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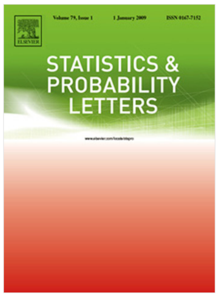

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Prabhakar Lévy Processes

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Abstract

We introduce here a generalization of the Mittag-Leffler Lévy process (with parameter α), obtained by extending its Lévy measure through the Prabhakar function (which is a Mittag-Leffler with the additional parameters β and γ). We prove that this so-called Prabhakar process, in the special case $\beta = 1$, can be represented as an α-stable process subordinated by an independent generalized gamma subordinator; thus it can be considered as an extension of the geometric stable process, to which it reduces for $\gamma = 1$. On the other hand, for $\alpha = \beta = 1$, it coincides with the generalized gamma process itself. Therefore, by suitably specifying the three parameters, the Prabhakar process turns out to represent an interpolation among various well-known and widely applied stochastic models.

Keywords: Mittag-Leffler distribution, subordinated, stochastic processes, Lévy density.
2010 MSC: 33E12, 26A33, 60G51, 60G52

1. Introduction

The widespread of fractional (non-integer order) calculus [1, 2, 3] in many areas of probability gave an impact for the development of new stochastic models: many new processes have been defined and investigated, for example, in the context of anomalous diffusion ([4, 5, 6, 7, 8] and references therein), relaxation phenomena ([9]), point processes ([10, 11, 12]), time-changed Brownian motion ([13], [14]). In particular, the relationship between Lévy stable processes (as well as their inverses) and fractional differential equations have been largely studied and extended to more general settings (see, among the others, [15, 16]).
A very important tool, in this context, is the well-known Mittag-Leffler (ML) function ([17, 18, 19]). Its relevance is confirmed by numerous generalizations ([19, 20]): recently the three parameter ML function (also called Prabhakar function) has been attracting more and more attention (see [21, 22]). Let us define this function, for any $z \in \mathbb{C}$, via the following infinite series representation

$$E^{\gamma}_{\alpha,\beta}(z) = \frac{1}{\Gamma(\gamma)} \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+k) z^k}{k!\Gamma(\alpha k+\beta)}, \quad z \in \mathbb{C}, \alpha, \beta, \gamma \in \mathbb{C},\ \Re(\alpha) > 0, \tag{1.1}$$

where $\Gamma(\cdot)$ is the Euler gamma function. It is well-known that $E^{\gamma}_{\alpha,\beta}(x)$ is an entire function of order $\rho = \frac{1}{\Re(\alpha)}$([23]) and it is a special case of more general Fox-Wright function ${}_p\Psi_q[.]$ (see [24, 3]). When $\gamma = 1$ we

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obtain the two parameters ML function, while, for $\beta = 1$ and $\gamma = 1$, we recover the classical ML function (for further properties, see [3, 19]).

The Prabhakar function is mostly known for its usefulness in the description of dielectric relaxation phenomena, i.e. the so called Havriliak-Negami model (see [25, 26]). Moreover, we recall also its applications to fractional viscoelasticity (in [27]) or to systems with anisotropy (in [28]). In mathematics, one can mention probability distributions based on Prabhakar function (in [29, 30]) and some related stochastic processes ([31, 32]), while, for various other applications and historical remarks, the interested reader is referred to [33].

The wide interest in the ML functions (in particular in the Prabhakar case) is the main motivation of the present paper. Here, however, instead of considering it as a tool for expressing distribution functions, we apply the ML function as a building block for defining a new class of stochastic processes via their Lévy measure ([34]). It is well-known that the ML process is defined as an α-stable subordinator $S_\alpha := S_\alpha(t)$, $t \geq 0$, time-changed by an independent gamma subordinator,and hence it is also called geometric stable in [35, 36] and [37], $G := G(t)$, $t \geq 0$, i.e. as $X_\alpha(t) := S_\alpha(G(t))$, $t \geq 0$ (see [38] for further properties). Its Lévy measure has the following density:

$$\pi_{X_\alpha}(x) = \frac{\alpha}{x} E_{\alpha,1}(-\lambda x^\alpha), \ \lambda > 0, \ x \geq 0, \ \alpha \in (0,1]. \tag{1.2}$$

We define here the so-called Prabhakar process $M_\alpha^{\beta,\gamma} = M_\alpha^{\beta,\gamma}(t)$, $t \geq 0$, by considering a Lévy measure expressed by means of the function (1.1), for $\alpha \in (0,1]$, $\beta \in [1, 1+\alpha\gamma]$ and $\gamma > 0$ (see formula (2.1) for details).

We prove that, in the special case $\beta = 1$, it can be represented as an α-stable process subordinated by an independent generalized gamma subordinator, thus it can be considered as an extension of the geometric stable process, to which it reduces for $\beta = \gamma = 1$. On the other hand, for $\alpha = \beta = 1$, it coincides with the generalized gamma process itself. We recall that the gamma Lévy subordinator has found applications in many areas ranging from finance ([37]) to physics ([39]). In particular, when it is used to subordinate the Brownian motion, it allows to define the so-called variance gamma process (or Laplace motion), which is widely used in option pricing, for a wider modelling of skewness and kurtosis than the Brownian motion (see [40]).

Therefore the Prabhakar Lévy process (PL) studied here turns out to represent an interpolation among various well-known and widely applied stochastic models, by suitably specifying its three parameters.

The rest of the paper has the following structure. In Section 2 we provide the definition and the main properties of the PL process. In Section 3 we investigate its subordination representation, together with its governing differential equation (at least in a special case). Indeed, when $\alpha = \beta = 1$, we prove that the transition density of the PL process (also called, in this case, generalized gamma subordinator) is governed by a partial differential equation with convolution-type space derivative: the latter is defined, in the Riemann-Liouville sense, by means of a fractional counterpart of the exponential integral. Section 4 provides brief informations on the simulation procedure of the PL process' sample paths, for further investigation of its behavior and for an interpretation of the parameters' role.

2. Main results

We start by giving the definition of the generalization of the ML process which we call Prabhakar Lévy (PL) process.

Definition 2.1 (PL process). *Let $\alpha \in (0,1]$, $\gamma > 0$ and $\beta \in [1, 1+\alpha\gamma)$ then the PL process $M_\alpha^{\beta,\gamma} = M_\alpha^{\beta,\gamma}(t)$, $t \geq 0$ is defined as a Lévy process with triplet $(0, 0, t\pi_{M_\alpha^{\beta,\gamma}})$, where $\pi_{M_\alpha^{\beta,\gamma}}(\cdot)$ is the following density* *of Lévy measure*

$$\pi_{M_\alpha^{\beta,\gamma}}(x) = \frac{\alpha\Gamma(\gamma)}{\Gamma\left(\gamma + \frac{1-\beta}{\alpha}\right)} x^{\beta-2} E_{\alpha,\beta}^{\gamma}(-\lambda x^{\alpha}), \ \lambda > 0, \ x \geq 0. \tag{2.1}$$

We note that the parameter α belongs to the interval $(0,1]$, as in the stable subordinator case, whereas the parameter β is assumed to vary in the interval $[1, 1+\alpha\gamma)$ so that $\Gamma\left(\gamma + \frac{1-\beta}{\alpha}\right)$ is positive and finite for any $\gamma > 0$.

Remark 2.1. *In the special case where $\beta = \gamma = 1$, formula (2.1) reduces to the well-known density of the Lévy measure of the ML process. Adding the further condition $\alpha = 1$ we obtain the gamma subordinator case. In Fig. 1 we present a comparison of the Lévy measures for gamma, ML and PL processes. We can see that the PL process, for certain parameters, occupies some intermediate place between gamma and ML processes. In Fig. 2 we further provide a description of the influence of α on the Lévy measure, which can be heuristically summarized as follows: that greater the parameter α the less bulky the Lévy measure.*

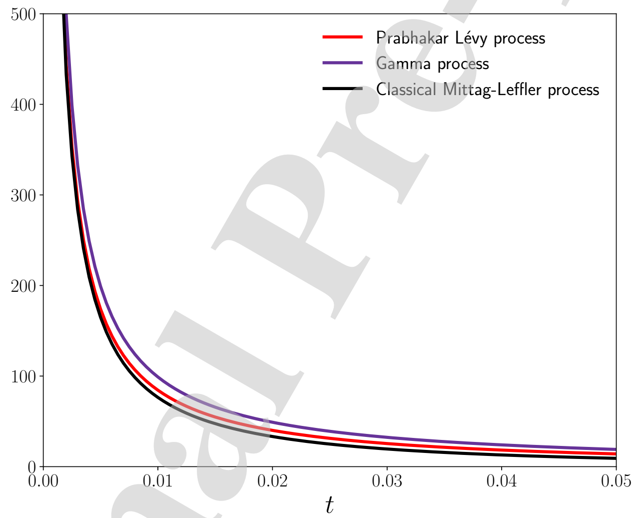

Figure 1: Behavior of the Lévy density of the PL process (for $\alpha = 0.9, \beta = 1, \gamma = 0.4, \lambda = 10$), Mittag-Leffler process (for $\alpha = 0.9, \beta = 1, \gamma = 1, \lambda = 10$) and gamma process (for $\alpha = 1, \beta = 1, \gamma = 1, \lambda = 10$)

Let us now show that (2.1) defines a proper Lévy density.

Proposition 2.1. *The density defined in Eq.(2.1) is a proper Lévy density.*

Proof. We need to check that the following condition holds

$$\int_0^\infty \frac{x}{1+x} \pi_{M_\alpha^{\beta,\gamma}}(x)\, dx < +\infty.$$

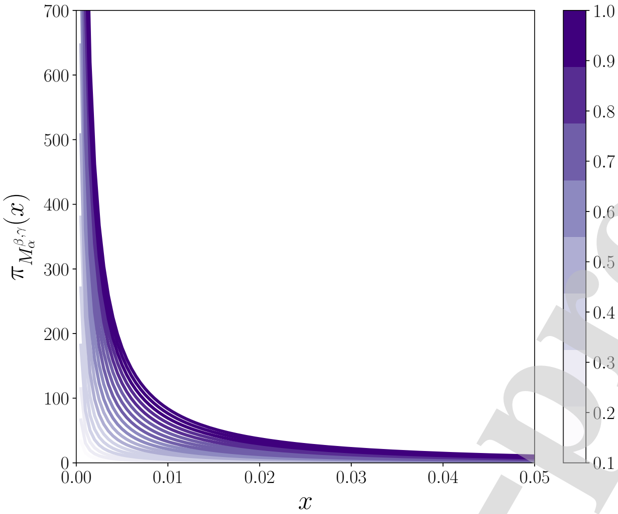

Figure 2: Behavior of the Lévy density of the PL process, for $\beta = 1, \gamma = 1, \lambda = 10$. The right colorbar represents the parameter α ranging from 0 to 1.

(see [34]). We have

$$\int_0^\infty \frac{x}{1+x}\pi_{M_\alpha^{\beta,\gamma}}(x)\,dx = \frac{\alpha\Gamma(\gamma)}{\Gamma\left(\gamma+\frac{1-\beta}{\alpha}\right)}\int_0^\infty \frac{x^{\beta-1}}{1+x}E^{\gamma}_{\alpha,\beta}\left(-\lambda x^{\alpha}\right)dx = \frac{\alpha\Gamma(\gamma)}{\Gamma\left(\gamma+\frac{1-\beta}{\alpha}\right)}\sum_{j=0}^{\infty}\frac{(\gamma)_j(-\lambda)^j}{j!\Gamma(\alpha j+\beta)}\int_0^\infty \frac{x^{\beta+\alpha j-1}}{1+x}dx$$

$$= \frac{\alpha\Gamma(\gamma)}{\Gamma\left(\gamma+\frac{1-\beta}{\alpha}\right)}\sum_{j=0}^{\infty}\frac{(\gamma)_j(-\lambda)^j}{j!\Gamma(\alpha j+\beta)}\int_0^\infty x^{\beta+\alpha j-1}\left(\int_0^\infty e^{-v(1+x)}dv\right)dx$$

$$= \frac{\alpha\Gamma(\gamma)}{\Gamma\left(\gamma+\frac{1-\beta}{\alpha}\right)}\sum_{j=0}^{\infty}\frac{(\gamma)_j(-\lambda)^j}{j!\Gamma(\alpha j+\beta)}\int_0^\infty e^{-v}\left(\int_0^\infty e^{-vx}x^{\beta+\alpha j-1}dx\right)dv$$

$$= \frac{\alpha\Gamma(\gamma)}{\Gamma\left(\gamma+\frac{1-\beta}{\alpha}\right)}\sum_{j=0}^{\infty}\frac{(\gamma)_j(-\lambda)^j}{j!}\int_0^\infty e^{-v}v^{-\beta-\alpha j}dv = \frac{\alpha}{\Gamma\left(\gamma+\frac{1-\beta}{\alpha}\right)}\sum_{j=0}^{\infty}\frac{\Gamma(\gamma+j)(-\lambda)^j}{j!}\Gamma(1-\alpha j-\beta)$$

$$= \frac{\alpha}{\Gamma\left(\gamma+\frac{1-\beta}{\alpha}\right)}{}_2\Psi_1\left[\begin{matrix}(\gamma,1)(1-\beta,-\alpha)\\ \end{matrix}\middle| -\lambda\right]$$

here ${}_p\Psi_q$ is the generalised Fox-Wright function which is absolutely convergent, by applying Theorem 1.5 in [3] (since $\delta = \alpha - 1 > -1$). □

Now we provide the expression of the Laplace exponent of the PL process, in terms of generalized Fox-Wright function.

Theorem 2.1. *The Laplace exponent of the Prabhakar process reads*

$$\psi_{M_\alpha^{\beta,\gamma}}(u) = \frac{u^{\alpha\gamma-\beta+1}}{\Gamma\left(\gamma+\frac{1-\beta}{\alpha}\right)\lambda^{\gamma}}\,{}_2\Psi_1\left[\begin{matrix}(\gamma,1)\left(\gamma+\frac{1-\beta}{\alpha},1\right)\\ \left(\gamma+\frac{1-\beta}{\alpha}+1,1\right)\end{matrix}\middle| -\frac{u^{\alpha}}{\lambda}\right],\ u \geq 0, \tag{2.2}$$

where ${}_p\Psi_q$ is the Fox-Wright function (see [3] for its definition).

Proof.

$$\psi_{M_\alpha^{\beta,\gamma}}(u) = -\frac{1}{t}\log\left(\mathbb{E}\left(e^{-uM^\gamma_{\alpha,\beta}(t)}\right)\right) = \int_0^\infty \left(1-e^{-ux}\right)\frac{\alpha\Gamma(\gamma)}{\Gamma\left(\gamma+\frac{1-\beta}{\alpha}\right)}\frac{1}{x}x^{\beta-1}E^\gamma_{\alpha,\beta}\left(-\lambda x^\alpha\right)dx$$

$$= \frac{\alpha\Gamma(\gamma)}{\Gamma\left(\gamma+\frac{1-\beta}{\alpha}\right)}\int_0^\infty\int_0^u e^{-vx}x^{\beta-1}E^\gamma_{\alpha,\beta}\left(-\lambda x^\alpha\right)dvdx$$

$$= \frac{\alpha\Gamma(\gamma)}{\Gamma\left(\gamma+\frac{1-\beta}{\alpha}\right)}\int_0^u\int_0^\infty e^{-vx}x^{\beta-1}E^\gamma_{\alpha,\beta}\left(-\lambda x^\alpha\right)dxdv$$

$$= \frac{\alpha\Gamma(\gamma)}{\Gamma\left(\gamma+\frac{1-\beta}{\alpha}\right)}\int_0^u \frac{v^{\alpha\gamma-\beta}}{(\lambda+v^\alpha)^\gamma}dv$$

$$= \frac{\Gamma(\gamma)}{\Gamma\left(\gamma+\frac{1-\beta}{\alpha}\right)\lambda^\gamma}u^{\alpha\gamma-\beta+1}\int_0^1 w^{\gamma-\frac{\beta}{\alpha}+\frac{1}{\alpha}-1}\left(1+\frac{wu^\alpha}{\lambda}\right)^{-\gamma}dw$$

$$= \frac{\Gamma(\gamma)}{\Gamma\left(\gamma+\frac{1-\beta}{\alpha}+1\right)\lambda^\gamma}u^{\alpha\gamma-\beta+1}{}_2F_1\left(\gamma,\gamma+\frac{1-\beta}{\alpha},\gamma+\frac{1-\beta}{\alpha}+1,-\frac{u^\alpha}{\lambda}\right).$$

Here ${}_2F_1$ is the Gauss hypergeometric function (see [3], p.27). In the last step we used its Euler representation (see [3], formula 1.6.2). By applying equation (1.11.26) in [3], we get (2.2). □

It is easy to see that, in the special case where $\gamma=\beta=1$, the Laplace exponent of the PL process reduces to that of the ML process, i.e.

$$\psi_{M_\alpha^{1,1}}(u) = \log\left(1+\frac{u^\alpha}{\lambda}\right).$$

In the following proposition we investigate the tail behavior of the PL process and the conditions for the existence of finite fractional moments.

Proposition 2.2. *The tail behavior for PL process for $\gamma<\beta/\alpha$ reads*

$$\lim_{x\to\infty} x^{1+\alpha\gamma-\beta}P\left(M_\alpha^{\beta,\gamma}(t)>x\right) = \frac{t\Gamma(\gamma)}{\Gamma\left(\gamma+\frac{1-\beta}{\alpha}+1\right)\Gamma\left(\beta-\alpha\gamma\right)\lambda^\gamma}. \tag{2.3}$$

Moreover the PL process has finite p-th order moments iff $0<p<1+\alpha\gamma-\beta$.

Proof. The tail behavior can be obtained from the Tauberian theorems (see [41], Theorem XIII, p.446) as follows

$$\int_0^\infty e^{-ux}P\left(M_\alpha^{\beta,\gamma}(t)>x\right)dx = \frac{1-E\left[e^{-uM_\alpha^{\beta,\gamma}(t)}\right]}{u} = \frac{1-e^{-t\psi_{M_\alpha^{\beta,\gamma}}(u)}}{u}$$

$$\underset{u\to0}{\approx}\frac{t\psi_{M_\alpha^{\beta,\gamma}}(u)}{u} = \frac{t\Gamma(\gamma)u^{\alpha\gamma-\beta}}{\Gamma\left(\gamma+\frac{1-\beta}{\alpha}+1\right)\lambda^\gamma},$$

since ${}_2F_1\left(\gamma,\gamma+\frac{1-\beta}{\alpha},\gamma+\frac{1-\beta}{\alpha}+1;0\right)=1$. Under the restriction $\gamma<\beta/\alpha$ from Theorem XIII 5.4 in [41] we obtain the desired result. Based on the tail behavior and considering that $EX^p=\int_0^\infty P(X^p>x)dx$, we can immediately derive the result on the finiteness of the p-th order moments. □

For $\beta=\gamma=1$ formula (2.3) reduces to the well known tail behavior of the ML process (see [42], in the inhomogeneous case).

We should mention here that the p-th order fractional moment (for $p\in(0,1)$) can be derived by fractional differentiation of the Laplace transform of $M_\alpha^{\beta,\gamma}$ (as proved in Lemma 1.1. in [43]), via the following formula

$$\begin{aligned} E\left(\left[M_\alpha^{\beta,\gamma}(t)\right]^p\right) &= -\frac{1}{\Gamma(1-p)}\int_0^\infty w^{-p}\frac{d}{dw}E\left[e^{-wM_\alpha^{\beta,\gamma}(t)}\right]dw \\ &= \frac{t}{\Gamma(1-p)}\int_0^\infty w^{-p}e^{-t\psi_{M_\alpha^{\beta,\gamma}}(w)}\frac{d}{dw}\psi_{M_\alpha^{\beta,\gamma}}(w)dw. \end{aligned}$$

However, due to this complicated form, it is hard to obtain explicit formulas. We provide instead the fractional counterpart of the cumulants, which we define as follows. For a Lévy process $X(t)$, $t\geq 0$, with a Lévy measure ν (in analogy to the integer order case, see [44]), let

$$C_p\left(X(t)\right):=t\int_{-\infty}^{+\infty}x^p\nu(dx), \tag{2.4}$$

for any p such that the integral converges. In our case we can obtain

$$\begin{aligned} C_p\left(M_\alpha^{\beta,\gamma}(t)\right) &= \frac{\alpha\Gamma(\gamma)t}{\Gamma\left(\gamma+\frac{1-\beta}{\alpha}\right)}\int_0^\infty x^{p+\beta-2}E_{\alpha,\beta}^{\gamma}\left(-\lambda x^\alpha\right)dx \\ &\quad \text{[by (1.137) in [23]]} \\ &= \frac{\alpha t}{\Gamma\left(\gamma+\frac{1-\beta}{\alpha}\right)}\int_0^\infty x^{p+\beta-2}H_{1,2}^{1,1}\left[\lambda x^\alpha\left|\begin{matrix}(1-\gamma,1) & \\ (0,1), & (1-\beta,\alpha)\end{matrix}\right.\right]dx \\ &= \frac{t}{\Gamma\left(\gamma+\frac{1-\beta}{\alpha}\right)}\int_0^\infty z^{\frac{p+\beta-1}{\alpha}-1}H_{1,2}^{1,1}\left[\lambda z\left|\begin{matrix}(1-\gamma,1) & \\ (0,1), & (1-\beta,\alpha)\end{matrix}\right.\right]dz \\ &= \frac{t\lambda^{\frac{1-\beta-p}{\alpha}}}{\Gamma\left(\gamma+\frac{1-\beta}{\alpha}\right)}\frac{\Gamma\left(\frac{p-1+\beta}{\alpha}\right)\Gamma\left(\gamma-\frac{p-1+\beta}{\alpha}\right)}{\Gamma(1-p)} \end{aligned}$$

provided that $p\in(1-\beta,\min\{\alpha\gamma+1-\beta,1\})$. Since we restrict ourselves to $\gamma<\beta/\alpha$, we can write $p\in(1-\beta,\alpha\gamma+1-\beta)$.

We now provide some results on the path behavior of the PL process.

Proposition 2.3. *Almost all the paths of the PL process have finite variation.*

Proof. We can write

$$\int_0^1 x\pi_{M_\alpha^{\beta,\gamma}}(x)\,dx = \frac{\alpha\Gamma(\gamma)}{\Gamma\left(\gamma+\frac{1-\beta}{\alpha}\right)}\int_0^1 x^{\beta-1}E_{\alpha,\beta}^{\gamma}(-\lambda x^\alpha)\,dx$$
$$= \frac{\alpha\Gamma(\gamma)}{\Gamma\left(\gamma+\frac{1-\beta}{\alpha}\right)}E_{\alpha,\beta+1}^{\gamma}(-\lambda),$$

where we have applied formula (5.1.19) of [19], in the last step. Then by Theorem 21.9 in [45], the sample function $M_\alpha^{\beta,\gamma}(t,\omega)$ has almost surely finite variation on $(0,t]$ for any $t\in(0,\infty)$. □

3. Subordination representation and fractional equations

We recall that the ordinary ML process can be represented as an α-stable subordinator S_α, with stability parameter $\alpha\in(0,1)$, time-changed by an independent gamma subordinator. Analogously, we prove that the process $M_\alpha^{\beta,\gamma}$ can be conveniently defined as a subordinated process. Let us recall that the Laplace transform of S_α is given by

$$\mathbb{E}\left(e^{-uS_\alpha(t)}\right) = e^{-tu^\alpha} \tag{3.5}$$

and thus the Laplace exponent of S_α is $\psi_{S_\alpha}(u)=u^\alpha$, $u\in\mathbb{R}^+$.

Theorem 3.1. *The PL process can be defined through the subordination of an α-stable subordinator as follows*

$$M_\alpha^{\beta,\gamma}(t) \overset{d}{=} S_\alpha\left(M_1^{1-\left(\frac{1-\beta}{\alpha}\right),\gamma}(t)\right), \tag{3.6}$$

where $\overset{d}{=}$ indicates equality of finite dimensional distributions and the process $M_1^{1-\left(\frac{1-\beta}{\alpha}\right),\gamma}$ is assumed to be independent of S_α.

Proof. First it is easy to check, from Theorem 2.1, that

$$\psi_{M_1^{1-\left(\frac{1-\beta}{\alpha}\right),\gamma}}(u) = \frac{u^{\gamma+\frac{1-\beta}{\alpha}}}{\Gamma\left(\gamma+\frac{1-\beta}{\alpha}\right)\lambda^\gamma}\,{}_2\Psi_1\left[\left.\begin{matrix}(\gamma,1)\,\left(\gamma+\frac{1-\beta}{\alpha},1\right)\\ \left(\gamma+\frac{1-\beta}{\alpha}+1,1\right)\end{matrix}\right| -\frac{u}{\lambda}\right]. \tag{3.7}$$

Now, by considering Eq. (3.5) together with Eq. (3.7) and using a conditioning argument (see also Prop. 1.3.27 in [34]), we have

$$\begin{aligned}
\psi_{S_\alpha\left(M_1^{1-\left(\frac{1-\beta}{\alpha}\right),\gamma}\right)}(u) &= -\frac{1}{t}\log\left(\mathbb{E}\left(e^{-uS_\alpha\left(M_1^{1-\left(\frac{1-\beta}{\alpha}\right),\gamma}(t)\right)}\right)\right)\\
&= -\frac{1}{t}\log\left(\mathbb{E}\left[\mathbb{E}\left(e^{-uS_\alpha\left(M_1^{1-\left(\frac{1-\beta}{\alpha}\right),\gamma}(t)\right)}\middle| M_1^{1-\left(\frac{1-\beta}{\alpha}\right),\gamma}(t)\right)\right]\right)\\
&= -\frac{1}{t}\log\left(\mathbb{E}\left(e^{-u^\alpha M_1^{1-\left(\frac{1-\beta}{\alpha}\right),\gamma}(t)}\right)\right)\\
&= \frac{u^{\alpha\gamma-\beta+1}}{\Gamma\left(\gamma+\frac{1-\beta}{\alpha}\right)\lambda^\gamma}\,{}_2\Psi_1\left[\left.\begin{matrix}(\gamma,1)\,\left(\gamma+\frac{1-\beta}{\alpha},1\right)\\ \left(\gamma+\frac{1-\beta}{\alpha}+1,1\right)\end{matrix}\right| -\frac{u^\alpha}{\lambda}\right]
\end{aligned}$$

□

Representation (3.6) is particularly interesting in the special case $\beta = 1$. Indeed, in this case, the inner process is independent of the parameter α and reduces to a generalized gamma subordinator. Let us denote $M_1^{1,\gamma}$ as G_γ for simplicity, then we have that

$$\pi_{G_\gamma}(x) = \frac{1}{x} E_{1,1}^{\gamma}\left(-\lambda x\right), \; x \geq 0. \tag{3.8}$$

It is immediate to check that the previous expression coincides with the Lévy measure of the gamma subordinator, for $\gamma = 1$. Therefore, for $\beta = 1$, formula (3.6) reduces to $M_\alpha^{1,\gamma} = S_\alpha(G_\gamma(t))$ and thus the Prabhakar process is, in this case, obtained by subordinating a stable Lévy process by an independent generalized gamma subordinator.

We are able to write explicitly the generator of the generalized gamma subordinator G_γ with Lèvy density (3.8), by defining a fractional counterpart of the exponential integral, as follows. Let us recall the standard exponential integral, i.e. $\mathcal{E}_1(x) := \int_x^{+\infty} z^{-1} e^{-z} dz$. Then we consider the following function, for $\gamma > 0$,

$$\mathcal{E}_1^{\gamma}(x) := \int_x^{+\infty} z^{-1} E_{1,1}^{\gamma}(-z) dz, \tag{3.9}$$

which reduces to $\mathcal{E}_1(\cdot)$, in the special case $\gamma = 1$. Another fractional generalization of the exponential integral, defined via the one-parameter ML function, has been already introduced and studied in [46].

We first check that the integral in (3.9) is convergent, by recalling the following asymptotic formula of the Prabhakar function

$$E_{\alpha,\beta}^{\gamma}(-ct^{\nu}) \simeq \frac{c^{-\gamma} t^{-\alpha\gamma}}{\Gamma\left(\beta - \alpha\gamma\right)}, \qquad t \to +\infty, \; c > 0,$$

(see [47]). Moreover, we can evaluate the Laplace transform of (3.9) as follows:

$$\begin{aligned} \int_0^{+\infty} e^{-\eta x} \mathcal{E}_1^{\gamma}(x) dx &= \int_0^{+\infty} z^{-1} E_{1,1}^{\gamma}(-z) \int_0^z e^{-\eta x} dx dz \\ &= \frac{1}{\eta} \int_0^{+\infty} (1 - e^{-\eta x}) z^{-1} E_{1,1}^{\gamma}(-z) dz \\ &= \frac{\psi_{G_\gamma}(\eta)}{\eta} \\ &= \frac{\eta^{\gamma-1}}{\Gamma(\gamma)} {}_2\Psi_1 \left[\begin{array}{cc} (\gamma, 1) & (\gamma, 1) \\ (\gamma+1, 1) & \end{array} \middle| -\eta \right], \end{aligned} \tag{3.10}$$

where, in the last step, we have applied formula (3.7), for $\alpha = \beta = 1$ and $\lambda = 1$. It is easy to check that, for $\gamma = 1$, the previous expression coincides with the Laplace transform of the exponential integral, i.e.

$$\int_0^{+\infty} e^{-\eta x} \mathcal{E}_1(x) dx = \frac{1}{\eta} \log\left(1 + \eta\right)$$

(see [46]). We now define the following convolution-type derivative (in the sense of Riemann-Liouville): let

$$\mathcal{D}_x^{\gamma} f(x) = \frac{d}{dx} \int_0^x f(x-y) \mathcal{E}_1^{\gamma}(y) dy,$$

for an absolutely continuous function f (see [48] and [16] for details on the convolution-type operators). Then, by Theorem 4.1 in [16], we can conclude that the transition density $f_\gamma(x,t)$, $x, t \geq 0$ of G_γ satisfies the following problem

$$\begin{cases} \frac{\partial}{\partial t} f_\gamma(x,t) = -\mathcal{D}_x^\gamma f_\gamma(x,t) \\ f_\gamma(0,t) = 0 \\ f_\gamma(x,0) = \delta(x) \end{cases},$$

where $\delta(\cdot)$ is the Dirac's delta function.

4. Simulation of sample paths

Simulation of the PL process can't be obtained by classical methods through generation of increments from some known distributions, due to its complicated representation. Thus, in this paper, we employ the procedure proposed in [49]. By numerical inversion of the Laplace transform we are able to generate the sample paths of the PL process. Sample trajectories are presented in Fig. 3, for $\alpha \in \{0.5, 0.6, 0.7, 0.8, 0.9, 1\}$. In Fig. 4, we show the influence of the parameter β, which, as can be inferred, is responsible for the amplitude of the jumps. The last Fig. 5 depicts the influence of the parameter γ, which, on the contrary, seems to have more effect on the jumps' frequency.

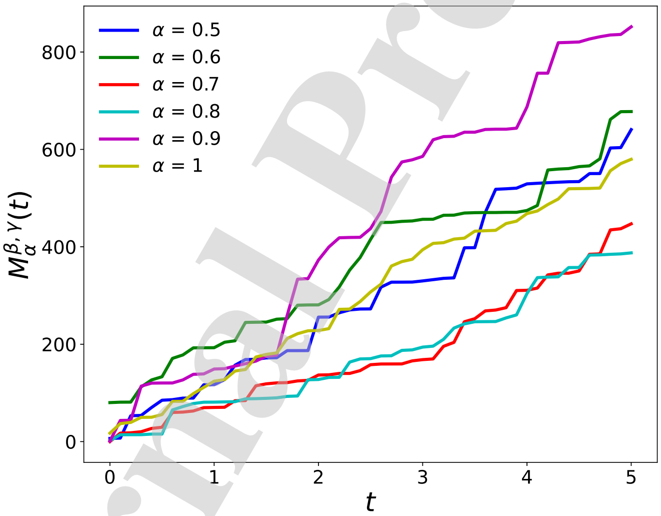

Figure 3: Sample paths for the PL process for $\alpha \in \{0.5, 0.6, 0.7, 0.8, 0.9, 1\}$ and $\beta = 1, \gamma = 0.8, \lambda = 1$.

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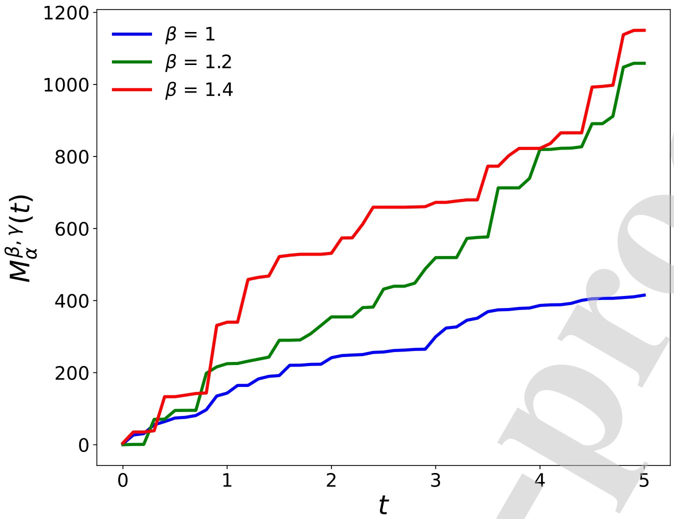

Figure 4: Sample paths for the PL process for $\beta \in \{1, 1.2, 1.4\}$ and $\alpha = 0.9, \gamma = 0.8, \lambda = 1$.

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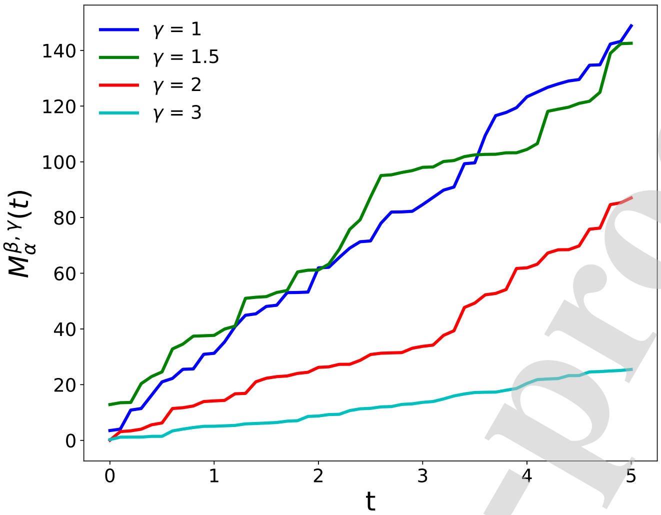

Figure 5: Sample paths for the PL process for $\gamma \in \{1, 1.5, 2, 3\}$ and $\alpha = 0.9, \beta = 1, \lambda = 1$.

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