomorphism of groups*

$$\mathrm{Aut}(\mathcal{C}_k(X)) \simeq \mathrm{Aut}_{\geq k}(X) \ltimes \mathrm{Pic}_{\geq k}(X).$$

The action in the semi-direct product is given by $(f, L) \mapsto (f^{-1})^* L$.

Let $G = \mathrm{Aut}_{\geq k}(X) \ltimes \mathrm{Pic}_{\geq k}(X)$. There is an obvious group homomorphism $G \to \mathrm{Aut}(\mathcal{C}_k)$, sending the pair (f, L) to the auto-equivalence $F \mapsto (f^{-1})^* F \otimes L$. On the other hand, an auto-equivalence Φ induces an automorphism of the locally ringed space $X_{\geq k}$, and hence a birational self-map f. This map is uniquely defined thanks to Lemma 3.3, which assures us that two automorphisms which are the same on all points of dimension k or greater have to be the same on a whole open subset whose complement has strictly lower dimension. Then we have a (set-theoretic, a priori) retraction $\Phi \mapsto (f, \Phi(\mathcal{O}_X))$, which shows that the map $\operatorname{Aut}(\mathcal{C}_k(X)) \simeq \operatorname{Aut}_{\geq k}(X) \ltimes \operatorname{Pic}_{\geq k}(X)$ is injective. Thus proving the theorem amounts to proving surjectivity.

To do this, we only need to show that an auto-equivalence τ such that the induced map on $X_{\geq k}$ is the identity and $\Phi(\mathcal{O}_X) = \mathcal{O}_X$ must be trivial, as then for any $\Phi \in \operatorname{Aut}(\mathcal{C}_k(X))$ there will always be a Φ' coming from $\operatorname{Aut}_{\geq k}(X) \ltimes \operatorname{Pic}_{\geq k}(X)$ such that $\Phi' \circ \Phi$ is the identity.

We do this by constructing a natural isomorphism $\tau \colon \Phi \to \operatorname{Id}_{\mathcal{C}_k(X)}$.

Lemma 4.2. *let M, M' be coherent sheaves on X and assume that $M \otimes \mathcal{O}_{X,P} = M_P = M'_P$ for a point $P \in X$. Then there exists an open neighborhood U of P and an isomorphism of coherent sheaves $M_U \simeq M'_U$.*

Proof. The same argument we used for Lemma 3.2 works here. ☐

Lemma 4.3. *let M, M' be coherent sheaves on X and let $f, g \colon M \to M'$ be two morphisms. If the localizations $f_x, g_x \colon M_x \to M'_x$ are equal for all points $x \in X_{\geq k}$, then there is an open subset $X_{\geq k} \subset U$ such that the restrictions $f|_U, g|_U$ are equal.*

Proof. The same argument as in Lemma 3.3 shows that given each such x there is an open neighborhood containing x where the maps are equal. We can then conclude by the fact that being 0 is an open property. ☐

Proof of Theorem 4.1. We wish to construct a natural isomorphism $\tau \colon \Phi \to \operatorname{Id}_{\mathcal{C}_k(X)}$, as explained earlier. To start, assume that X is affine. Let M be a coherent sheaf on X. Then

$$\operatorname{Hom}_{\mathcal{C}_k(X)}(\mathcal{O}_X, M) = \operatorname{Hom}_{\mathcal{C}_k(X)}(\Phi(\mathcal{O}_X), \Phi(M)) = \operatorname{Hom}_{\mathcal{C}_k(X)}(\mathcal{O}_X, \Phi(M)).$$

Fix a representative $M' \in \operatorname{Coh}(X)$ of $\Phi(M)$, and let $m'_1, \dots, m'_r$ generators for M'. To each of these corresponds a map $s'_i \colon \mathcal{O}_X \to M'$. Let $s_1, \dots, s_r$ be the inverse images of these maps in $\operatorname{Hom}_{\mathcal{C}_k(X)}(\mathcal{O}_X, M)$. By Lemma 1.5 we can pick morphisms $S_1, \dots, S_r \in \operatorname{Hom}_U(\mathcal{O}_U, M_U)$ where U is an open subset of X whose complement has dimension at most $k-1$, which in turn give us global sections $m_1, \dots, m_r$ of M_U.

Note that the assignment $m'_i \mapsto m_i$ induces an isomorphism between the local modules M'_P and M_P for all points P of dimension k. By Lemma 4.2, up to restricting U by removing a subset of dimension at most $k-1$, we can assume that the sheaf of relations between $m'_1, \dots, m'_r$ is isomorphic to the sheaf of relations between $m_1, \dots, m_r$, and thus the assignment $m'_i \mapsto m_i$ induces a morphism $M'_U \to M_U$ (note that this works because every open subset is sent to itself by the induced morphism on $X_{\geq k}$).

Now fix an affine covering $U_1, \dots, U_s$ of U. The assignment $m'_1, \dots, m'_r \mapsto m_1, \dots, m_r$ induces a morphism $M'_{U_i} \to M_{U_i}$. Thanks to Lemma 4.3, up to restricting U further, we can assume that these maps glue to a morphism $M'_U \to M_U$. Note now that $m_1, \dots, m_r$ are generators of M on an appropriate open subset due to the exact sequence

$$\mathcal{O}_X^r \xrightarrow{\oplus s'_i} M' \to 0$$

being sent to

$$\mathcal{O}_X^r \xrightarrow{\oplus s_i} M \to 0$$

by ψ^{-1}. So applying the same process we can construct a map $M_V \to M'_V$ for some open subset V of X whose complement has dimension at most $k-1$. On $U \cap V$ these two maps are inverse to each other, so we have constructed an isomorphism $\tau_M\colon M \simeq M'$ in $\mathcal{C}_k(X)$.

Note now that on local rings this isomorphism is uniquely determined by the isomorphism $\mathrm{Hom}_{\mathcal{C}_P}(\mathcal{O}_X, M) \simeq \mathrm{Hom}_{\mathcal{C}_P}(\mathcal{O}_X, M')$ for each minimal object P. Using Lemmas 4.3, and 1.5 we conclude that the isomorphism τ_M is unique and functorial. We have thus proven the claim for affine schemes. Now note that by taking an affine covering X_i of a general scheme X of finite-type over $\mathbf{k}$, the maps $\tau_{M,i}$ we construct as above will glue (up to removing closed subsets of dimension at most $k-1$), giving rise to the sought after natural isomorphism τ. ☐

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