Twisted Cohomotopy implies level quantization of the full 6d Wess-Zumino term of the M5-brane

Domenico Fiorenza, Hisham Sati, Urs Schreiber

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Abstract

The full 6d Wess-Zumino term in the action functional for the M5-brane is anomalous as traditionally defined. What has been missing is a condition implying the higher analogue of level quantization familiar from the 2d Wess-Zumino term. We prove that the anomaly cancellation condition is implied by the hypothesis that the C-field is charge-quantized in twisted Cohomotopy theory. The proof follows by a twisted/parametrized generalization of the Hopf invariant, after identifying the full 6d Wess-Zumino term with a twisted homotopy Whitehead integral formula, which we establish.

Contents

1 Introduction and results

The expected but elusive quantum theory of M5-branes in M-theory (see [Duf99, Sec. 3], [HSS18, Sec. 2]) has come to be widely regarded as a core open problem in string theory, already in its decoupling limit of an expected 6-dimensional superconformal quantum field theory (see [Mo12], [HR18]). Most attempts to understand at least aspects of this theory have been based on analogies (such as with the known M2-brane theory) and consistency checks (such as from implications of the expected superconformal structure). But a systematic derivation of the theory from deeper principles had not been possible, since these deeper principles must be those of the ambient M-theory, whose formulation is itself a wide open problem (see [Mo14, Sec. 12]).

Recently in [FSS19a], following [Sa13], we motivated, from rigorous analysis of the super homotopy theory of super p-branes initiated in [FSS13b], a hypothesis about the mathematical foundations of microscopic M-theory:

Hypothesis H. The M-theory C-field is charge-quantized in J-twisted Cohomotopy theory (Def. 4.1).

We proved in [FSS19b] that this hypothesis implies a list of subtle consistency conditions that had informally been argued to be necessary for M-theory to exist. This suggests that *Hypothesis H* could indeed be a correct assumption about the mathematical principles underlying microscopic M-theory. If this is the case, further aspects of M-theory must be systematically derivable, by rigorous mathematical deduction.

Here we prove that *Hypothesis H* implies global consistency of the full 6d Wess-Zumino-type term (WZ-term) that appears in the action functional for the Green-Schwarz-type action functional of the M5-brane.

The open problem. The full 6d WZ-term of the M5-brane, originally proposed in [Ah96, p. 11] and fully established by [BLNPST97, (1)] (and reviewed in detail below in §2) is a functional of fields on a 6d worldvolume Σ^6 which may be expressed in terms of auxiliary extended fields on a cobounding extended worldvolume $\widehat{\Sigma}^7$ as follows (full generality and details in Def. 2.5 below):

$$\widehat{S}^{1\mathrm{M5}}_{\mathrm{WZ}} = 2\widehat{S}^{\mathrm{M5}}_{\mathrm{WZ}} := 2\int\limits_{\widehat{\Sigma}^7}\Big(\tfrac{1}{2}\widehat{H}_3\wedge\widehat{f}^*G_4^{\mathrm{int}} + \widehat{f}^*G_7\Big) \qquad (1)$$

$$\exp\Big(2\pi i\big(\widehat{S}^{1\mathrm{M5}}_{\mathrm{WZ}}\big)\Big) \in U(1) \qquad (2)$$

$\widehat{\Sigma}^7$	Extended worldvolume
$\widehat{f}$	Extended sigma-model field
$\widehat{H}_3$	Extended worldvolume higher gauge field
G_4^{int}	Shifted background C-field flux
G_7	Dual background C-field flux

The open problem is to show that this expression (1) is well-defined, in that it is independent of the choice of extensions, or at least independent up to integer shifts, so that at least the exponentiated Wess-Zumino action functional (2) is well-defined.

Partial solution in the literature. A suggestive partial solution to this problem was proposed in [In00] by

- **(i)** assuming that G_4 is not only the form datum underlying a topological cocycle in rational Cohomotopy, but even that of an actual smooth function c_{smth} to the smooth 4-sphere [In00, (5.3)];
- **(ii)** focusing on the first summand [In00, (2.4)] and disregarding the second summand in (1), leaving its understanding for later [In00, top of p. 16].

With these simplifications imposed, expression (1) reduces on oriented difference manifolds $\widetilde{\Sigma}^7 := \widehat{\Sigma}^7_1 - \widehat{\Sigma}^7_2$ to the classical Whitehead integral formula [Wh47] (see [BT82, Prop. 17.22]) for the Hopf invariant $\mathrm{HI}(c_{\mathrm{smth}}\circ\widetilde{f})$ of maps to the 4-sphere. Since the Hopf invariant is an integer by its homotopy-theoretic definition (recalled as Def. 4.2 below), [In00] suggests that (2) is satisfied and thus refers to the first summand in (1) as the *Hopf-Wess-Zumino term*, a terminology that was used for other sigma-models before [WZ83][TN89] and has become widely adopted for the M5 since (e.g. [KS03, Sec. 3.2][HN11][Ar18, Sec. 4.1]). But, since assumption **(i)** is not supposed to be generally satisfied, so that disregarding the second term **(ii)** is not generally possible, this is a partial solution, and the full problem of showing consistency of (1) by demonstrating (2) had remained open.

Solution by homotopy periods in Cohomotopy. We identify the Whitehead product/formula as the right setting and observe that Haefliger [Ha78, p. 17] already remarked that the strict Whitehead integral formula [Wh47] should generalize to a homotopy-invariant formula with integrands the "functional cup products" of Steenrod [St49]. We note that these secondary characteristic classes descending from the intersection pairing are expressions just of the full form seen in (1)! For maps from the 3-sphere to the 2-sphere, this was worked out in [GM81, 14.5]. A more general statement appears in [SW08, Ex. 1.9] under the name *homotopy period*-expressions.

Our **first main result** here (Theorem 3.2 below) is a transparent proof that the full 6d WZ term (1) (including both summands) is a homotopy period/homotopy Whitehead integral in this sense, which reduces to the Whitehead integral formula for the Hopf invariant in the respective special cases (Remark 4.5 below). In fact, we prove a more general statement which incorporates also the topological twists that account for the half-integral shift by $\frac{1}{4}p_1$ demanded by flux quantization of the background C-field (see [FSS19b, Section 3.4]).

This shows, in particular, that the two summands in (1) can not be invariantly separated, and hence that it is really the full term (1) which deserves to be called the *Hopf-Wess-Zumino term*. Thereby resolves the puzzlement expressed in [In00, top of p. 8]: the first summand of (1) by itself does not actually qualify as a Wess-Zumino term, since it is not (the pullback of) a cocycle. The full term *is* a cocycle, and in fact a cocycle in integral cohomology if *Hypothesis H* is satisfied, by the proof of our second main result:

Our **second main result** (Theorem 4.6 below) shows that under *Hypothesis H* the 6d Wess-Zumino term (1) is generally integral, even in its topologically twisted generalization. This topologically twisted/parametrized generalization of the Hopf invariant thus establishes (2) and hence proves anomaly cancellation for the 6d Wess-Zumino term of the M5-brane.

Consequences. We briefly highlight some consequences of and conclusions drawn from this result:

1. Level quantization. A key argument of [In00, (2.8)] was that the mathematical incarnation of N coinciding M5-branes is that the bare Hopf-WZ term (1) $\widetilde{S}^{\mathrm{M5}}_{\mathrm{WZ}} = \int \frac{1}{2} H_3 \wedge \widehat{f}^* G_4 + \cdots$ is to be multiplied by $N(N+1)$, at least in its first summand. But, since by our result the two summands cannot be invariantly separated, this means that the full term has to be multiplied this way, hence that for N coincident M5-branes (1) generalizes to

$$\widehat{S}^{N\mathrm{M5}}_{\mathrm{WZ}} := N(N+1) \int_{\widehat{\Sigma}^7} \Big(\tfrac{1}{2} \widehat{H}_3 \wedge \widehat{f}^* G_4^{\mathrm{int}} + \widehat{f}^* G_7 \Big) \qquad \boxed{N \,\|\, \text{Number of coincident M5-branes}} \tag{3}$$

with the factor of 2 in (1) being the case of $N = 1$. But, since $N(N+1)$ is even for all N, the condition that (2) is well-defined up to an integral shift, by Theorems 3.2 and 4.6, implies that

$$\exp\Big(2\pi i \big(\widetilde{S}^{N\mathrm{M5}}_{\mathrm{WZ}}\big)\Big) \in U(1) \tag{4}$$

is also well-defined, for all N. Thus the factor $N(N+1)$ plays the role of the *level* of the 6d Wess-Zumino term of the M5-brane, and its special even integral form is the *level quantization* for the 6d Wess-Zumino term of the M5-brane, in analogy with integral levels of ordinary Wess-Zumino terms [Wi83].

2. Dimensional generalization of the Hopf invariant one theorem. The full 6d Wess-Zumino term of the M5-brane (1) is evidently the special case $k = 1$ of a sequence of Wess-Zumino terms $S^{1B(4k+1)}_{\mathrm{WZ}}$ that exist for all $k \in \mathbb{N}$ on higher gauged p-brane sigma-model fields with $p = 4k+1$ (hence precisely in those worldvolume dimensions that admit self-dual higher gauge fields). For trivial topological twist τ in (29) the proof of Theorem 3.2 generalizes verbatim to this infinite hierarchy, simply by generalizing the degree of the generator ω_4 in (29) to $2(k+1)$ and the degree of the generator ω_7 to $4k+3$. Similarly, Prop. 4.4 generalizes verbatim and shows that for all $k \in \mathbb{N}$ the anomaly functionals $\widetilde{S}^{1B(4k+1)}_{\mathrm{WZ}}$ of these Wess-Zumino terms compute, in the absence of topological twists and under *Hypothesis H*, the Hopf invariant of the composite of the brane's sigma-model field with the cocycle of the background field in Cohomotopy.

It is interesting to note that, from this perspective, we take the classical *Hopf invariant one theorem* [Ad60] to say that if the oriented difference of extended worldvolumes is the $(4k+1)$-sphere $\widetilde{\Sigma}^{4k+1} = S^{4k+1}$, then for almost all values of $k \in \mathbb{N}$ the anomaly functional $\widetilde{S}^{1B(4k+1)}_{\mathrm{WZ}}$ is an even integer, in that the only values of k for which it may take odd integer values are precisely those corresponding to branes that actually appear in string/M-theory:

$k =$	0	1	2
$(4k+1)$-brane	string	five-brane	nine-brane

Hypothesis H* with the *Hopf invariant one theorem singles out the worldvolume dimensions $p+1 \in \{2, 6, 10\}$ among p-branes admitting self-dual higher gauge fields, as those whose Wess-Zumino anomaly functional $\widetilde{S}^{1B(4k+1)}_{\mathrm{WZ}}$ is integrally *in*divisible.

3. Unifying role of the quaternionic Hopf fibration. It is noteworthy that the proofs of our main results (Theorem 3.2 and Theorem 4.6) proceed entirely by characterizing lifts in Cohomotopy through the quaternionic Hopf fibration, observing that it is such lifts which reflect, under *Hypothesis H*, the higher gauge field H_3 on the worldvolume of the M5-brane [FSS19b, Proposition 3.20]. This tightly connects the discussion of the 6d Wess-Zumino term here to the analogous cohomotopical discussion of its supersymmetric completion in [FSS15] and to the anomaly cancellation conditions on the background fields in [FSS19b], all rigorously derived from first principles; and thus suggests that a complete derivation of the elusive quantum M5-brane may exist guided by *Hypothesis H*.

4. Outlook. It is well known that the definition of WZW- and CS-terms by field extensions over a cobounding manifold is, while an elegant method when it applies, not the most general definition of these terms: in cases where such field extensions do not exist, the WZW- and CS-terms may still exist, now defined as hypervolume holonomies of some cocycle in a *differential* generalized cohomology theory. For the ordinary WZW- and CS-term this differential cohomology theory is differential ordinary cohomology, represented equivalently as Cheeger-Simons differential characters or as Deligne cohomology or as bundle gerbes with connections, or as $B^nU(1)$-principal connections. The reader may find a review in [FSS13a].

For the case of the 6dWZW term of the M5-branes, the results of this article show that the appropriate differential cohomology theory that generalizes the construction by field extension presented here must be a differential refinement of Cohomotopy cohomology theory. We had constructed one version of such a *differential Cohomotopy cohomology theory* in [FSS15, 4], further discussed in [GS20, 3]. Ultimately one should use this, or possibly some variant, to generalize the results we present here to situations where extensions of fields over cobounding manifolds may not exist.

Outline. In §2 we make precise the 6d Wess-Zumino term and its anomaly, including topological twisting. In §3 we establish that the full WZ term is a homotopy period/homotopy Whitehead integral. In §4 we prove that *Hypothesis H* implies anomaly cancellation of the full 6d Wess-Zumino term.

2 The full 6d Wess-Zumino term of the M5-brane

In this section we present a precise definition, paraphrasing from the informal literature, of the 6d Wess-Zumino term of the M5-brane, generalize it to include topological twists reflecting the shifted flux quantization condition on the C-field flux, and then prove that the corresponding anomaly functional is a homotopy invariant.

First we state (in Def. 2.3 below) the 6d WZ term for "small" sigma-model fields as found in the original articles [Ah96, p. 11][BLNPST97, (1)], then we consider its globalization via extension to cobounding extended worldvolumes as in [In00, (5.4)] (Def. 2.5 below), where we generalize to include the half-integral shift of G_4 by $\frac{1}{4}p_1$, demanded by the flux quantization of the C-field (see [FSS19b, Section 3.4]). Finally we discuss the corresponding anomaly functional (Def. 2.8 below) and show that it is a homotopy invariant on the space of gauged sigma-model fields (Lemma 2.9).

To be precise and reasonably self-contained, we begin by introducing the relevant ingredients:

Definition 2.1 (Background C-field and higher gauged sigma-model fields)**.**
(i) Let X^8 be a smooth 8-manifold which is connected, simply connected[1] and spin, to be called the *target spacetime*[2].
(ii) Let Σ be a smooth manifold, which is compact and oriented, to be called
(a) the *worldvolume* if it is 6-dimensional $\Sigma := \Sigma^6$ without boundary;
(b) the *extended worldvolume* if it is 7-dimensional $\Sigma := \widehat{\Sigma}^7$, with collared boundary

$$\Sigma^6 = \partial\widehat{\Sigma}^7 \xhookrightarrow{(\mathrm{id},0)} \big(\partial\widehat{\Sigma}^7\big) \times [0,1) \hookrightarrow \widehat{\Sigma}^7\,. \tag{5}$$

(c) the *oriented difference of extended worldvolumes* if it is 7-dimensional $\Sigma := \widetilde{\Sigma}^7$ and arising as the oriented difference

$$\widetilde{\Sigma}^7 \;=\; \widehat{\Sigma}^7_1 - \widehat{\Sigma}^7_2 \;:=\; \widehat{\Sigma}^7_1 \cup_{\Sigma^6} \big(\widehat{\Sigma}^7_2\big)^{\mathrm{op}} \tag{6}$$

(where $(-)^{\mathrm{op}}$ denotes orientation reversal) of two collared coboundary extension $\widehat{\Sigma}^7_{1,2}$ (5) of the same worldvolume $\partial\widehat{\Sigma}^7_{1,2} = \Sigma^6$; in particular $\widetilde{\Sigma}^7$ is without boundary.

(iii) A *background field* configuration on X^8 is

[1] All results in the following readily generalize to non-connected X, but nothing essential is gained thereby. The assumption that X is simply connected is to allow the use of Sullivan model analysis in §3 and §4 (as in [FSS19b, Remark 2.6]). For this it would be sufficient to assume that X is *simple* in that it has abelian fundamental group acting trivially on homotopy and homology groups of its universal cover. This assumption should not be necessary, but without it all proofs will become much more involved.

[2] This pertains to M-theory on 8-manifolds, see [FSS19b, Remark 3.1]. We will often just write X for X^8.

(a) an affine Spin(8)-connection ∇ on the tangent bundle[3] TX^8;

(b) a pair of differential forms

$$\begin{array}{r} G_4 \in \Omega^4(X^8) \\ 2G_7 \in \Omega^7(X^8) \end{array} \quad \text{such that} \quad \begin{array}{l} d \quad G_4 = 0\,, \\ d\,2G_7 = -G_4 \wedge G_4 + \big(\tfrac{1}{4}p_1(\nabla)\big) \wedge \big(\tfrac{1}{4}p_1(\nabla)\big) \end{array} \tag{7}$$

where the Pontrjagin 4-form is given by

$$p_1(\nabla) := \langle R_\nabla \wedge R_\nabla \rangle\,. \tag{8}$$

In terms of the shifted flux form

$$G_4^{\text{int}} \;:=\; G_4 + \tfrac{1}{4}p_1(\nabla) \tag{9}$$

the condition in (7) equivalently reads

$$d\,2G_7 = -\big(G_4^{\text{int}} \wedge G_4^{\text{int}} - \tfrac{1}{2}p_1(\nabla) \wedge G_4^{\text{int}}\big). \tag{10}$$

(iv) A *higher gauged*[4] *sigma-model field* is a pair of

$$\big(f, H_3\big) \;=\; \big(\Sigma \xrightarrow{f \text{ smooth}} X,\; dH_3 \;=\; f^*\big(G_4 - \tfrac{1}{4}p_1(\nabla)\big)\big) \tag{11}$$

(a) with a smooth function f from the (extended) worldvolume to spacetime,

(b) a smooth differential 3-form H_3 on the (extended) worldvolume, which trivializes the pullback along f of the difference between G_4 from (7) and $\frac{1}{4}p_1(\nabla)$ from (8),

both required to have *sitting instants* on any collared boundary (5), in that in some neighborhood of the boundary they are constant in the direction perpendicular to it [FSS10, Def. 4.2.1];

(v) A *homotopy* between two higher gauged sigma-model fields (11)

$$\big(f_0, (H_3)_0\big) \xRightarrow{\left(\eta,(H_3)_{[0,1]}\right)} \big(f_1, (H_3)_1\big) \tag{12}$$

is a pair of a smooth homotopy η from f_0 to f_1 and a differential 3-form $(H_3)_{[0,1]} \in \Omega^3\big(\Sigma \times [0,1]\big)$ gauging η and restricting to $(H_3)_{0,1}$ at the boundaries of the interval:

$$\begin{array}{ccc} \Sigma & \xrightarrow{f_0} & \\ {\scriptstyle (\mathrm{id},0)}\downarrow & & \\ \Sigma \times [0,1] & \xrightarrow{\eta \text{ smooth}} & X \\ {\scriptstyle (\mathrm{id},1)}\uparrow & & \\ \Sigma & \xrightarrow{f_1} & \end{array} \qquad \begin{array}{c} (\widetilde{H}_3)_0 \\ \uparrow {\scriptstyle (\mathrm{id},0)^*} \\ (H_3)_{[0,1]} \\ \downarrow {\scriptstyle (\mathrm{id},1)^*} \\ (H_3)_1 \end{array} \qquad d\big((H_3)_{[0,1]}\big) = \eta^*\big(G_4 - \tfrac{1}{4}p_1(\nabla)\big). \tag{13}$$

(vi) We write

$$\mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}(\Sigma, X) \;:=\; \big\{\big(f, H_3\big)\big\}\,, \qquad \pi_0\big(\mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}(\Sigma, X)\big) \;:=\; \big\{\big(f, H_3\big)\big\}_{/\sim_{\text{homotopy}}} \tag{14}$$

for the sets[5] of higher gauged sigma-model fields (11) and of their *homotopy classes* (12), respectively.

[3] The theorems below hold, as general statements about the 6d WZ term, for ∇ a connection on any Spin-bundle. But application to the actual M5-brane system requires ∇ to be a tangent connection on spacetime.

[4] This is the higher analog of abelian gauging of 2d WZW model fields (e.g. [Fo03, (5)]), making the 6d Wess-Zumino term the action functional of a *higher gauged Wess-Zumino model* [FSS13b].

[5] The inclined reader will notice (see [FSS13b] for exposition) that the set $\mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}(\Sigma, X)$ is of course the underlying set of global sections of the atlas for the smooth *moduli 2-stack* of higher gauged sigma-model fields on Σ, and $\pi_0\big(\mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}(\Sigma, X)\big)$ is the set of connected components of the geometric realization of this moduli 2-stack. All of the following discussion lifts to the higher differential geometry of moduli stacks of fields, but for the sake of brevity we will not further consider this here.

As an important example, we offer the following.

Lemma 2.2 (The 7-sphere as extended worldvolume)**.** *In the situation of Def. 2.1 let the oriented difference of extended worldvolumes be the 7-sphere:* $\Sigma := \widetilde{\Sigma}^7 := S^7$. *Then the set* (14) *of homotopy classes of extended gauged sigma-model fields is the set underlying the 7th homotopy group of target spacetime* X*:*

$$\pi_0\Big(\mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}\big(S^7, X\big)\Big) \;\simeq\; \pi_7(X)\,. \tag{15}$$

Proof. Since homotopy classes of continuous functions between smooth manifolds are given by smooth homotopy classes of smooth functions (e.g. [BT82, Cor. 17.8.1]) it follows that already smooth homotopy classes of ungauged sigma-model fields are in bijection to $\pi_7(X)$ (since the target spacetime X is assumed to be connected there is no dependence on a basepoint). Hence it only remains to show that, for any extended sigma-model field $\widetilde{f}$ there exist at least a gauging $\widetilde{H}_3$, and that for any $\big(\widetilde{f}, (\widetilde{H}_3)_0\big)$ and $\big(\widetilde{f}, (\widetilde{H}_3)_1\big)$ two gaugings (11) of the same extended sigma-model field $\widetilde{f}$, there exists a gauged homotopy (12)

$$\big(\widetilde{f}, (\widetilde{H}_3)_0\big) \xrightarrow{\;\big(\widetilde{\eta}, (\widetilde{H}_3)_{[0,1]}\big)\;} \big(\widetilde{f}, (\widetilde{H}_3)_1\big)$$

between them. For the existence of the gauging $\widetilde{H}_3$ for a given $\tilde{f}$, we only need to notice that as $H^7_{\mathrm{dR}}(S^7) \cong H^4(S^7;\mathbb{R}) = 0$, we have $f^*[G_4 - \frac{1}{4}p_1(\nabla)] = 0$ and so there exists $\widetilde{H}_3 \in \Omega^3\big(S^7\big)$ such that $d\widetilde{H}_3 = f^*\big(G_4 - \frac{1}{4}p_1(\nabla)\big)$. Similarly, given two gaugings $(\widetilde{H}_3)_0$ and $(\widetilde{H}_3)_1$ of $\tilde{f}$, since $H^3_{\mathrm{dR}}\big(S^7\big) = 0$ and $(\widetilde{H}_3)_1 - (\widetilde{H}_3)_0 \in \Omega^3\big(S^7\big)$ is closed by assumption, there exists

$$\alpha \in \Omega^2\big(S^7\big) \qquad \text{such that} \qquad d\alpha = (\widetilde{H}_3)_1 - (\widetilde{H}_3)_0\,.$$

Then

$$\Big(\widetilde{\eta} : (x,s) \longmapsto \widetilde{f}(x),\, (\widetilde{H}_3)_{[0,1]} \;:=\; (\widetilde{H}_3)_1 + (s-1)\cdot d\alpha + (ds)\wedge\alpha\Big)$$

constitutes a homotopy as required. □

We now consider the 6d WZ term in its various incarnations, surveyed in Table A.

$\mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}\big(\Sigma^6, X^{11}\big) \xrightarrow{\;S\;} \mathbb{R}$	WZ action functional on worldvolume Σ^6	Def. 2.3
$\mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}\big(\widehat{\Sigma}^7, X^{11}\big) \xrightarrow{\;\widehat{S}\;} \mathbb{R}$	Extended WZ functional on coboundary $\partial\widehat{\Sigma}^7 = \Sigma^6$	Def. 2.5
$\mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}\big(\widetilde{\Sigma}^7, X^{11}\big) \xrightarrow{\;\widetilde{S}\;} \mathbb{R}$	WZ anomaly functional on oriented difference $\widetilde{\Sigma}^7 = \widehat{\Sigma}^7_1 - \widehat{\Sigma}^7_2$	Def. 2.8

Table A – Incarnations of the WZ term The 6d WZ term functional $S := S^{\mathrm{M5}}_{\mathrm{WZ}}$ is a priori defined on gauged sigma-model fields on Σ^6. Its global definition involves an extension $\widehat{S}$ to extended fields on a coboundary $\widehat{\Sigma}^7$. The difference of any two extensions is the anomaly functional $\widetilde{S}$ on fields on the oriented difference $\widetilde{\Sigma}^7 = \widehat{\Sigma}^7_1 - \widehat{\Sigma}^7_2$.

Definition 2.3 (6d WZ term for small sigma-model fields)**.** In the setting of Def. 2.1, let $U \subset X^8$ be a chart (a contractible open subset). For Σ^6 any closed orientable 6-manifold, write $\mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}\big(\Sigma^6, U\big) \subset \mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}\big(\Sigma^6, X\big)$ for the subset of those higher gauged sigma-model fields (14) which factor through $U \subset X$ (the "U-small sigma-model fields"). As the cohomology of U is trivial in positive degree, we can choose local potentials $C^U_3 \in \Omega^3\big(U\big)$ for $\iota_U^*(G_4 + \frac{1}{4}p_1(\nabla))$ and $2C^U_6 \in \Omega^6\big(U\big)$ for $\iota_U^* 2G_7 + C^U_3 \wedge \iota_U^*\big(G_4 - \frac{1}{4}p_1(\nabla)\big)$.

$$\begin{array}{ccc} & & U \\ & {\scriptstyle f_U}\nearrow & \big\downarrow {\scriptstyle \iota_U} \\ \Sigma^6 & \xrightarrow{\;\;f\;\;} & X^8 \end{array} \qquad \begin{aligned} dC^U_3 &= \iota_U^* G_4 \\ d\,2C^U_6 &= \iota_U^* 2G_7 + C^U_3 \wedge \iota_U^*\big(G_4 - \tfrac{1}{4}p_1(\nabla)\big) \\[2em] dG_4 &= 0 \\ d\,2G_7 &= -G_4 \wedge G_4 + \big(\tfrac{1}{4}p_1(\nabla)\big) \wedge \big(\tfrac{1}{4}p_1(\nabla)\big) \end{aligned} \tag{16}$$

Then the *M5 6d Wess-Zumino term action functional* on these small fields is the function

$$\begin{array}{ccc} \mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}\big(\Sigma^6, U\big) & \xrightarrow{\quad S^{\mathrm{M5}}_{\mathrm{WZ}} \quad} & \mathbb{R} \\ \big(f_U, H_3\big) & \longmapsto & S^{\mathrm{M5}}_{\mathrm{WZ}}\big(f_U, H_3\big) \;:=\; \tfrac{1}{2}\int\limits_{\Sigma^6}\big(-H_3\wedge f_U^* C_3^U + f_U^* 2C_6^U\big). \end{array} \tag{17}$$

Lemma 2.4 (Independence of choices)**.** *The functional $S^{\mathrm{M5}}_{\mathrm{WZ}}\big(f_U, H_3\big)$ is well defined, i.e., it does not depend on the choice of the local potentials C_3^U and C_6^U.*

Proof. A different choice of local potentials is of the form $(C_3^U + \alpha_3^U, 2C_6^U + 2\alpha_6^U)$, with derivatives $d\alpha_U^3 = 0$ and $d2\alpha_6^U = \alpha_3^U \wedge \iota_U^*(G_4 - \frac{1}{4}p_1(\nabla)\big)$. As the local chart U is contractible, this is equivalent to $\alpha_3^U = d\alpha_2^U$ and $2\alpha_6^U = \alpha_2^U \wedge \iota_U^*(G_4 - \frac{1}{4}p_1(\nabla) + d\alpha_5^U$. Therefore, we have

$$\begin{aligned} \int_{\Sigma^6}\big(-H_3\wedge f_U^*(C_3^U+\alpha_3^U) + f_U^* 2(C_6^U+\alpha_6^U)\big) - \int_{\Sigma^6}\big(-H_3\wedge f_U^* C_3^U + f_U^* 2C_6^U\big) \\ = \int_{\Sigma^6} -H_3\wedge d f_U^*\alpha_2^U + f_U^*\big((\alpha_2^U \wedge \iota_U^*(G_4 - \tfrac{1}{4}p_1(\nabla)\big) + d f_U^*\alpha_5^U \\ = \int_{\Sigma^6} -H_3\wedge d f_U^*\alpha_2^U + f_U^*\alpha_2^U \wedge f^*(G_4 - \tfrac{1}{4}p_1(\nabla) + d f_U^*\alpha_5^U \\ = \int_{\Sigma^6} -H_3\wedge d f_U^*\alpha_2^U + f_U^*\alpha_2^U \wedge dH_3 + d f_U^*\alpha_5^U \\ = \int_{\Sigma^6} d\big(H_3\wedge f_U^*\alpha_2^U + f_U^*\alpha_5^U\big) = 0. \end{aligned}$$

□

Now we globalize this definition, following the well-known procedure originally introduced in the 2-dimensional case in [Wi83].

Definition 2.5 (Global 6d Wess-Zumino term via extended worldvolumes)**.** In the situation of Def. 2.1, for Σ^6 a given worldvolume, let $\widehat{\Sigma}^7$ be a compact oriented smooth collared cobounding 7-manifold[6] according to (5)

$$\Sigma^6 := \partial\widehat{\Sigma}^7\,. \tag{18}$$

Then we say that the corresponding *extended action functional* for the 6d WZ-term on the closed manifold Σ^6 is the function

$$\begin{array}{ccc} \mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}\big(\widehat{\Sigma}^7, X\big) & \xrightarrow{\quad \widehat{S}^{\mathrm{M5}}_{\mathrm{WZ}} \quad} & \mathbb{R} \\ \big(\widehat{f}, \widehat{H}_3\big) & \longmapsto & \widehat{S}^{\mathrm{M5}}_{\mathrm{WZ}}\big(\widehat{f}, \widehat{H}_3\big) \;:=\; \tfrac{1}{2}\int\limits_{\widehat{\Sigma}^7}\Big(\widehat{H}_3\wedge \widehat{f}^*\big(G_4 + \tfrac{1}{4}p_1(\nabla)\big) + \widehat{f}^* 2G_7\Big) \end{array} \tag{19}$$

on the set of extended gauged sigma-model fields (11).

Lemma 2.6 (Global WZW term restricts to local WZW term)**.** *In the situation of Def. 2.3, consider a worldvolume Σ^6. Then, for every choice of extended worldvolume $\widehat{\Sigma}^7$* (18) *the corresponding* extended action functional $\widehat{S}$ *(Def. 2.5) coincides, for any chart $U \subset X$ with $p_1(\nabla)_{|U} = 0$, on U-small extended sigma-model fields $\widehat{f} = \iota_U \circ \widehat{f}_U$* (16) *with the local action functional S (Def. 2.3) evaluated on the boundary values $f := \widehat{f}\,|_{\Sigma^6}$ of the extended fields:*

$$\begin{array}{ccc} \mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}\big(\widehat{\Sigma}^7, U\big) & \xrightarrow{\widehat{S}^{\mathrm{M5}}_{\mathrm{WZ}}} & \mathbb{R} \\ \big\downarrow{\scriptstyle (-)|_{\partial\widehat{\Sigma}^7}} & \nearrow{\scriptstyle S^{\mathrm{M5}}_{\mathrm{WZ}}} & \\ \mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}\big(\Sigma^6, U\big) & & \end{array} \qquad \widehat{S}^{\mathrm{M5}}_{\mathrm{WZ}}\big(\widehat{f}, \widehat{H}_3\big) \;=\; S^{\mathrm{M5}}_{\mathrm{WZ}}\Big(f := \widehat{f}\,|_{\partial\Sigma^7},\, H_3 := \big(\widehat{H}_3\big)|_{\partial\widehat{\Sigma}^7}\Big)\,.$$

[6] This always exists, since the oriented cobordism ring in dimension 6 is trivial and by the collar neighbourhood theorem.

Proof. Observe with (11) and (16) that

$$
\begin{aligned}
d\big(-\widehat{H}_3 \wedge \widehat{f}_U^{\,*} C_3^U + \widehat{f}_U^{\,*} 2 C_6^U\big) &= -\underbrace{d\widehat{H}_3}_{\widehat{f}^*(G_4 - \frac{1}{4}p_1(\nabla))} \wedge \widehat{f}_U^{\,*} C_3^U + \widehat{H}_3 \wedge \underbrace{f_U^* dC_3^U}_{\widehat{f}^*(G_4+\frac{1}{4}p_1(\nabla))} + \underbrace{\widehat{f}_U^{\,*} d2C_6^U}_{\substack{\widehat{f}^*(G_4-\frac{1}{4}p_1(\nabla))\wedge \widehat{f}_U^{\,*} C_3^U \\ +\widehat{f}^* 2G_7}} \\
&= \widehat{H}_3 \wedge \widehat{f}^*\big(G_4 + \tfrac{1}{4}p_1(\nabla)\big) + \widehat{f}^* 2G_7\,.
\end{aligned}
\tag{20}
$$

With this, the claim follows by Stokes' theorem:

$$
\begin{aligned}
\widehat{S}_{\mathrm{WZ}}^{\mathrm{M5}}\big(\widehat{f}, \widehat{H}_3\big) &:= \tfrac{1}{2}\int_{\widehat{\Sigma}^7} \big(\widehat{H}_3 \wedge \widehat{f}^*\big(G_4+\tfrac{1}{4}p_1(\nabla)\big) + \widehat{f}_U^{\,*} 2G_7\big) \\
&= \tfrac{1}{2}\int_{\widehat{\Sigma}^7} d\big(-\widehat{H}_3 \wedge \widehat{f}_U^{\,*} C_3^U + \widehat{f}_U^{\,*} 2C_6^U\big) \\
&= \tfrac{1}{2}\int_{\partial\widehat{\Sigma}^7} \big(-\widehat{H}_3 \wedge \widehat{f}_U^{\,*} C_3^U + \widehat{f}_U^{\,*} 2C_6^U\big)\big|_{\partial\Sigma^7} \\
&= \tfrac{1}{2}\int_{\Sigma^6} \big(-H_3 \wedge f_U^* C_3^U + f_U^* 2C_6^U\big) \\
&=: S_{\mathrm{WZ}}^{\mathrm{M5}}\big(f_U, H_3\big)\,.
\end{aligned}
\tag{21}
$$

□

Example 2.7 (Coboundaries for $\Sigma^6 = S^3 \times S^3$)**.** In the situation of Def. 2.1, consider as worldvolume the product manifold of two 3-spheres (this is considered in [MS15, Example 2] in the non-commutative setting):

$$
\Sigma^6 \;=\; S^3 \times S^3\,.
$$

In this case there is a canonical choice of cobounding manifold $\widehat{\Sigma}^7$ (18) given by the Cartesian product of the 4-disk D^4 (the closed 4-dimensional ball) with the 3-sphere, in either order (as in [Sa13]):

$$
\widehat{\Sigma}_L^7 \;:=\; D^4 \times S^3 \quad \text{and} \quad \widehat{\Sigma}_R^7 \;:=\; \big(S^3 \times D^4\big)^{\mathrm{op}}\,.
\tag{22}
$$

Here we are equipping each of

$$
\left.
\begin{array}{lcl}
S^3 \times S^3, & & \\
D^4 \times S^3 & = & D^4 \times \big(\partial D^4\big), \\
S^3 \times D^4 & = & \big(\partial D^4\big) \times D^4\,,
\end{array}
\right\} \subset D^4 \times D^4 \subset \mathbb{R}^8
$$

with the orientation induced from the canonical embedding into $\mathbb{R}^8$, which implies, by the odd-dimensionality of S^3, that the boundary of $S^3 \times D^4$ is $\big(S^3 \times S^3\big)^{\mathrm{op}}$ (opposite orientation). This way, with (22) we indeed have

$$
\partial \widehat{\Sigma}^7_{L,R} \;=\; \Sigma^6 := S^3 \times S^3
$$

as oriented manifolds. Observe that the union of one of these coboundaries with the orientation reversal of the other is the 7-sphere (as considered in Lemma 2.2): [7]

$$
\begin{aligned}
\widetilde{\Sigma}^7 \;:=\; \widehat{\Sigma}_L^7 \cup \big(\widehat{\Sigma}_R^7\big)^{\mathrm{op}} &= D^4 \times \big(\partial D^4\big) \,\cup\, \big(\partial D^4\big) \times D^4 \\
&= \partial\big(D^4 \times D^4\big) \\
&\simeq \partial D^8 \\
&= S^7\,.
\end{aligned}
\qquad
\begin{array}{ccc}
S^3 \times S^3 & \longrightarrow & D^4 \times S^3 \\
\big\downarrow & \text{(po)} & \big\downarrow \\
S^3 \times D^4 & \longrightarrow & S^7
\end{array}
\tag{23}
$$

[7] Note that a different manipulation treats these as manifolds with corners [Sa14][Sa13].

While Def. 2.5 gives global meaning to the local WZW term (Def. 2.3), by Lemma 2.6, this potentially comes at the cost that the global definition depends on the choice of coboundary (18). The following definition measures this potential dependency:

Definition 2.8 (WZ anomaly functional)**.** In the situation of Def. 2.1, with given worldvolume Σ^6, consider in Def. 2.5 two choices $\widehat{\Sigma}^7_{L,R}$ of collared cobounding extended worldvolumes (5) $\partial\Sigma^7_{L,R} = \Sigma^6$. This makes their oriented difference (6) a smooth closed 7-manifold $\widetilde{\Sigma}^7 := \widehat{\Sigma}^7_L - \widehat{\Sigma}^7_R := \widehat{\Sigma}^7_L \cup_{\Sigma^6} \big(\widehat{\Sigma}^7_R\big)^{\mathrm{op}}$. Then for

$$
\begin{array}{ccc}
\widehat{\Sigma}^7_L & & \\
{\scriptstyle \iota_{\partial L}}\uparrow & \searrow^{\widehat{f}_L} & \\
\Sigma^6 & \xrightarrow{\;f\;} & X \\
{\scriptstyle \iota_{\partial R}}\downarrow & \nearrow_{\widehat{f}_R} & \\
\widehat{\Sigma}^7_R & &
\end{array}
\qquad\qquad
\begin{array}{c}
d\big(\widehat{H}_3\big)_L = \widehat{f}_L^{\,*} G_4 \\
\downarrow {\scriptstyle \iota^*_{\partial L}} \\
d H_3 = f^* G_4 \\
\uparrow {\scriptstyle \iota^*_{\partial R}} \\
d\big(\widehat{H}_3\big)_R = \widehat{f}_R^{\,*} G_4
\end{array}
$$

any pair of gauged extended sigma-model fields (11), extending the same ordinary sigma-model field f over the two choices of coboundaries, respectively, we obtain a gauged extended sigma-model field $\big(\widetilde{f}, \widetilde{H}_3\big)$ on the closed 7-manifold $\widetilde{\Sigma}^7$ (6) (which is smooth by the assumption of sitting instants in (11)):

$$
\begin{array}{ccc}
\widehat{\Sigma}^7_L & & \\
{\scriptstyle \iota_L}\downarrow & \searrow^{\widehat{f}_L} & \\
\widetilde{\Sigma}^7 & \xrightarrow{\;\widetilde{f}\;} & X \\
{\scriptstyle \iota_R}\uparrow & \nearrow_{\widehat{f}_R} & \\
\widehat{\Sigma}^7_R & &
\end{array}
\qquad\qquad
\begin{array}{c}
d\big(\widehat{H}_3\big)_L = \widehat{f}_L^{\,*} G_4 \\
\uparrow {\scriptstyle \iota^*_L} \\
d \widetilde{H}_3 = \widetilde{f}^* G_4 \\
\downarrow {\scriptstyle \iota^*_R} \\
d\big(\widehat{H}_3\big)_R = \widehat{f}_R^{\,*} G_4
\end{array}
\tag{24}
$$

In terms of this, the difference between the two extended action functionals (Def. 2.5) corresponding to the two choices of coboundaries may be expressed as a single integral over $\widetilde{\Sigma}^7$:

$$
\begin{aligned}
\widetilde{S}\big(\widetilde{f}, \widetilde{H}_3\big) &:= \widehat{S}\big(\widehat{f}_L, (\widehat{H}_3)_L\big) - \widehat{S}\big(\widehat{f}_R, (\widehat{H}_3)_R\big) \\
&= \tfrac{1}{2}\int\limits_{\widetilde{\Sigma}^7} \big(\widetilde{H}_3 \wedge \widetilde{f}^*\big(G_4 + \tfrac{1}{4}p_1(\nabla)\big) + \widetilde{f}^* 2G_7\big).
\end{aligned}
\tag{25}
$$

We call expression (25) the *anomaly functional* of the 6d Wess-Zumino term.

Lemma 2.9 (Anomaly functional is homotopy invariant)**.** *In the situation of Def. 2.1, let $\Sigma := \widetilde{\Sigma}^7$ be a closed 7-manifold. Then the anomaly functional* (25) *is well-defined on the set* (14) *of homotopy-classes of higher gauged sigma-model fields:*

$$
\begin{array}{ccc}
\pi_0\Big(\mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}\big(\widetilde{\Sigma}^7, X\big)\Big) & \xrightarrow{\qquad\quad \widetilde{S} \quad\qquad} & \mathbb{R} \\
\big[\widetilde{f}, \widetilde{H}_3\big] & \longmapsto & \tfrac{1}{2}\int\limits_{\widetilde{\Sigma}^7} \Big(\widetilde{H}_3 \wedge \widetilde{f}^*\big(G_4 + \tfrac{1}{4}p_1(\nabla)\big) + \widetilde{f}^* 2G_7\Big)
\end{array}
\tag{26}
$$

in that the integral on the right is independent of the choice of representative $\big(\widetilde{f}, \widetilde{H}_3\big)$ in its homotopy class.

Proof. Consider a homotopy (12) between two extended gauged sigma-model fields

$$
\big(\widetilde{f}_0, (\widetilde{H}_3)_0\big) \xRightarrow{\big(\widetilde{\eta}, (\widetilde{H}_3)_{[0,1]}\big)} \big(\widetilde{f}_1, (\widetilde{H}_3)_1\big).
$$

We need to show that then $\widetilde{S}\big([\widetilde{f}_1,(\widetilde{H}_3)_1]\big) = \widetilde{S}\big([\widetilde{f}_0,(\widetilde{H}_3)_0]\big)$. With the data (13) and using Stokes' theorem we directly compute as follows:

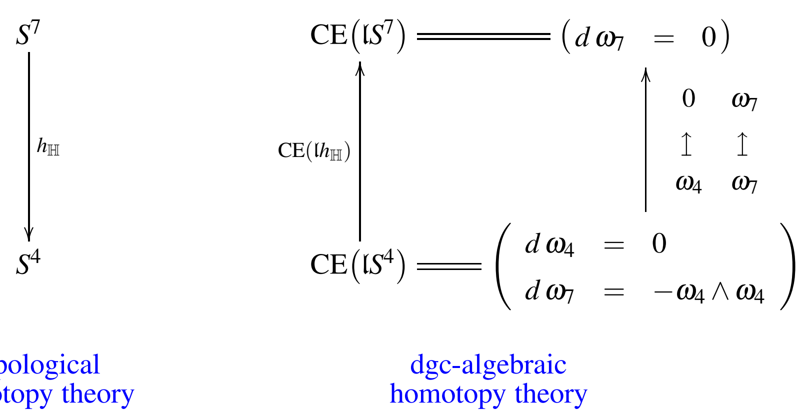
$$
\begin{aligned}
\widetilde{S}\Big([\widetilde{f}_1,(\widetilde{H}_3)_1]\Big) - \widetilde{S}\Big([\widetilde{f}_0,(\widetilde{H}_3)_0]\Big) &= \tfrac{1}{2}\int\limits_{\partial\left(\widetilde{\Sigma}^7\times[0,1]\right)} \Big(\big(\widetilde{H}_3\big)_{[0,1]}\wedge\widetilde{\eta}^*\big(G_4+\tfrac{1}{4}p_1(\nabla)\big)+\widetilde{\eta}^*2G_7\Big) \\
&= \tfrac{1}{2}\int\limits_{\widetilde{\Sigma}^7\times[0,1]} d\Big(\big(\widetilde{H}_3\big)_{[0,1]}\wedge\widetilde{\eta}^*\big(G_4+\tfrac{1}{4}p_1(\nabla)\big)+\widetilde{\eta}^*2G_7\Big) \\
&= \tfrac{1}{2}\int\limits_{\widetilde{\Sigma}^7\times[0,1]} \Big(\underbrace{d\Big(\big(\widetilde{H}_3\big)_{[0,1]}\Big)}_{\widetilde{\eta}^*\left(G_4-\frac{1}{4}p_1(\nabla)\right)}\wedge\widetilde{\eta}^*\big(G_4+\tfrac{1}{4}p_1(\nabla)\big)+\widetilde{\eta}^*d2G_7\Big) \\
&= \tfrac{1}{2}\int\limits_{\widetilde{\Sigma}^7\times[0,1]} \widetilde{\eta}^*\Big(\underbrace{\big(G_4-\tfrac{1}{4}p_1(\nabla)\big)\wedge\big(G_4+\tfrac{1}{4}p_1(\nabla)\big)+d2G_7}_{=0}\Big) \\
&= 0\,,
\end{aligned}
$$

where in the last step, under the brace, we used the condition (7). □

3 The full M5 WZ anomaly is a homotopy Whitehead integral

We first recall from [FSS19b] how the background C-field $(G_4, 2G_7)$ is a cocycle in twisted rational Cohomotopy, as Remark 3.1 below. Then we prove in Theorem 3.2, that the WZ anomaly functional from §2 is equivalently a lift in rational Cohomotopy through the equivariant quaternionic Hopf fibration, hence in particular a homotopy invariant of both the gauged sigma-model fields and the background fields in Cohomotopy. Below in §4 we identify this as a twisted/parametrized generalization of a homotopy Whitehead integral.

Notions from rational homotopy theory. In the following we freely make use of Sullivan models in rational homotopy theory (i.e., what in supergravity are called "FDA"s [FSS13b]); see [Hes06] for introduction, [GM81] for a standard textbook account, and see [FSS16, FSS19a] for review in our context. As in these references, for X a simply connected topological space of finite rational type, we write $\mathrm{CE}(\mathfrak{l}X)$ for its minimal Sullivan model differential graded-commutative algebra (dgc-algebra), indicating that this is the Chevalley-Eilenberg algebra of the minimal L_∞-algebra $\mathfrak{l}X$ corresponding to the loop group of X. For making Sullivan models explicit we display the list of differential relations on each generator, thereby declaring what the generators are. For example, the Sullivan model of the plain quaternionic Hopf fibration we denote as follows (see [FSS19b, Lemma 3.18]):



We take the base field to be $\mathbb{R}$ instead of $\mathbb{Q}$, so that our "rational homotopy groups" are actually "real homotopy groups" $\pi(X)\otimes_{\mathbb{Z}}\mathbb{R}$; but, since this makes no essential difference in our context, we still speak of "rational" models. We denote the real de Rham dg-algebras of a smooth manifold X by $\Omega^\bullet(X)$.

Remark 3.1 (Background C-field is cocycle in rational twisted Cohomotopy)**.** The Sullivan model ("FDA") for the 4-sphere is free on generators ω_4 and ω_7 (in degrees 4 and 7, respectively) subject to differential relations as on the right of the following (27). Then the background field data (7) is identified, in the case that $p_1(\nabla) = 0$ (8), with a map to the rationalized 4-sphere $(S^4)_{\mathbb{R}}$, hence with a cocycle in rational *Cohomotopy* [Sa13, Sec. 2.5] (see [FSS16]):

$$X \xrightarrow{(G_4, 2G_7)} (S^4)_{\mathbb{R}} \qquad\qquad \Omega^\bullet(X) \xleftarrow[\;2G_7 \,\leftarrow\!\shortmid\, \omega_7\;]{G_4 \,\leftarrow\!\shortmid\, \omega_4} \left(\begin{array}{ll} d\,\omega_4 & = 0 \\ d\,\omega_7 & = -\omega_4 \wedge \omega_4 \end{array} \right). \tag{27}$$

topological cocycle   dg-algebra homomorphism

More generally [FSS19b, Prop. 3.20], if the tangent bundle of X is equipped with topological $\mathrm{Sp}(2) \times \mathrm{Sp}(1) \to \mathrm{Spin}(8)$-structure τ, and the corresponding Euler-form trivializes

$$\Theta_7 \in \Omega^7(X) \quad \text{s.t.} \quad d\Theta_7 = \chi_8(\nabla) := \mathrm{Pf}(F_\nabla) \tag{28}$$

then the general background field data (7) is identified with a cocycle in rational τ*-twisted Cohomotopy*

$$\begin{array}{ccc} X & \xrightarrow{(G_4, 2G_7)} & \big(S^4 /\!\!/ (\mathrm{Sp}(2) \times \mathrm{Sp}(1))\big)_{\mathbb{R}}, \\ & {\scriptstyle \tau} \searrow \quad \Leftarrow \quad \swarrow & \\ & \big(B(\mathrm{Sp}(2) \cdot \mathrm{Sp}(1))\big)_{\mathbb{R}} & \end{array} \qquad \begin{array}{ccc} \Omega^\bullet(X) & \xleftarrow[\;2G_7 - \Theta_7 \,\leftarrow\!\shortmid\, \omega_7\;]{G_4 \,\leftarrow\!\shortmid\, \omega_4} & \left(\begin{array}{l} d\,\omega_4 = 0 \\ d\,\omega_7 = -\omega_4 \wedge \omega_4 + \tfrac{1}{4} p_1 \wedge \tfrac{1}{4} p_1 \\ \qquad - \chi_8 \end{array} \right) \\ {\scriptstyle \begin{array}{l} p_1(\nabla) \leftarrow\!\shortmid p_1 \\ p_2(\nabla) \leftarrow\!\shortmid p_2 \\ \chi_8(\nabla) \leftarrow\!\shortmid \chi_8 \end{array}} \nwarrow & & \nearrow {\scriptstyle \begin{array}{l} p_1 \mapsto p_1 \\ p_2 \mapsto p_2 \\ \chi_8 \mapsto \chi_8 \end{array}} \\ & \mathrm{CE}\big(\mathfrak{l}B(\mathrm{Sp}(2) \cdot \mathrm{Sp}(1))\big) & \end{array} \tag{29}$$

Notice that the assumption (28) does not restrict the generality of the physics setup, as it is satisfied after removing the singular loci of black M2-branes from spacetime [FSS19b, Section 2.5].

The following Theorem 3.2 says that not only does rational twisted Cohomotopy naturally encode the background C-field, via Remark 3.1 [FSS19a, FSS19b], but that it also naturally encodes the gauging (11) of the M5-brane sigma-model fields (as in [FSS19b, Rem. 3.17]) as well as the anomaly functional of the 6d Wess-Zumino term (Def. 2.8) as a homotopy invariant (Lemma 2.9):

Theorem 3.2 (6d WZ anomaly functional is lift through $h_{\mathbb{H}}$)**.** *In the situation of Def. 2.1, consider a closed extended worldvolume* $\Sigma := \widetilde{\Sigma}^7$*. Then, under the identification of the background field with a cocycle* c *in rational twisted Cohomotopy, via Remark 3.1, we have:*

(i) *The homotopy classes* (14) *of gaugings* $\widetilde{H}_3$ (11) *of an extended sigma-model field* $\widetilde{f}$ *are in bijection to homotopy classes of homotopy lifts* $\widehat{c \circ \widetilde{f}}$ *through the quaternionic Hopf fibration* $h_{\mathbb{H}}$ *of the composite* $c \circ \widetilde{f}$ *with the classifying map* c (29) *of the background C-field:*

$$\pi_0\big(\mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}(\Sigma, X)\big)\big\vert_{\widetilde{f}} \;\simeq\; \left\{ \begin{array}{ccc} \widetilde{\Sigma}^7 & \overset{\widehat{c \circ \widetilde{f}}}{\dashrightarrow} & \big(S^7 /\!\!/ (\mathrm{Sp}(2) \cdot \mathrm{Sp}(1))\big)_{\mathbb{R}} \\ & {\scriptstyle c \circ \widetilde{f}} \searrow \quad \Leftarrow \quad \swarrow {\scriptstyle \left(h_{\mathbb{H}} /\!\!/ (\mathrm{Sp}(2) \cdot \mathrm{Sp}(1))\right)_{\mathbb{R}}} & \\ & \big(S^4 /\!\!/ (\mathrm{Sp}(2) \cdot \mathrm{Sp}(1))\big)_{\mathbb{R}} & \end{array} \right\} \Big/ \sim_{\mathrm{homotopy}} \tag{30}$$

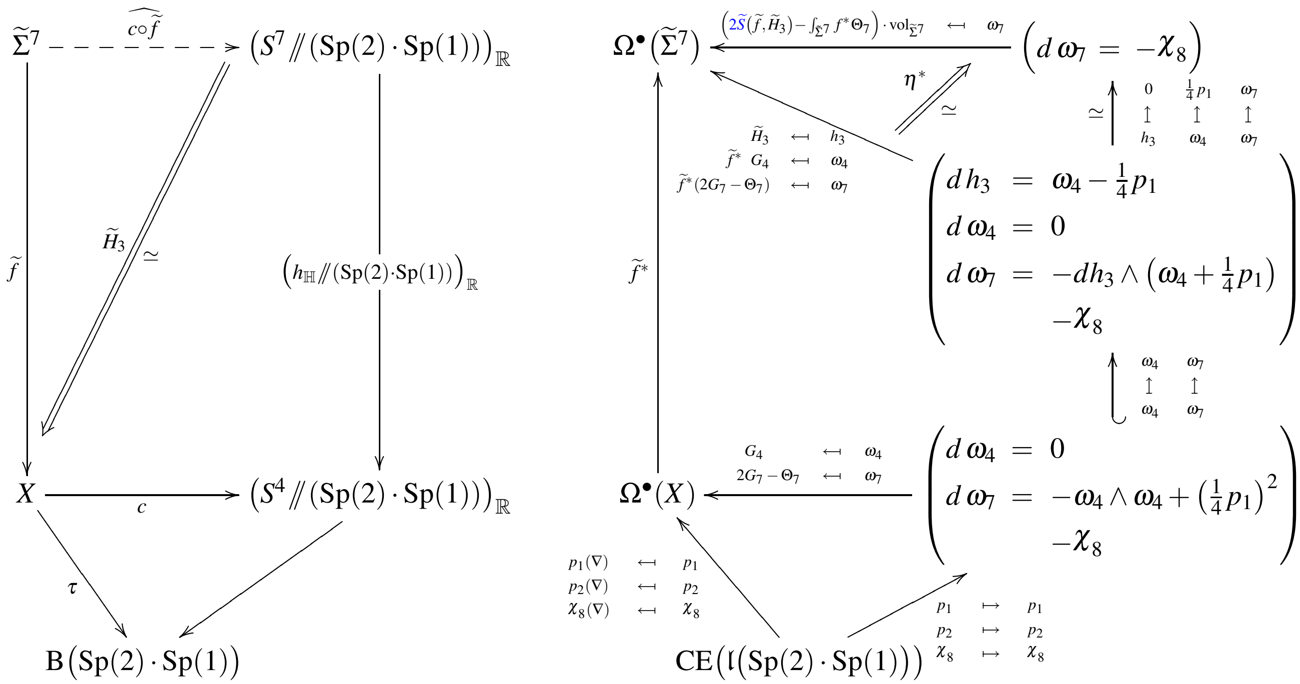
(ii) *Under the bijection of* (30) *twice the anomaly functional (Def. 2.8) equals the correction by the Euler-potential* Θ_7 (28) *of the integral*

$$2\widetilde{S}\big(\widetilde{f}, \widetilde{H}_3\big) \;=\; \int\limits_{\widetilde{\Sigma}^7} \big(\widehat{c \circ \widetilde{f}}\,\big)^*(\omega_7) + f^*\Theta_7$$

of the pullback of the angular cochain ω_7 on the universal 7-spherical fibration which is fiberwise the unit volume form on S^7 and which trivializes minus the universal Euler form:

$$\langle \omega_7, S^7 \rangle = 1\,, \qquad d\omega_7 = -\chi_8\,. \tag{31}$$

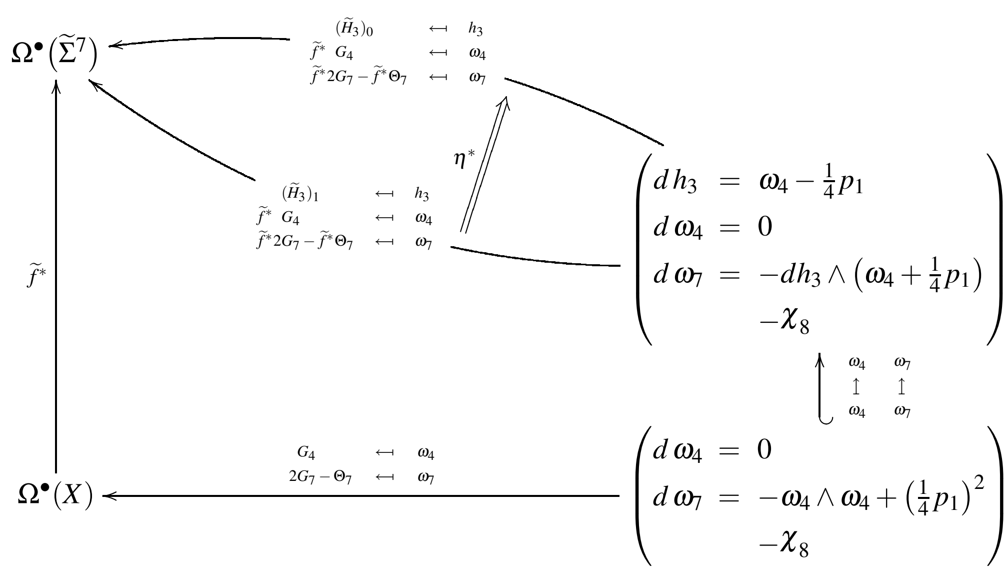
Proof. By [FSS19b, Lemma 3.19] the dgc-algebra model for the situation is as shown on the right in the following diagram, where the generator ω_7 in the top right satisfies (31) by [FSS19b, Prop. 2.5 (39)]:



(32)

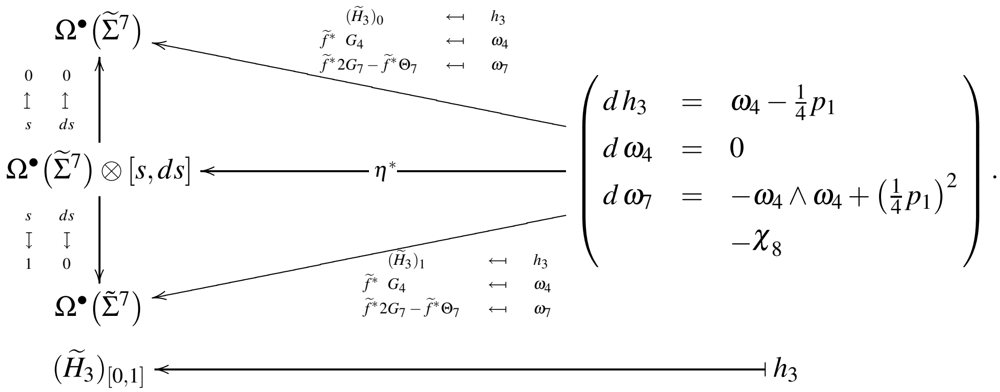
Here the right vertical morphism exhibits the minimal relative cofibration resolution of the parametrized quaternionic Hopf fibration, and hence any homotopy as on the left in (32) is represented by a homotopy η^* as shown on the right.

(i) With this cofibration resolution, the diagonal map on the right of (32) manifestly exhibits a choice of gauging $\widetilde{H}_3$ of $\widetilde{f}$. It just remains to see that this establishes a bijection on homotopy classes. But a homotopy of homotopy lifts is now of the form



(33)

Hence, since path objects of de Rham dgc-algebras are given by tensoring with $\mathbb{R}[s,ds] := \Omega^\bullet([0,1])$, this is equivalently a dgc-algebra homomorphism making the following diagram commute:



(34)

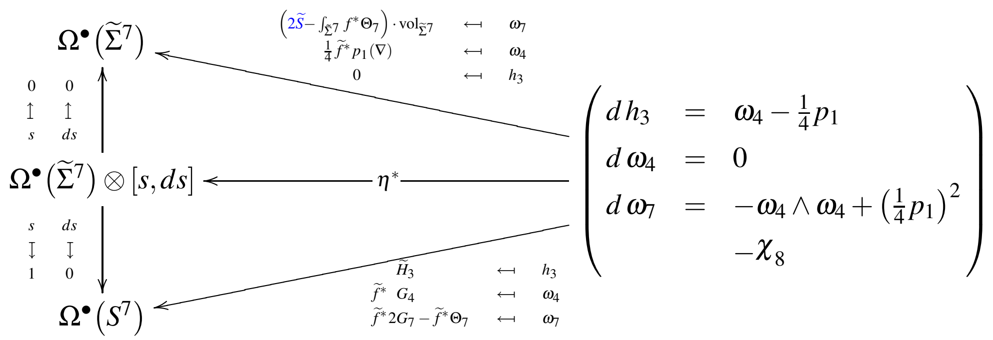
This diagram (34) exhibits manifestly the same data and conditions as in (13) for a homotopy of gaugings of a sigma-model field $\widetilde{f}$:

$$\big(\widetilde{f},(H_3)_0\big) \xRightarrow{\left(\mathrm{id},(\widetilde{H}_3)_{[0,1]}\right)} \big(\widetilde{f},(H_3)_1\big)\,.$$

(ii) Consider in the following any 7-form on $\widetilde{\Sigma}^7$ of unit volume:

$$\mathrm{vol}_{\widetilde{\Sigma}^7} \;\in\; \Omega^7\big(\widetilde{\Sigma}^7\big) \quad \text{such that} \quad \int\limits_{\widetilde{\Sigma}^7} \mathrm{vol}_{\widetilde{\Sigma}^7} \;=\; 1\,. \tag{35}$$

Since path objects of de Rham dgc-algebras are given by tensoring with $\mathbb{R}[s,ds] := \Omega^\bullet([0,1])$, the homotopy η^* on the right in (32) is a dgc-algebra homomorphism that makes the following diagram commute:



(36)

We claim that such an η^* is given by:

$$
\begin{array}{lcl}
s\widetilde{H}_3 & \longleftarrow & h_3 \\
ds\wedge\widetilde{H}_3 + s\cdot\widetilde{f}^*G_4 + (1-s)\tfrac{1}{4}\widetilde{f}^*p_1(\nabla) & \overset{\eta^*}{\longleftarrow} & \omega_4 \\
s\cdot\big(\widetilde{f}^*2G_7 - \widetilde{f}^*\Theta_7\big) + \Big(2\widetilde{S} - \int_{\widetilde{\Sigma}^7}\widetilde{f}^*\Theta_7\Big)\cdot(1-s)\cdot\mathrm{vol}_{\widetilde{\Sigma}^7} + s(1-s)\cdot\widetilde{H}_3\wedge\widetilde{f}^*\big(G_4 - \tfrac{1}{4}p_1(\nabla)\big) + ds\wedge Q_6 & \longleftarrow & \omega_7
\end{array}
$$

where $Q_6 \in \Omega^6\big(\widetilde{\Sigma}^7\big)$ is any differential form which satisfies

$$dQ_6 \;=\; \Big(\widetilde{H}_3\wedge\widetilde{f}^*\big(G_4 + \tfrac{1}{4}p_1(\nabla)\big) + \widetilde{f}^*\big(2G_7 - \Theta_7\big)\Big) - \Bigg(\overbrace{\int\limits_{\widetilde{\Sigma}^7}\Big(\widetilde{H}_3\wedge\widetilde{f}^*\big(G_4 + \tfrac{1}{4}p_1(\nabla)\big) + \widetilde{f}^*\big(2G_7 - \Theta_7\big)\Big)}^{=:2\widetilde{S} - \int_{\widetilde{\Sigma}^7}\widetilde{f}^*\Theta_7}\Bigg)\cdot\mathrm{vol}_{S^7}\,.$$

This exists by (35) and because cohomology classes of differential forms in top degree on compact connected manifolds are in bijection with the values of their integrals (e.g. [La15, Sec. 7.3, Thm. 7.5]). It is clear that

η^* thus defined satisfies the boundary conditions in (36), as a linear map. Hence it only remains to check that it is indeed a dgc-algebra homomorphism, in that it respects the differentials on the generators. This is verified by direct computation:

$$
\begin{aligned}
d\eta^*(h_3) &= d\big(s\widetilde{H}_3\big) \\
&= ds \wedge \widetilde{H}_3 + s \cdot \big(\widetilde{f}^* G_4 - \tfrac{1}{4}\widetilde{f}^* p_1(\nabla)\big) \\
&= \eta^*(\omega_4) - \tfrac{1}{4}\widetilde{f}^* p_1(\nabla) \\
&= \eta^*\big(dh_3\big). \\
d\eta^*(\omega_4) &= d\big(ds \wedge \widetilde{H}_3 + s \cdot \widetilde{f}^* G_4 + (1-s)\tfrac{1}{4}\widetilde{f}^* p_1(\nabla)\big) \\
&= -ds \wedge \big(\widetilde{f}^* G_4 - \tfrac{1}{4}\widetilde{f}^* p_1(\nabla)\big) + ds \wedge \big(\widetilde{f}^* G_4 - \tfrac{1}{4}\widetilde{f}^* p_1(\nabla)\big) \\
&= 0 \\
&= \eta^*\big(d\omega_4\big).
\end{aligned}
$$

$$
\begin{aligned}
d\eta^*(\omega_7) &= d\Big(s \cdot (\widetilde{f}^* 2G_7 - \widetilde{f}^*\Theta_7) + \Big(2\widetilde{S} - \int_{\tilde{\Sigma}^7} \tilde{f}^*\Theta_7\Big) \cdot (1-s) \cdot \mathrm{vol}_{S^7} + s(1-s) \cdot \widetilde{H}_3 \wedge \widetilde{f}^*\big(G_4 - \tfrac{1}{4}p_1(\nabla)\big) + ds \wedge Q_6\Big) \\
&= ds \wedge \big(\widetilde{f}^* 2G_7 - \widetilde{f}^*\Theta_7\big) - ds \wedge \Big(2\widetilde{S} - \int_{\tilde{\Sigma}^7} \tilde{f}^*\Theta_7\Big) \cdot \mathrm{vol}_{S^7} + ds \wedge \widetilde{H}_3 \wedge \widetilde{f}^*\big(G_4 - \tfrac{1}{4}p_1(\nabla)\big) \\
&\qquad - 2s \cdot ds \wedge \widetilde{H}_3 \wedge \widetilde{f}^*\big(G_4 - \tfrac{1}{4}p_1(\nabla)\big) - ds \wedge dQ_6 \\
&= ds \wedge \Big((\widetilde{f}^* 2G_7 - \widetilde{f}^*\Theta_7) + \widetilde{H}_3 \wedge \widetilde{f}^*\big(G_4 - \tfrac{1}{4}p_1(\nabla)\big) - 2\widetilde{S} \cdot \mathrm{vol}_{S^7} - dQ_6\Big) \\
&\quad + \underbrace{(2-2) \cdot ds \wedge \widetilde{H}_3 \wedge \tfrac{1}{4}\widetilde{f}^* p_1(\nabla)}_{\text{insert } 0} - 2s \cdot ds \wedge \widetilde{H}_3 \wedge \widetilde{f}^*\big(G_4 - \tfrac{1}{4}p_1(\nabla)\big) \\
&= ds \wedge \underbrace{\Big(\widetilde{f}^*(2G_7 - \Theta_7) + \widetilde{H}_3 \wedge \widetilde{f}^*\big(G_4 + \tfrac{1}{4}p_1(\nabla)\big) - \Big(2\widetilde{S} - \int_{\tilde{\Sigma}^7} \tilde{f}^*\Theta_7\Big) \cdot \mathrm{vol}_{S^7} - dQ_6\Big)}_{=0} \\
&\qquad - 2 \cdot ds \wedge \widetilde{H}_3 \wedge \widetilde{f}^*\big(s \cdot G_4 + (1-s)\tfrac{1}{4}p_1(\nabla)\big) \\
&= -\eta^*(\omega_4) \wedge \eta^*(\omega_4) \\
&= \eta^*\big(d\omega_7\big).
\end{aligned}
$$

Notice that in the last two steps we used the fact that all 8-forms on $\widetilde{\Sigma}^7$ vanish, hence that $\widetilde{f}^* G_4 \wedge \widetilde{f}^* G_4 = 0$, $\widetilde{f}^* p_1(\nabla) \wedge \widetilde{f}^* p_1(\nabla) = 0$, and $\widetilde{f}^* G_4 \wedge \widetilde{f}^* p_1(\nabla) = 0$. □

Remark 3.3 (Interpretation)**.** The appearance of the correction term $-\int_{\tilde{\Sigma}^7} f^*\Theta_7$ has a transparent interpretation in terms of dgca's. Namely, as we are assuming the Euler form on X comes with the trivialization $d\Theta_7 = \chi_8(\nabla)$ from equation (28), the lift $\widehat{c \circ \tilde{f}} \colon \tilde{\Sigma}^7 \to \big(\mathrm{Sp}(2) \cdot \mathrm{Sp}(1))\big)_{\mathbb{R}}$ is actually a lift $\widehat{c \circ \tilde{f}} \colon \tilde{\Sigma}^7 \to E^7_{\mathbb{R}}$, where $E^7_{\mathbb{R}}$ is the rational space obtained from $\big(\mathrm{Sp}(2) \cdot \mathrm{Sp}(1))\big)_{\mathbb{R}}$ by universally trivializing the closed element χ_8, i.e., the rational space whose dgca $\mathrm{CE}(\mathfrak{l}E^7_{\mathbb{R}})$ is obtained from $\mathrm{CE}(\mathfrak{l}(\mathrm{Sp}(2) \cdot \mathrm{Sp}(1))_{\mathbb{R}})$ by adding a single generator θ_7 and the relation $d\theta_7 = \chi_8$. Now, in $\mathrm{CE}(\mathfrak{l}E^7_{\mathbb{R}})$ the element $\omega_7 + \theta_7$ is a cocycle and so $\int_{\tilde{\Sigma}^7} \varphi^*(\omega_7 + \theta_7)$ is a homotopy invariant of $\varphi \colon \tilde{\Sigma}^7 \to E^7_{\mathbb{R}}$. As φ is constrained by $\varphi^*\theta_7 = f^*\Theta_7$, this homotopy invariant is

$$
\int_{\tilde{\Sigma}^7} \varphi^*\omega_7 + \int_{\tilde{\Sigma}^7} f^*\Theta_7 .
$$

Theorem 3.2 precisely shows that $\widehat{c \circ \tilde{f}}$ is in in the same homotopy class of a morphism φ with

$$
\varphi^*\omega_7 = \Big(2\widetilde{S}(\widetilde{f}, \widetilde{H}_3) - \int_{\tilde{\Sigma}^7} f^*\Theta_7\Big) \cdot \mathrm{vol}_{\tilde{\Sigma}^7},
$$

so the homotopy invariant of $\widehat{c \circ \widetilde{f}}$ is

$$\int_{\widetilde{\Sigma}^7} \left(2\widetilde{S}(\widetilde{f}, \widetilde{H}_3) - \int_{\widetilde{\Sigma}^7} f^* \Theta_7 \right) \cdot \mathrm{vol}_{\widetilde{\Sigma}^7} + \int_{\widetilde{\Sigma}^7} f^* \Theta_7 \; = \; 2\widetilde{S}(\widetilde{f}, \widetilde{H}_3) .$$

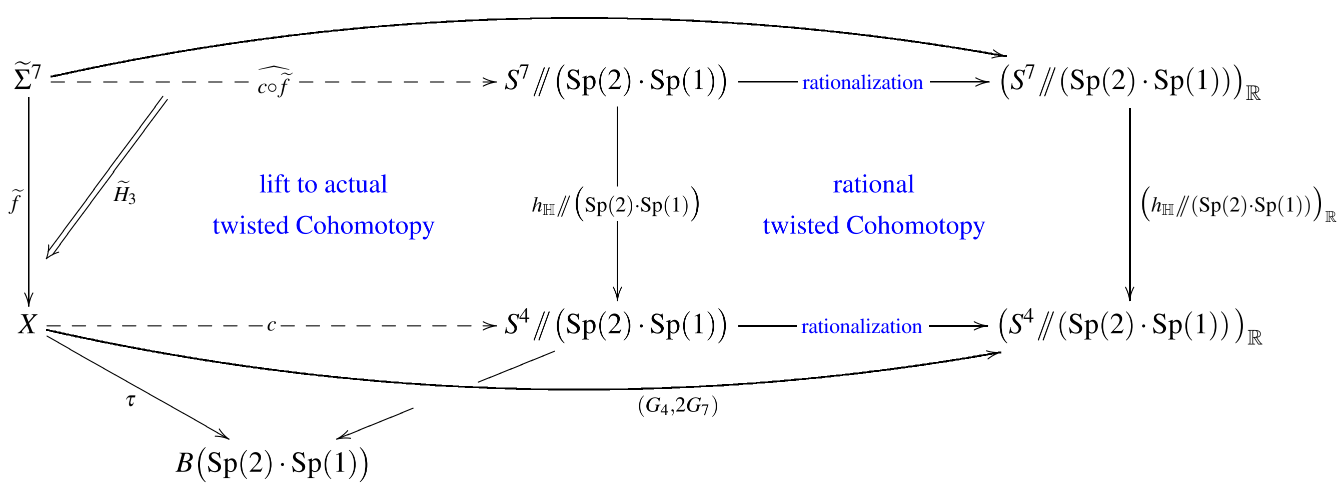
We will see this situation brought out also in integral cohomology, below in Theorem 4.6.

4 Hypothesis H implies M5 WZ anomaly cancellation

In view of the rational cohomotopical interpretation of background C-field (Remark 3.1) and of the 6d WZ anomaly functional (Theorem 3.2) it is natural to hypothesize that the topological sector of the background C-field should be required to be a cocycle in actual twisted Cohomotopy. This charge-quantization condition is called *Hypothesis H* in [FSS19b]; we recall the precise statement as Def. 4.1 below.

We observe in Prop. 4.4 that, under *Hypothesis H* and in the absence of topological twisting, Theorem 3.2 exhibits the M5 WZ anomaly functional as the homotopy Whitehead integral formula (see Remark 4.5 below) for the Hopf invariant (recalled in Def. 4.2 below). This proves the anomaly cancellation (2) for the special case of oriented differences of extended worldvolumes being the 7-sphere and for vanishing topological twist. Finally we establish a twisted/parametrized generalization of the integral Hopf invariant in Theorem 4.6, which proves the anomaly cancellation condition (2) generally.

Definition 4.1 (*Hypothesis H* [FSS19b])**.** In the situation of Def. 2.1 we say that:
(i) the background fields $(G_4, 2G_7)$ (7) *satisfy Hypothesis H* if they are classified as in [FSS19b, Def. 3.5] by an actual cocycle c in twisted Cohomotopy [FSS19b, Section 2.1], hence if their classifying map in rational twisted Cohomotopy from Remark 3.1 factors, up to homotopy, through the homotopy quotient of the 4-sphere canonically acted on by the central product group $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$;
(ii) the (extended or not) higher gauged sigma-model fields $(\widetilde{f}, \widetilde{H}_3)$ (11) *satisfy Hypothesis H* if the corresponding lift (32) through the rationalized parametrized quaternionic Hopf fibration, which classifies them by Theorem 3.2, factors by a lift through the actual parametrized quaternionic Hopf fibration $h_{\mathbb{H}}$:



(37)

Before analyzing the implications of *Hypothesis H*, we recall the definition of the Hopf invariant (e.g. [MT86, p. 33]):

Definition 4.2 (Hopf invariant)**.** For $k \in \mathbb{N}$ with $k \geq 1$, let

$$S^{4k-1} \xrightarrow{\phi} S^{2k} \tag{38}$$

be a continuous function between higher dimensional spheres, as shown. Then the homotopy cofiber space of ϕ has integral cohomology given by

$$H^p\big(\mathrm{cofib}(\phi),\mathbb{Z}\big) \;\simeq\; \left\{ \begin{array}{ccc} \mathbb{Z} & \mid & p \in \{2k, 4k\} \\ 0 & \mid & \text{otherwise} \end{array} \right.$$

Hence, with generators denoted

$$\omega_{2k} \;:=\; \pm 1 \in \mathbb{Z} \;\simeq\; H^{2k}\big(\mathrm{cofib}(\phi),\mathbb{Z}\big)\,, \quad \omega_{4k} \;:=\; \pm 1 \in \mathbb{Z} \;\simeq\; H^{4k}\big(\mathrm{cofib}(\phi),\mathbb{Z}\big),$$

there exists a unique integer

$$\mathrm{HI}(\phi) \;\in\; \mathbb{Z}\,, \qquad \text{s.t.} \qquad \omega_{2k} \cup \omega_{2k} \;=\; \mathrm{HI}(\phi)\cdot \omega_{4k} \tag{39}$$

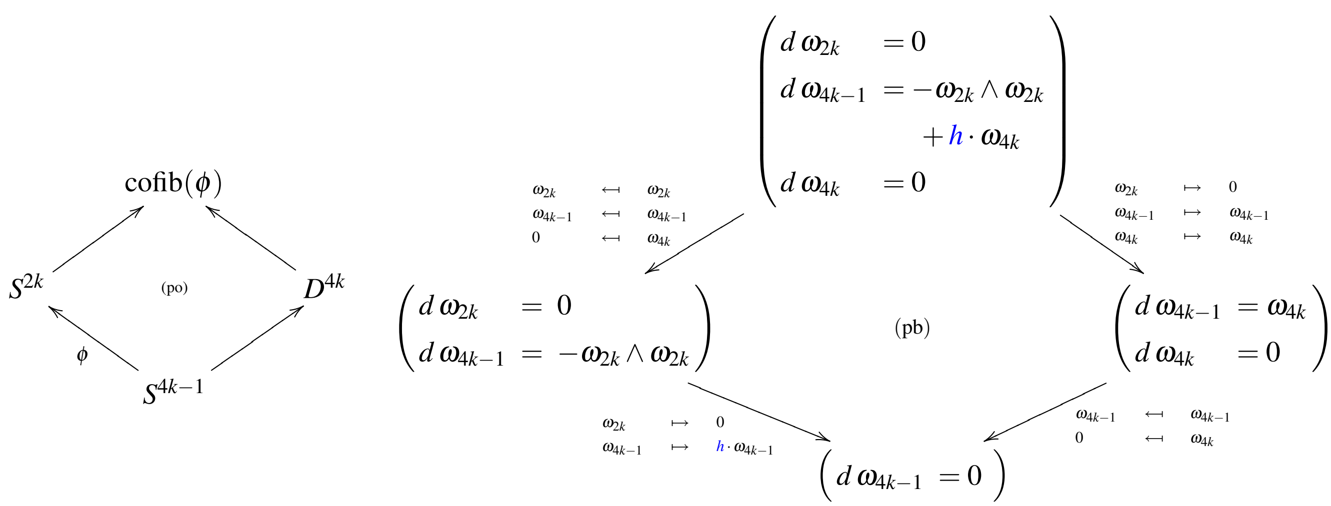
relating the cup-product square of the first to a multiple of the second. This integer is called the *Hopf invariant* $\mathrm{HI}(\phi)$ of ϕ. It depends on the choice of generators only up to a sign.

We make the following basic observation:

Lemma 4.3 (Recognition of Hopf invariants from Sullivan models)**.** *The unique coefficient in the minimal Sullivan model for a map ϕ of spheres as in* (38) *is the Hopf invariant* $\mathrm{HI}(\phi)$ *(Def. 4.2):*

$$\begin{array}{ccc} S^{4k-1} & \xrightarrow{\quad\quad\phi\quad\quad} & S^{2k} \\ \big(d\,\omega_{4k-1} = 0\big) & \xleftarrow{\;\begin{smallmatrix} \mathrm{HI}(\phi)\cdot\omega_{4k-1} \,\longleftarrow\!\shortmid\, \omega_{4k-1} \\ 0 \,\longleftarrow\!\shortmid\, \omega_{2n} \end{smallmatrix}\;} & \left(\begin{array}{lcl} d\,\omega_{4k-1} & = & -\omega_{2k}\wedge\omega_{2k} \\ d\,\omega_{4k} & = & 0 \end{array} \right). \end{array} \tag{40}$$

Proof. The homotopy cofiber is represented by the ordinary pushout of topological spaces as shown in the following diagram on the left. This is algebraically represented by the pullback of dgc-algebras as shown on the right:



One reads off the pullback dgc-algebra at the top by inspection, with the coefficient h as shown, inherited from the Sullivan model for ϕ in the bottom left. By the fact that Sullivan models compute the non-torsion cohomology groups, comparison with (39) shows that $h = \mathrm{HI}(\phi)$. ∎

Using this, we obtain the following corollary of Theorem 3.2:

Proposition 4.4 (Recovering the homotopy Whitehead formula)**.** *In the situation of Def. 2.1, consider the special case when:*
(i) *the background C-field* (7) *satisfies Hypothesis H (Def. 4.1);*
(ii) *the extended worldvolume is the 7-sphere* $\Sigma := \widetilde{\Sigma}^7 := S^7$ *(as in Lemma 2.2 and Example 22);*
(iii) *the* $\mathrm{Spin}(8)$*-bundle over* X *is trivial, as well as the* $\mathrm{Spin}(8)$*-connection* ∇*, and the trivial trivialization* $\Theta_7 = 0$ *of* $\chi_8(\nabla)$ (28) *is chosen.*

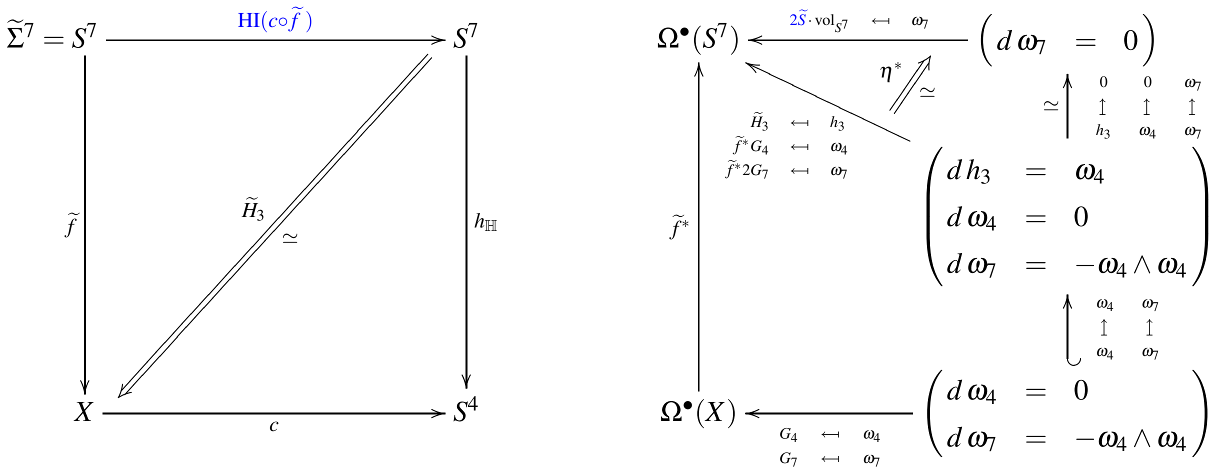
Then twice the WZ anomaly functional $2\widetilde{S}$ *(Def. 2.8, Lemma 2.9) is equal to the Hopf invariant* $\mathrm{HI}\big(c \circ \widetilde{f}\,\big)$ *(Def. 4.2) of the composite*

$$S^7 \xrightarrow{\;\widetilde{f}\;} X \xrightarrow{\;c\;} S^4$$

of the extended sigma-model field $\widetilde{f}$ (11) *with the (untwisted) Cohomotopy cocycle* c (37) *that classifies the background fields:*

$$2\widetilde{S}\big(\widetilde{f},\widetilde{H}_3\big) \;=\; \int_{S^7} \Big(\widetilde{H}_3 \wedge \widetilde{f}^* G_4 + \widetilde{f}^* 2G_4\Big) \;=\; \mathrm{HI}\big(c \circ \widetilde{f}\,\big) \;\in\; \mathbb{Z}\,. \tag{41}$$

Proof. Under the given assumption, the diagram (32) in Theorem 3.2 reduces to



(42)

This identifies the top horizontal map with the Hopf invariant, as shown, by Lemma 4.3. ☐

Remark 4.5 (Whitehead integral formulas in the literature)**.** The statement of Prop. 4.4 is essentially that of [Ha78, p. 17] [GM81, 14.5], the integrand being the *functional cup product*-expression of [St49], recalled as a *homotopy period*-expression in [SW08, Exmpl. 1.9]; but the proof as a special case of Theorem 3.2 is new and more conceptual. In the special case that $G_7 = 0$ (which is given if the classifying map $X \xrightarrow{c} S^4$ (37) is a smooth function) the statement of Prop. 4.4 further reduces to that of the classical *Whitehead integral formula* for the Hopf invariant [Wh47] (see [Ha78] [BT82, Prop. 17.22]).

The final theorem generalizes this situation to arbitrary oriented differences $\widetilde{\Sigma}^7$ of extended worldvolumes and to non-trivial topological twists:

Theorem 4.6 (6d WZ anomaly functional is integral)**.** *Hypothesis H (Def. 4.1) implies that twice the general 6d Wess-Zumino anomaly functional of the M5-brane (Def. 2.8, Lemma 2.9) takes values in the integers*

$$\begin{array}{ccccc} \pi_0\Big(\mathrm{Maps}^{\mathrm{ggd}}_{\mathrm{smth}}\big(\widetilde{\Sigma}^7, X\big)\Big) & \longrightarrow & \mathbb{Z} & \hookrightarrow & \mathbb{R} \\ \big[\widetilde{f},\widetilde{H}_3\big] & \longmapsto & \displaystyle\int_{\widetilde{\Sigma}^7} \Big(\widetilde{H}_3 \wedge \widetilde{f}^*\big(G_4 + \tfrac{1}{4}p_1(\nabla)\big) + \widetilde{f}^* 2G_7\Big) & & \end{array} \qquad (\text{composite: } 2\widetilde{S}) \tag{43}$$

and hence that the exponentiated action (2) *of the 6d WZ term of the M5-brane (Def. 2.5) is anomaly-free and well-defined.*

Proof. For readability we state the proof for $G := \mathrm{Sp}(2)$-structure; but the generalization to $\mathrm{Sp}(2)\times\mathrm{Sp}(1)$-structure is immediate, as it just amounts to observing, by the Künneth theorem, that (48) below still holds in this generality.

By Theorem 3.2, the anomaly term is characterized as a homotopy lift through the rationalization of the parametrized quaternionic Hopf fibration, and by *Hypothesis H* it is a lift through the actual parametrized Hopf fibration $h_{\mathbb{H}} /\!\!/ G$. Moreover, the chosen trivialization (28) of $\chi_8(\nabla)$ means that the twist τ factors through the homotopy fiber space

$$\widehat{BG} \xrightarrow{\mathrm{hofib}(\chi_8)} BG \xrightarrow{\chi_8} K(\mathbb{Z},8) \tag{44}$$

which carries a universal 7-cochain

$$d\theta_7 \;=\; \chi_8 \tag{45}$$

that pulls back under τ to the chosen 7-cochain (28) $\Theta_7 \leftarrow\!\shortmid \theta_7$. We write

$$\begin{array}{ccc} E^7 & \longrightarrow & S^7 /\!\!/ G \\ {\scriptstyle h_{\mathbb{H}}/\!\!/G}\downarrow & \text{(pb)} & \downarrow{\scriptstyle h_{\mathbb{H}}/\!\!/G} \\ E^4 & \longrightarrow & S^4 /\!\!/ G \\ \downarrow & \text{(pb)} & \downarrow{\scriptstyle \rho_{S^4}} \\ \widehat{BG} & \xrightarrow[\mathrm{hofib}(\chi_8)]{} & BG \end{array} \tag{46}$$

for the corresponding pullback of the parametrized Hopf fibration, which we will still denote by the same symbol, for brevity. The algebraic model for the left column in (46) is obtained from that for the right column, shown in (32), by adding the generator θ_7 and the differential relation (45).

This construction thus exhibits the anomaly polynomial of Def. 2.8 as the integral of a pullback of the following real cohomology 7-class on E^7:

$$h_3 \wedge \big(\omega_4 + \tfrac{1}{4}p_1\big) + \big(\omega_7 + \theta_7\big) \;\in\; H^7\big(E^7, \mathbb{R}\big)\,. \tag{47}$$

Therefore, it is sufficient to show that the rational class (47) is the rational image of an integral class. This is what we prove now.

First, consider the Gysin sequence (see e.g. [Sw75, 15.30]) for the universal 4-spherical fibration

$$S^4 \xrightarrow{\mathrm{hofib}(\rho_{S^4})} S^4 /\!\!/ G \xrightarrow{\rho_{S^4}} BG\,.$$

Since for the integral cohomology groups of the classifying space we have (see e.g. [Pi][Ka06, (12)])

$$H^\bullet\big(B\mathrm{Sp}(2), \mathbb{Z}\big) \;\simeq\; \mathbb{Z}\big[\tfrac{1}{2}p_1, \chi_8\big] \qquad \text{are non-torsion groups concentrated in even degrees}\,, \tag{48}$$

the 5-class controlling this Gysin sequence vanishes, and so it breaks up into short exact sequences of the form

$$0 \longrightarrow H^\bullet\big(BG,\mathbb{Z}\big) \xrightarrow{\rho_{S^4}^*} H^\bullet\big(S^4 /\!\!/ G, \mathbb{Z}\big) \xrightarrow{\int_{S^4}} H^{\bullet-4}\big(BG,\mathbb{Z}\big) \longrightarrow 0\,. \tag{49}$$

Since, moreover, the integral cohomology groups (48) have no torsion, these short exact sequences imply that also

$$H^\bullet\big(S^4 /\!\!/ G, \mathbb{Z}\big) \qquad \text{are non-torsion groups}\,. \tag{50}$$

Now observe, by [FSS19b, Prop. 3.13], that

$$\Gamma_4^{\mathrm{int}} \;:=\; \omega_4 + \tfrac{1}{4}p_1 \;\in\; H^4\big(S^4 /\!\!/ BG, \mathbb{Z}\big) \longrightarrow H^4\big(S^4 /\!\!/ BG, \mathbb{R}\big) \tag{51}$$

is an integral class, being the universal integral shifted C-field flux density (9). Hence, by (50), the rational trivialization from (32)

$$\begin{aligned} d\omega_7 &= -\omega_4 \wedge \omega_4 + \left(\tfrac{1}{4}p_1\right)^2 - \chi_8 \\ &= -\left(\Gamma_4^{\mathrm{int}} \wedge \Gamma_4^{\mathrm{int}} - \tfrac{1}{2}p_1 \wedge \Gamma_4^{\mathrm{int}}\right) - \chi_8 \end{aligned}$$

implies that also this integral cohomology class vanishes:

$$\left[\Gamma_4^{\mathrm{int}} \cup \Gamma_4^{\mathrm{int}} - \tfrac{1}{2}p_1 \cup \Gamma_4^{\mathrm{int}} + \chi_8\right] \;=\; 0 \;\in\; H^8\left(S^4 /\!\!/ G, \mathbb{Z}\right). \tag{52}$$

Consider next the integral Gysin sequence corresponding to the 3-spherical fibration which is the parametrized quaternionic Hopf fibration:

$$S^3 \xrightarrow{\mathrm{hofib}(h_{\mathbb{H}} /\!\!/ G)} S^7 /\!\!/ G \xrightarrow{h_{\mathbb{H}} /\!\!/ G} S^4 /\!\!/ G\,. \tag{53}$$

Since, rationally, $S^7 /\!\!/ G$ is obtained from $S^4 /\!\!/ G$ by adding the relation

$$\begin{aligned} dh_3 &= \omega_4 - \tfrac{1}{4}p_1 \\ &= \Gamma_4^{\mathrm{int}} - \tfrac{1}{2}p_1 \end{aligned} \tag{54}$$

by (32), it follows from [FSS19b, Prop. 2.5 (44)] that $\omega_4 - \frac{1}{4}p_1$ is the rational image of the integral Euler class of (53). Consequently,

$$\Gamma_4^{\mathrm{int}} - \tfrac{1}{2}p_1 \;\in\; H^4\left(S^4 /\!\!/ G, \mathbb{Z}\right) \tag{55}$$

is the integral 4-class controlling the Gysin sequence of (53), which therefore reads:

$$\cdots \longrightarrow H^7\left(S^4 /\!\!/ G, \mathbb{Z}\right) \xrightarrow{(h_{\mathbb{H}} /\!\!/ G)^*} H^7\left(S^7 /\!\!/ G, \mathbb{Z}\right) \xrightarrow{\int_{S^3}} H^4\left(S^4 /\!\!/ G, \mathbb{Z}\right) \xrightarrow{\cup(\Gamma_4^{\mathrm{int}} - \frac{1}{2}p_1)} H^8\left(S^4 /\!\!/ G, \mathbb{Z}\right) \longrightarrow \cdots \tag{56}$$

Now consider pulling back this situation along the homotopy fiber of χ_8 (44) to yield the sequence of spherical fibrations (46). After this pullback the Euler class from (52) disappears, in that we obtain vanishing of the class

$$\left[\Gamma_4^{\mathrm{int}} \cup \Gamma_4^{\mathrm{int}} - \tfrac{1}{2}p_1(TX) \cup \Gamma_4^{\mathrm{int}}\right] \;=\; 0 \in H^8\left(E^4, \mathbb{Z}\right). \tag{57}$$

With this, the integral Gysin sequence of the pullback 3-spherical fibration from the left of (46)

$$S^3 \xrightarrow{\mathrm{hofib}(h_{\mathbb{H}} /\!\!/ G)} E^7 \xrightarrow{h_{\mathbb{H}} /\!\!/ G} E^4$$

is found to be

$$\begin{array}{ccccccccc} \cdots \longrightarrow H^7\left(E^4, \mathbb{Z}\right) & \xrightarrow{(h_{\mathbb{H}} /\!\!/ G)^*} & H^7\left(E^7, \mathbb{Z}\right) & \xrightarrow{\int_{S^3}} & H^4\left(E^4, \mathbb{Z}\right) & \xrightarrow{\cup\left(\Gamma_4^{\mathrm{int}} - \frac{1}{2}p_1(TX)\right)} & H^8\left(E^4, \mathbb{Z}\right) & \longrightarrow \cdots \\ & & 2\widetilde{\mathbf{S}} & \longmapsto & \Gamma_4^{\mathrm{int}} & \longmapsto & \underbrace{\Gamma_4^{\mathrm{int}} \cup \Gamma_4^{\mathrm{int}} - \tfrac{1}{2}p_1(TX) \cup \Gamma_4^{\mathrm{int}}}_{=0} & \end{array} \tag{58}$$

Here, in the bottom row, we observe that the image of Γ_4^{int} (51) under forming cup product with the 4-class (55) is just the vanishing class (57), which by exactness of the Gysin sequence implies that there exists an integral 7-class

$$2\widetilde{\mathbf{S}} \;\in\; H^7\left(E^7, \mathbb{Z}\right) \tag{59}$$

whose integration over the S^3-fibers is Γ_4^{int}, as shown. Since, by (54) and [FSS19b, Prop. 2.5 (45)] the fiberwise volume form is h_3, this is, rationally, the same fiber integration as that of (47), which by the exactness of the Gysin sequence (58), now with rational coefficients

$$\begin{array}{ccccccccc} \cdots \longrightarrow H^7\left(E^4, \mathbb{R}\right) & \xrightarrow{(h_{\mathbb{H}} /\!\!/ G)^*} & H^7\left(E^7, \mathbb{R}\right) & \xrightarrow{\int_{S^3}} & H^4\left(E^4, \mathbb{R}\right) & \xrightarrow{\cup\left(\Gamma_4^{\mathrm{int}} - \frac{1}{2}p_1(TX)\right)} & H^8\left(E^4, \mathbb{R}\right) & \longrightarrow \cdots \\ D & \longmapsto & \left(\begin{smallmatrix} h_3 \wedge (\omega_4 + \frac{1}{4}p_1) + \omega_7 + \theta_7 \\ -2\widetilde{\mathbf{S}} \end{smallmatrix}\right) & \longmapsto & 0 & & & \end{array} \tag{60}$$

implies that the integral class $2\widetilde{\mathbf{S}}$ (59) differs from the rational class (47) by a 7-class D pulled back from E_4, as shown. But by (48) and (32) there is no non-trivial 7-class on E^4. Hence the equality $2\widetilde{\mathbf{S}} = h_3 \wedge \Gamma_4^{\mathrm{int}} + (\omega_7 + \theta_7)$ holds, and so the anomaly integrand (47) is indeed the rational image of an integral class (59) and hence has itself integral periods:

$$
\begin{array}{ccccccc}
2\widetilde{\mathbf{S}} & & H^7\big(E^7,\mathbb{Z}\big) & \xrightarrow{\left(\widetilde{c\circ f}\right)^*} & H^7\big(\widetilde{\Sigma}^7,\mathbb{Z}\big) & \xrightarrow{\int_{\widetilde{\Sigma}^7}} & \mathbb{Z} \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
 & & H^7\big(E^7,\mathbb{R}\big) & \xrightarrow{\left(\widetilde{c\circ f}\right)^*} & H^7\big(\widetilde{\Sigma}^7,\mathbb{R}\big) & \xrightarrow{\int_{\widetilde{\Sigma}^7}} & \mathbb{R} \\
\begin{smallmatrix} h_3\wedge(\omega_4+\frac{1}{4}p_1) \\ +\omega_7+\theta_7 \end{smallmatrix} & \longmapsto & & & \begin{smallmatrix} \widetilde{H}_3\wedge\widetilde{f}^*G_4^{\mathrm{int}} \\ +\widetilde{f}^*2G_7 \end{smallmatrix} & \longmapsto & 2\widetilde{S}_{\mathrm{WZ}}^{\mathrm{M5}}\big(\widetilde{f},\widetilde{H}_3\big) = \widetilde{S}_{\mathrm{WZ}}^{1\mathrm{M5}}\big(\widetilde{f},\widetilde{H}_3\big).
\end{array}
\tag{61}
$$

□

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Domenico Fiorenza, *Dipartimento di Matematica, La Sapienza Universita di Roma, Piazzale Aldo Moro 2, 00185 Rome, Italy.*

Hisham Sati, *Mathematics, Division of Science, New York University Abu Dhabi, UAE.*

Urs Schreiber, *Mathematics, Division of Science, New York University Abu Dhabi, UAE, on leave from Czech Academy of Science, Prague.*